# Nonlinear Differential Equations

and The Beauty of Chaos

# Examples of nonlinear equations

Simple harmonic oscillator (linear ODE)

$$m\frac{d^2x(t)}{dt} = -kx(t)$$

More complicated motion (nonlinear ODE)

$$m\frac{d^2x(t)}{dt} = -kx(t)(1-\alpha x(t))$$

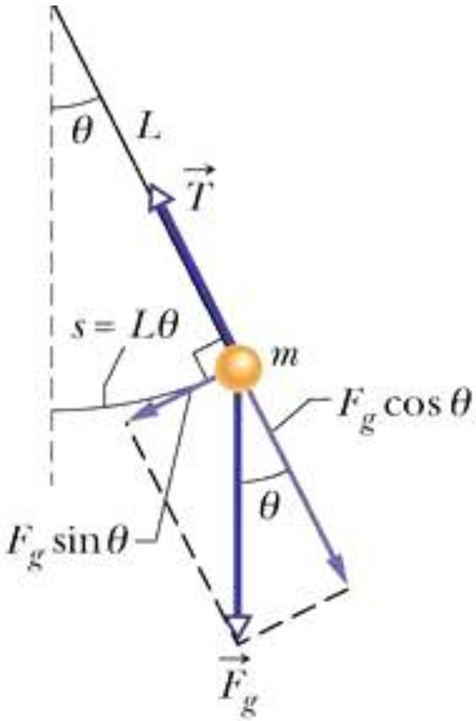
Other examples: weather patters, the turbulent motion of fluids

Most natural phenomena are essentially nonlinear.

# What is special about nonlinear ODE?

- For solving nonlinear ODE we can use the same methods we use for solving linear differential equations
- What is the difference?
- Solutions of nonlinear ODE may be simple, complicated, or chaotic
- Nonlinear ODE is a tool to study nonlinear dynamic: chaos, fractals, solitons, attractors

# A simple pendulum




Model: 3 forces

- gravitational force
- frictional force is proportional to velocity
- periodic external force

$$I\frac{d^2\theta}{dt^2} = \tau_g + \tau_f + \tau_{ext}$$

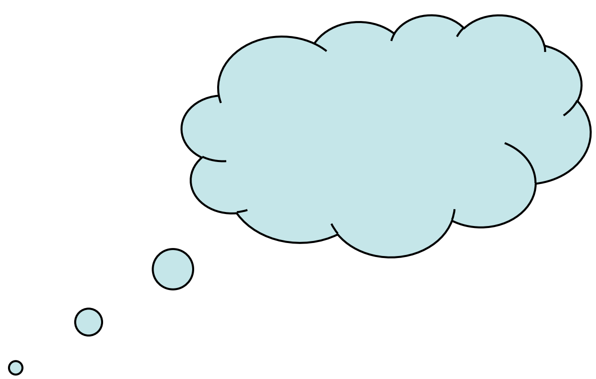
$$\tau_g = -mgL\sin(\theta), \quad \tau_f = -\beta\frac{d\theta}{dt},. \quad \tau_{ext} = F\cos(\omega t)$$

# Equations

$$\frac{d^2\theta}{dt^2} = -\omega_0^2 \sin(\theta) - \alpha \frac{d\theta}{dt} + f \cos(\omega t)$$

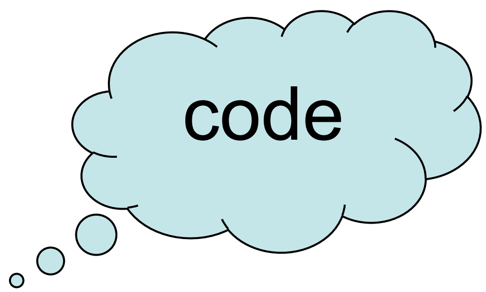
$$\omega_0^2 = \frac{mgL}{I} = \frac{g}{L}, \quad \alpha = \frac{\beta}{mL^2}, \quad f = \frac{F}{mL^2}$$

