Problem 1. True (T) or false (F):

1. Schrödinger equation tells us how the wave function depends on time. **T**
2. Eigenfunction of the Hamiltonian describe systems for which physical observables do not depend on time. **T**
3. Heisenberg uncertainty relation states that $\Delta p \Delta x$ is always equal to $\hbar/2$. **F**
4. The following is a possible eigenstate of the hydrogen atom: n=3,l=2,m=-3. **F**

Problem 2.

Two electrons are accelerated from rest through a potential differences 1V and 1kV, respectively. What is the ratio of their de Broglie wavelengths?

Solution

$$mc^2 + eU = \sqrt{p^2c^2 + m^2c^4} \quad \Rightarrow \quad p^2c^2 \; = \; e^2U^2 + 2eUmc^2 \quad \rightarrow \quad p \; = \; \sqrt{2meU + \frac{e^2U^2}{c^2}}$$

so

$$\frac{\lambda_1}{\lambda_2} = \frac{p_2}{p_1} = \sqrt{\frac{2meU_2 + \frac{e^2U_2^2}{c^2}}{2meU_1 + \frac{e^2U_1^2}{c^2}}} = \sqrt{\frac{2mc^2eU_2 + e^2U_2^2}{2mc^2eU_1 + e^2U_1^2}} = \sqrt{\frac{1.02 \times 10^9 + 10^6}{1.02 \times 10^6 + 1}} \simeq \sqrt{1000} \simeq 31.6$$

Actually, since in both cases $eU \ll mc^2$ one can use non-relativistic formula $p = \sqrt{2meU}$ with the same result.

Problem 3.

A particle in a one-dimensional box of size L(with infinite walls) is in equal admixture of two lowest states so that at $t = 0$

$$\psi(x, 0) \; = \; \frac{1}{\sqrt{2}}(\psi_1(x) + \psi_2(x))$$

where $\psi_n = \sqrt{\frac{2}{L}} \sin \frac{\pi n x}{L}$. Find the probability that the particle will be found in the left-hand half of the wall at a later time t.

Integrals

$$\int_0^{\frac{\pi}{2}} dx \;\; \sin^2 x \; = \; \int_0^{\frac{\pi}{2}} dx \;\; \sin^2 2x \; = \; \frac{\pi}{4}$$
$$\int_0^{\frac{\pi}{2}} dx \;\; \sin x \sin 2x \; = \; \frac{2}{3}$$

Solution

$$\int_0^{\frac{L}{2}} dx\ |\psi(x)|^2 \ = \ \frac{1}{L}\int_0^{\frac{L}{2}} dx\ |\sin\frac{\pi x}{L}e^{-iE_1t} + \sin\frac{2\pi x}{L}e^{-iE_2t}|^2$$
$$= \ \frac{1}{\pi}\int_0^{\frac{\pi}{2}} dy\ |e^{-iE_1t}\sin y + e^{-iE_2t}\sin 2y|^2 \ = \ \frac{1}{\pi}\int_0^{\frac{\pi}{2}} dy\ (e^{-iE_1t}\sin y + e^{-iE_2t}\sin 2y)(e^{iE_1t}\sin y + e^{iE_2t}\sin 2y)$$
$$= \ \frac{1}{\pi}\int_0^{\frac{\pi}{2}} dy\ [\sin^2 y + (e^{-iE_2t} + e^{iE_2t})\sin y \sin 2y + \sin^2 2y] \ = \ \frac{1}{2} + \frac{4}{3\pi}\cos(E_2 - E_1)t$$

Problem 4.

The electron in a hydrogen atom is in the state described by wave function

$$\frac{1}{\sqrt{2}}(\psi_{100}e^{-i\frac{E_1}{\hbar}t} + \psi_{211}e^{-i\frac{E_2}{\hbar}t}), \qquad\qquad \psi_{nlm} \ = \ R_{nl}(r)Y_{lm}(\theta,\phi)$$

(here we disregard spin of the electron). What is the expectation value of L_z in this state at time t?

Solution

Solution #1

By definition, $\langle\hat{L}_z\rangle \ = \ \int d^3x\ \psi^*(\vec{r},t)\hbar(-i\frac{\partial}{\partial\phi})\psi(\vec{r},t)$

$$\langle\hat{L}_z\rangle \ = \ \frac{1}{2}\int d^3x(\psi^*_{100}e^{i\frac{E_1}{\hbar}t} + \psi^*_{211}e^{i\frac{E_2}{\hbar}t})\hbar(-i\frac{\partial}{\partial\phi})(\psi_{100}e^{-i\frac{E_1}{\hbar}t} + \psi_{211}e^{-i\frac{E_2}{\hbar}t})$$
$$= \ -\frac{i\hbar}{2}\int d^3x\ \psi^*_{211}\frac{\partial}{\partial\phi}\psi_{211} \ = \ -\frac{i\hbar}{2}\int r^2dr\int_0^{\pi}d\theta\ \sin\theta\int_0^{2\pi}d\phi\ R_{21}(r)Y^*_{11}(\theta\phi)\frac{\partial}{\partial\phi}R_{21}(r)Y_{11}(\theta\phi)$$
$$= \ -\frac{i\hbar}{2}\int r^2dr\ \frac{r^2}{24a_0^5}e^{-\frac{r}{a_0}}\int_0^{\pi}d\theta\frac{3}{8\pi}\sin^3\theta\int_0^{2\pi}d\phi\ e^{-i\phi}\frac{\partial}{\partial\phi}e^{i\phi} \ = \ \frac{\hbar}{2}\int dr\ \frac{r^4}{24a_0^5}e^{-\frac{r}{a_0}}\ \frac{3}{4}\int_0^{\pi}d\theta\sin^3\theta \ = \ \frac{\hbar}{2}$$

Solution #2

$\hat{L}_z\psi_{nlm} \ = \ m\hbar\psi_{nlm} \Rightarrow \hat{L}_z\psi_{100} \ = \ 0,\ \hat{L}_z\psi_{211} \ = \ \hbar\psi_{211}$. Recalling orthogonality of the eigenfunctiopns (Eq. (59) of Supplemental note “Schrödinger Eqn. in 3 dimensions”) we get

$$\langle\hat{L}_z\rangle \ = \ \frac{1}{2}\int d^3x(\psi^*_{100}e^{i\frac{E_1}{\hbar}t}+\psi^*_{211}e^{i\frac{E_2}{\hbar}t})\hat{L}_z(\psi_{100}e^{-i\frac{E_1}{\hbar}t}+\psi_{211}e^{-i\frac{E_2}{\hbar}t}) \ = \ \frac{1}{2}\int d^3x\ \hbar|\psi_{211}(\vec{r})|^2 \ = \ \frac{\hbar}{2}$$