CHAPTER 1
Rutherford Scattering

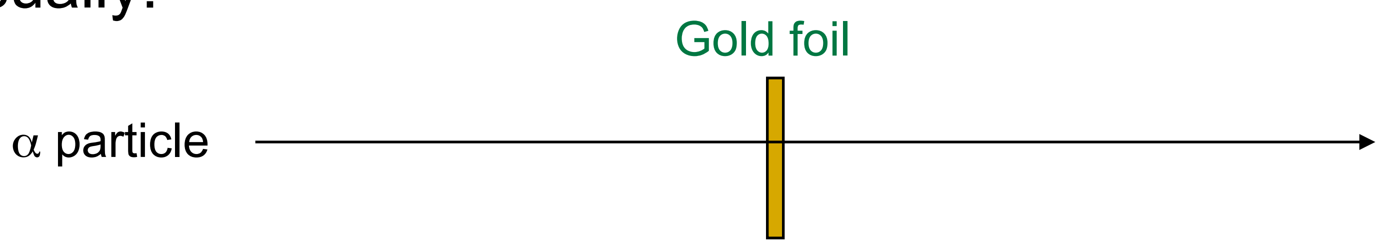
Lecture Notes For
PHYS 415
Introduction to Nuclear and Particle Physics

To Accompany the Text
Introduction to Nuclear and Particle Physics, 2nd Ed.
A. Das and T. Ferbel
World Scientific

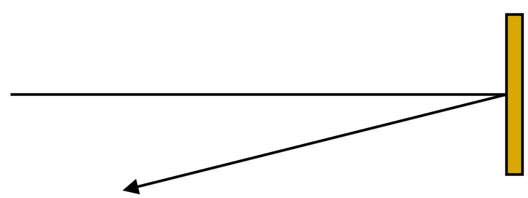
Rutherford Scattering

Scattered α particles from thin foils

Usually:



But occasionally (rarely):



Effect of atomic electrons

Mass of $\alpha \approx 4\times10^3$ MeV/c^2

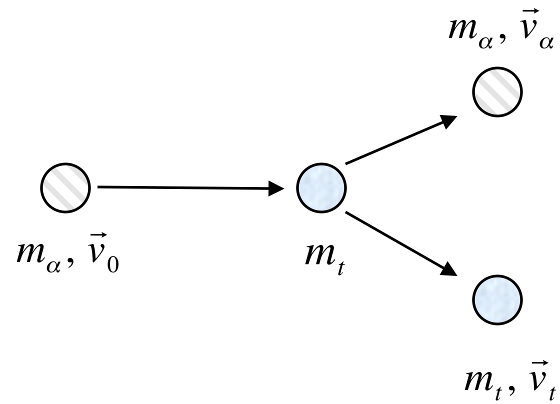
Mass of $e^- \approx 0.5$ MeV/c^2

Electrons cannot cause large deviations.

Further, if mass of the atom is spread throughout the atomic volume, only minor deflections would be observed.

Let's analyze more carefully …

Analysis of the collision



Momentum conservation:

$$m_\alpha \vec{v}_0 = m_\alpha \vec{v}_\alpha + m_t \vec{v}_t$$

$$\Rightarrow \vec{v}_0 = \vec{v}_\alpha + \frac{m_t}{m_\alpha} \vec{v}_t$$

Energy conservation:

$$\frac{1}{2} m_\alpha v_0{}^2 = \frac{1}{2} m_\alpha v_\alpha{}^2 + \frac{1}{2} m_t v_t{}^2$$

$$\Rightarrow v_0{}^2 = v_\alpha{}^2 + \frac{m_t}{m_\alpha} v_t{}^2$$

Analysis, cont'd.

Combining these equations, gives

$$v_t^{\,2}\left(1-\frac{m_t}{m_\alpha}\right)=2\vec{v}_\alpha\cdot\vec{v}_t$$

- For $m_\alpha >> m_t$, LHS > 0 and motion of α is along incident direction.
- For $m_\alpha << m_t$, LHS < 0 and motion of α is along backward direction.
- For electrons as target, the first condition applies and hence backward scattering does not occur.