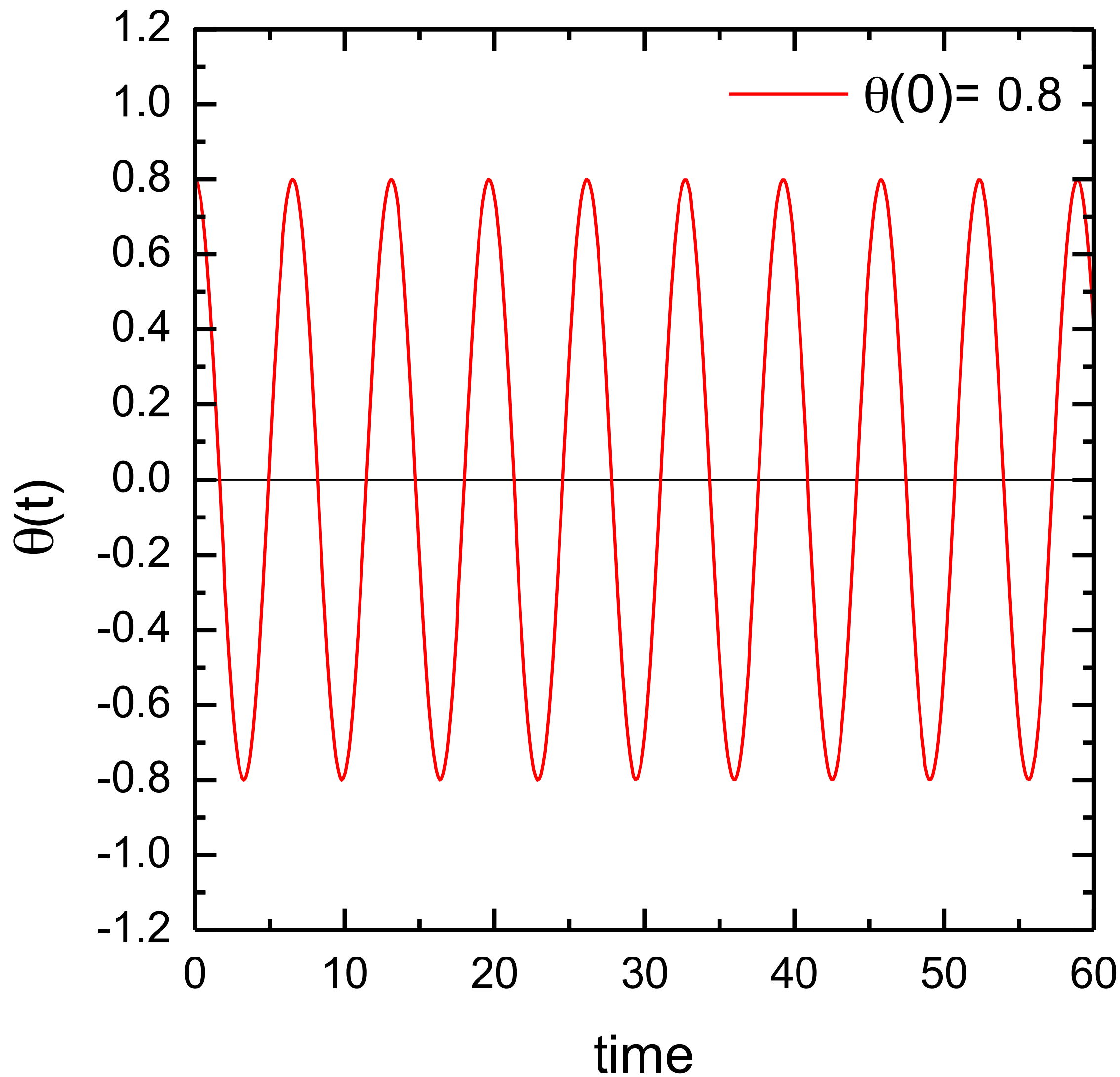
Computer simulation: there are very many web sites with Java animation for the simple pendulum



# Case 1: A very simple pendulum

$$\frac{d^2\theta}{dt^2} = -\omega_0^2 \sin(\theta)$$



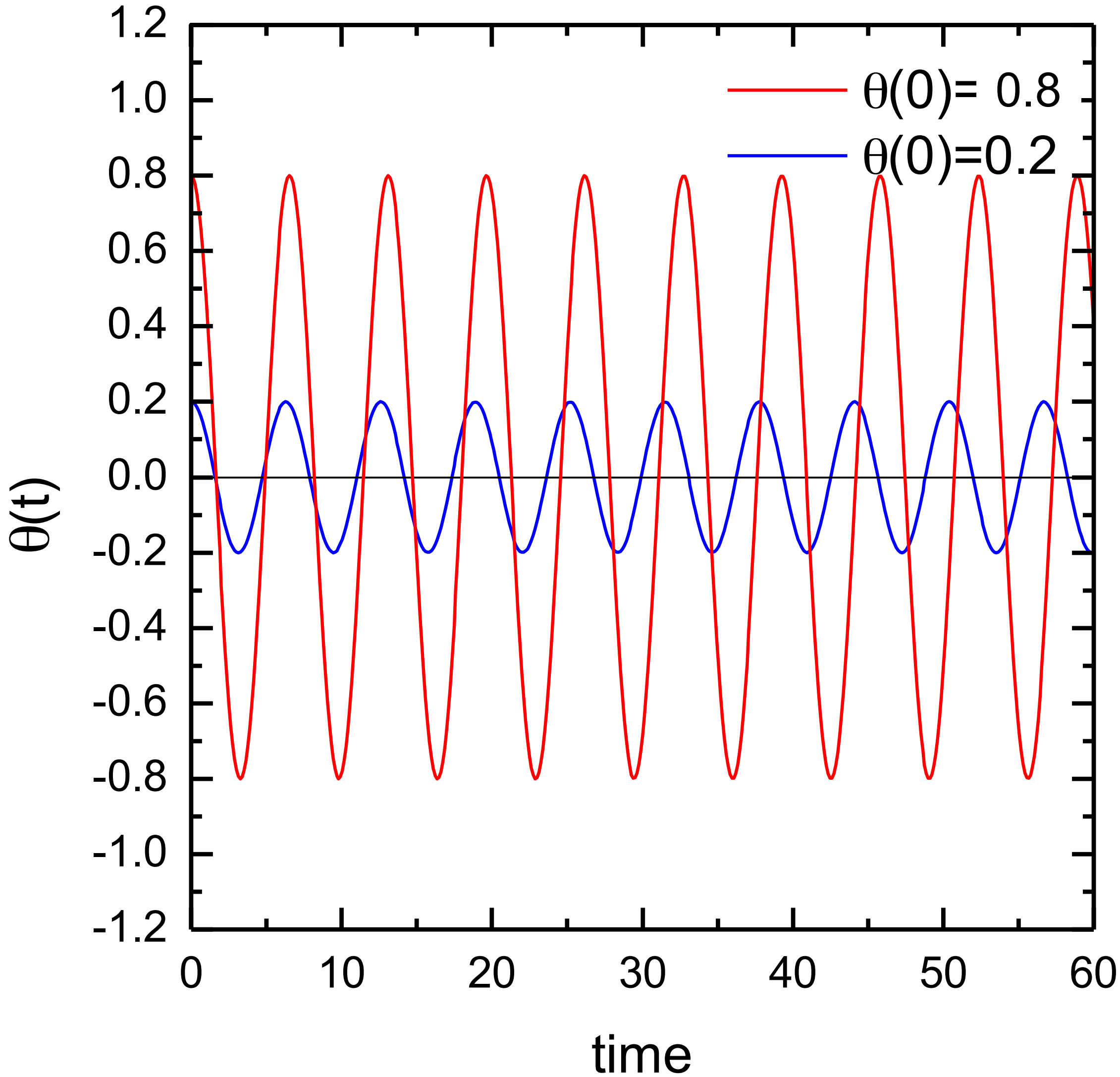

θ(0)= 0.8
θ(t)
time
1.2
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
-0.8
-1.0
-1.2
0
10
20
30
40
50
60

