High-energy QCD and Wilson lines

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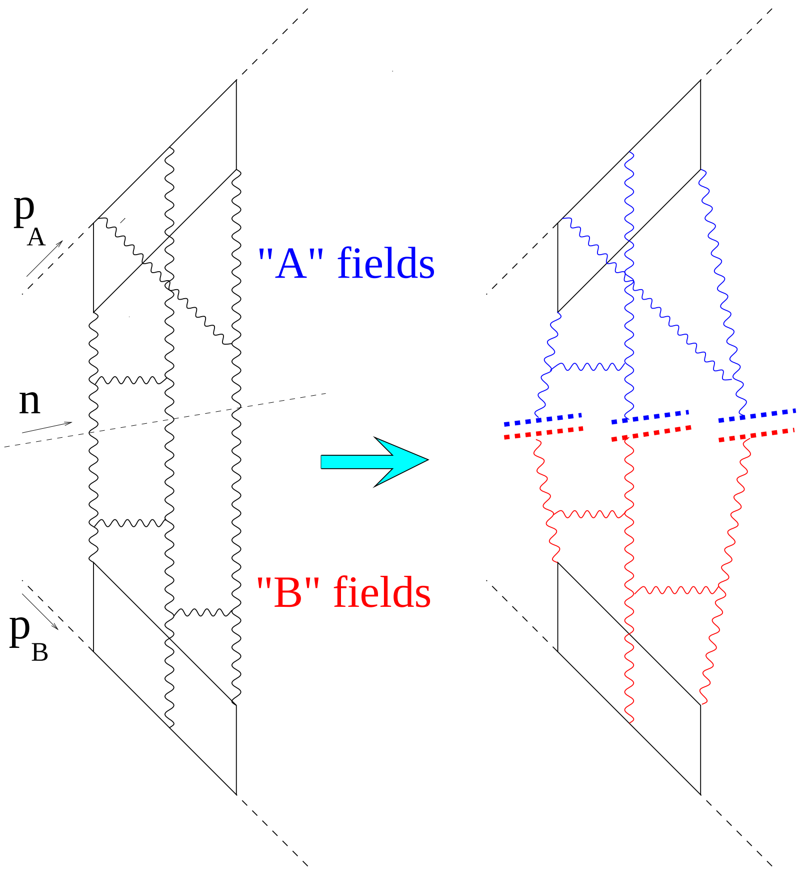
Plan

- Wilson lines as effective degrees of freedom in high-energy processes.
- A search for the 2+1 effective theory.
- Small-x DIS as an evolution of Wilson lines
- Non-linear evolution equation
- Effective field theory for the small-x evolution.
- DIS from heavy nuclei: saturation
- High-energy effective action from scattering of QCD shock waves
- Conclusions and outlook

Wilson lines $\equiv$ effective degrees of freedom



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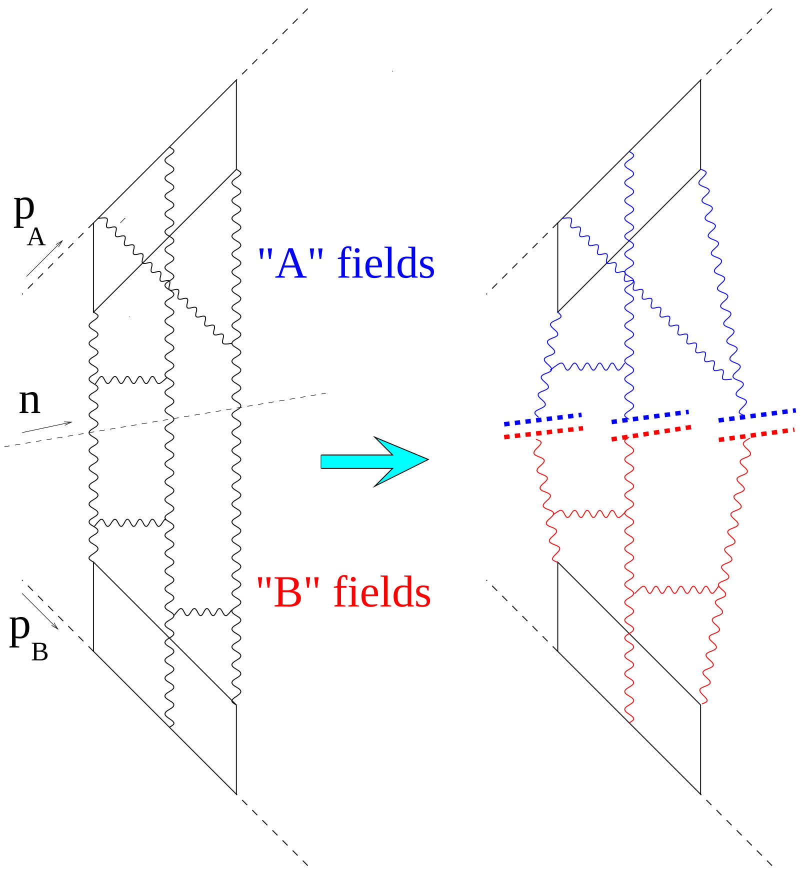
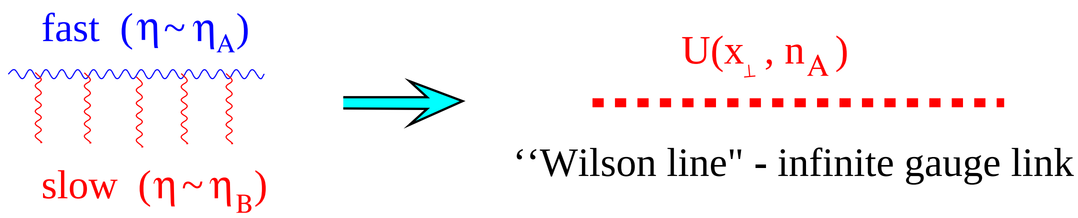
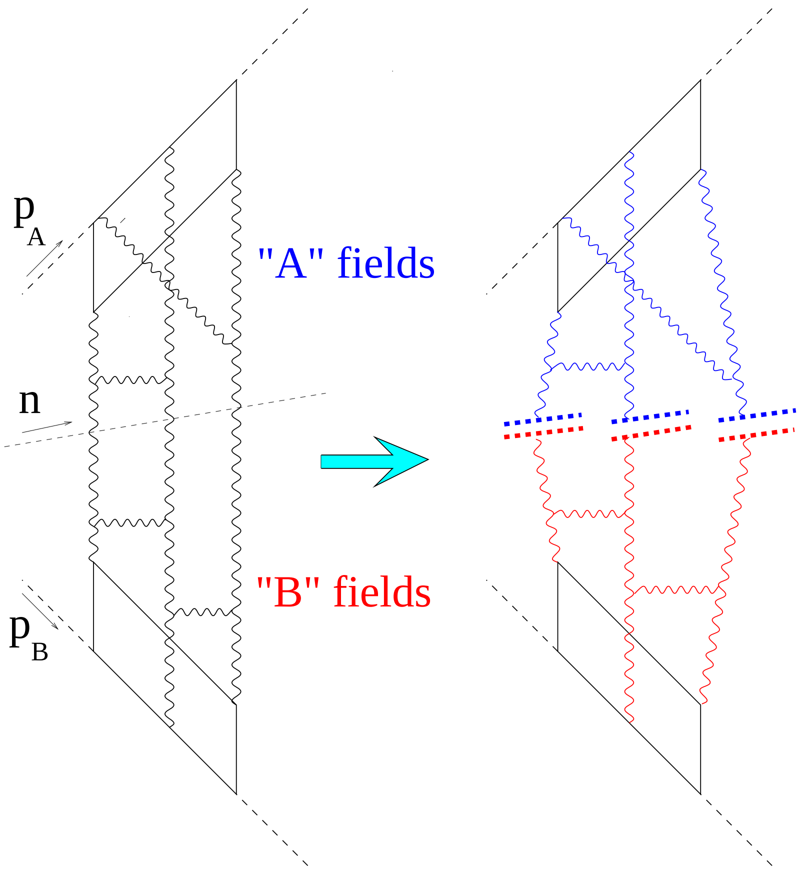


Particles from clusters with different rapidity
perceive each other as Wilson lines.
In the target frame



Wilson lines ≡ effective degrees of freedom



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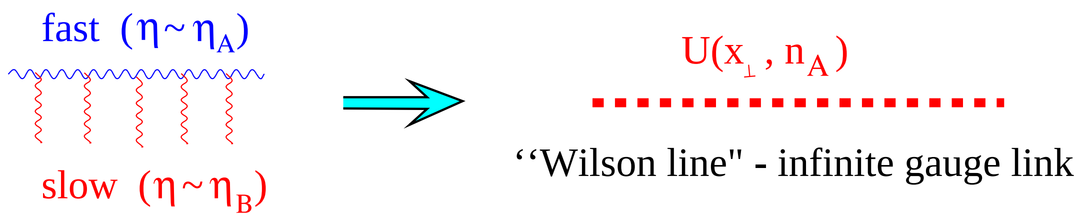
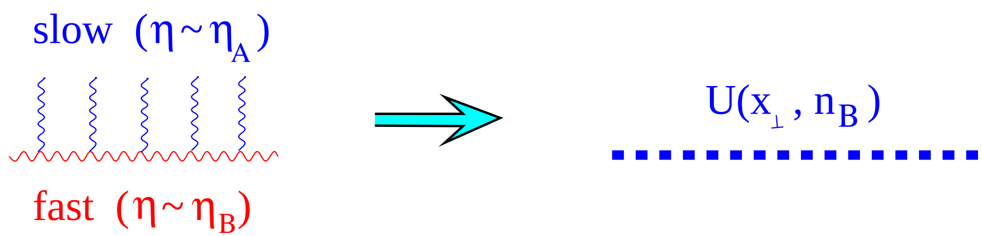


In the spectator frame



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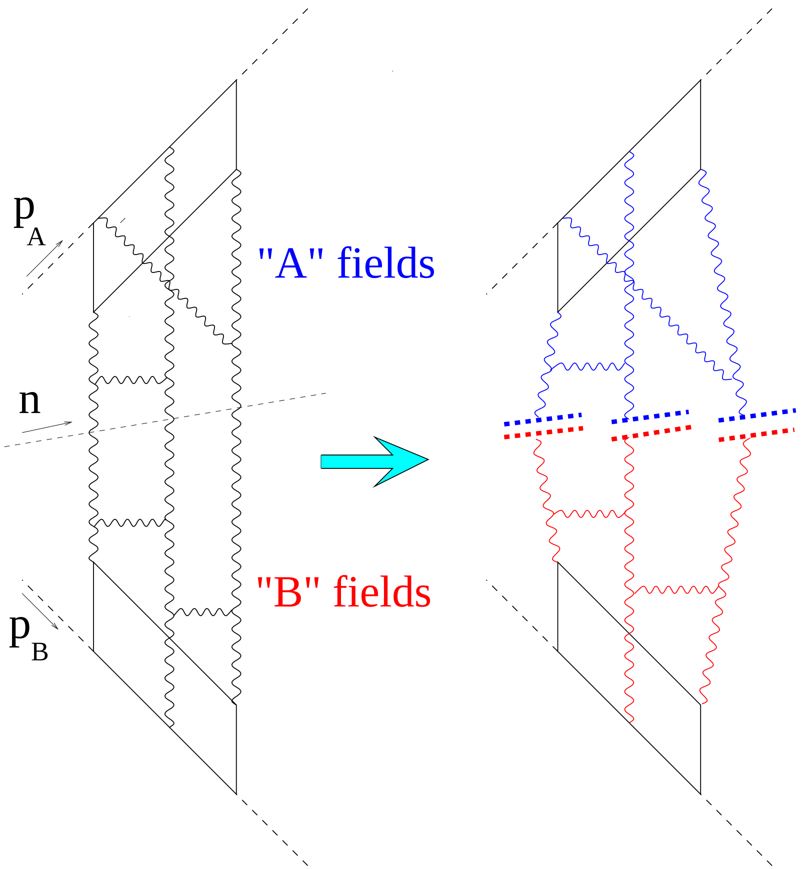

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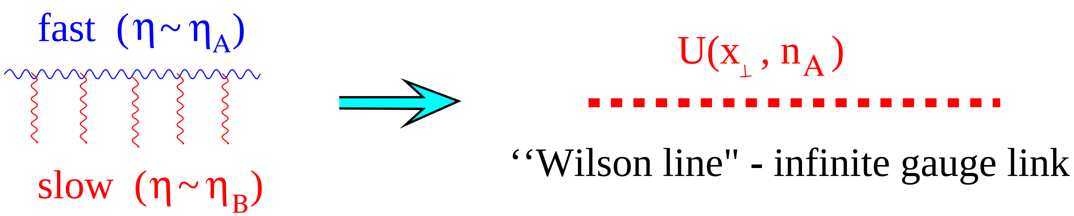

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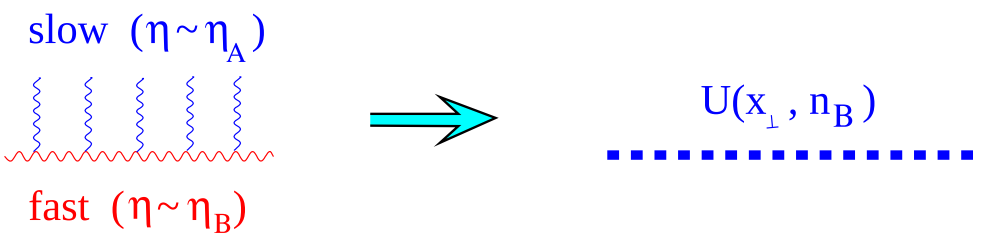


$\Rightarrow U(x_\perp, \eta)$ ($\eta \equiv$ slope) is the relevant degree of freedom for high-energy scattering.

$$U(x_\perp, \eta) \equiv [-\infty n + x_\perp, \infty n + x_\perp]$$

$$[x, y] \equiv P e^{ig\int_0^1 du (x-y)_\mu A^\mu(ux+(1-u)y)}$$

A search for the 2+1 effective theory

Wanted:

$$\langle e^{i\rho_i^A U_i(\eta_A)} e^{i\rho_i^B U_i(\eta_B)} \rangle_{\text{QCD}} \stackrel{s\to\infty}{\longrightarrow}$$

$$\int \mathcal{D}U(z_\perp, \eta)\ e^{i\rho_i^A U_i(\eta_A)} e^{i\rho_i^B U_i(\eta_B)} \exp\left\{ i \int_{\eta_B}^{\eta_A} d\eta \int d^2 z_\perp \mathcal{L}(U) \right\}$$

$U_i \equiv U^\dagger \frac{i}{g} \partial_i U$ and $e^{i\rho_i^A U_i(\eta_A)} \equiv e^{\int d^2 z \rho_i^A(z_\perp) U_i(z_\perp, \eta_A)}$

$\rho_i^A(z_\perp)$ and $\rho_i^B(z_\perp)$ - sources for the Wilson-line operators ($\rho_i \equiv \rho^\dagger \frac{i}{g} \partial_i \rho$, $\rho \in SU_3$).

How to approach this goal?

- pQCD
- sQCD (s for semiclassical)
- symmetries?

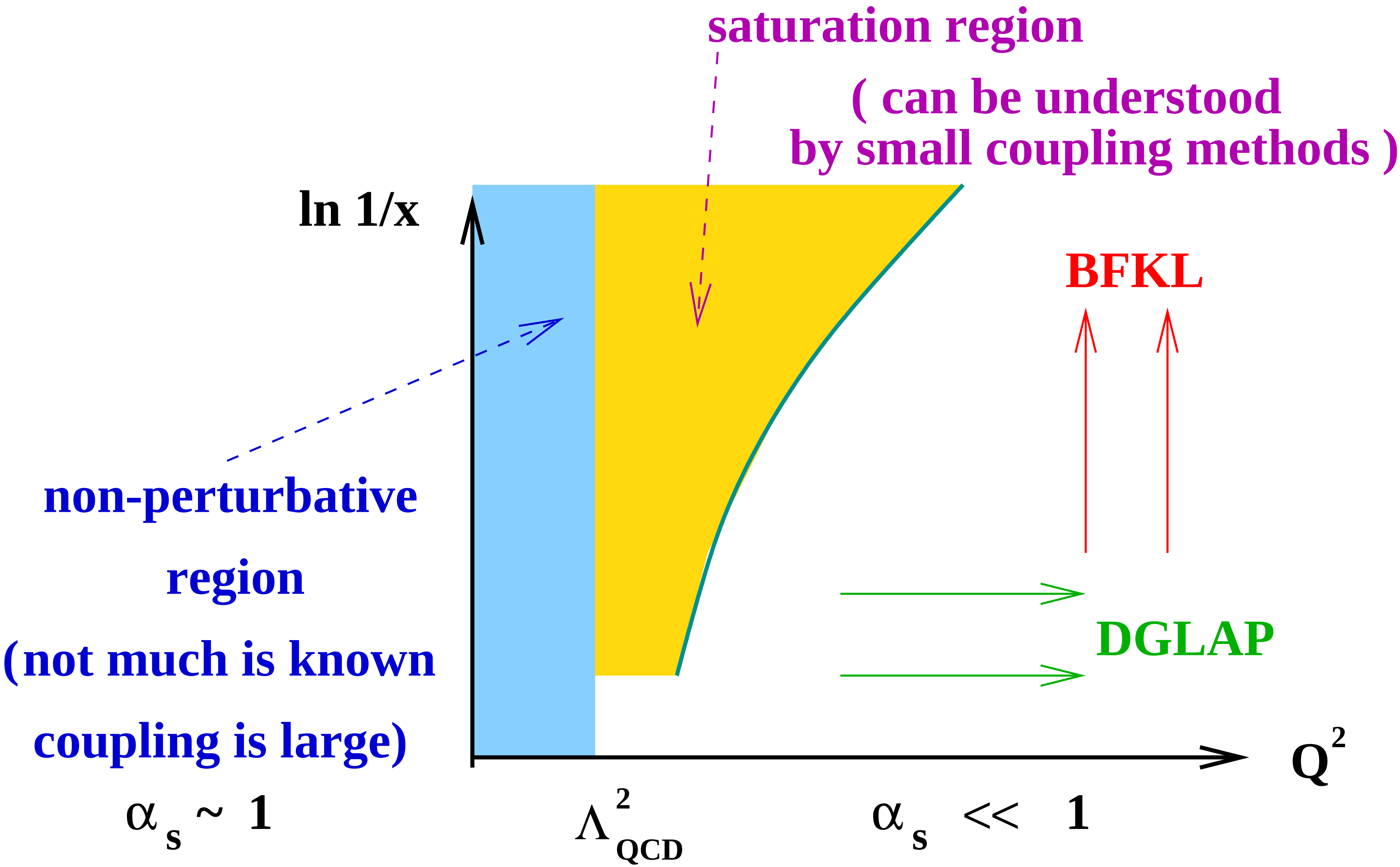
Power counting for sources in the LLA

In pQCD, the parameters of the expansion are g^2, $g^2\eta$, and $g^2[\rho_i^A,\rho_j^B]\eta$ ($\eta \equiv \ln s$).
LLA: $g^2 \ll 1,\ g^2\eta \sim 1$ (NLO is $\sim g^4\eta \ll 1$).