Other comments on the scattering

- If mass and charge are evenly distributed, the theory predicts only small scattering angles with prob(θ >90) ~ 10^{-3000}
- Experiment revealed prob(θ >90º) ~ 10^{-4}
- If one assumes scattering from a single electron in the material, θ_{max}=0.016º. Considering the # of atoms (~2300) across the thickness of the foil, this number increases to ($\sqrt{2300}$) × 0.016º = 0.8º (assuming one electron per atom can scatter).
- Even if the α particle scattered from all 79 electrons in each atom of gold, $\langle\theta\rangle_{\text{total}} = 6.8°$
- Conclusion: **Only a massive concentrated "nuclear" center can give rise to observed large angle scattering.**

Coulomb Scattering

- Energy/Momentum conservation gives the correct *asymptotic* values for the particles involved in the scattering.
- To go further, we must consider the Coulomb force between the α and the atomic nucleus.
- Both are positively charged, so the force is repulsive.
- First, an aside on **UNITS** …

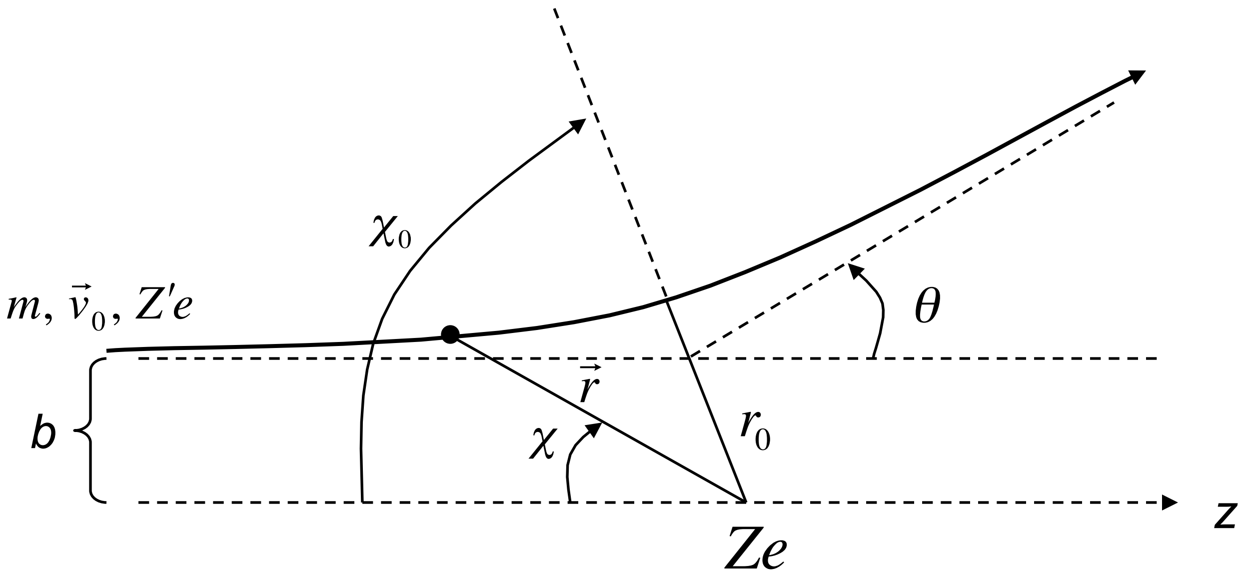
Units Conventions (with some exceptions)

- Mass, Energy, Momentum:
  - eV/c^2, eV, eV/c
  - MeV/c^2, MeV, MeV/c
- Otherwise: “cgs” units: cm, g, s
- Electromagnetic formulae take on simpler forms, but the units of charge are now:

  1esu = 1Stat-Coulomb = 3.34×10^{-10} C

  $\Rightarrow \; e = 4.8 \times 10^{-10}$ esu

Coulomb Scattering Analysis



- Assume nucleus is infinitely massive $\Rightarrow$ no recoil
- Use non-relativistic kinematics
- Target is thin $\Rightarrow$ only one scattering
- Projectile and nucleus are point-like objects
- Consider only the Coulomb force