Is there any difference between the nonlinear pendulum

$$\frac{d^2\theta}{dt^2} = -\omega_0^2 \sin(\theta)$$

and the linear pendulum?

$$\frac{d^2\theta}{dt^2} = -\omega_0^2 \theta$$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0.0
-0.2
-0.4
-0.6
-0.8
-1.0
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# Amplitude dependence of frequency

- For small oscillations the solution for the nonlinear pendulum is periodic with

$$\omega = \omega_0 = \sqrt{g/L}$$

- For large oscillations the solution is still periodic but with frequency

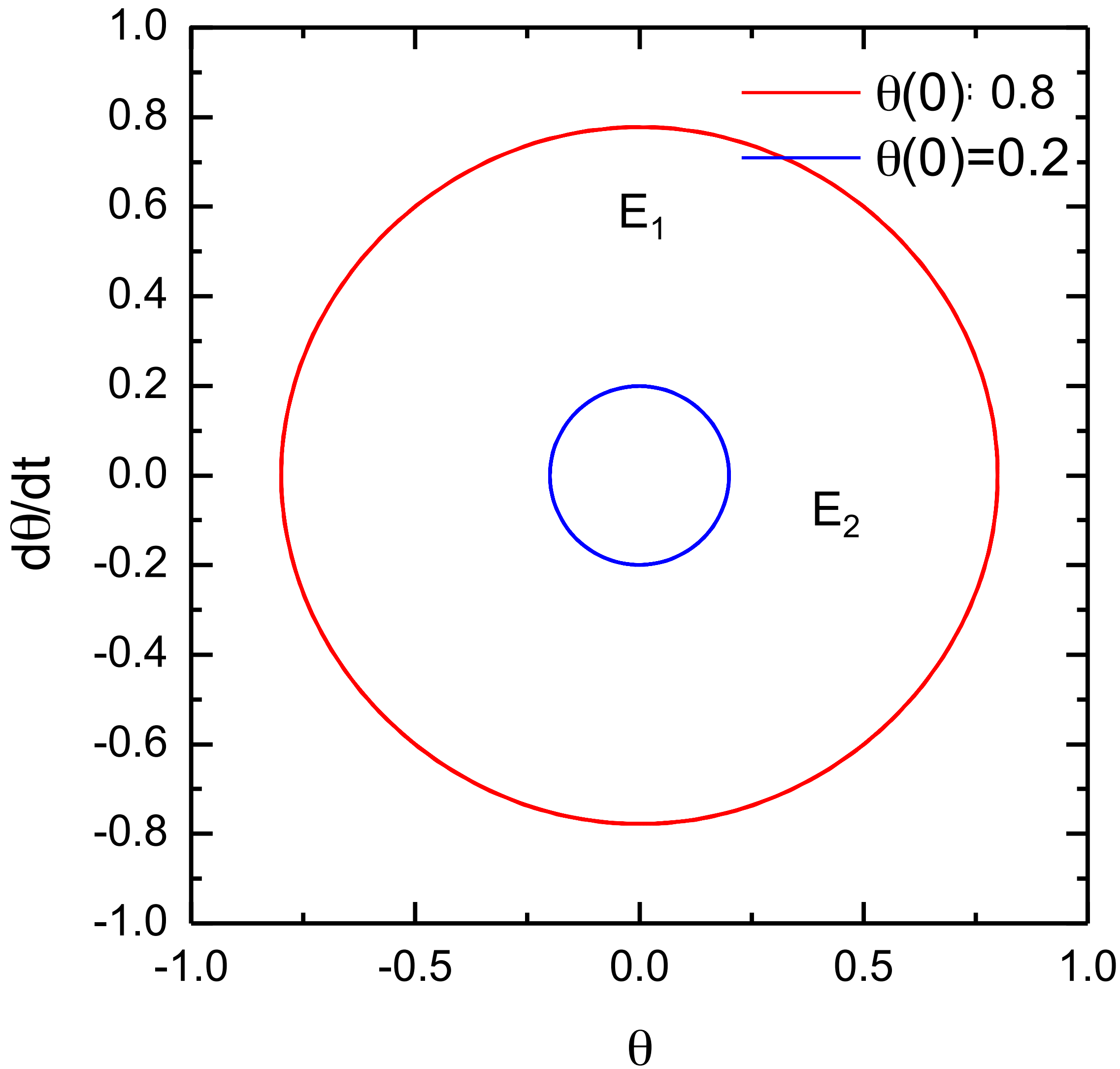
$$\omega < \omega_0 = \sqrt{g/L}$$

- explanation:

$$\sin(\theta) \approx \theta - \frac{1}{2}\theta^2 + \dots$$

$$\sin(\vartheta) < \theta$$

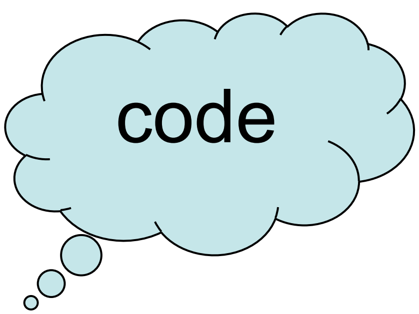
# Phase-Space Plot
# velocity versus position