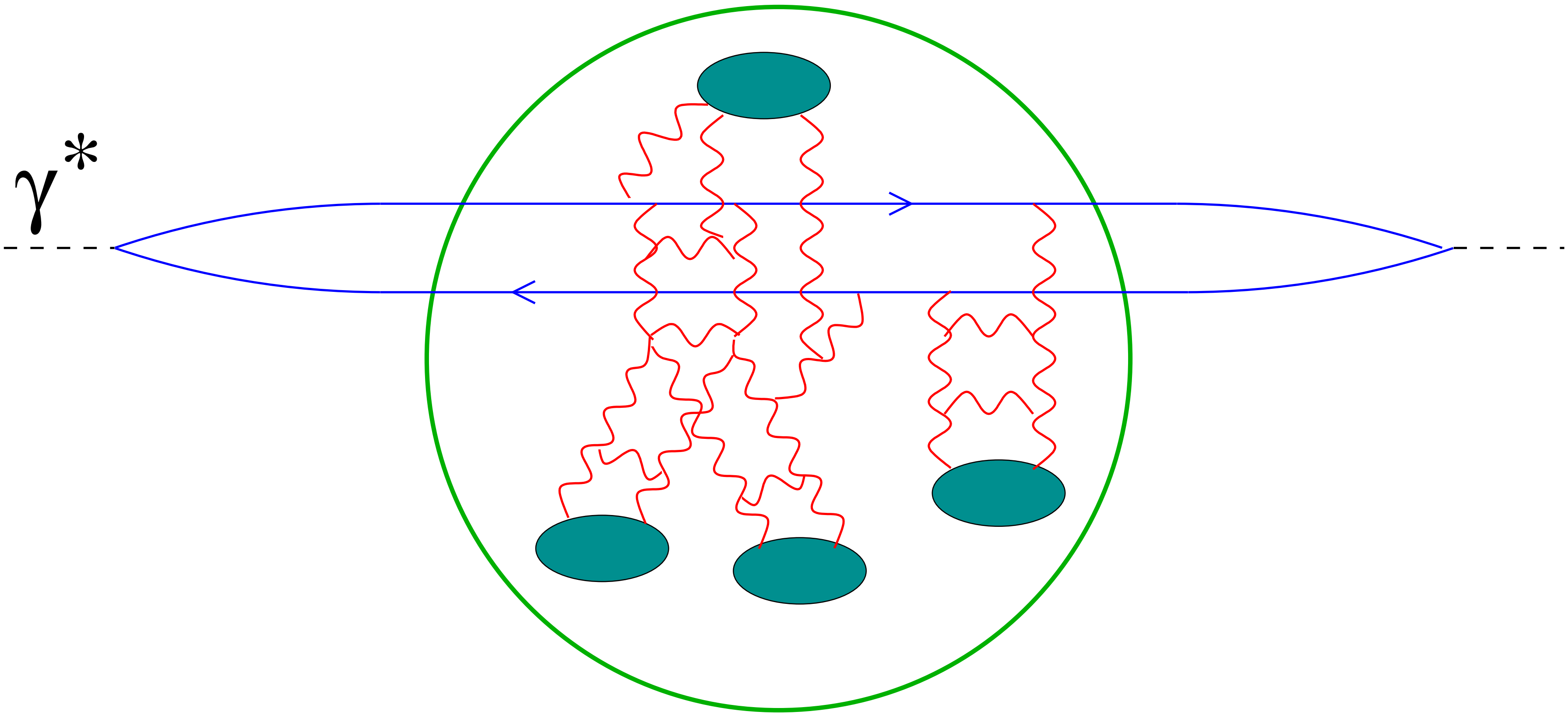
3 regimes:

- $\rho_i \sim 1$ ($\gamma^*\gamma^*$ scattering) $\xrightarrow{\mathrm{LLA}}$ BFKL pomeron
- $\rho_i \sim \frac{1}{g} \gg 1$ ($\rho \sim A^{1/3}$ for the heavy-ion collisions) $\Rightarrow g^2\rho_i^A\rho_j^B\eta \gg 1$ $\Rightarrow g^4\rho_i^A\rho_j^B\eta \not\ll 1 \Rightarrow$ beyond the LLA.
  Best hope for this region is sQCD.
- $\rho_i^A \sim 1$, $\rho_i^B \sim \frac{1}{g}$ (DIS from the heavy nuclei) $\Rightarrow g^2\rho_i^A\rho_j^B\eta \gg 1$ but $g^4\rho_i^A\rho_j^B\eta \ll 1 \Rightarrow$ LLA is OK.

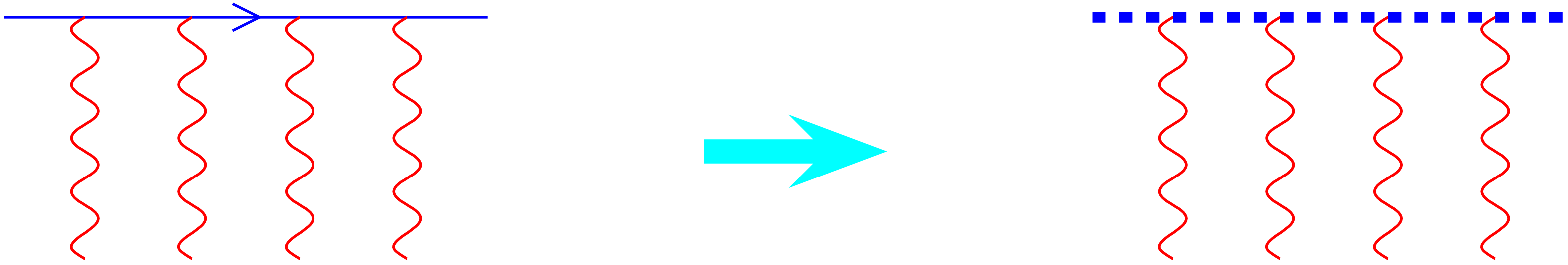
“Phase diagram” for DIS



Small-x DIS from the nucleus

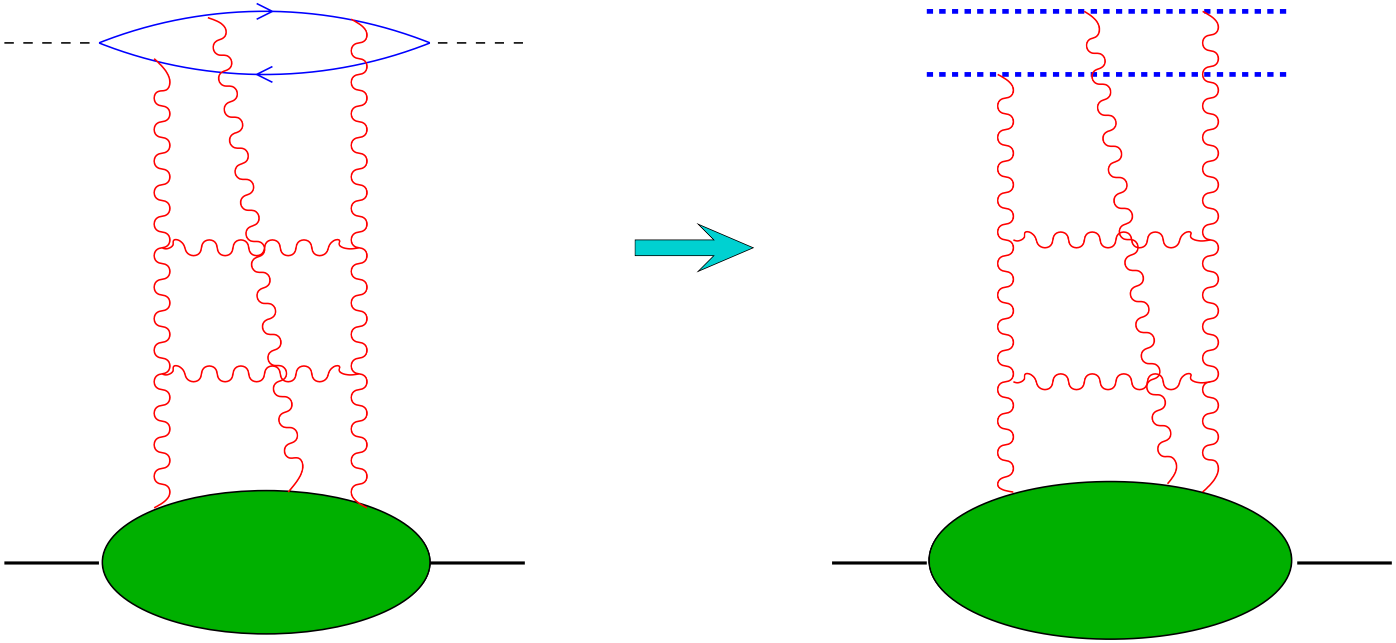


Fast quark moves along the straight line $\Rightarrow$



quark propagator reduces to the Wilson line collinear to quark's velocity

At high energies, the amplitude of $\gamma^* A \to \gamma^* A$ scattering reduces to the matrix element of a two-Wilson-line operator ("color dipole").



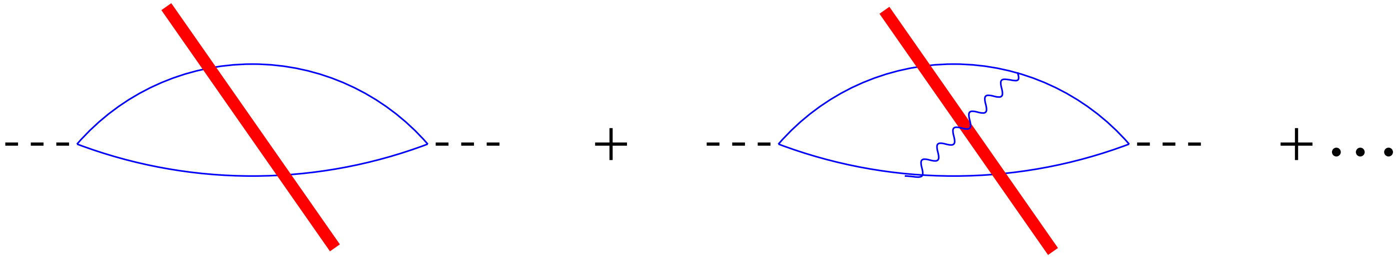
$$A(s) = \int \frac{d^2k_\perp}{4\pi^2} I(k_\perp) \langle A|\mathrm{Tr}\{U^{\eta_A}(k_\perp) U^{\dagger\eta_A}(-k_\perp)\}|A\rangle + \ldots$$

Energy dependence of the amplitude $A(s)$ is determined by the dependence of the Wilson lines on the rapidity η_A defined by the slope of the line.

In the spectator frame

Technically, it is more convenient to use the spectator frame

High-speed nucleus shrinks to a “pancake” $\Rightarrow$



= Feynman diagrams in a shock-wave background.

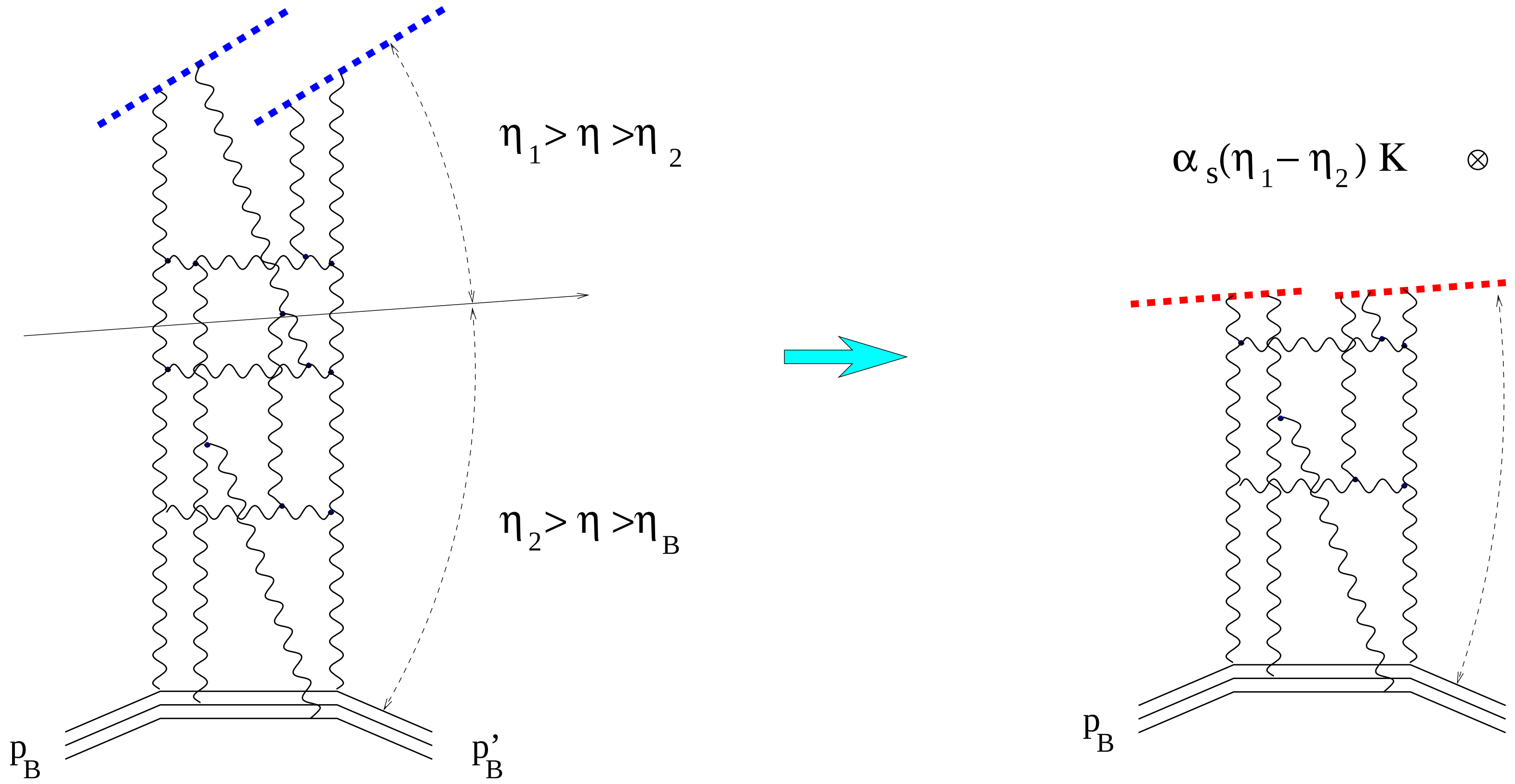
Quarks (and gluons) do not have time to deviate in the transverse direction $\Rightarrow$

$$\int d^4x d^4z\, e^{-ip_A\cdot x}\langle T\{j_A(x+z)j'_A(z)\}\rangle_A$$

$$= \int \frac{d^2k_\perp}{4\pi^2} I^A(k_\perp)\mathrm{Tr}\{U^{\eta_A}(k_\perp)U^{\dagger\eta_A}(-k_\perp)\} + \ldots \;\Rightarrow$$

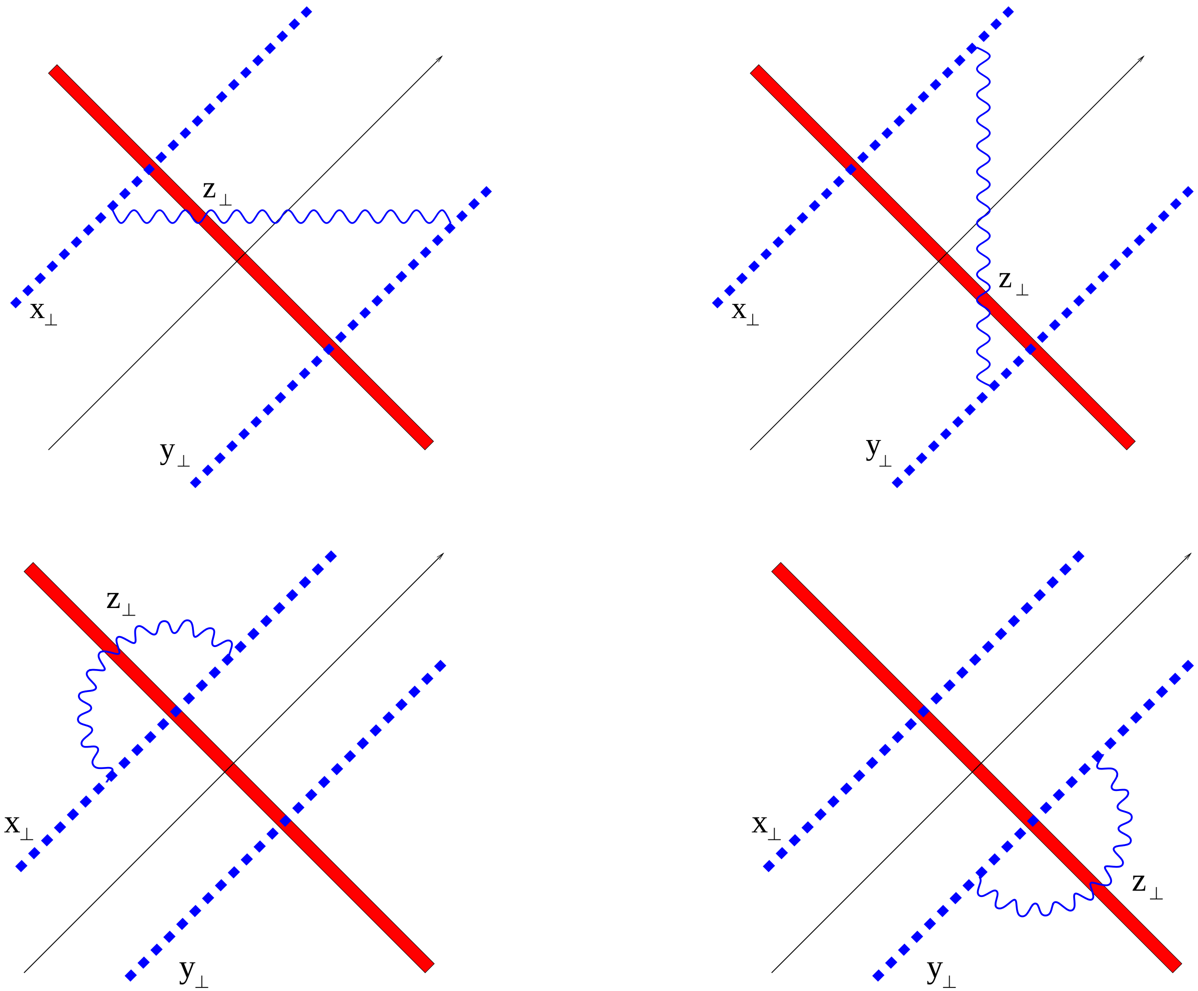
$$A(s) = \int \frac{d^2k_\perp}{4\pi^2} I(k_\perp)\langle A|\mathrm{Tr}\{U^{\eta_A}(k_\perp)U^{\dagger\eta_A}(-k_\perp)\}|A\rangle + \ldots$$

To get the evolution equation, consider the dipole with the slope $\parallel\ \eta_1$ and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with the slope corresponding to η_2).



