I. A FEW WORDS ABOUT MEASUREMENTS

A. Overlap integral

At any given time, if a particle is in the state described by wavefunction $\psi_a(x)$, the probability to discover it in the state described by $\psi_b(x)$ is

$$P = |A_{ab}|^2, \qquad A_{12} \equiv \int dx\ \psi_b^*(x)\psi_a(x) \tag{1}$$

The amplitude A_{ab} is sometimes called an overlap amplitude or overlap integral.

B. Example

Problem:

A free particle is described by a Gaussian wave packet $\psi(x,t)$ such that at $t \to 0-$ it has the form

$$\psi_g(x,0-) = \frac{(2\pi)^{-1/4}}{\sqrt{\Delta x}} e^{-\frac{x^2}{4\Delta x^2}} e^{ikx} \tag{2}$$

where $k = \frac{p}{\hbar}$. Suddenly, at $t = 0$ some interaction switches on so the particle find itself in the harmonic oscillator potential $V(x) = \frac{m\omega^2 x^2}{2}$. Find the probability that the particle will end up in the first excited state of harmonic oscillator.

Solution:

The wavefunction of the first excited state of harmonic oscillator has the form

$$\psi_1(x) = \sqrt{\frac{2m\omega}{\hbar}} \Big(\frac{m\omega}{\pi\hbar}\Big)^{\frac{1}{4}} x e^{-\frac{m\omega}{2\hbar}x^2} \tag{3}$$

so the overlap integral is

$$\begin{aligned} A &= \int dx\ \psi_g^*(x)\psi_1(x) = \int dx\ \frac{(2\pi)^{-1/4}}{\sqrt{\Delta x}} e^{-\frac{x^2}{4\Delta x^2}} e^{ikx} \sqrt{\frac{2m\omega}{\hbar}} \Big(\frac{m\omega}{\pi\hbar}\Big)^{\frac{1}{4}} x e^{-\frac{m\omega}{2\hbar}x^2} \\ &= \frac{(2\pi)^{-1/4}}{\sqrt{\Delta x}} \sqrt{\frac{2m\omega}{\hbar}} \Big(\frac{m\omega}{\pi\hbar}\Big)^{\frac{1}{4}} \int dx\ x e^{-x^2\left(\frac{1}{4\Delta x^2} + \frac{m\omega^2}{2}\right) + ikx} \end{aligned} \tag{4}$$

As usual, the gaussian integral of the type $\int dx\ e^{-ax^2+bx}$ is calculated by shift $x \to x + \frac{b}{2a}$

$$\begin{aligned} \int dx\ x e^{-ax^2+bx} &= \int dx\ x e^{-a\left(x-\frac{b}{2a}\right)^2 + \frac{b^2}{4a}} \\ &= e^{\frac{b^2}{4a}} \int dx\ \Big(x + \frac{b}{2a}\Big) e^{-ax^2} = \frac{b}{2a} e^{\frac{b^2}{4a}} \int dx\ e^{-ax^2} = \frac{b}{2a}\sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}} \end{aligned} \tag{5}$$

For the integral in the r.h.s. of Eq. (4) this yields

$$\int dx\ xe^{-x^2\left(\frac{1}{4\Delta x^2}+\frac{m\omega^2}{2}\right)+ikx} = \frac{ik\sqrt{\pi}}{2\left(\frac{1}{4\Delta x^2}+\frac{m\omega^2}{2}\right)^{\frac{3}{2}}}e^{-\frac{k^2\Delta x^2}{1+m\omega^2\Delta x^2}} \tag{6}$$

and therefore the overlap integral is

$$A = \frac{ik\sqrt{\pi}}{2\left(\frac{1}{4\Delta x^2}+\frac{m\omega^2}{2}\right)^{\frac{3}{2}}}\frac{(2\pi)^{-1/4}}{\sqrt{\Delta x}}\sqrt{\frac{2m\omega}{\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}e^{-\frac{k^2\Delta x^2}{1+m\omega^2\Delta x^2}} \tag{7}$$

Correspondingly, the amplitude to discover particle in the first excited state of the harmonic oscillator at $t = 0$ is

$$P = |A|^2 = \frac{k^2}{\left(\frac{1}{4\Delta x^2}+\frac{m\omega^2}{2}\right)^3}\frac{1}{\sqrt{2}\Delta x}\left(\frac{m\omega}{\hbar}\right)^{\frac{3}{2}}e^{-\frac{2k^2\Delta x^2}{1+m\omega^2\Delta x^2}} \tag{8}$$

Since eigenstates of harmonic oscillator are stable, the probability to discover the particle in the first excited state will be given by Eq. (8) at any $t \geq 0$.

C. Measurement

P.A.M. Dirac: "A measurement always causes the system to jump in the eigenstate of the dynamical variable being measured".

Before a measurement of observable $\mathcal{O}$ is made, the system is assumed to be represented by some linear combination of eigenfunctions of operator $\hat{\mathcal{O}}$

$$\Psi(x,t) = \sum c_n\psi_n(x,t), \qquad \hat{\mathcal{O}}\psi_n(x,t) = O_n\psi_n(x,t) \tag{9}$$

When the measurement is performed, the system is "thrown into" one of the eigenstates of $\hat{\mathcal{O}}$, say $\psi_m(x,t)$ of the observable $\mathcal{O}$:

$$\Psi(x,t) \xrightarrow{\text{measurement}} \psi_m(x,t) \tag{10}$$

A measurement usually changes the wave function of the state. The only exception is when the state is already in one of the eigenstates of the observable being measured, in which case

$$\psi_m(x,t) \xrightarrow{\text{measurement}} \psi_m(x,t) \tag{11}$$

When the measurement causes the state $\Psi(x,t)$ to change to $\psi_m(x,t)$, it is said that $\mathcal{O}$ is measured to be $\mathcal{O}_m$.

$\Rightarrow$ The result of the measurement yields one of the eigenvalues of the observable being measured and the particle is in the corresponding eigenstate after the measurement.