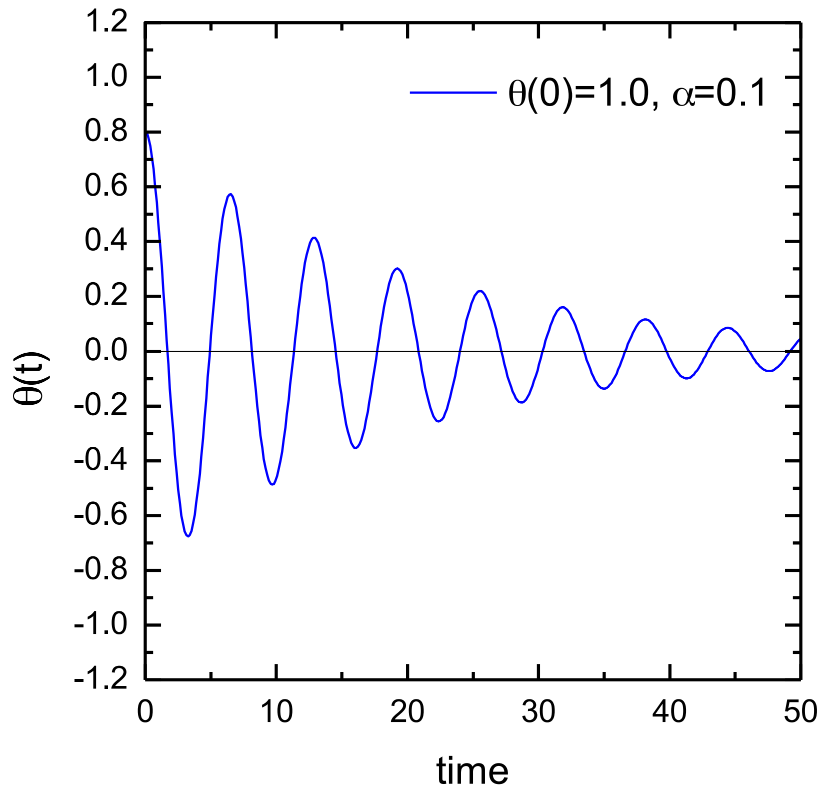



phase-space plot is a very good way to explore the dynamic of oscillations

# Case 2: The pendulum with dissipation

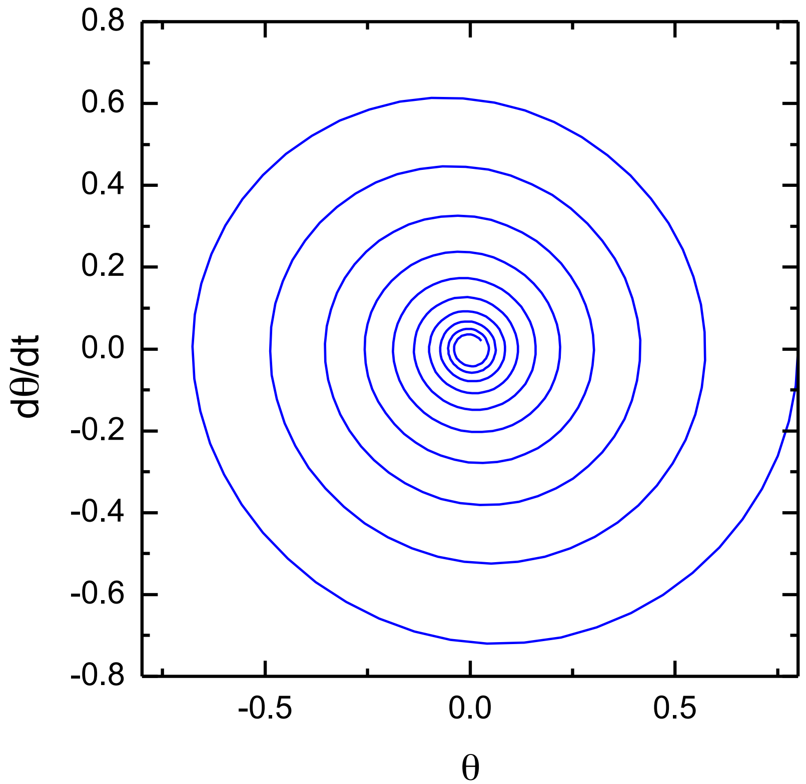
$$\frac{d^2\theta}{dt^2} = -\omega_0^2 \sin(\theta) - \alpha \frac{d\theta}{dt}$$

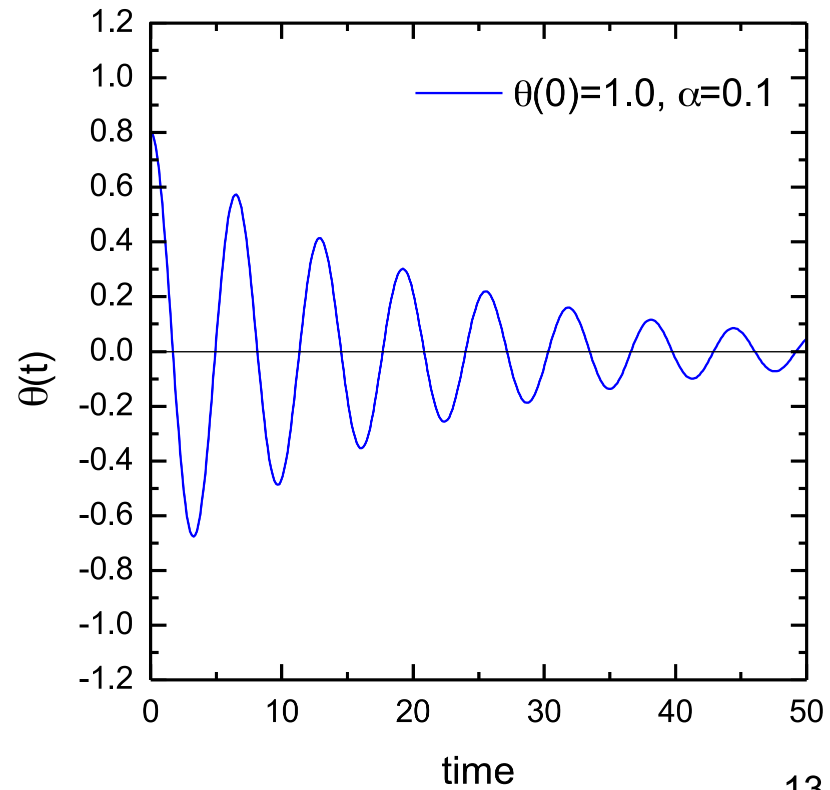



How about frequency in this case?