In the frame $\parallel\ \eta_1$ the gluons with $\eta < \eta_2$ are seen as a pancake $\Rightarrow$

One-loop evolution



The structure is

[$x \longrightarrow z$: free propagation]
$\times$
$[U^{ab}(z_\perp)$ - instantaneous interaction with the $\eta < \eta_2$ shock wave$]$
$\times$
$[z \longrightarrow y$: free propagation]

$$U_z^{ab} = \mathrm{Tr}\{t^a U_z t^b U_z^\dagger\} \;\Rightarrow (U_x U_y^\dagger)^{\eta_1} = (U_x U_y^\dagger)^{\eta_1} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

$\Rightarrow$ non-linear evolution

$$\frac{\partial}{\partial\eta}\mathcal{U}(x_\perp, y_\perp) \ =$$

$$-\frac{\bar{\alpha}}{4\pi}\int dz_\perp \frac{(\vec{x}-\vec{y})^2_\perp}{(\vec{x}_\perp - \vec{z}_\perp)^2(\vec{z}_\perp - \vec{y}_\perp)^2}$$

$$\times \left\{\mathcal{U}(x_\perp, z_\perp) + \mathcal{U}(z_\perp, y_\perp) - \mathcal{U}(x_\perp, y_\perp) - \mathcal{U}(x_\perp, z_\perp)\mathcal{U}(z_\perp, y_\perp)\right\}$$

$$\mathcal{U}(x_\perp, y_\perp) \equiv \frac{1}{N_c}(\mathrm{Tr}\{U(x_\perp)U^\dagger(y_\perp)\} - N_c)$$

LLA for DIS in pQCD $\Rightarrow$ BFKL
LLA for DIS in sQCD $\Rightarrow$ NL eqn
(s for semiclassical)

$$\frac{\partial}{\partial\eta}\mathcal{U}(x_\perp, y_\perp) =$$

$$-\frac{\bar{\alpha}}{4\pi}\int dz_\perp \frac{(\vec{x}-\vec{y})_\perp^2}{(\vec{x}_\perp-\vec{z}_\perp)^2(\vec{z}_\perp-\vec{y}_\perp)^2}$$

$$\times\left\{\underbrace{\mathcal{U}(x_\perp, z_\perp)+\mathcal{U}(z_\perp, y_\perp)-\mathcal{U}(x_\perp, y_\perp)}-\mathcal{U}(x_\perp, z_\perp)\mathcal{U}(z_\perp, y_\perp)\right\}$$

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$$\mathcal{U}(x_\perp, y_\perp)\equiv\frac{1}{N_c}(\mathrm{Tr}\{U(x_\perp)U^\dagger(y_\perp)\}-N_c)$$

LLA for DIS in pQCD $\Rightarrow$ BFKL
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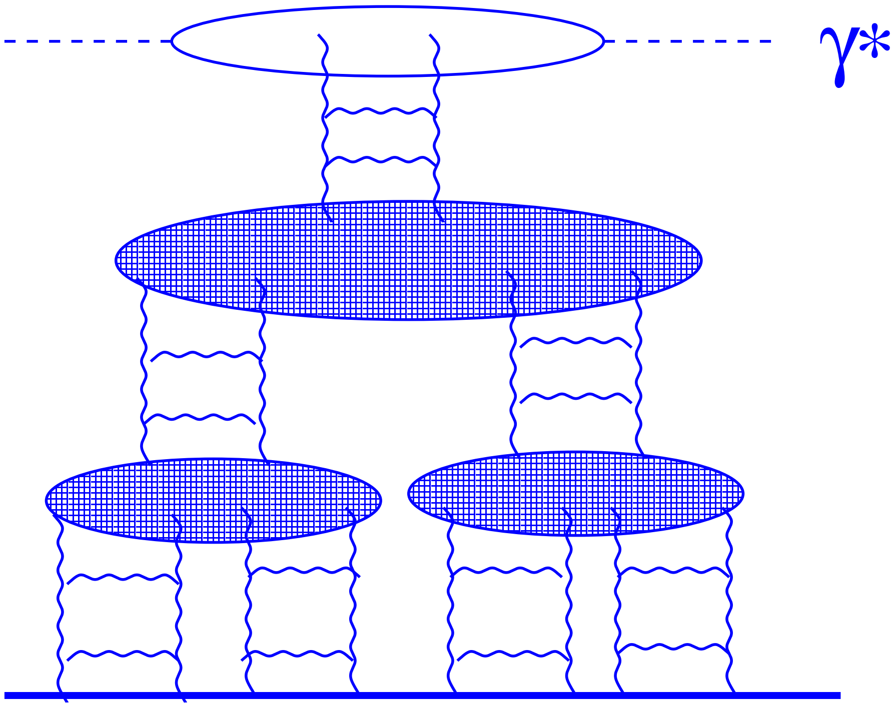
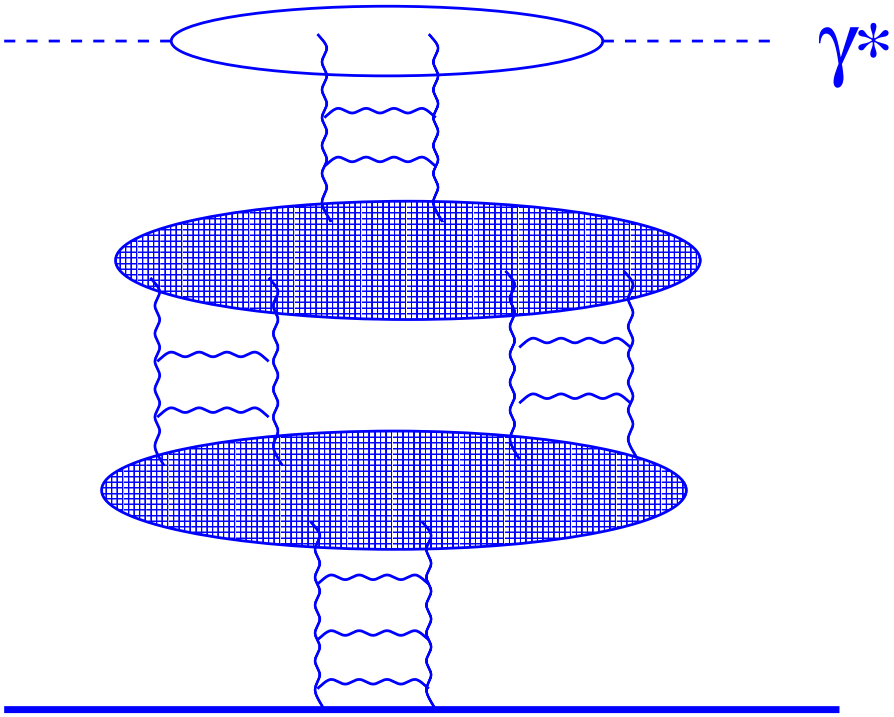
Example - LLA for the structure functions of large nuclei: $\alpha_s\ln\frac{1}{x} \sim 1,$ $\alpha_s^2 A^{1/3}\sim 1$

Non-linear equation sums up the "fan" diagrams



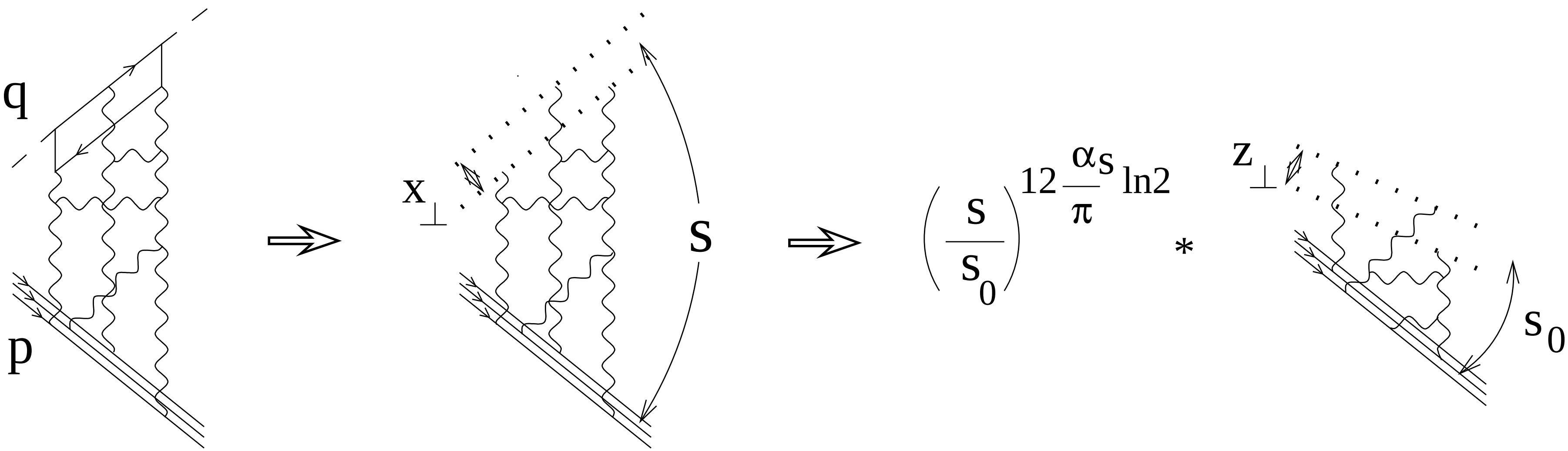
Example of the diagrams left behind by the NL eqn: pomeron loops



- Gribov, Levin, Ryskin (1983) - GLR eqn suggested
- Mueller, Qui (1986) - DLA limit of GLR eqn proved
- I.B. (1996) - the above eqn derived
- Kovchegov (1999) - the above eqn rederived (in the dipole model) and used for the large nuclei
- Braun (M.A.) (2000) - NL = GLR + 3-pomeron vertex from Bartels *et. al.*
- JIMWLK(2000) - obtained from the RG eqn for Color Glass Condensate

Initial conditions for the small-x evolution

BFKL evolution in terms of Wilson lines:



$\langle N|\mathrm{Tr}\{U(z_\perp,\eta_0)U^\dagger(0,\eta_0)\}|N\rangle$ - color dipole with the energy $s_0 = s\omega_0$

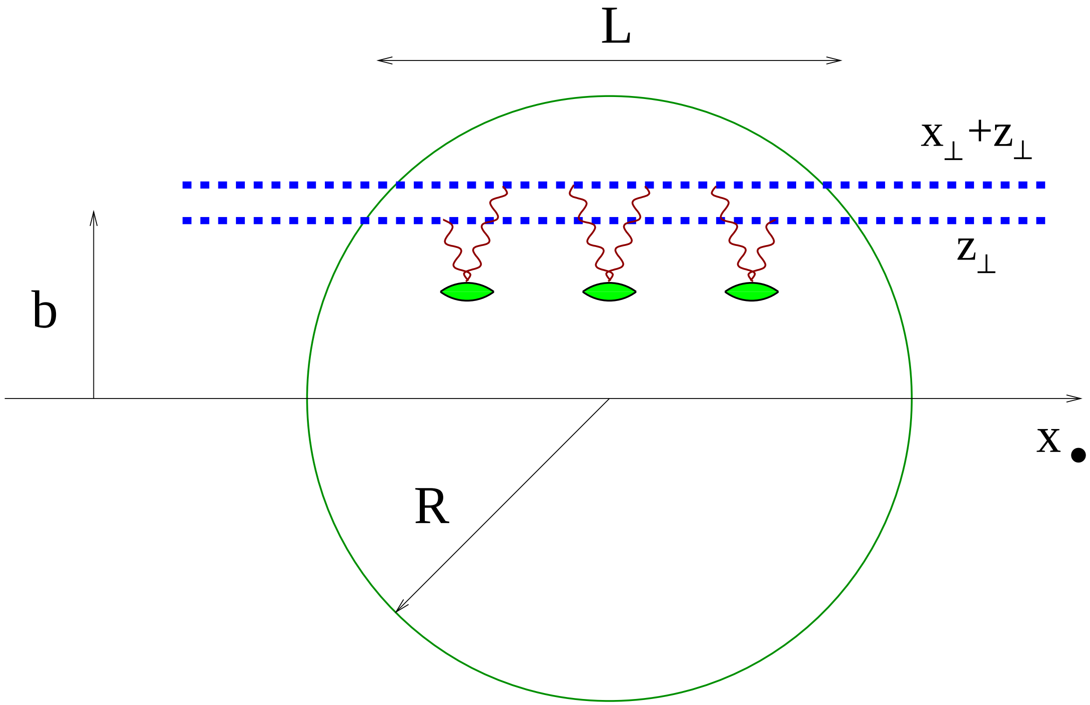
Initial point of the evolution s_0:

- small from the high-energy viewpoint: $\alpha_s \ln\frac{s_0}{m^2} \ll 1$
- large from the viewpoint of the low-energy physics: $\frac{s_0}{m^2} \gg 1$

For small dipoles (size $\ll$ 1fm)

$$\langle N|\mathrm{Tr}\{U(z_\perp,\eta_0)U^\dagger(0,\eta_0)\}|N\rangle \simeq z_\perp^2\omega_0 G(\omega_0,\mu^2 = \frac{1}{z_\perp^2})$$

Large nuclei: initial conditions $\Leftarrow$ Mueller-Glauber formula.



$$N_\eta(x_\perp, b_\perp) \equiv \frac{1}{N_c}\langle A|\mathrm{Tr}U(x_\perp + z)U^\dagger(z)|A\rangle$$

$$N_\eta(x_\perp, b_\perp) = \left[1 - e^{-g^2 c_F \mathcal{G}(x_\perp^2) L_b}\right]$$

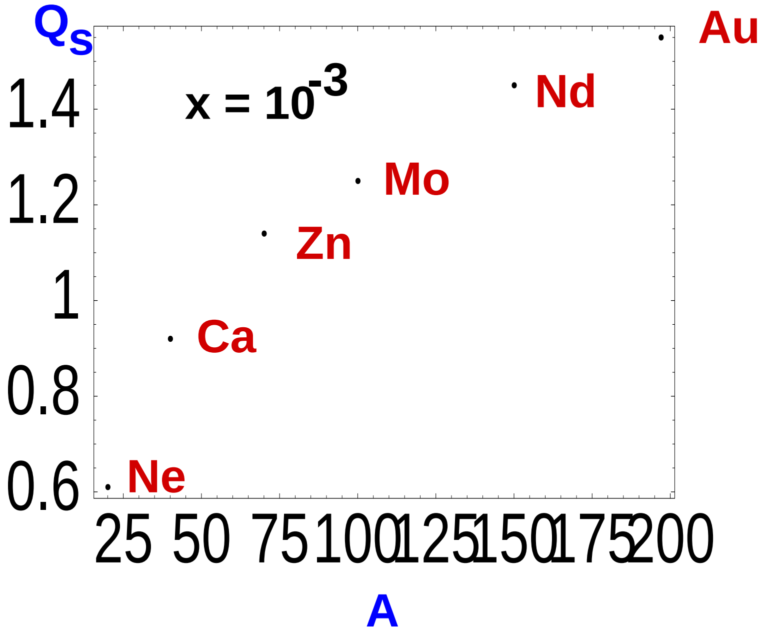
$$\mathcal{G}(x_\perp^2) \equiv \frac{\pi x_\perp^2}{4(N_c^2 - 1)}\rho\sigma_0 G(\sigma_0, \mu^2 = \frac{1}{x_\perp^2}).$$

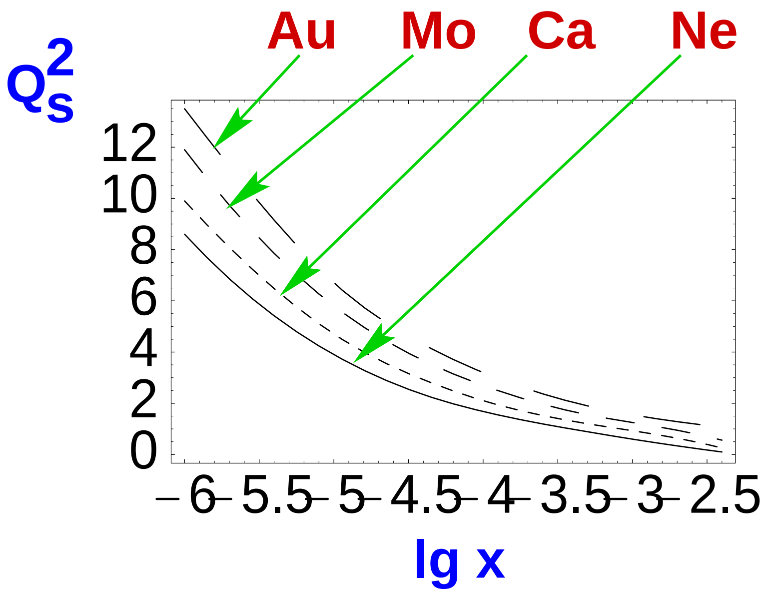
Saturation scale: $Q_s^2 = 1/x_{s\perp}^2$ such that $S_\eta(x_{s\perp}, b_\perp) = 1 - N(x_{s\perp}, b_\perp) \sim 1$

Theoretical estimates yield $Q_s(\eta) \sim Q_0 e^{\lambda \alpha_s \eta}$

x_B decreases $\Rightarrow \alpha_s(Q_s) \to 0.$

Numerical results (E.Levin, from THERA book)