Scattering from a central potential V(r)

$$\frac{d\chi}{dt} = \frac{\ell}{mr^2}. \tag{1.16}$$

The energy is identical at every point of the trajectory, and can be written as

$$\begin{aligned} E &= \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}mr^2\left(\frac{d\chi}{dt}\right)^2 + V(r) \\ &= \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}mr^2\left(\frac{\ell}{mr^2}\right)^2 + V(r), \end{aligned}$$

$$\text{or} \quad \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 = E - \frac{\ell^2}{2mr^2} - V(r),$$

$$\text{or} \quad \frac{dr}{dt} = -\left[\frac{2}{m}\left(E - V(r) - \frac{\ell^2}{2mr^2}\right)\right]^{\frac{1}{2}}. \tag{1.17}$$

$$
\begin{aligned}
\frac{dr}{dt} &= -\left[\frac{2}{m}\frac{\ell^2}{2mr^2}\left\{\frac{2mEr^2}{\ell^2}\left(1-\frac{V(r)}{E}\right)-1\right\}\right]^{\frac{1}{2}} \\
&= -\frac{\ell}{mr}\left[\frac{r^2}{b^2}\left(1-\frac{V(r)}{E}\right)-1\right]^{\frac{1}{2}} \\
&= -\frac{\ell}{mrb}\left[r^2\left(1-\frac{V(r)}{E}\right)-b^2\right]^{\frac{1}{2}}. \qquad (1.18)
\end{aligned}
$$

$$
\begin{aligned}
d\chi &= \frac{\ell}{mr^2}\,dt = \frac{\ell}{mr^2}\,\frac{dt}{dr}\,dr \\
&= -\frac{\ell}{mr^2}\frac{dr}{\frac{\ell}{mrb}\left[r^2\left(1-\frac{V(r)}{E}\right)-b^2\right]^{\frac{1}{2}}},
\end{aligned}
$$

$$
\text{or} \quad d\chi = -\frac{b\,dr}{r\left[r^2\left(1-\frac{V(r)}{E}\right)-b^2\right]^{\frac{1}{2}}}. \qquad (1.19
$$

$$\int_0^{\chi_0} d\chi = -\int_\infty^{r_0} \frac{b\,dr}{r\left[r^2\left(1-\frac{V(r)}{E}\right)-b^2\right]^{\frac{1}{2}}},$$

$$\text{or}\quad \chi_0 = b\int_{r_0}^{\infty} \frac{dr}{r\left[r^2\left(1-\frac{V(r)}{E}\right)-b^2\right]^{\frac{1}{2}}}. \tag{1.20}$$

The point of closest approach is determined by noting that, as the particle approaches from infinity, its velocity decreases continuously (assuming the repulsive potential for the case of an α-particle approaching a nucleus), until the point of closest approach, where the radial velocity $\left(\frac{dr}{dt}\right)$ vanishes and subsequently changes sign. That is, beyond this point, the velocity of the particle increases again. Therefore, at the distance of closest approach, when $r = r_0$, both the radial and the absolute velocities attain a minimum, and we have

$$\left.\frac{dr}{dt}\right|_{r=r_0} = 0,$$

which, from Eqs. (1.17) and (1.18), means that

$$E - V(r_0) - \frac{\ell^2}{2mr_0^2} = 0,$$

$$\text{or}\quad r_0^2\left(1-\frac{V(r_0)}{E}\right) - b^2 = 0. \tag{1.21}$$

Scattering angle as a function of the impact parameter

Thus, given a specific form of the potential, we can determine r_0, and therefore χ_0, as a function of the impact parameter b.[3] Defining the scattering angle θ as the change in the asymptotic angles of the trajectory, we get

$$\theta = \pi - 2\chi_0 = \pi - 2b \int_{r_0}^{\infty} \frac{dr}{r\left[r^2\left(1 - \frac{V(r)}{E}\right) - b^2\right]^{\frac{1}{2}}}. \tag{1.22}$$

Impact parameter

Coulomb force: $$V(r) = \frac{ZZ'e^2}{r}$$

This leads to a relation between the impact parameter, *b*, and the incident energy, $E=1/2\, mv_0^2$, and scattering angle, θ:

$$b = \frac{ZZ'e^2}{2E}\cot\frac{\theta}{2}$$

Small *b* corresponds to large angle scattering $\Rightarrow$ large θ is possible since nucleus is treated as point object and we get a large force close to the nucleus.