# Phase-space plot
# for the pendulum with dissipation



# Case 3: Resonance and beats

$$\frac{d^2\theta}{dt^2} = -\omega_0^2 \sin(\theta) + f\cos(\omega t)$$

- When the magnitude of the force is very large – the system is overwhelmed by the driven force (*mode locking*) and the are no beats
- When the magnitude of the force is comparable with the magnitude of the natural restoring force the beats may occur

code

# Beats

- In beating, the natural response and the driven response add:

$$\theta \approx \theta_0 \sin(\omega t) + \theta_0 \sin(\omega_0 t) = 2\theta_0 \cos\left(\frac{\omega - \omega_0}{2} t\right) \sin\left(\frac{\omega + \omega_0}{2} t\right)$$

mass is oscillating at the average frequency $(\omega + \omega_0)/2$ and an amplitude is varying at the slow frequency $(\omega - \omega_0)/2$

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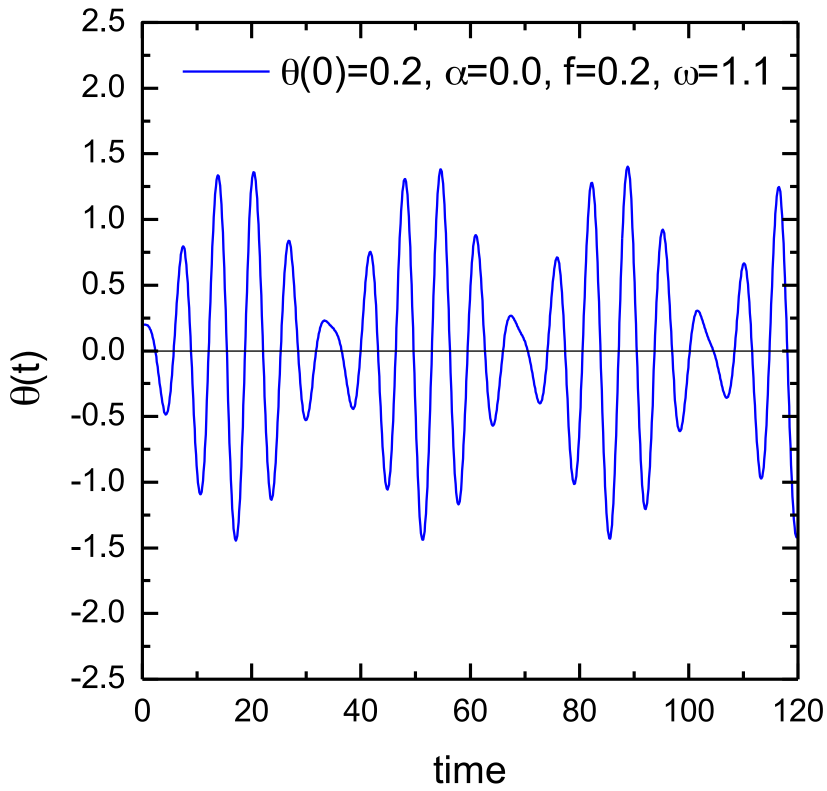
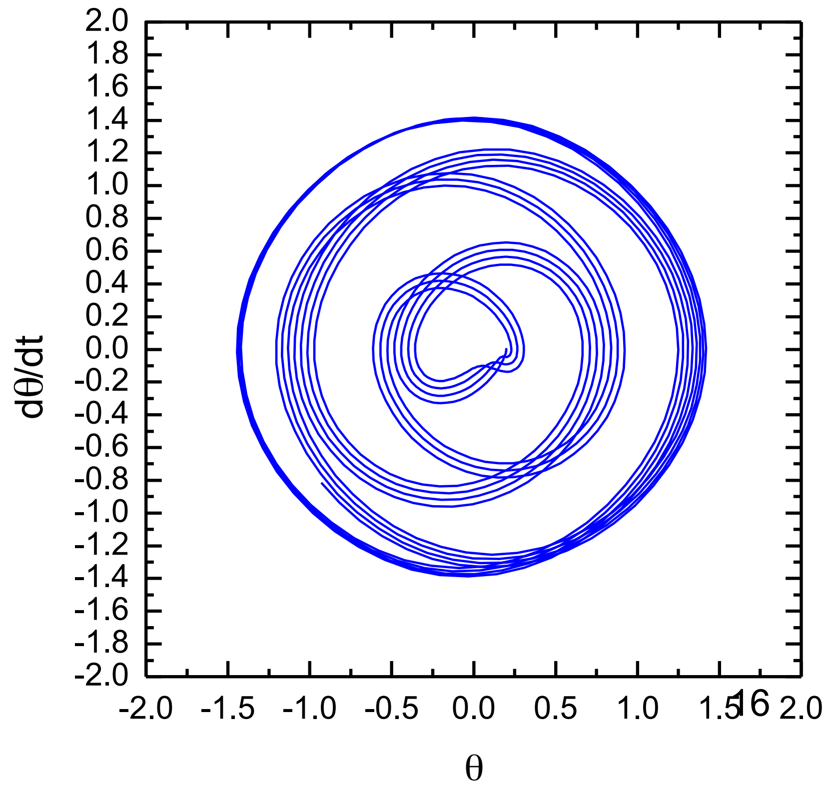
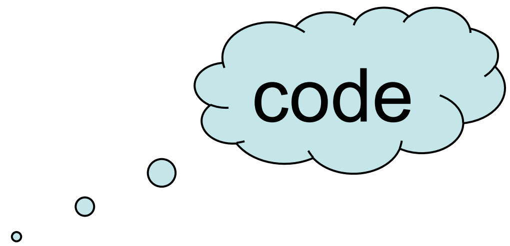
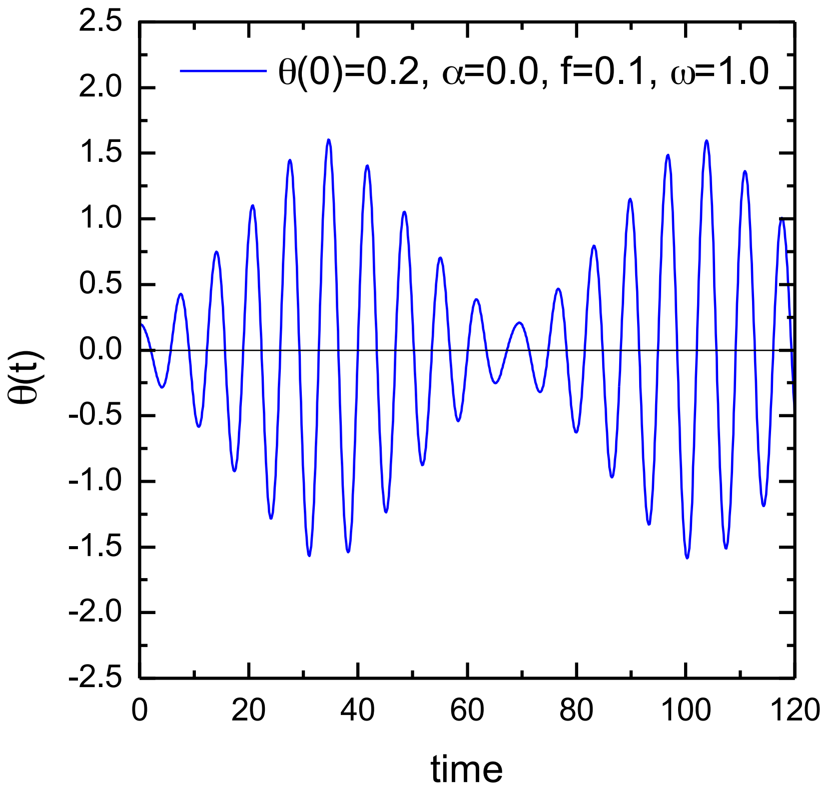
# Example: beats