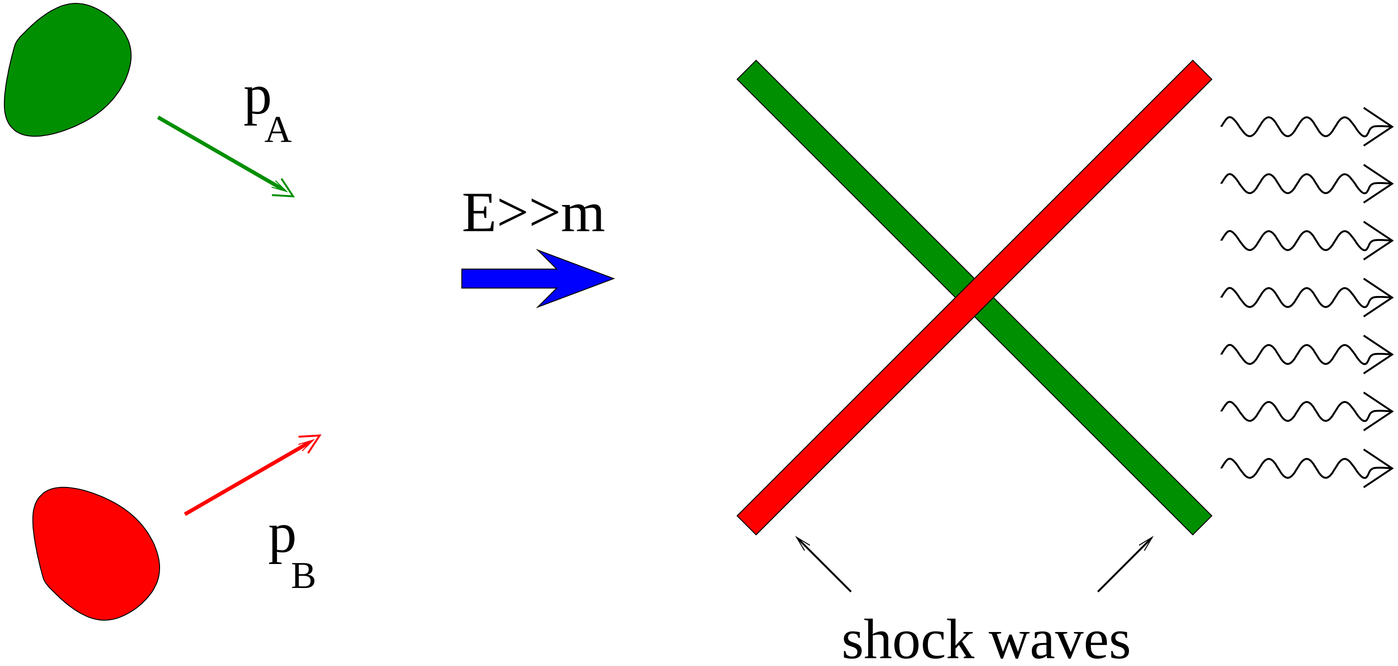


Both numerical and theoretical estimates show that even if we start from small target fields (e.g. γ^*) with $S_\eta(x_\perp) \ll 1$

$$x_B \to 0 \overset{\text{BFKL}}{\Rightarrow} S_\eta(x_\perp) \text{ increases} \overset{\text{NL}}{\Rightarrow} \text{saturation} \Rightarrow CGC?$$

High-energy scattering as a collision of shock waves

A typical hadron-hadron collision viewed from the c.m. frame has the form of scattering of two shock waves. Regge limit: $E \gg$ everything else



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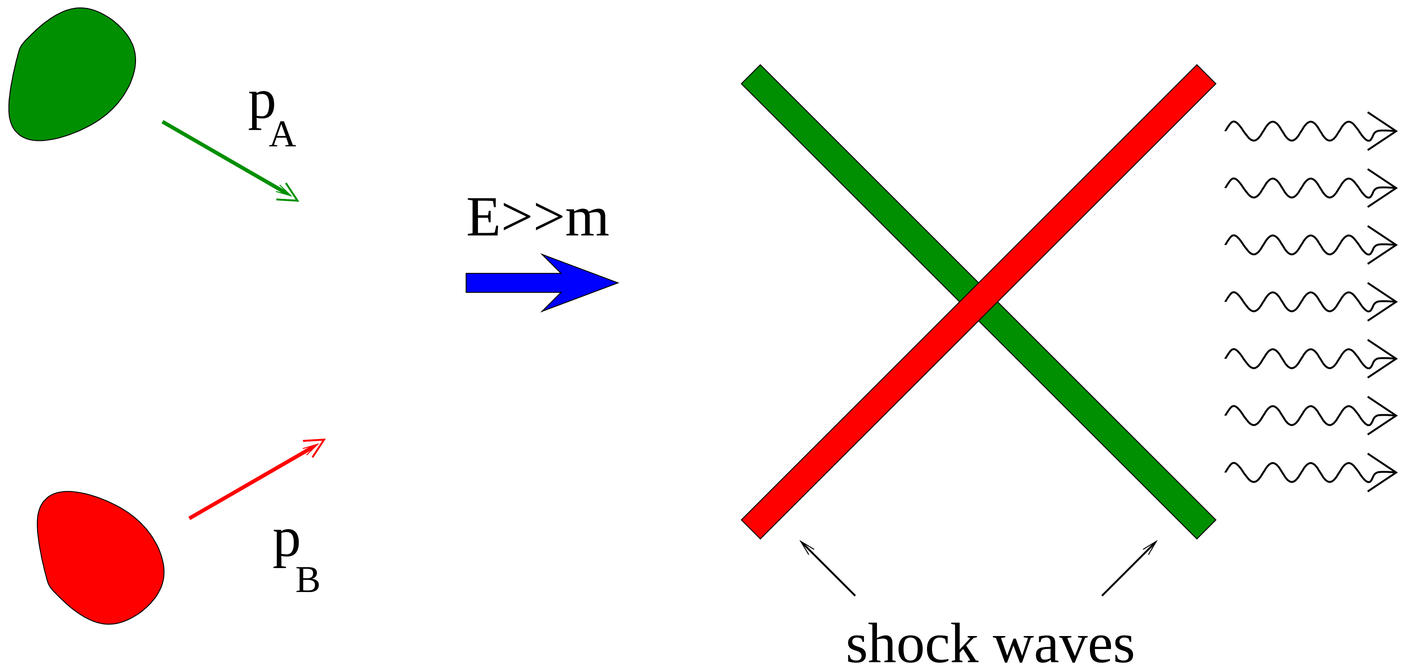

- **Big Q: Produced particles/fields $\Leftrightarrow$ S$_{\text{eff}}$?**

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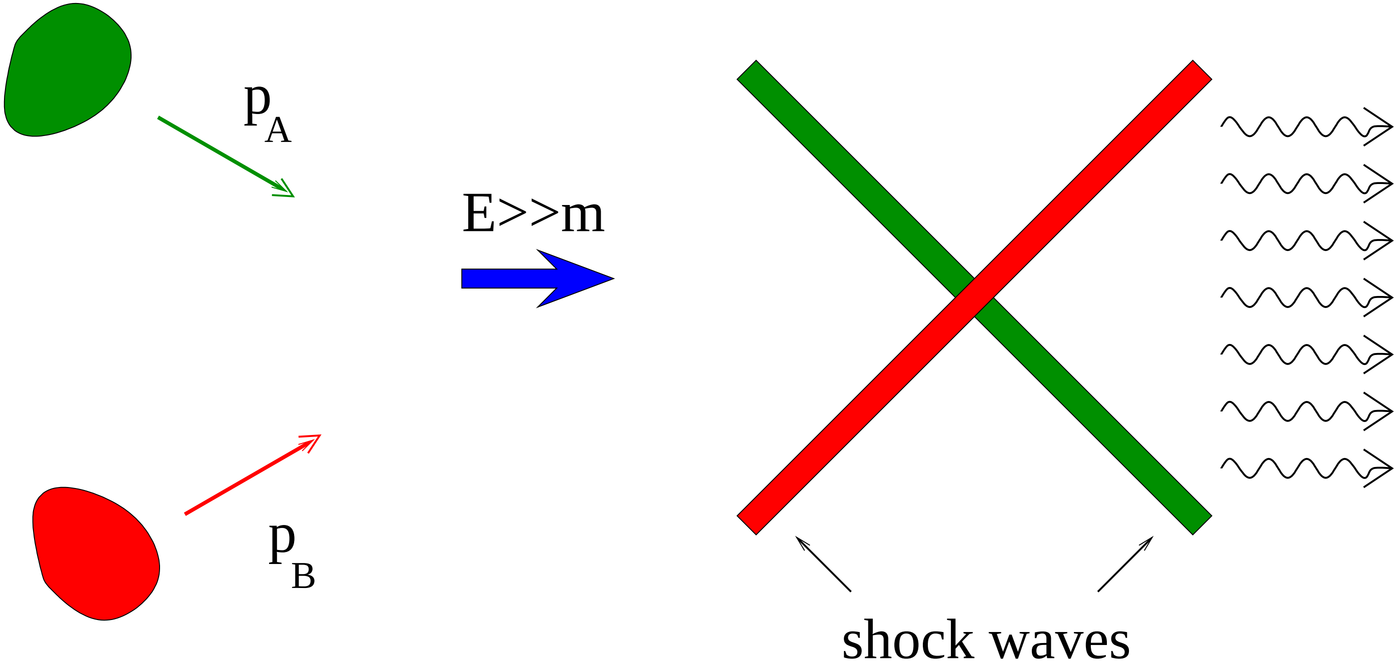
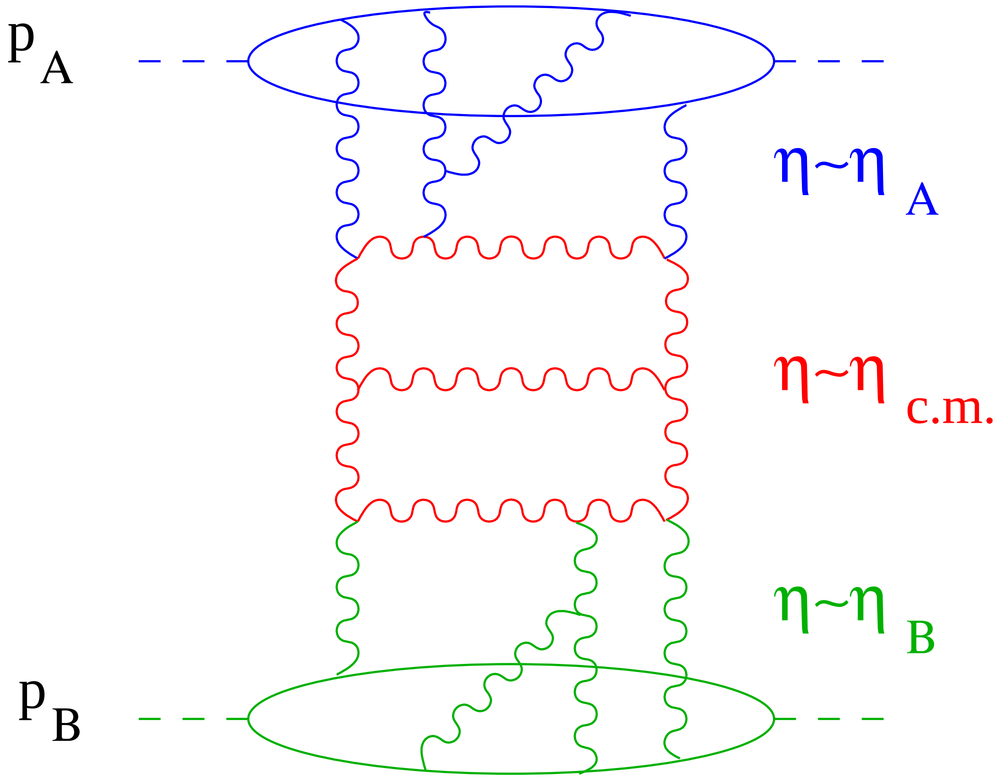
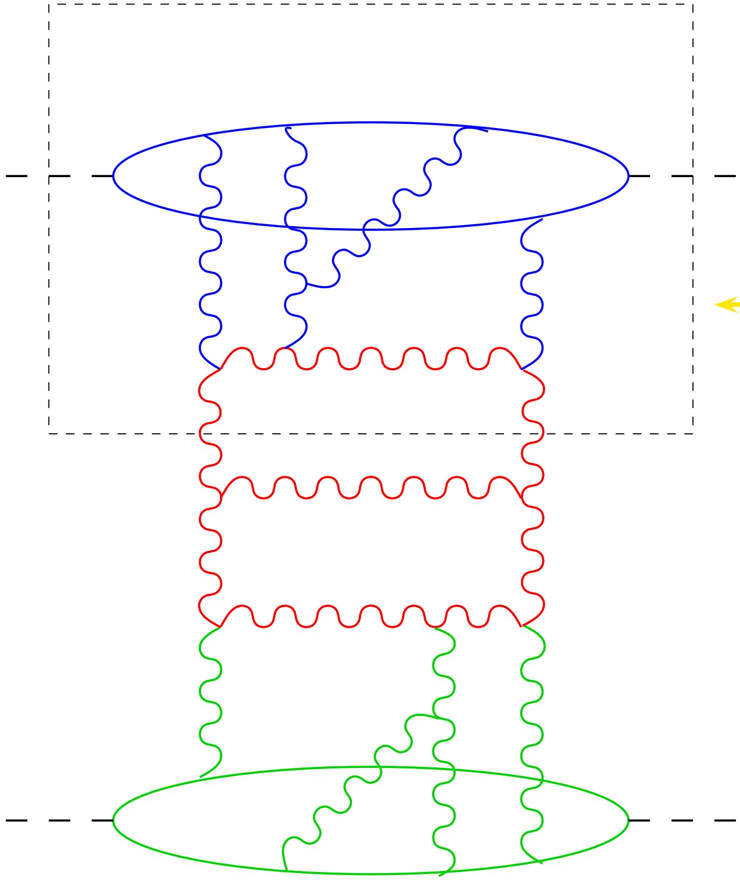


- **Q # 0: What is a scattering of two QCD shock waves?**

- **Big Q: Produced particles/fields $\Leftrightarrow$ $S_{\rm eff}$?**

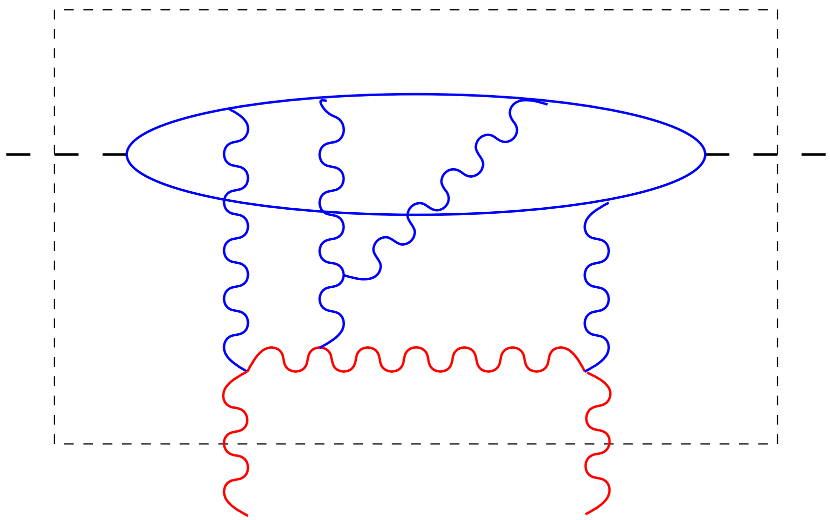


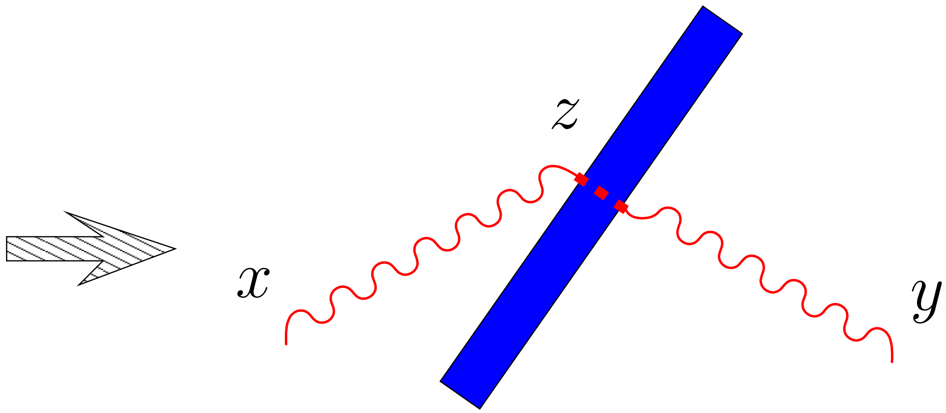
At first, we integrate over "red" gluons moving with rapidities in the central region $\eta \sim \eta_{\rm c.m.}$.
They interact with the "external" fileds (to be integrated over later) with rapidities $\eta \sim \eta_A$ and $\eta \sim \eta_B$



Consider the propagation of the red gluon in the background of blue gluons with greater rapidity

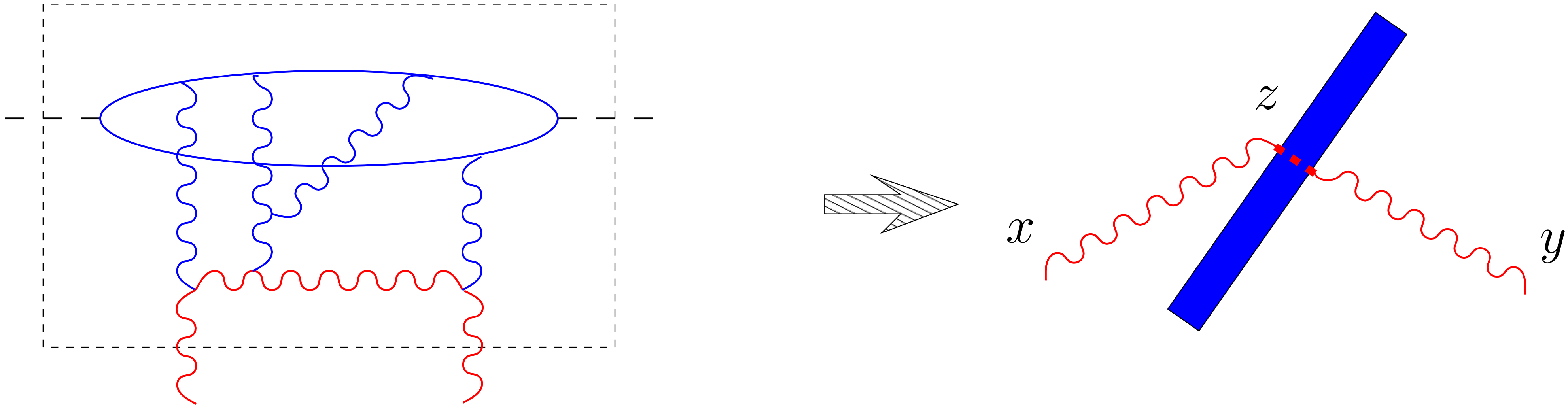
Fast-moving hadron $\Rightarrow$ QCD shock wave





Fast-moving (blue) fileds shrink into a pancake $A_+ \sim \delta(x_-)$.