Distance of closest approach

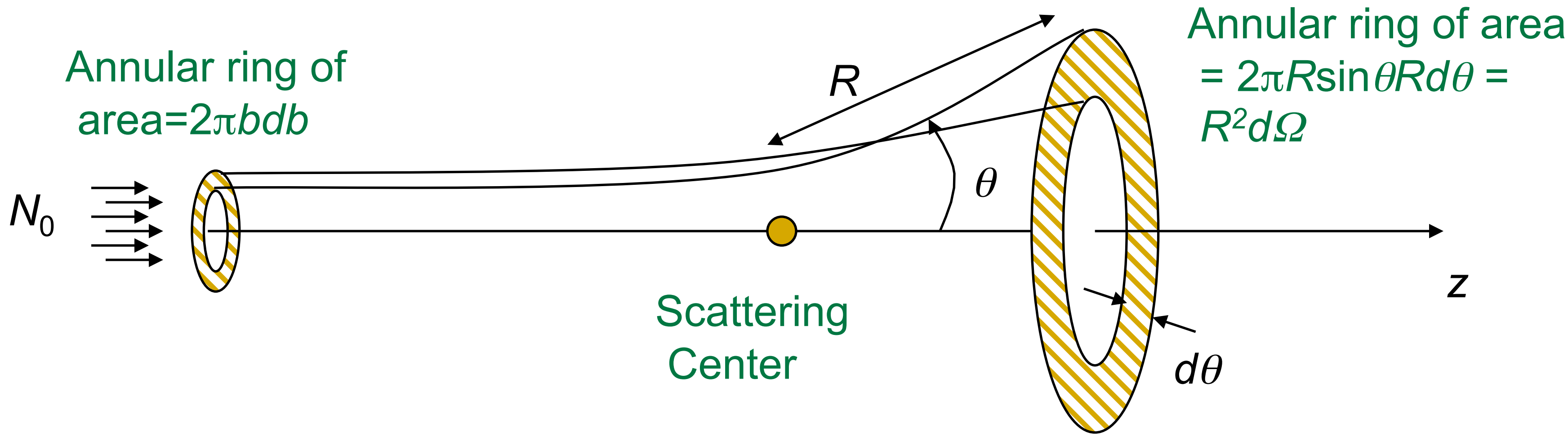
- It can also be shown that the distance of closest approach, r_0, is given by

$$r_0 = \frac{ZZ'e^2}{2E}\left(1 + \sqrt{1 + \frac{4b^2E^2}{\left(ZZ'e^2\right)^2}}\right) \xrightarrow{E\to\infty} b$$

- The previous slide shows that for non-zero θ, $b \to 0$ as $E \to \infty$. Therefore $r_0 \to 0$ as $E \to \infty$.
- Thus, at high enough energy we can approach the nucleus as closely as we wish. The assumption of a point-like nucleus can then be tested.

Scattering Cross Section

- In an actual experiment, detectors are positioned to cover a range of scattering angles and, therefore, impact parameters.



Differential Cross Section

Effective area for scattering into $d\theta$:

$$\Delta\sigma(\theta,\phi) = b\ db\ d\phi = -\frac{d\sigma}{d\Omega}(\theta,\phi)d\Omega = -\frac{d\sigma}{d\Omega}(\theta,\phi)\sin\theta\ d\theta\ d\phi$$

$$\xrightarrow{\text{no } \phi \text{ dependence}} \frac{d\sigma}{d\Omega}(\theta) = -\frac{b}{\sin\theta}\ \frac{db}{d\theta}$$

Entire effect of scattering is given in the θ dependence of the differential cross section, or of the yield.

We can get the explicit dependence on observable parameters for the case of Rutherford scattering (i.e. Coulomb force) …

Rutherford Cross Section

$$\boxed{b = \frac{ZZ'e^2}{2E}\cot\frac{\theta}{2}}$$

$$\boxed{\frac{db}{d\theta} = -\frac{1}{2}\frac{ZZ'e^2}{2E}\operatorname{cosec}^2\frac{\theta}{2}}$$

$$\boxed{\frac{d\sigma}{d\Omega}(\theta) = -\frac{b}{\sin\theta}\frac{db}{d\theta} = \left(\frac{ZZ'e^2}{4E}\right)^2\frac{1}{\sin^4\frac{\theta}{2}}}$$

Total Cross Section

We can get the total cross section by integrating:

$$\sigma_{\mathrm{TOT}} = \int \frac{d\sigma}{d\Omega}(\theta)\, d\Omega = 2\pi \int_0^{\pi} d\theta \sin\theta \frac{d\sigma}{d\Omega}(\theta)$$

$$= 8\pi \left(\frac{ZZ'e^2}{4E} \right)^2 \int_0^1 d\left(\sin\frac{\theta}{2} \right) \frac{1}{\sin^3 \frac{\theta}{2}} \to \infty$$

The infinite result reflects the infinite range of the Coulomb potential. Normally, the detectors exclude very small scattering angles.