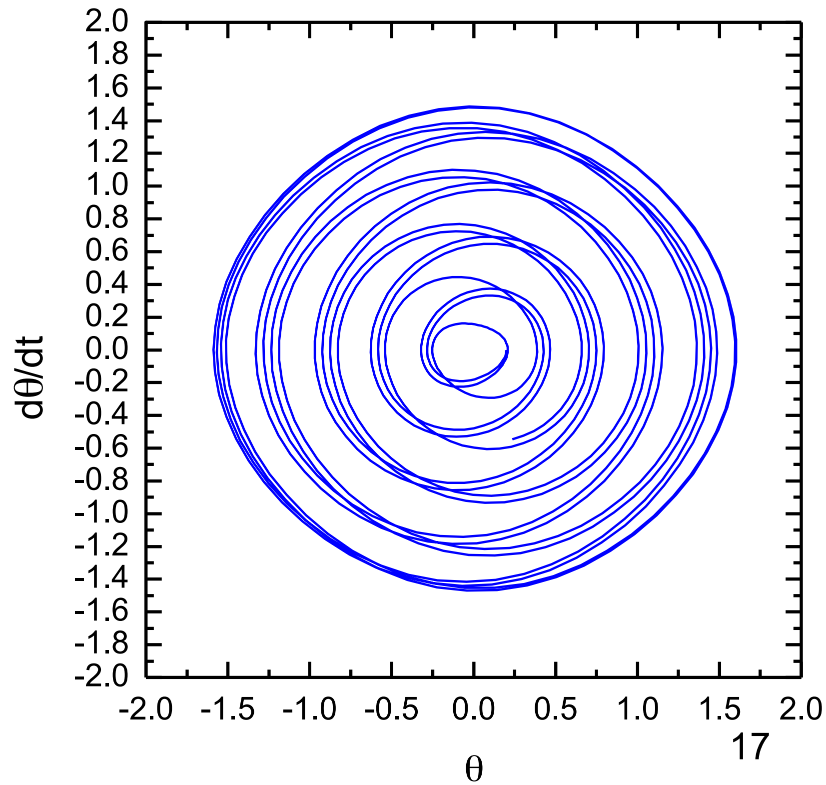



# Resonance




For a simple harmonic oscillator the amplitude of oscillations increases without bound




That is not true for the nonlinear oscillator

# Case 4: Complex Motion

$$\frac{d^2\theta}{dt^2} = -\omega_0^2 \sin(\theta) - \alpha \frac{d\theta}{dt} + f \cos(\omega t)$$

- We have to compare the relative magnitude of the natural restoring force, the driven force and the frictional force
- The most complex motion one would expect when the three forces are comparable