Fast-moving hadron $\Rightarrow$ QCD shock wave

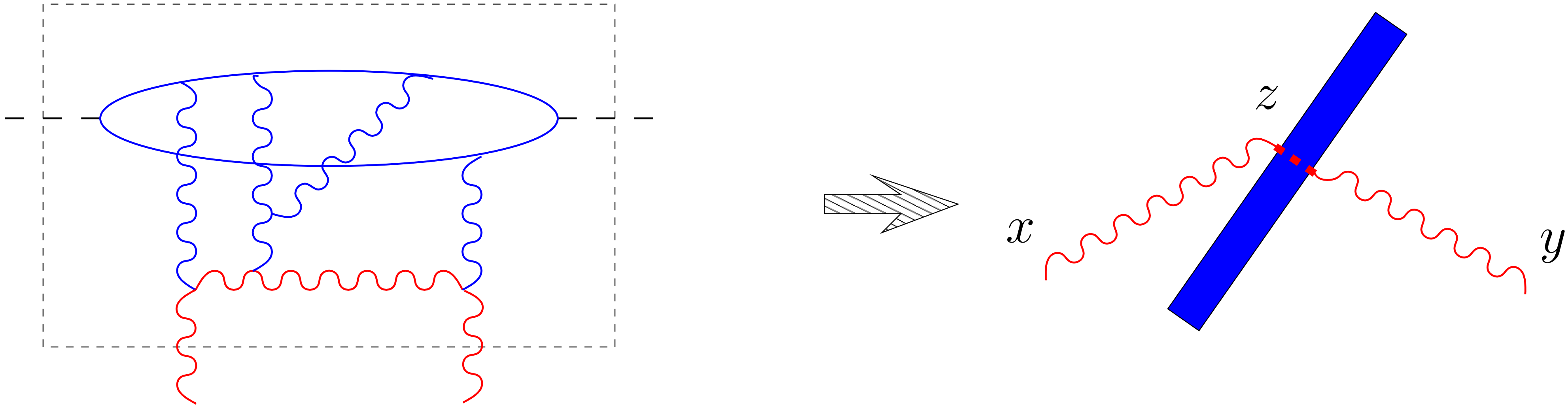


Fast-moving (blue) fileds shrink into a pancake $A_+ \sim \delta(x_-)$.

Interaction with the shock wave is instantaneous

$\Rightarrow$ no time to deviate in transverse plane

Fast-moving hadron ⇒ QCD shock wave



Fast-moving (blue) fileds shrink into a pancake $A_+ \sim \delta(x_-)$.

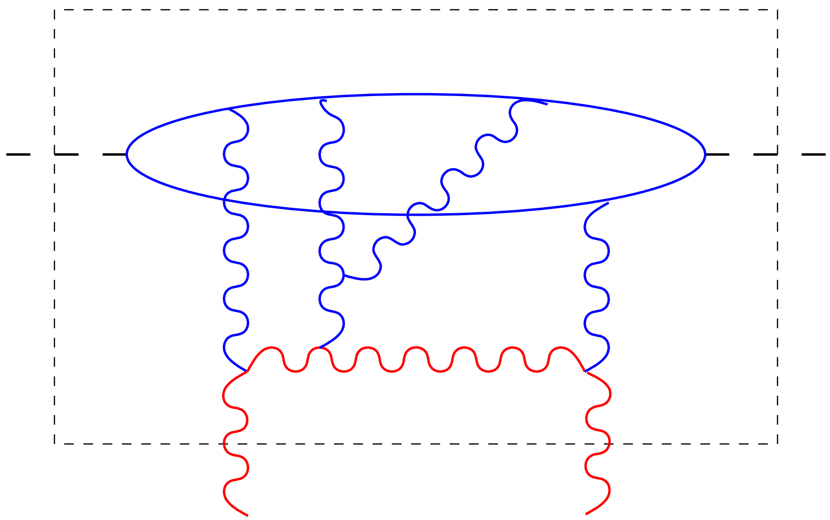
Interaction with the shock wave is instantaneous

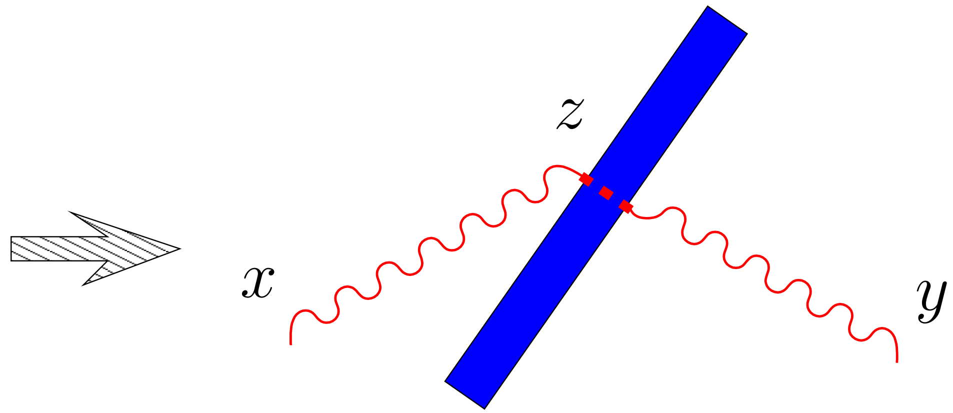
⇒ no time to deviate in transverse plane

⇒ the interaction is described by the *Wilson line*

$$V_z = [\infty p_2 + z_\perp, -\infty p_2 + z_\perp], \qquad [x,y] \equiv P e^{ig\int_0^1 du (x-y)^\mu A_\mu(ux+(1-u)y)}$$

Fast-moving hadron ⇒ QCD shock wave





$$G(x,y) = \int dz \frac{1}{(x-z)^2} V(z_\perp)\delta(z_-)\frac{1}{(z-y)}$$

Fast-moving (blue) fileds shrink into a pancake $A_+ \sim \delta(x_-)$.

Interaction with the shock wave is instantaneous

⇒ no time to deviate in transverse plane

⇒ the interaction is described by the *Wilson line*

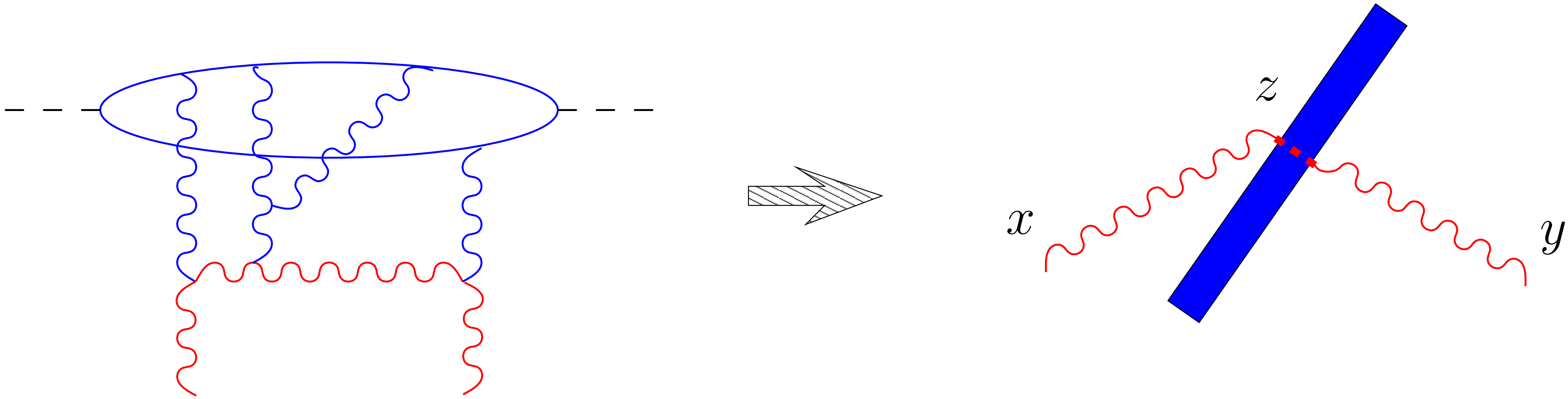
$$V_z = [\infty p_2 + z_\perp, -\infty p_2 + z_\perp], \qquad [x,y] \equiv P e^{ig\int_0^1 du (x-y)^\mu A_\mu(ux+(1-u)y)}$$

Propagator in the shock-wave background = (free propagator)

$\times$(instantaneous interaction with the shock wave $\sim$ V)$\times$(free propagator)

Covariant vs axial gauge

Covariant gauges: the shock wave is a pancake: $A_+ \sim \delta(x_-)$, $A_- = A_i = 0$.

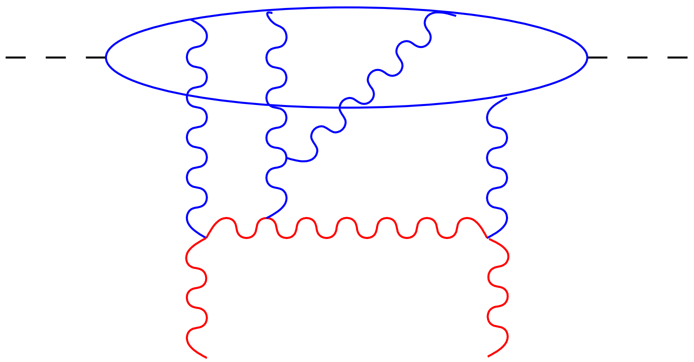


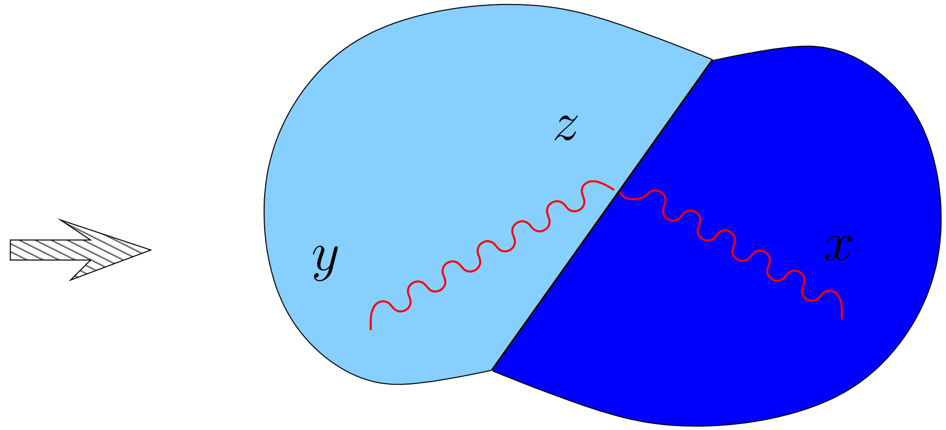
Covariant vs axial gauge

Covariant gauges: the shock wave is a pancake: $A_+ \sim \delta(x_-)$, $A_- = A_i = 0$.

Axial (temporal) gauges: the shock wave is a piece-wise pure gauge

$$A^i = \mathcal{V}_1^i(z_\perp)\theta(z_+) + \mathcal{V}_2^i(z_\perp)\theta(-z_+), \qquad A_+ = A_- = 0, \qquad \mathcal{V}_i \equiv V^\dagger \frac{i}{g}\partial_i V$$



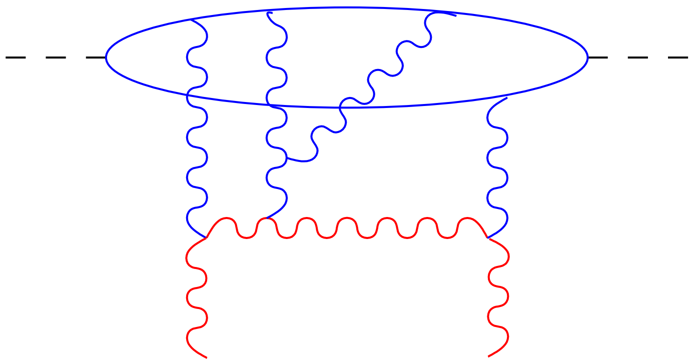


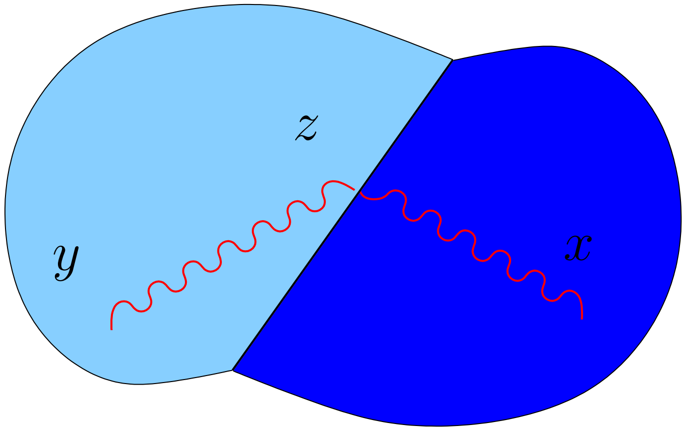
$$G(x,y) = \int dz V_1^\dagger(x_\perp)\frac{1}{(x-z)^2}V_1(z_\perp)\delta(z_-)V_2^\dagger(z_\perp)\frac{1}{(z-y)^2}V_2(y_\perp)$$

Covariant vs axial gauge

Covariant gauges: the shock wave is a pancake: $A_+ \sim \delta(x_-)$, $A_- = A_i = 0$.
Axial (temporal) gauges: the shock wave is a piece-wise pure gauge

$$A^i = \mathcal{V}^i_1(z_\perp)\theta(z_+) + \mathcal{V}^i_2(z_\perp)\theta(-z_+), \qquad A_+ = A_- = 0, \qquad \mathcal{V}_i \equiv V^\dagger \frac{i}{g}\partial_i V$$



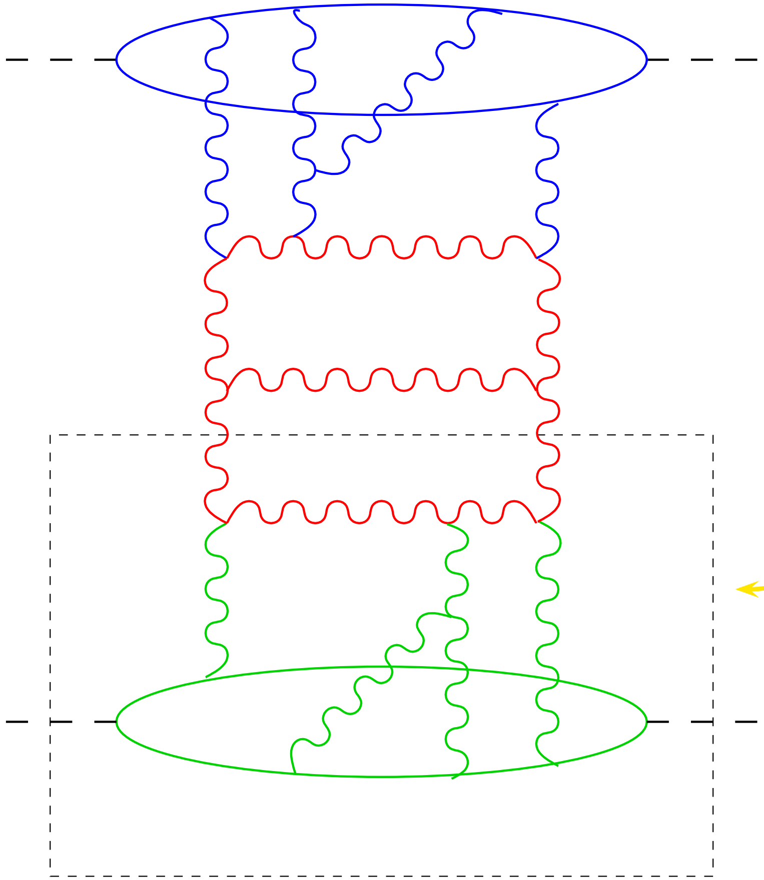


$$G(x,y) = \int dz V_1^\dagger(x_\perp)\frac{1}{(x-z)^2}V_1(z_\perp)\delta(z_-)V_2^\dagger(z_\perp)\frac{1}{(z-y)^2}V_2(y_\perp)$$

The source for such a field is

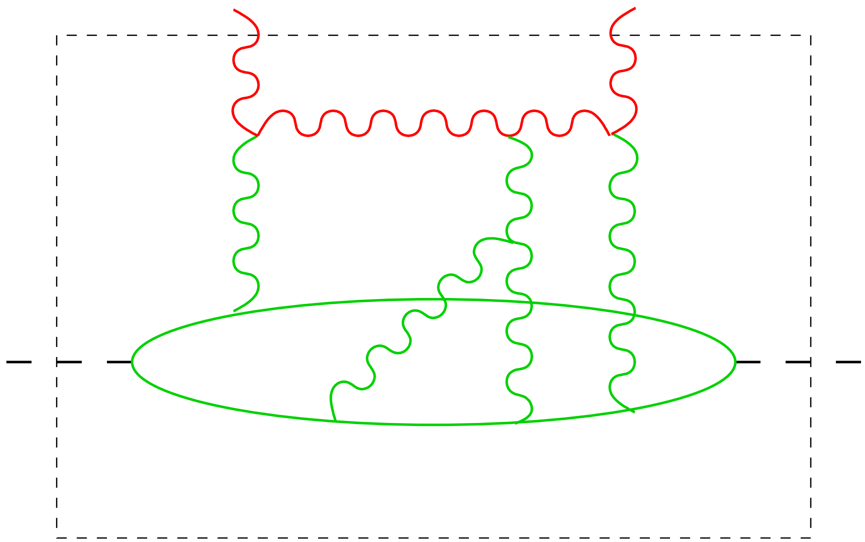
$$\exp\{i\int d^2 z_\perp\{\mathcal{V}^i_1(z_\perp) - \mathcal{V}^i_2(z_\perp)\}(0, F_{-i}, 0)_z\}$$

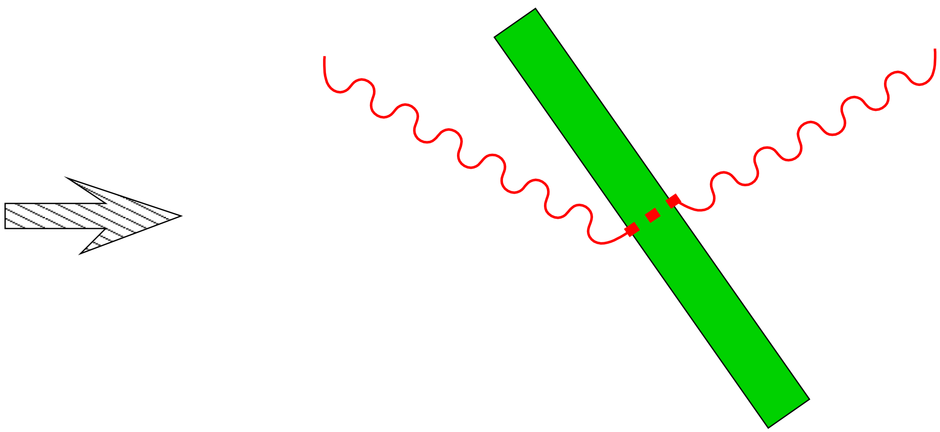
$$(0, F_{-i}, 0)_z \equiv \int du[z_\perp, up_1 + z_\perp]F_{-i}(up_1 + z_\perp)[up_1 + z_\perp, z_\perp]$$