Cross Section Units

1 bn = 1 barn = 10^{-24} cm^2

(Typical nuclear radius ~ 10^{-12} cm = 10 fm)

Differential cross section, for example, may be expressed in **mb/sr** units

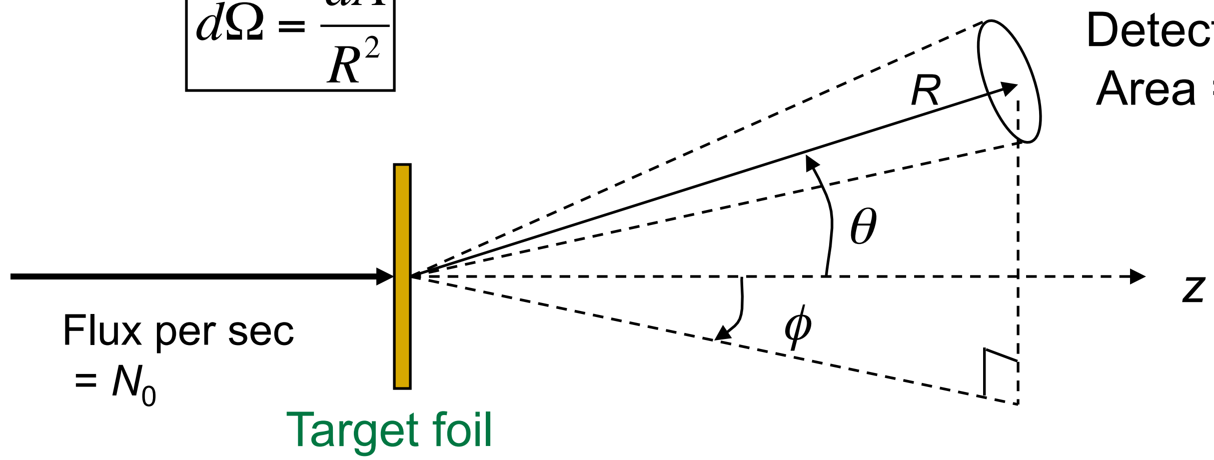
(4π sr = full solid angle about a point)

(1 mb = 1 millibarn = 10^{-3} bn)

Measuring Cross Sections

Consider *dn* particles per unit time, scattering into a solid angle $d\Omega$ at a given θ and ϕ

$$\boxed{d\Omega = \frac{dA}{R^2}}$$



Measuring Cross Sections, cont'd.

nuclei (scatterers) per unit area:

$$\frac{N}{S} = \frac{\rho t}{A} A_0$$

$$\text{where} \begin{cases} \rho = \text{density of foil (g/cm}^3\text{)} \\ t = \text{thickness of foil (cm)} \\ A = \text{mass \# (g/mole)} \\ A_0 = \text{Avogadro's \#} = 6.02\times10^{23}\ \text{mole}^{-1} \\ S = \text{area of foil (cm}^2\text{)} \end{cases}$$

incident particles per unit time = N_0

Measuring Cross Sections, cont'd.

- The cross section has a statistical interpretation, though it's measured in units of area:

$$\boxed{dn = N_0 \frac{N}{S} \frac{d\sigma}{d\Omega}(\theta,\phi)\, d\Omega = N_0 \frac{\rho t}{A} A_0 \frac{d\sigma}{d\Omega}(\theta,\phi)\, d\Omega}$$

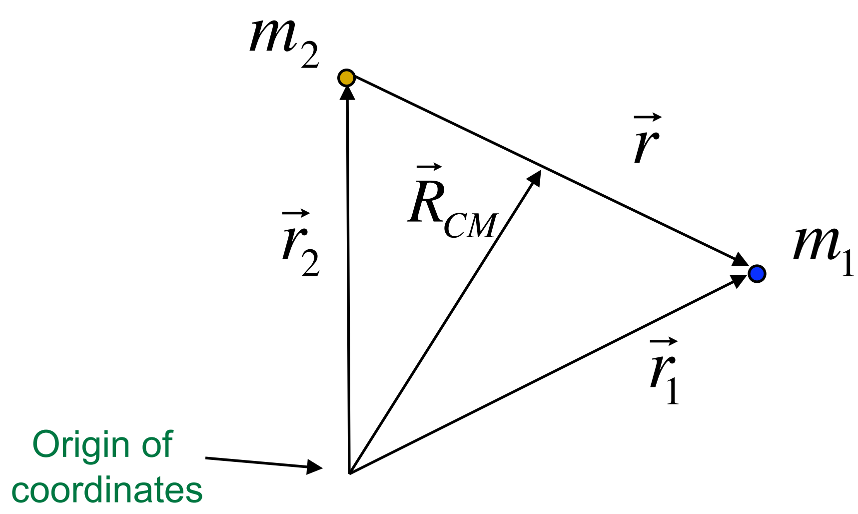
- Geiger and Marsden made such measurements and verified the Rutherford prediction.
- This gave convincing evidence of the hypothesis of a nuclear center, but provided no information about the nature of the nuclear force: the scattering is entirely due to the Coulomb repulsion, as the α particles never penetrated the nucleus.

