Problem 1.1 *Using Eq. (1.38) calculate the approximate total cross sections for Rutherford scattering of a* 10 MeV *α-particle from a lead nucleus for impact parameters b less than* 10^{-12}, 10^{-10} *and* 10^{-8} cm. *How well do these agree with the values of πb^2?*

There are various ways of doing this problem. We will list below two very simple methods.

Method I. In general, the total cross section for Rutherford scattering is given by (see Eq. (1.38) in the text)

$$\sigma_{\mathrm{TOT}} = 8\pi \left(\frac{ZZ'e^2}{4E}\right)^2 \int_0^1 \frac{d\left(\sin\frac{\theta}{2}\right)}{\left(\sin\frac{\theta}{2}\right)^3}. \tag{1.1}$$

However, if the impact parameter is restricted to a finite range, say $b \leq b_0$, then we can write the total cross section as

$$\sigma_{\mathrm{TOT}}(b_0) = 8\pi \left(\frac{ZZ'e^2}{4E}\right)^2 \int_{\theta_{b_0}}^1 \frac{d\left(\sin\frac{\theta}{2}\right)}{\left(\sin\frac{\theta}{2}\right)^3}, \tag{1.2}$$

where θ_{b_0} is the scattering angle corresponding to the impact parameter b_0 and is given by (see Eq. (1.32) of the text)

$$b_0 = \frac{ZZ'e^2}{2E} \cot\frac{\theta_{b_0}}{2}. \tag{1.3}$$

Carrying out the integration in (1.2), we obtain

$$\begin{aligned}\sigma_{\mathrm{TOT}}(b_0) &= 8\pi \left(\frac{ZZ'e^2}{4E}\right)^2 \left(-\frac{1}{2}\right)\left(1 - \operatorname{cosec}^2\frac{\theta_{b_0}}{2}\right)\\ &= 4\pi \left(\frac{ZZ'e^2}{4E}\right)^2 \cot^2\frac{\theta_{b_0}}{2}\\ &= \pi \left(\frac{ZZ'e^2}{2E}\cot\frac{\theta_{b_0}}{2}\right)^2 = \pi b_0^2,\end{aligned} \tag{1.4}$$

where we have used the identification in (1.3). It follows, therefore, that

b_0 (cm)	$\sigma_{\mathrm{TOT}}(b_0) = \pi b_0^2$ (cm^2)
10^{-12}	3.2×10^{-24}
10^{-10}	3.2×10^{-20}
10^{-8}	3.2×10^{-16}

Method II. An alternative method to obtain the same result is to note that the total cross section for Rutherford scattering can be written as

$$\begin{aligned}\sigma_{\mathrm{TOT}} &= 8\pi \left(\frac{ZZ'e^2}{4E}\right)^2 \int_0^1 \frac{d\left(\sin\frac{\theta}{2}\right)}{\left(\sin\frac{\theta}{2}\right)^3}\\ &= 4\pi \left(\frac{ZZ'e^2}{4E}\right)^2 \int_0^1 d\theta \cot\frac{\theta}{2}\operatorname{cosec}^2\frac{\theta}{2}.\end{aligned} \tag{1.5}$$

This can be converted into an integral over the impact parameters using the defining relationship (see Eqs. (1.32) and (1.36))

$$b = \frac{ZZ'e^2}{2E}\cot\frac{\theta}{2}, \quad \frac{db}{d\theta} = -\frac{ZZ'e^2}{4E}\operatorname{cosec}^2\frac{\theta}{2}, \tag{1.6}$$

so that we can write

$$\begin{aligned}\sigma_{\mathrm{TOT}} &= 4\pi \left(\frac{ZZ'e^2}{4E}\right)^2 \int_0^\infty db \left(\frac{ZZ'e^2}{4E}\right)^{-1} b \left(\frac{ZZ'e^2}{2E}\right)^{-1}\\ &= 2\pi \int_0^\infty db\, b.\end{aligned} \tag{1.7}$$

This is true in general and can also be deduced from the definition of the cross section in Eq. (1.33) or (1.34) of the text. If impact parameters are smaller than some fixed value, say b_0, then the total cross section takes the form

$$\sigma_{\mathrm{TOT}}(b_0) = 2\pi \int_0^{b_0} db\, b = \pi b_0^2, \tag{1.8}$$

which is the same result as derived earlier.