Consider now the propagation of the red gluon in the background of green gluons with greater rapidity

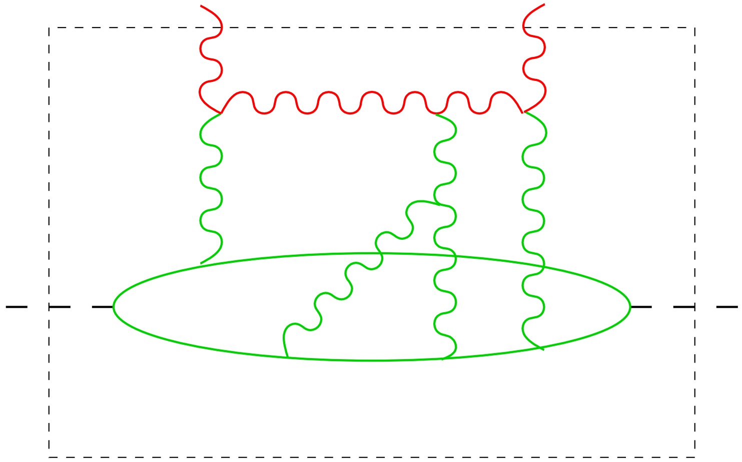
Covariant gauges:

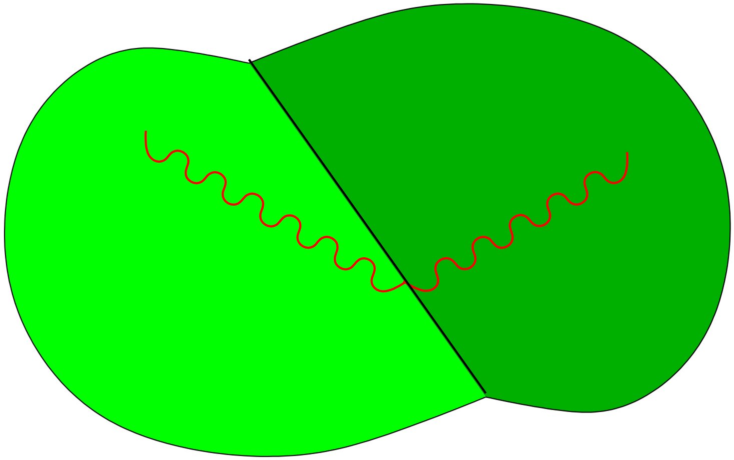




Axial gauges:

$$\mathcal{U}_i \equiv U^\dagger \frac{i}{g}\partial_i U$$





The source is

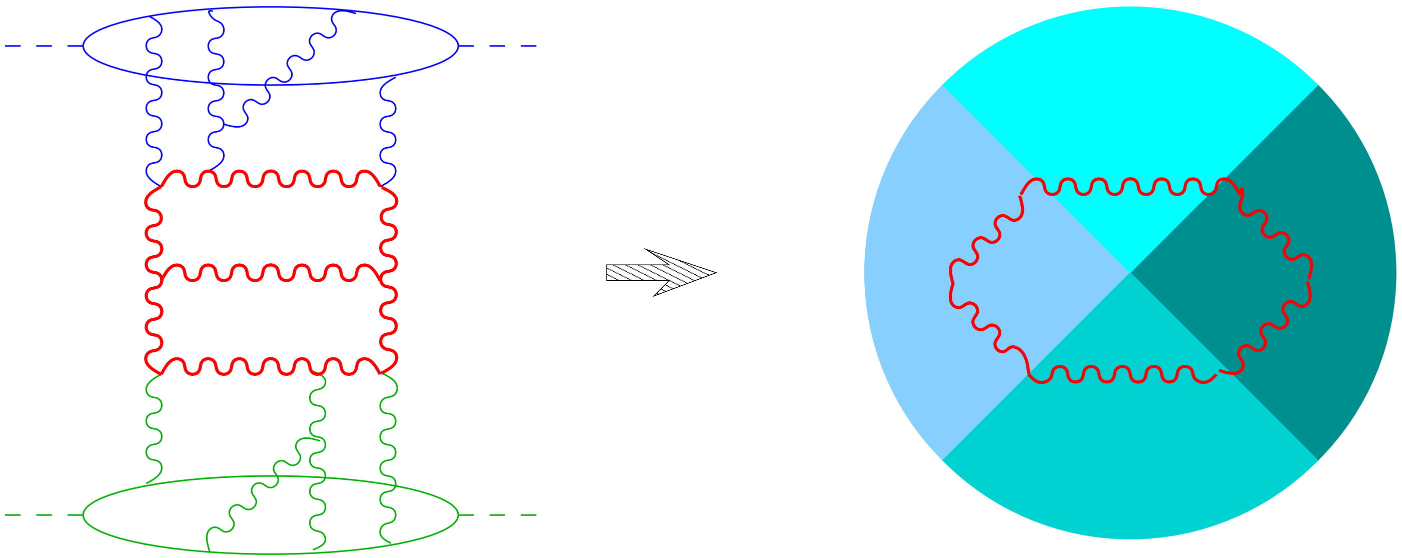
$$\exp\{i\int d^2z_\perp(\mathcal{U}_1^i - \mathcal{U}_2^i)(z_\perp)(0,F_{+i},0)_z\}$$

$$[0,F_{+i},0] \equiv \int du[z_\perp, up_2+z_\perp]F_{+i}(up_2+z_\perp)[up_2+z_\perp, z_\perp]$$
$$= \;[0,\infty p_2]_z i\frac{\partial}{\partial z_i}[\infty p_2, 0]_z + [0,-\infty p_2]_z i\frac{\partial}{\partial z_i}[-\infty p_2, 0]_z$$

Scattering of two shock waves

Gluons in the central region of rapidity move in the “external” fields of two shock waves

In the axial gauges



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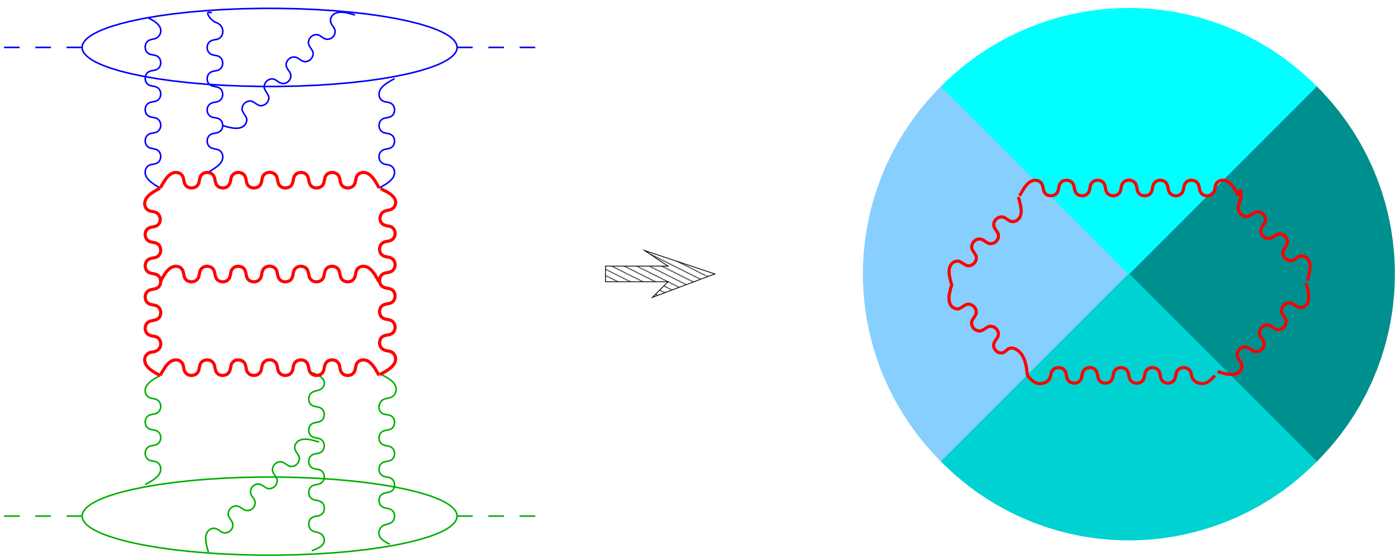
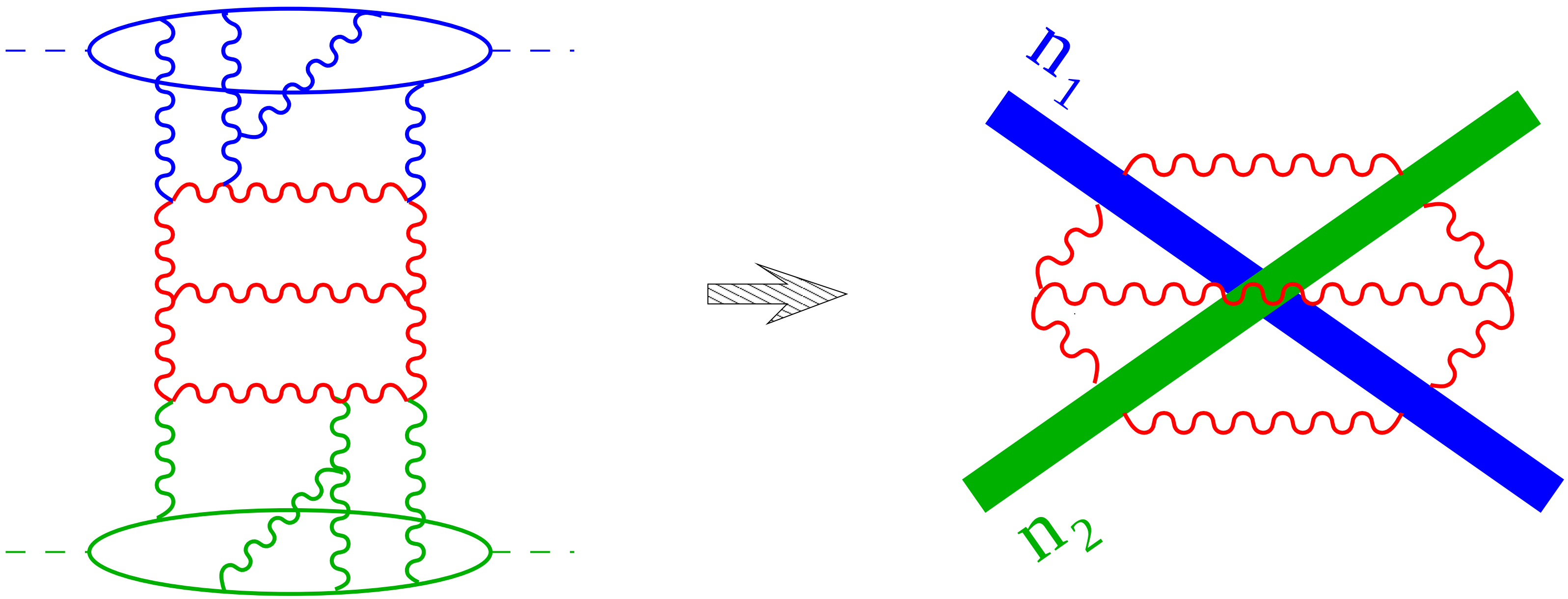


Integration over A fields gives the effective action

$$e^{iS_{\rm eff}(U_i,V_i,\Delta\eta)} = \int DA \; e^{iS_{\rm QCD}(A)+i\int d^2z_\perp\{(\mathcal{V}_1^i-\mathcal{V}_2^i)_z(0,F_{-i},0)_z+(\mathcal{U}_1^i-\mathcal{U}_2^i)_z[0,F_{+i},0]_z\}}$$

Rapidity $\Leftrightarrow$ slope of the Wilson line $\qquad \Delta\eta = \eta_1 - \eta_2$



$$e^{iS_{\rm eff}(U_i, V_i, \Delta\eta)} = \int DA\ e^{iS_{\rm QCD}(A) + i\int d^2z_\perp\{(\mathcal{V}_1^i - \mathcal{V}_2^i)_z(0, F_{-i}, 0)_z + (\mathcal{U}_1^i - \mathcal{U}_2^i)_z[0, F_{+i}, 0]_z\}}$$

$$(0, F_{-i}, 0)_z = (0, \infty n_1)_z i\partial_i(\infty n_1, 0)_z + (0, -\infty n_1)_z i\partial_i(-\infty n_1, 0)_z,$$
$$[0, F_{-i}, 0]_z = [0, \infty n_2]_z i\partial_i[\infty n_2, 0]_z + [0, -\infty n_2]_z i\partial_i[-\infty n_2, 0]_z$$

$S_{\rm eff}$ gives the small-x evolution of the Wilson-line operators

BASIC IDEA: $\alpha_s = \alpha_s(Q_s) \ll 1 \Rightarrow$ SEMICLASSICS IS RELEVANT

McLerran & Venugopalan

Classical YM equation

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$$D^\mu F_{\mu\nu} = \frac{\partial}{\partial A_\mu}(\text{sources})$$

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Two methods of the solution on the market:

- Numerical simulations. Venugopalan & Krasnitz
- Perturbative expansion in strength of one of the shock waves McLerran et al, Kovchegov & Mueller

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  McLerran et al, Kovchegov & Mueller

$\Leftrightarrow$ expansion in powers of commutators $[U,V]$ (calculated up to $[U,V]^2$)

If $[U,V]=0$

$$\bar{A}_+ = \bar{A}_- = 0, \quad \bar{A}^i = \mathcal{U}_1^i \theta(x_+) + \mathcal{U}_2^i \theta(-x_+) + \mathcal{V}_1^i \theta(x_-) + \mathcal{V}_2^i \theta(-x_-)$$

= piece-wise pure gauge .

QED-like: no interaction $\Rightarrow$ no particle production

If $[U, V] = 0$

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= piece-wise pure gauge .
QED-like: no interaction $\Rightarrow$ no particle production

If $[U, V] \neq 0$ one can take this ansatz

$$\bar{A}_+^{(0)} = \bar{A}_-^{(0)} = 0, \quad \bar{A}^{i(0)} = \mathcal{U}_1^i \theta(x_+) + \mathcal{U}_2^i \theta(-x_+) + \mathcal{V}_1^i \theta(x_-) + \mathcal{V}_2^i \theta(-x_-)$$

as a trial configuration for the classical solution and improve it order by order in $[U, V]$ by calculating Feynman diagrams in the background of the trial configuration.

The expansion in commutators

Linear term (source) for the trial configuration

$$T_\mu \;=\; -D_k^{(0)}([\mathcal{U}^1_{\mu\perp}, \mathcal{V}^k_1] - \mu \leftrightarrow k)\theta(z_+)\theta(z_-) + 3\,\text{similar terms}$$

$$\bar{A}_\mu = \mathcal{U}^\perp_{1\mu}\theta(x_+) + \mathcal{U}^\perp_{2\mu}\theta(-x_+) + \mathcal{V}^\perp_{1\mu}\theta(x_-) + \mathcal{V}^\perp_{2\mu}\theta(-x_-) + Q_\mu$$

The expansion in commutators

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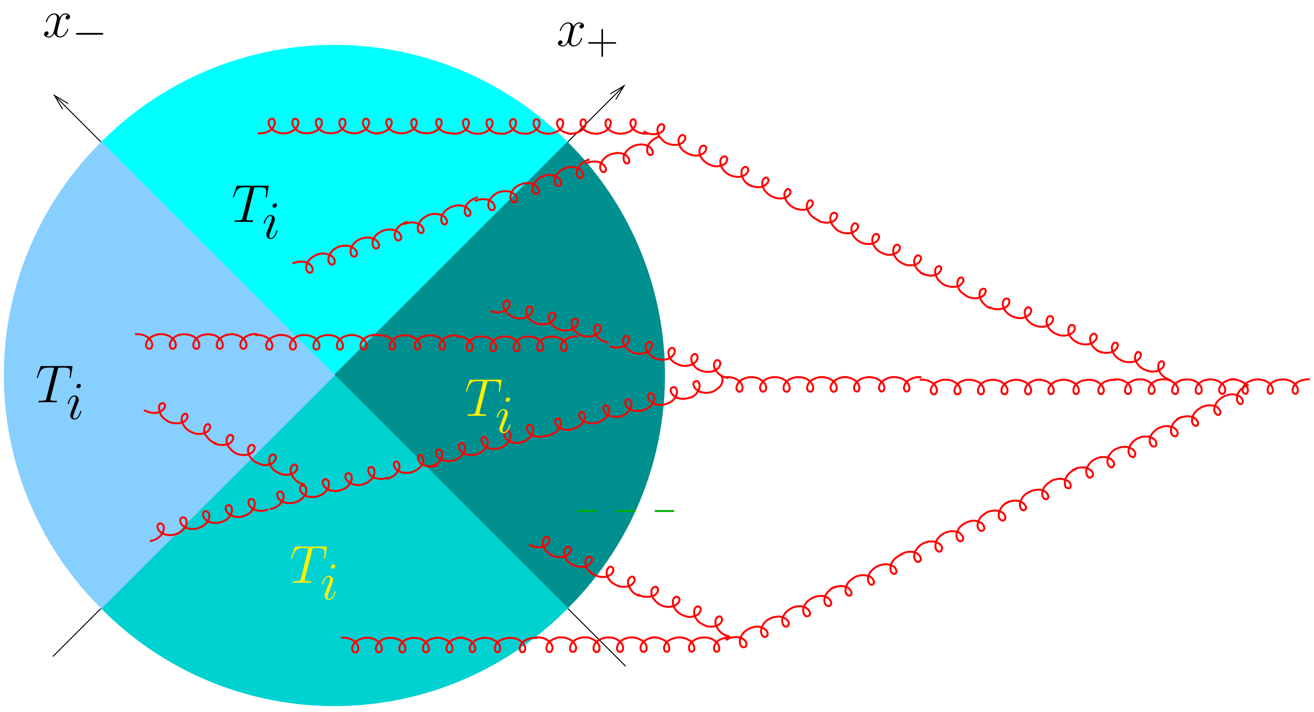
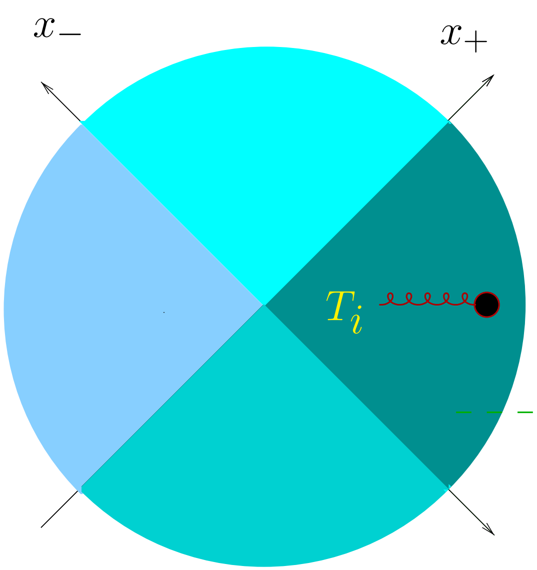
$$\bar{A}_\mu=\mathcal{U}^\perp_{1\mu}\theta(x_+)+\mathcal{U}^\perp_{2\mu}\theta(-x_+)+\mathcal{V}^\perp_{1\mu}\theta(x_-)+\mathcal{V}^\perp_{2\mu}\theta(-x_-)+Q_\mu$$