Laboratory vs. CoM Frame

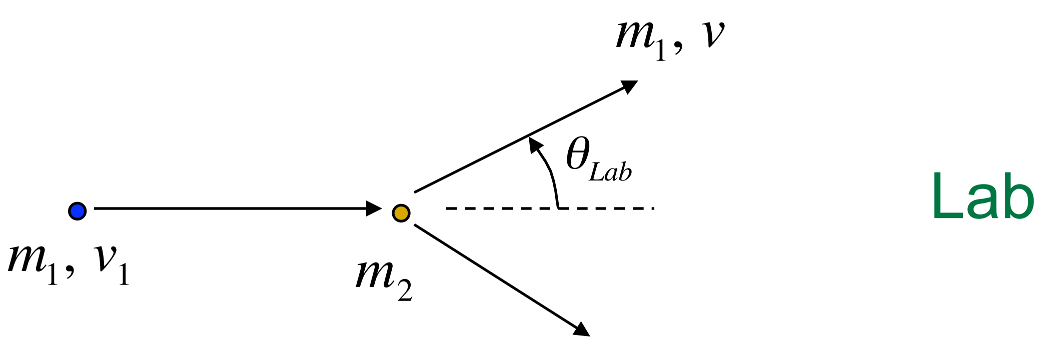
- We have ignored nuclear recoil, supposing that the mass of the target was infinite.
- We can treat this 2-body scattering problem in terms of relative and center-of-mass coordinates.
- The scattering can then be separated for central potentials:
  - The CoM moves at constant velocity.
  - The relative motion can be treated exactly as before, except that the projectile mass is replaced by the “reduced mass”.
- This is especially useful in treating colliding beam experiments.

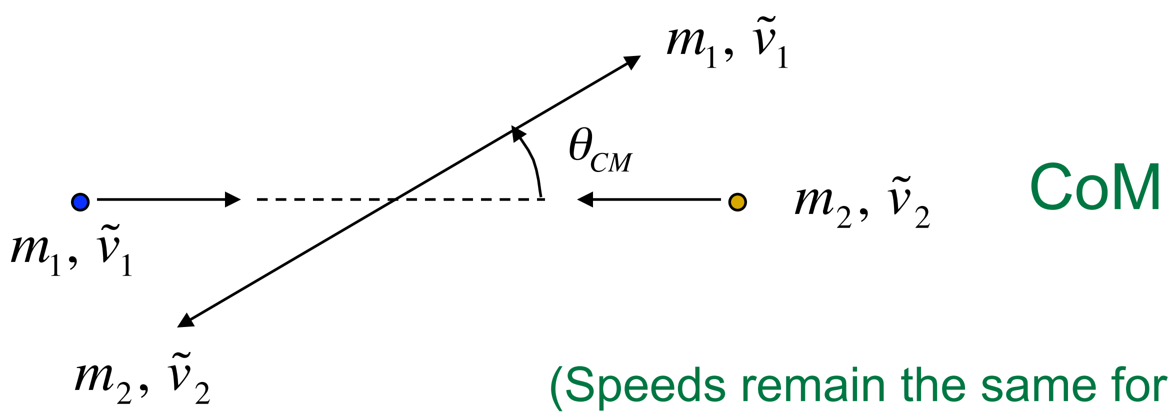
Definitions

Define
$$\begin{cases} \vec{r} = \vec{r}_1 - \vec{r}_2 \;, \quad \vec{R}_{CM} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \\ M = m_1 + m_2 = \text{ total mass} \\ \mu = \dfrac{m_1 m_2}{m_1 + m_2} = \text{ reduced mass} \end{cases}$$



Transforming between Frames





(Speeds remain the same for elastic scattering.)