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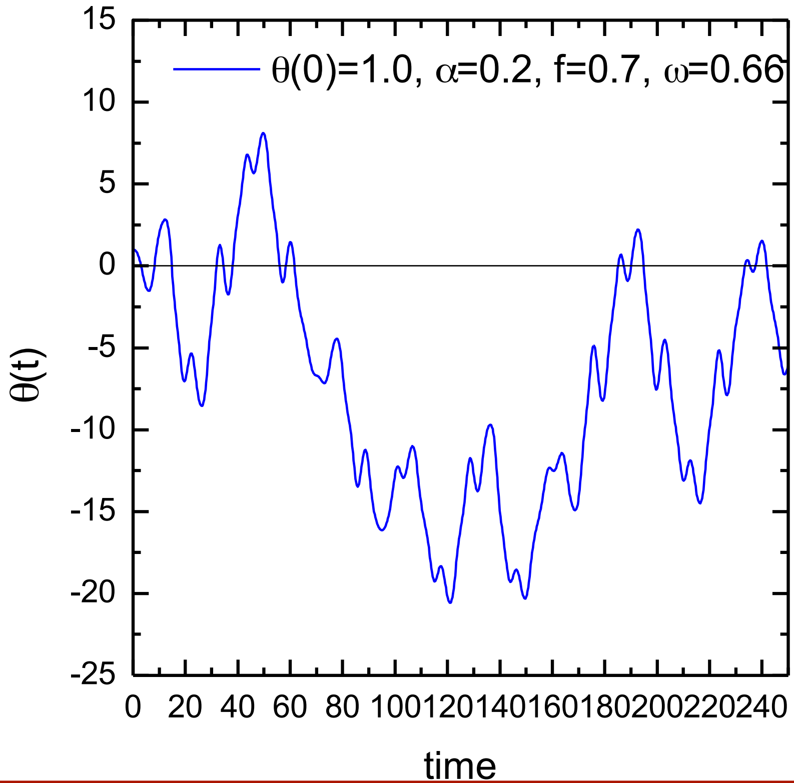
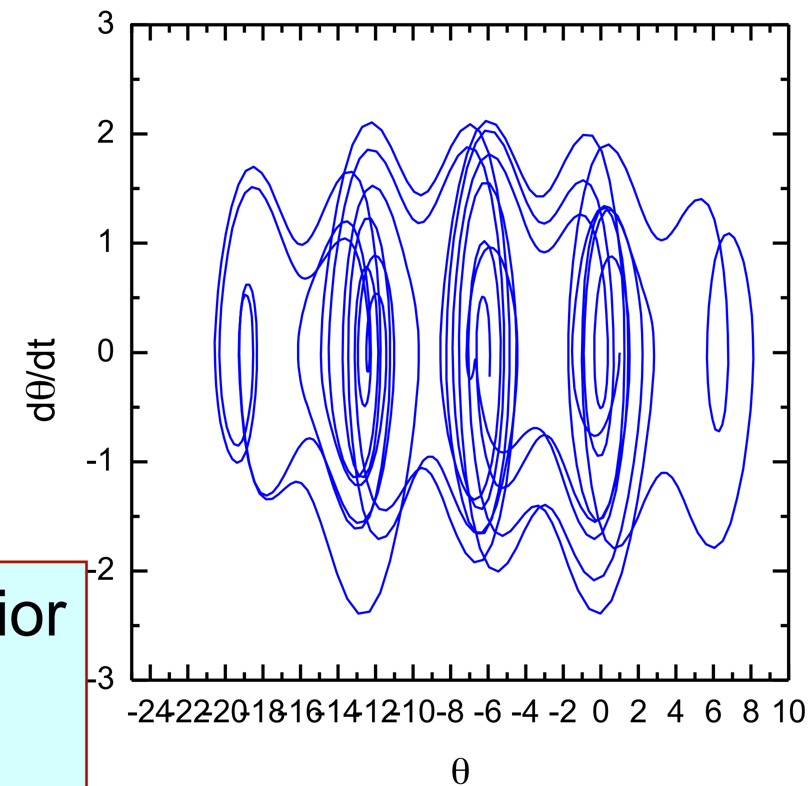
# Case 4: Chaotic Motion

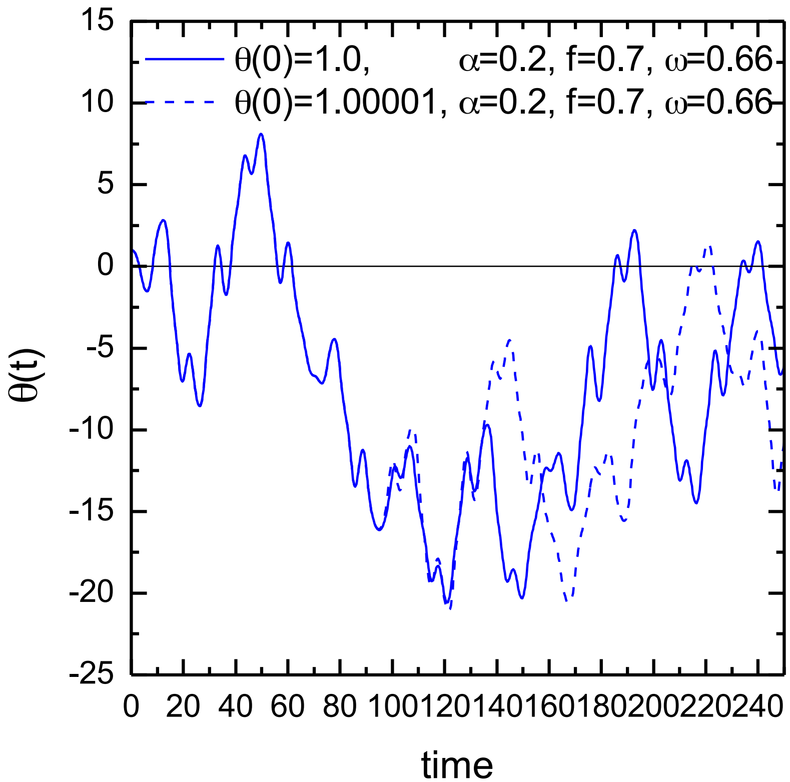



Chaotic motion is not random!