Solve the YM eqn for $Q_\mu(x)$ by iterations $\Leftrightarrow$ calculate Feynman diagrams in the external field $\bar{A}^{(0)}$



1/2-order approximation: a piece-wise pure gauge field



$$
\begin{aligned}
\bar{A}_+ &= \bar{A}_- = 0 \\
\bar{A}^i &= \mathcal{W}^i_F \theta(x_+)\theta(x_-) + \mathcal{W}^i_L \theta(-x_+)\theta(x_-) \\
&+ \mathcal{W}^i_R \theta(x_+)\theta(-x_-) + \mathcal{W}^i_B \theta(-x_+)\theta(-x_-)
\end{aligned}
$$

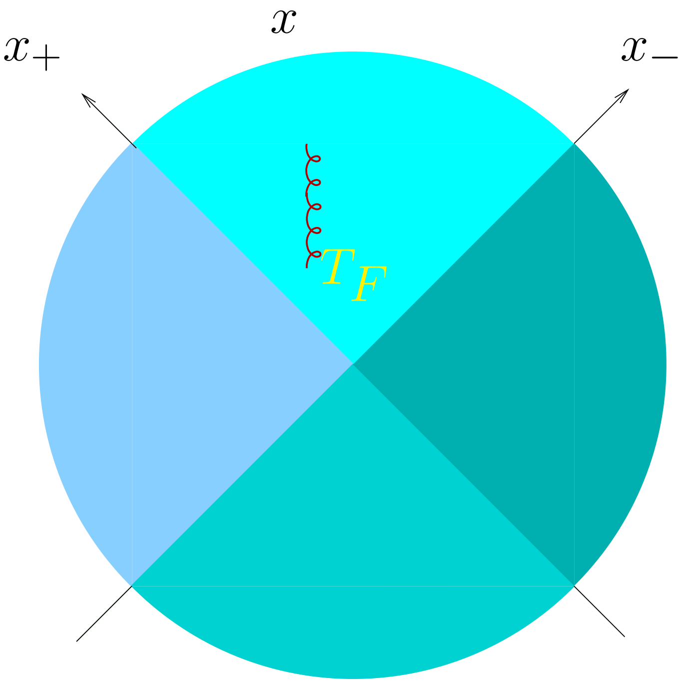
$$
\begin{aligned}
\mathcal{W}^i_F(x_\perp) &= \mathcal{U}^i_1 + \mathcal{V}^i_1 + E^i_F = \text{pure gauge} \\
\mathcal{W}^i_L(x_\perp) &= \mathcal{U}^i_2 + \mathcal{V}^i_1 + E^i_L = \ldots \\
\mathcal{W}^i_R(x_\perp) &= \mathcal{U}^i_1 + \mathcal{V}^i_2 + E^i_R = \ldots \\
\mathcal{W}^i_B(x_\perp) &= \mathcal{U}^i_2 + \mathcal{V}^i_2 + E^i_B = \ldots
\end{aligned}
$$

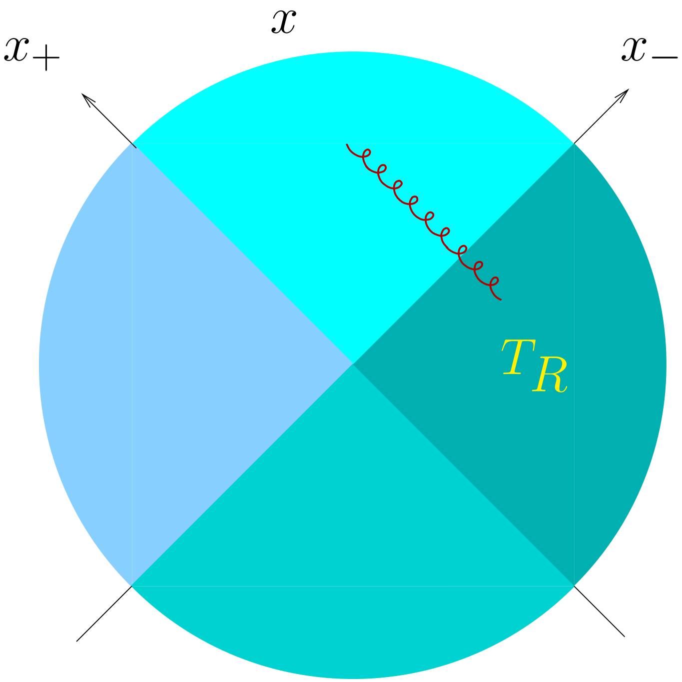
In the first order

$$
E^a_i(x_\perp) = ig \int d^2z (U_x U^\dagger_z + V_x V^\dagger_z - 1)^{ab} \frac{(x-z)^k_\perp}{2\pi^2 (x-z)^2_\perp} ([\mathcal{U}_i, \mathcal{V}_k]_z - i \leftrightarrow k)^b
$$

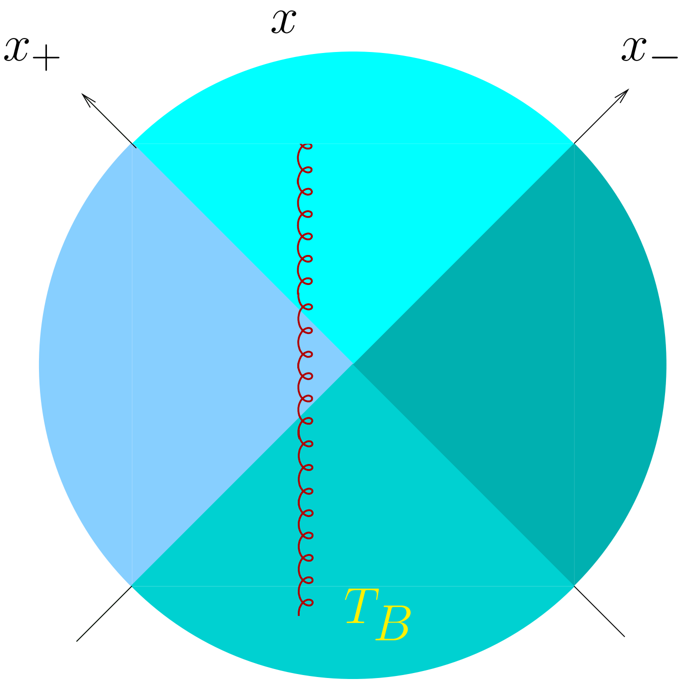
bF gauge $D^\mu Q_\mu = 0 \to (i\partial_i + g[\mathcal{U}_i + \mathcal{V}_i,)E^i = 0$

Q_μ in the first order

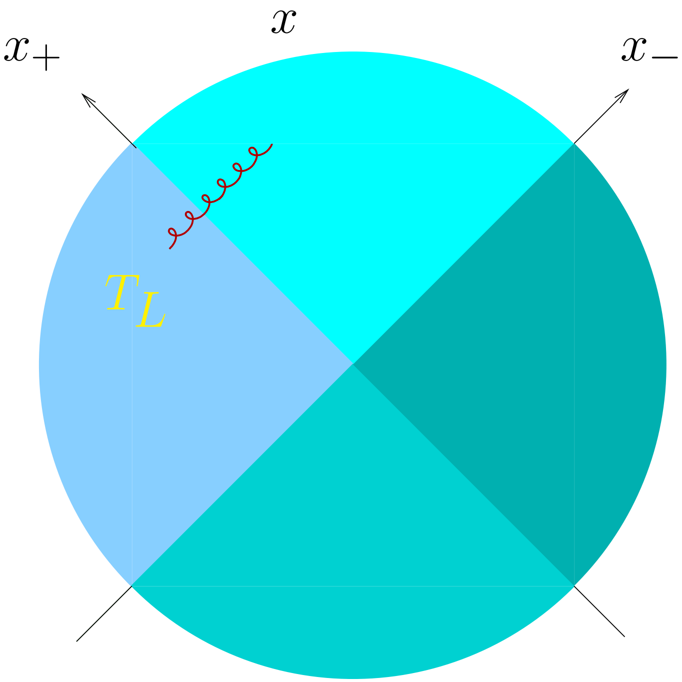



x
x_+
x_-
T_R

Q_μ in the first order



Q_μ in the first order



$$Q^{\mathcal{W}_F}_{(1)\mu}(k) \equiv W_F Q^{\mathcal{W}_F}_{(1)\mu} W_F^\dagger =$$

$$\frac{1}{k^2}\Big\{2\frac{p_{1\mu}}{k_-}[\mathcal{V}_{1i} - \mathcal{V}_{2i}, E^i_R - E^i_B] + 2\frac{p_{2\mu}}{k_+}[\mathcal{U}_{1i} - \mathcal{U}_{2i}, E^i_L - E^i_B] + \ 2E^\perp_\mu\Big\}$$

$$E^i \equiv E^i_F - E^i_L - E^i_R + E^i_B = \mathcal{W}^i_F - \mathcal{W}^i_L - \mathcal{W}^i_R + \mathcal{W}^i_B$$

Check: bF gauge condition $(i\partial^\mu + g[\mathcal{W}^\mu_F,)Q_\mu = 0$

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Check: bF gauge condition $(i\partial^\mu + g[\mathcal{W}^\mu_F,\ )Q_\mu = 0$

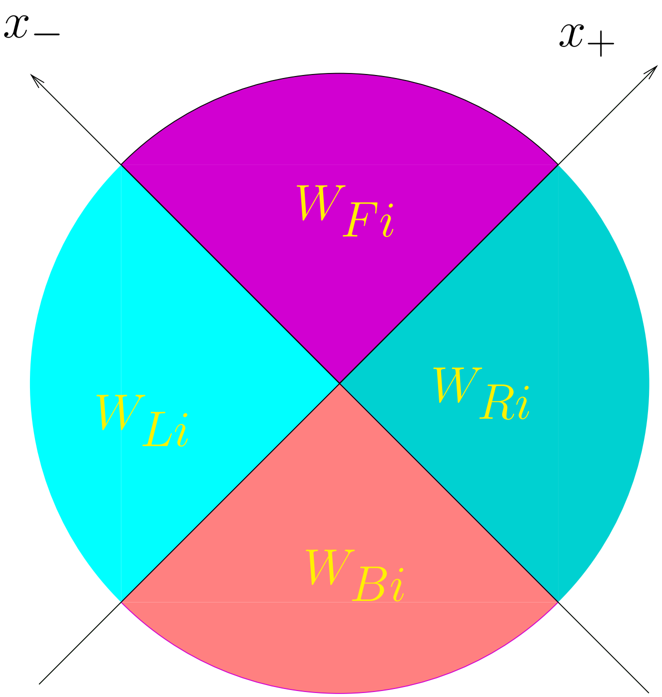
Lipatov vertex (effective vertex of gluon emission):

$$L^{(1)}_\mu(k) = k^2 Q^{\mathcal{W}_F}_{(1)\mu}(k)\Big|_{k^2=0} =$$
$$2E^\mu_\perp + 2\frac{p^\mu_1}{k_-}[\mathcal{V}_{1i}-\mathcal{V}_{2i},E^i_R-E^i_2]+2\frac{p^\mu_2}{k_+}[\mathcal{U}_{1i}-\mathcal{U}_{2i},E^i_L-E^i_2]$$

Effective action = product of two Lipatov vertices.

In the $[U,V]^2$ order

$$L^a_\mu L^{a\mu} = 4E^i_a E^{ai}$$



The trial configuration:

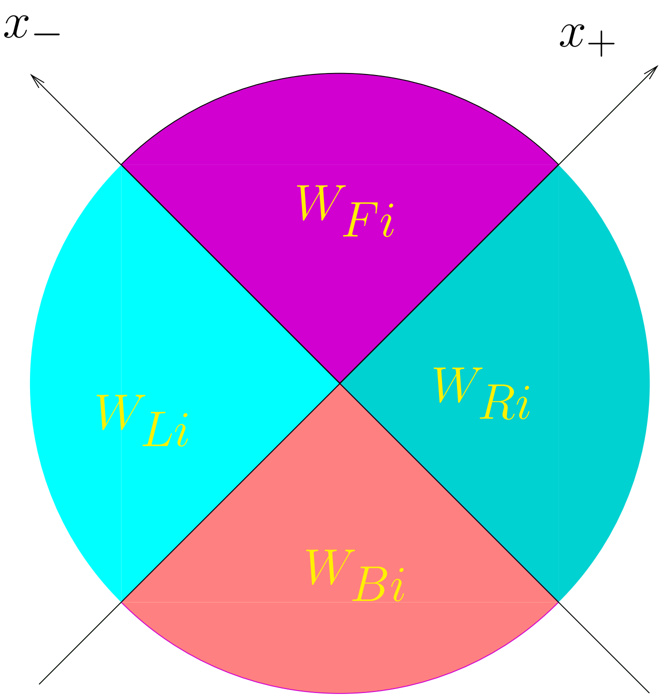
$A_- = A_+ = 0$ and

$$A_i = \theta(x_+)\theta(x_-)\mathcal{W}_{Fi} + \theta(x_+)\theta(-x_+)\mathcal{W}_{R}$$
$$+\, \theta(-x_+)\theta(x_-)\mathcal{W}_{Li} + \theta(-x_+)\theta(-x_-)\mathcal{W}_{B}$$

In each of the four quadrants of the space the field is a pure gauge

$$\mathcal{W}_{Fi} = \mathcal{U}_{1i} + \mathcal{V}_{1i} + E_{Fi}$$
$$\mathcal{W}_{Li} = \mathcal{U}_{2i} + \mathcal{V}_{1i} + E_{Li}$$
$$\mathcal{W}_{Ri} = \mathcal{U}_{1i} + \mathcal{V}_{2i} + E_{Ri}$$
$$\mathcal{W}_{Bi} = \mathcal{U}_{2i} + \mathcal{V}_{2i} + E_{Fi}$$

Shortcut to the effective action



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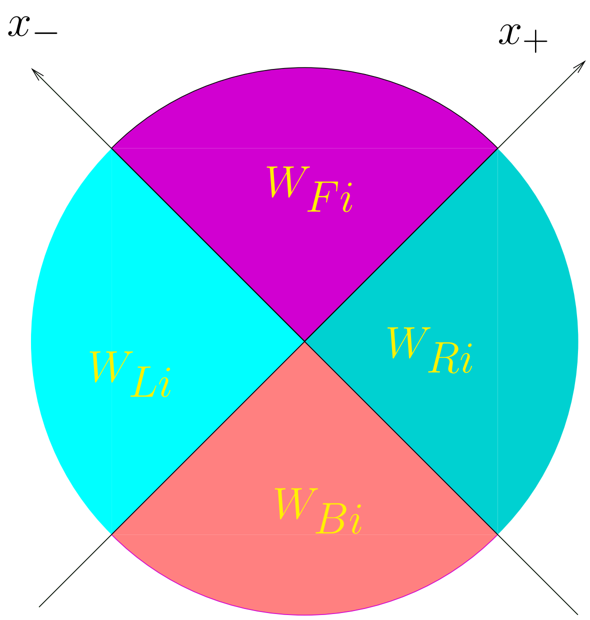
$$\mathcal{W}_{Bi} = \mathcal{U}_{2i} + \mathcal{V}_{2i} + E_{Fi}$$

$$T_i = 2\delta(x_+)\delta(x_-)E_i \;\;\Rightarrow$$

$$S_{\text{eff}} = \int dz dz' T_i^a(z) T^{ai}(z') \simeq \alpha_s \Delta\eta \int d^2 z_\perp E_i^a(z_\perp) E^{ai}(z_\perp)$$

Gauge-invariant form of the effective action

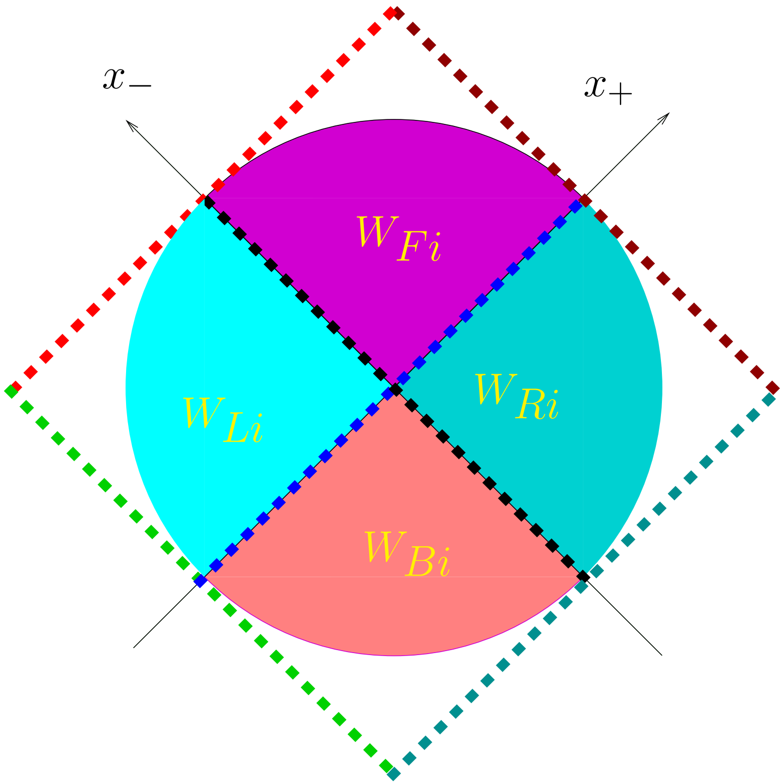


$$S_{\text{eff}}(V_1, V_2, U_1, U_2; \Delta\eta) \; = $$

$$(\mathcal{V}_1 - \mathcal{V}_2)^{ai}(\mathcal{U}_1 - \mathcal{U}_2)^a_i \; + \; \frac{\alpha_s \Delta\eta}{4} L^a_i L^{ai}$$

$$\begin{aligned} L^a_i \; &= \; 2(\mathcal{W}_F - \mathcal{W}_L - \mathcal{W}_R + \mathcal{W}_B)^a_i \\ &= \; 2(E_F - E_L - E_R + E_B)^a_i \end{aligned}$$