Transformation Equations

$$v_{CM} = \dot{R}_{CM} = \frac{m_1 v_1}{m_1 + m_2}$$

$$\tilde{v}_1 = v_1 - v_{CM} = \frac{m_2 v_1}{m_1 + m_2}$$

$$\tilde{v}_2 = v_{CM} = \frac{m_1 v_1}{m_1 + m_2}$$

$$\tan\theta_{Lab} = \frac{\sin\theta_{CM}}{\cos\theta_{CM} + \zeta} \quad \text{with } \zeta = \frac{m_1}{m_2}$$

$$\frac{d\sigma}{d\Omega_{Lab}}(\theta_{Lab}) = \frac{d\sigma}{d\Omega_{CM}}(\theta_{CM})\frac{d(\cos\theta_{CM})}{d(\cos\theta_{Lab})}$$

$$= \frac{d\sigma}{d\Omega_{CM}}(\theta_{CM})\frac{(1 + 2\zeta\cos\theta_{CM} + \zeta^2)^{3/2}}{\left|1 + \zeta\cos\theta_{CM}\right|}$$

Relativistic Variables

- So far we have ignored relativity. For most modern nuclear and particle physics experiments, we must treat the kinematics relativistically.
- First, we need the center-of-mass velocity:

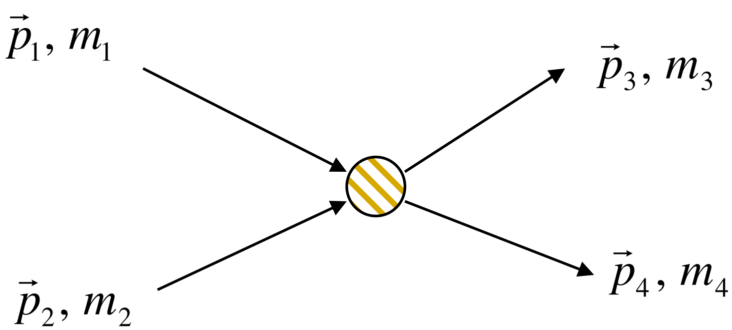
$$\frac{\vec{v}_{CM}}{c} = \vec{\beta}_{CM} = \frac{\vec{P}_1 + \vec{P}_2}{E_1 + E_2} c \xrightarrow{\text{in Lab frame}} \frac{\vec{P}_1 c}{E_1 + m_2 c^2}$$

- Which gives:

$$\gamma_{CM} = \left(1 - \beta_{CM}{}^2\right)^{-1/2} = \frac{E_1 + m_2 c^2}{\left(m_1{}^2 c^4 + m_2{}^2 c^4 + 2 E_1 m_2 c^2\right)^{1/2}}$$

Relativistic Invariants

- Certain quantities are frame-independent; they can therefore be evaluated in any (convenient) frame. Examples are the "Mandelstam" variables, *s*, *t* and *u.*



$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2$$

$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2$$

$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2$$

where the p's are 4 - vectors

and the squares imply 4 - vector dot products

Total CoM Energy

- For 2-body scattering, the total CoM energy is:

$$
\begin{aligned}
s &= \left(E_1 + E_2\right)^2 - \left(\vec{P}_1 + \vec{P}_2\right)^2 c^2 \\
&= m_1{}^2 c^4 + m_2{}^2 c^4 + 2E_1 m_2 c^2 \qquad \text{(in Lab)} \\
&= (E_{1CM} + E_{2CM})^2 = (E_{CM}^{TOT})^2 \quad \text{(in CoM)}
\end{aligned}
$$

- Also: $\gamma_{CM} = \dfrac{E_1 + m_2 c^2}{E_{CM}^{TOT}} = \dfrac{E_{Lab}^{TOT}}{E_{CM}^{TOT}}$

- The total CoM energy may be regarded as the available energy to produce particles, since the motion of the CoM itself is unchanged after scattering.