Gauge-invariant form of the effective action



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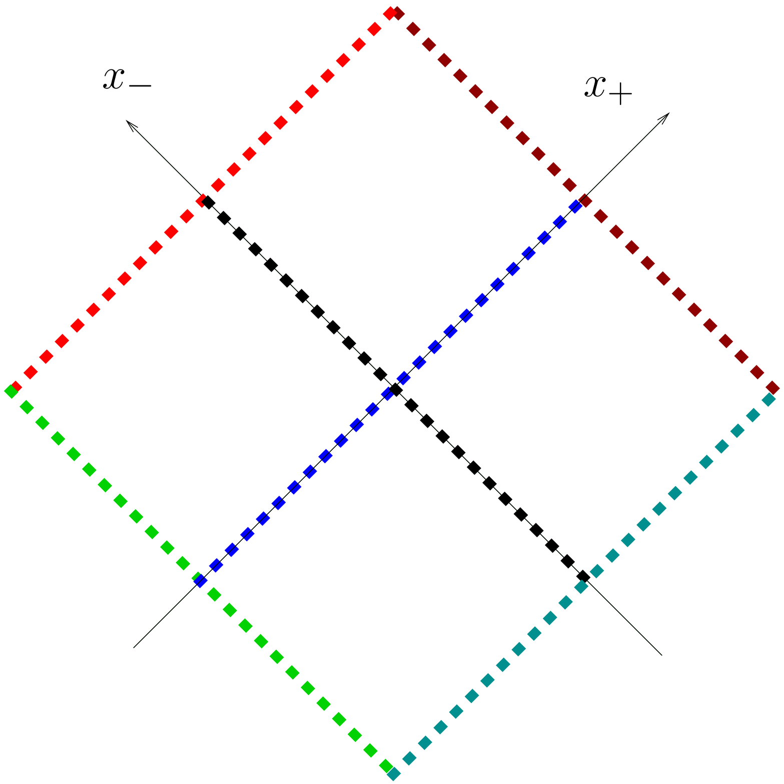
Gauge invariant representation (SMITH):

$$\frac{1}{4} L^{ai} L^a_i = \mathrm{tr}[\infty p_1, F_{-i}, -\infty p_1]_{\infty p_2}[\infty p_2, F_{+i}, -\infty p_2]_{\infty p_1}$$
$$\times \; [\infty p_1, -\infty p_1]_{-\infty p_2}[-\infty p_2, \infty p_2]_{-\infty p_1} \; + \; \text{cyclic perm.}$$

$$[\infty p_1, F_{-i}, -\infty p_1] \equiv \int_{-\infty}^{\infty} du [\infty p_1, u p_1] F_{-i}(u p_1)[u p_1, -\infty p_1]$$

Gauge-invariant form of the effective action



$$S_{\text{eff}}(V_1, V_2, U_1, U_2; \Delta\eta) \;=$$

$$(\mathcal{V}_1 - \mathcal{V}_2)^{ai}(\mathcal{U}_1 - \mathcal{U}_2)^a_i \;+\; \frac{\alpha_s \Delta\eta}{4} L^a_i L^{ai}$$

$$L^a_i \;=\; 2(\mathcal{W}_F - \mathcal{W}_L - \mathcal{W}_R + \mathcal{W}_B)^a_i$$
$$\;=\; 2(E_F - E_L - E_R + E_B)^a_i$$

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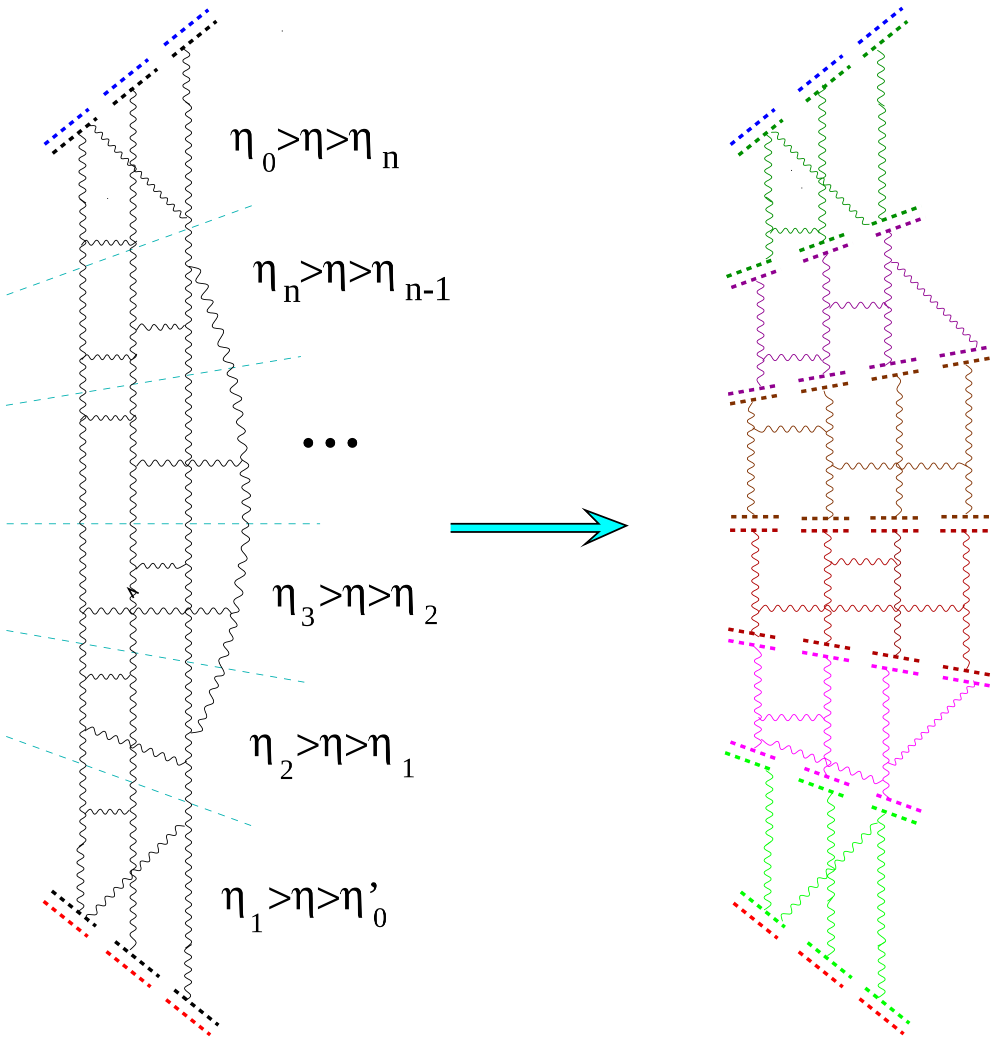
$$\times\; [\infty p_1, -\infty p_1]_{-\infty p_2}[-\infty p_2, \infty p_2]_{-\infty p_1} \;+\; \text{cyclic perm.}$$

$$[\infty p_1, F_{-i}, -\infty p_1] \equiv \int_{-\infty}^{\infty} du [\infty p_1, up_1] F_{-i}(up_1)[up_1, -\infty p_1]$$

Functional integral over the Wilson-line variables



$$e^{(\mathcal{V}_1^{\eta_n}-\mathcal{V}_2^{\eta_n})^a_i(\mathcal{U}_1^{\eta_n}-\mathcal{U}_2^{\eta_n})^{ai}}$$

$$=\frac{1}{\mathcal{N}}\int D\tilde{U}_1^{\eta_n}\,D\tilde{U}_2^{\eta_n}\,D\tilde{V}_1^{\eta_n}\,D\tilde{V}_2^{\eta_n}$$

$$\exp\Big\{(\mathcal{V}_1^{\eta_n}-\mathcal{V}_2^{\eta_n})^a_i(\tilde{\mathcal{U}}_1^{\eta_n}-\tilde{\mathcal{U}}_2^{\eta_n})$$

$$-(\tilde{\mathcal{U}}_1^{\eta_n}-\tilde{\mathcal{U}}_2^{\eta_n})^{ai}(\tilde{\mathcal{V}}_1^{\eta_n}-\tilde{\mathcal{V}}_2^{\eta_n})^{ai}$$

$$+(\tilde{\mathcal{V}}_1^{\eta_n}-\tilde{\mathcal{V}}_2^{\eta_n})^{ai}(\mathcal{U}_1^{\eta_n}-\mathcal{U}_2^{\eta_n})^{ai}\Big\}$$

$$\int_{\eta_{n-1}}^{\eta_n} DA\,e^{iS+i(\tilde{\mathcal{V}}_1^{\eta_n}-\tilde{\mathcal{V}}_2^{\eta_n})^a_i(\mathcal{U}_1^{\eta_n}-\mathcal{U}_2^{\eta_n})^{ai}+i(\mathcal{V}_1^{\eta_{n-1}}-\mathcal{V}_2^{\eta_{n-1}})^a_n(\tilde{\mathcal{U}}_1^{n-1}-\tilde{\mathcal{U}}_2^{n-1})^{ai}}$$

$$=\ e^{i(\tilde{\mathcal{V}}_1^{\eta_n}-\tilde{\mathcal{V}}_2^{\eta_n})^a_i(\tilde{\mathcal{U}}_1^{n-1}-\tilde{\mathcal{U}}_2^{n-1})^{ai}+\frac{\alpha_s}{4}L_i(\tilde{\mathcal{V}}^{\eta_n},\tilde{\mathcal{U}}^{n-1})L_i(\tilde{\mathcal{V}}^{\eta_n},\tilde{\mathcal{U}}^{n-1})}$$

Functional integral over the Wilson-line variables

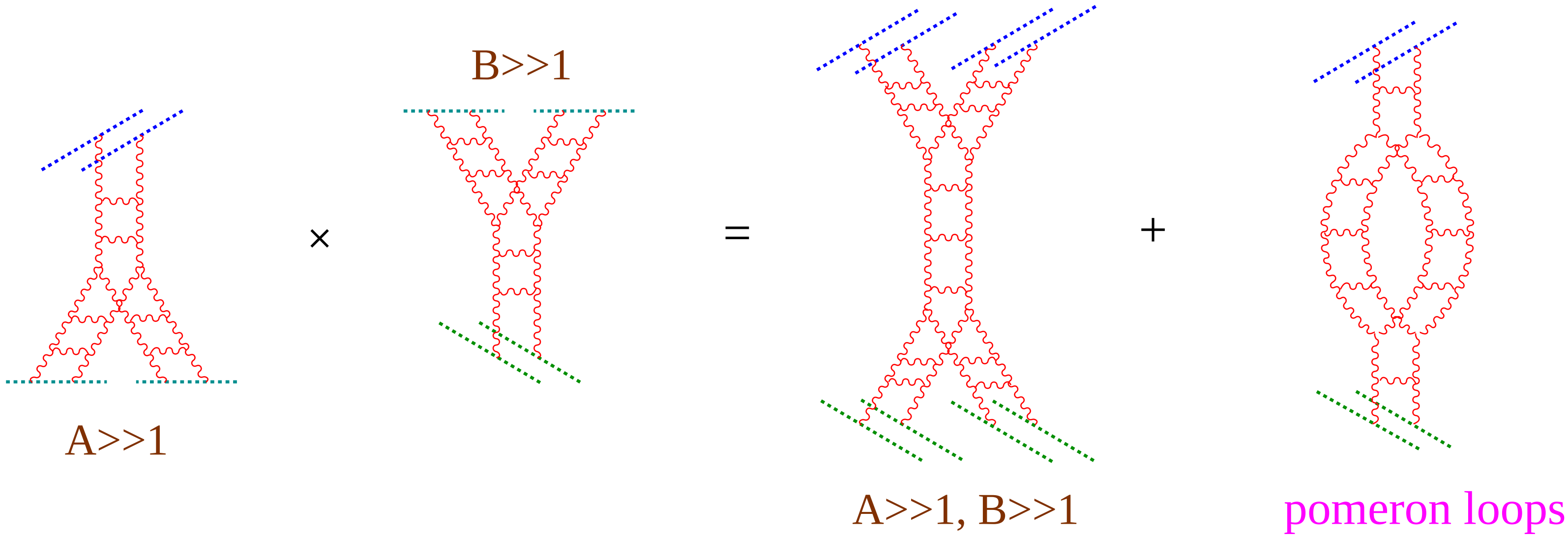
$$e^{iS_{\rm eff}(U_1,U_2,V_1,V_2;\eta_1-\eta_2)} =$$

$$\int_{U_{\eta_2}=U} DU^\eta_{1x} DU^\eta_{2x} DV^\eta_{1x} DV^\eta_{2x}\; e^{i\int d^2x_\perp [(\mathcal{V}_1-\mathcal{V}_2)^a_i(\mathcal{U}_1^{\eta_1}-\mathcal{U}_2^{\eta_1})^{ai}+\int_{\eta_2}^{\eta_1} d\eta\,\mathcal{L}(U_1,U_2,V_1,V_2,\eta)]}$$

$$\mathcal{L}(U_k,V_k,\eta) = -(\mathcal{V}_1^\eta - \mathcal{V}_2^\eta)^a_i \frac{\partial}{\partial\eta}(\mathcal{U}_1^\eta - \mathcal{U}_2^\eta)^{ai} - i\frac{\alpha_s}{4} L^a_i(U,V) L^{ai}(U,V)]\Big\}$$

L_i is local in terms of W's but unfortunately non-local in terms of U and V.

This formula contains both “upside down” and “bottom up” small-x “fan” evolutions $\Rightarrow$ pomeron loops



$\mathrm{S}_{\mathrm{eff}} \Rightarrow$ non-linear evolution eqn: proof

At small V

$$L_i = -2(U_1^\dagger \frac{p_i p^k}{p_\perp^2} U_1 - U_2^\dagger \frac{p_i p^k}{p_\perp^2} U_2)^{ab} (\mathcal{V}_1 - \mathcal{V}_2)_k^b$$

$\Rightarrow$ the integral over V is gaussian.

Rewrite

$$e^{iS_{\mathrm{eff}}(U_1,U_2,V_1,V_2;\eta_1-\eta_2)} =$$

$$\int_{U_{\eta_2}=U} DU_{1x}^\eta DU_{2x}^\eta DV_{1x}^\eta DV_{2x}^\eta D\lambda_i^{a,\eta}\; e^{i\int d^2x_\perp (\mathcal{V}_1-\mathcal{V}_2)_i^a (\mathcal{U}_1^{\eta_1}-\mathcal{U}_2^{\eta_1})^{ai}}$$

$$e^{[i(\mathcal{V}_1-\mathcal{V}_2)_i^a(\mathcal{U}_1^{\eta_1}-\mathcal{U}_2^{\eta_1})^{ai}+\int_{\eta_2}^{\eta_1}d\eta\int d^2x_\perp\{-i(\mathcal{V}_1^\eta-\mathcal{V}_2^\eta)_i^a\frac{\partial}{\partial\eta}(\mathcal{U}_1^\eta-\mathcal{U}_2^\eta)^{ai}-\alpha_s\lambda_\eta^{ai}\lambda_i^{a,\eta}+\alpha_s\lambda_\eta^{ai}L_i^a(U,V)\}]}$$

λ_η^{ai} = "white noise" : $\qquad 2\alpha_s\langle\lambda_\eta^{ai}(x_\perp)\lambda_{\eta'}^{ai}(y_\perp)\rangle = \delta^2(x-y)_\perp\delta(\eta-\eta')$

Integral over $V \Rightarrow$

$$\delta(\partial_i[\dot{\mathcal{U}}_1^i - \dot{\mathcal{U}}_2^i - \frac{ig}{2\pi}(U_1^\dagger\frac{\partial_i\partial_k}{\partial_\perp^2}U_1 - U_2^\dagger\frac{\partial_i\partial_k}{\partial_\perp^2}U_2)\lambda^k])$$

$$\dot{\mathcal{U}}^{ai}_{1\eta}-\dot{\mathcal{U}}^{ai}_{2\eta}=\frac{ig}{2\pi}\Big(U^\dagger_{1\eta}\frac{\partial_i\partial_k}{\partial^2_\perp}U_{1\eta}-U^\dagger_{2\eta}\frac{\partial_i\partial_k}{\partial^2_\perp}U_{2\eta}\Big)^{ab}\lambda^{kb}_\eta$$

Solution: $U^\eta_i=e^{2\alpha_s t^a\int_{\eta_B}^\eta d\eta'\,\frac{\partial_i}{\partial^2_\perp}(U^{ab}_{i\eta'}\lambda^{ib}_{\eta'})}U_i$

For the dipole

$$\mathrm{tr}(U_{1x}U^\dagger_{2x}U_{2y}U^\dagger_{1y})^\eta=\int D\lambda^{a,\eta}_i\; e^{-\alpha_s\int_{\eta_B}^\eta d\eta\int d^2x_\perp\lambda^{ai}_\eta\lambda^{a,\eta}_i}$$
$$\times\,\mathrm{tr}\{e^{2\alpha_s t^a\int_{\eta_B}^\eta d\eta'\,\frac{\partial_i}{\partial^2_\perp}(U^{ab}_{1x\eta'}\lambda^{ib}_{\eta'})}U_{1x}U^\dagger_{2x}e^{-2\alpha_s t^a\int_{\eta_B}^\eta d\eta'\,\frac{\partial_i}{\partial^2_\perp}(U^{ab}_{2x\eta'}\lambda^{ib}_{\eta'})}$$
$$\times\, e^{2\alpha_s t^a\int_{\eta_B}^\eta d\eta'\,\frac{\partial_i}{\partial^2_\perp}(U^{ab}_{2y\eta'}\lambda^{ib}_{\eta'})}U_{2y}U^\dagger_{1y}e^{-2\alpha_s t^a\int_{\eta_B}^\eta d\eta'\,\frac{\partial_i}{\partial^2_\perp}(U^{ab}_{1y\eta'}\lambda^{ib}_{\eta'})}\}\quad\Rightarrow$$

$$\frac{\partial}{\partial\eta}\mathrm{tr}(U_{1x}U^\dagger_{2x}U_{2y}U^\dagger_{1y})=\int dz_\perp\frac{(x-y)^2}{(x-z)^2(z-y)^2}\Big[-N_c\mathrm{tr}(U_{1x}U^\dagger_{2x}U_{2y}U^\dagger_{1y})$$
$$+\,\mathrm{tr}(U_{1x}U^\dagger_{2x}U_{2z}U^\dagger_{1z})\mathrm{tr}(U_{1z}U^\dagger_{2z}U_{2y}U^\dagger_{1y})\Big]$$

Conclusions

- High-energy hadron-hadron scattering $\Leftrightarrow$ collision of two QCD shock waves (Color Glass Condensates?)
- For two nuclei, A and B, the expansion in commutators of Wilson lines is a symmetic expansion in both $\frac{B}{A}$ or $\frac{A}{B}$ parameters.
- $\mathcal{L}(U,V) \ \ni$ pomeron loops ($\Rightarrow$ unitarity?)

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Outlook

- The $[U,V]^2$ term in $\mathcal{L}$
- Big Q: What is the field produced by the collision (in all orders in $[U,V]$)?
- $\Leftrightarrow$ Big Q: $S_{\rm eff}$ in (in all orders in $[U,V]$) ?

Addendum

An example of Feynman diagram not taken into account by the diamond action