Four-Momentum Transfer

- The Mandelstam variable, *t*, is just the square of the four-momentum transferred to the target:

$$
\begin{aligned}
t &= \left(E_1^{\,f} - E_1^{\,i}\right)^2 - \left(\vec{P}_1^{\,f} - \vec{P}_1^{\,i}\right)^2 c^2 \\
&= \left(E_2^{\,f} - E_2^{\,i}\right)^2 - \left(\vec{P}_2^{\,f} - \vec{P}_2^{\,i}\right)^2 c^2
\end{aligned}
$$

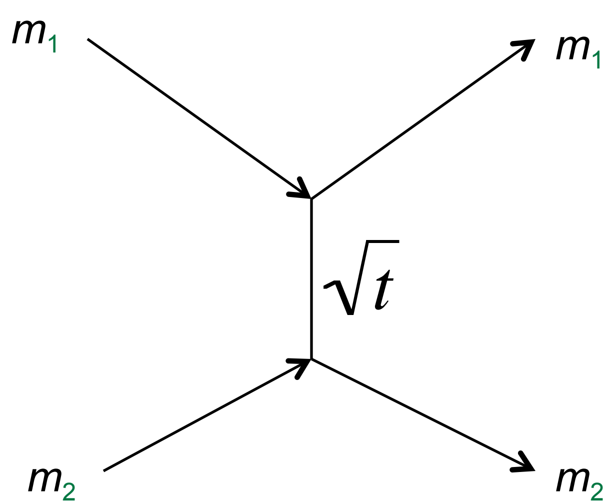
- In elastic scattering viewed from the CoM, each particle's speed, and therefore energy, does not change. Therefore:

$$
\begin{aligned}
t &= -\left({P_{1CM}^{f}}^{2} + {P_{1CM}^{i}}^{2} - 2\vec{P}_{1CM}^{f} \cdot \vec{P}_{1CM}^{i}\right)c^2 \\
&= -2P_{CM}^2 c^2 \left(1 - \cos\theta_{CM}\right) < 0
\end{aligned}
$$

Feynman Diagrams

- From the definition of *t,* we can consider the scattering to take place by the exchange of a particle, of mass *m*, where $t = m^2$.
- Since $t < 0$, the particle has an imaginary rest mass and is therefore called **virtual**.
- Feynman invented pictorial representations of scattering processes, in which each picture has a precise mathematical meaning: a **Feynman diagram**.

Feynman Diagrams, cont'd.



- Such diagrams can be used to calculate scattering amplitudes and cross sections.
- Feynman invented them to calculate processes in QED (quantum electrodynamics) via the technique of perturbation theory.

Interpretation of Four-momentum Transfer

- Define $q^2c^2 = -t$. We can show that:

$$q^2 = 2m_2 T^f_{2Lab} \xrightarrow{v_2 << c} (m_2 v_2)^2$$

- Therefore q is related to the momentum transferred to the target and reflects the "hardness" of the collision. Long-range collisions (soft collisions) are characterized by small q and vice versa:

$$\boxed{R \approx \frac{\hbar}{q}}$$

- Also, for small θ_{CM}, $q^2 \approx P_{CM}^2 \, \theta_{CM}^2 \approx p_T^2$ = square of transverse momentum due to collision.

Back to Rutherford Scattering

- For a massive target, we can rewrite the Rutherford cross section as:

$$\frac{d\sigma}{dq^2} = \frac{4\pi\left(ZZ'e^2\right)^2}{v^2}\frac{1}{q^4}$$

- The q^{-4} divergence reflects the $1/r$ dependence of the Coulomb potential.
- The average momentum transfer for all angles is small.

Quantum Treatment of Rutherford Scattering

- So far the treatment of Rutherford scattering has been classical.
- We can calculate this process in QM using *Fermi's Golden Rule*. The transition probability is:

$$P = \frac{2\pi}{\hbar}\left|H_{fi}\right|^2 \rho\left(E_f\right)$$

$$\text{where } \; H_{fi} = \langle f|H|i\rangle = \int d^3r\,\psi_f^*(\vec{r})H(r)\psi_i(\vec{r})$$

Quantum Treatment, cont'd.

- Using plane waves for the incident and scattered particle and using the Coulomb potential energy for *H*, and defining wave vectors:

$$H_{fi} \approx \int d^3 r \exp\left(i\vec{k}' \cdot \vec{r}\right) V(r) \exp\left(-i\vec{k} \cdot \vec{r}\right) = \int d^3 r V(r) \exp\left(\frac{i}{\hbar} \vec{q} \cdot \vec{r}\right)$$

- This is the Fourier transform of *V*(*r*).
- After performing the integral and calculating the density of final states, we obtain the same expression as given before.
- The classical calculation gives the correct quantum mechanical result (when spin is ignored).