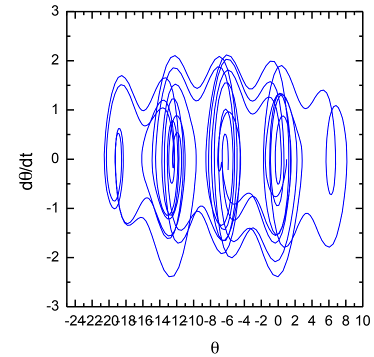

Chaos is the deterministic behavior of a system displaying no discernable regularity

# Case 4: Chaotic Motion




- A chaotic system is one with an extremely high sensitivity to parameters or initial conditions
- The sensitivity to even miniscule changes is so high that, in practice, it is impossible to predict the long range behavior unless the parameters are known to infinite precision (which they never are in practice)

# Measuring Chaos




How do we know if a system is chaotic?

The most important characteristic of chaos is sensitivity to initial conditions.

Sensitivity to initial conditions implies that our ability to make numerical predictions of its trajectory is limited.

# How can we quantify this lack of predictably?

This divergence of the trajectories can be described by the Lyapunov exponent λ, which is defined by the relation:

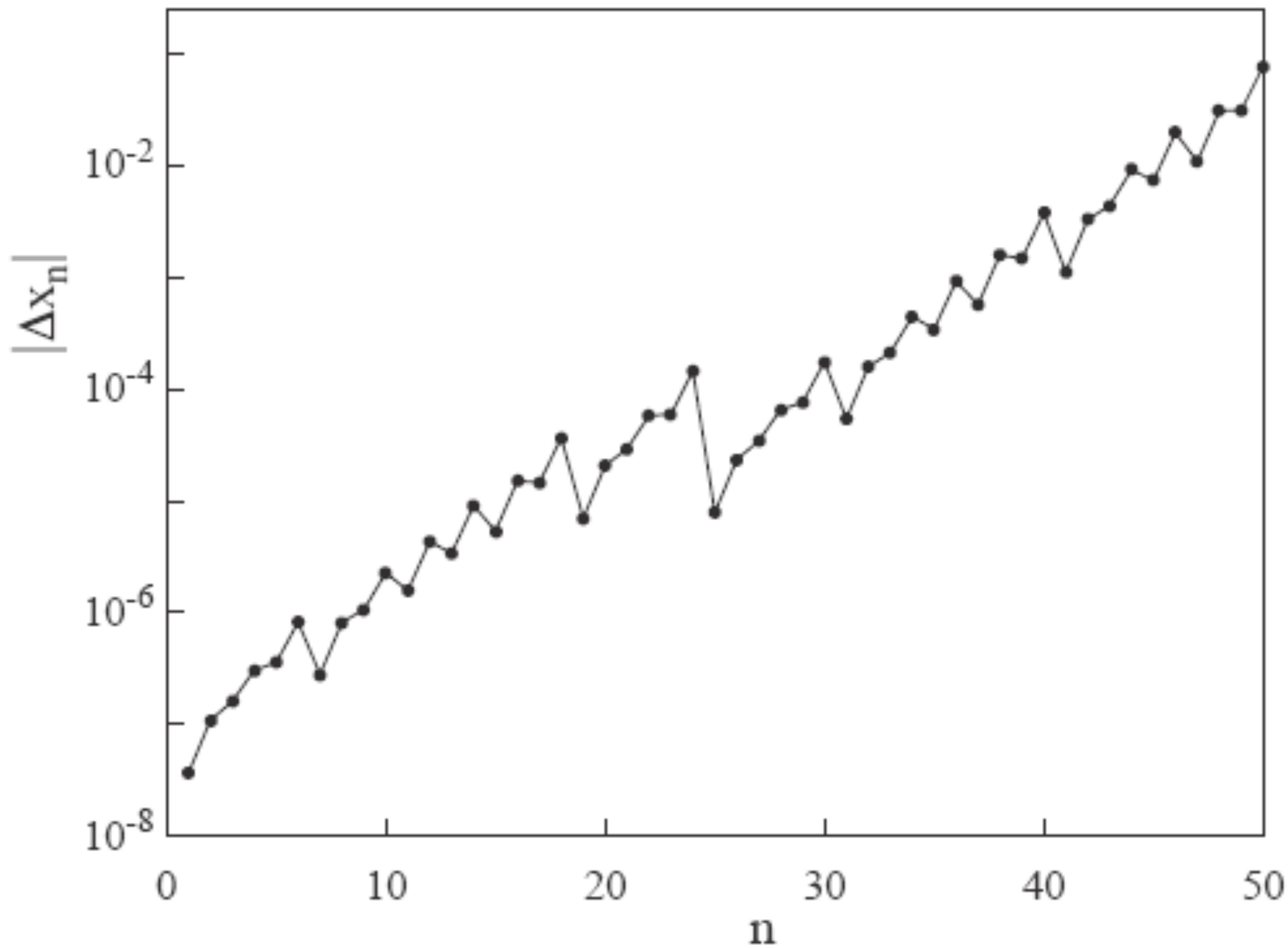
$$\left|\Delta x_n\right| = \left|\Delta x_0\right| e^{\lambda_n}$$

where Δxn is the difference between the trajectories at time n.

If the Lyapunov exponent λ is positive, then nearby trajectories diverge exponentially.

Chaotic behavior is characterized by the exponential divergence of nearby trajectories.

$$\left|\Delta x_n\right| = \left|\Delta x_0\right| e^{\lambda_n}$$

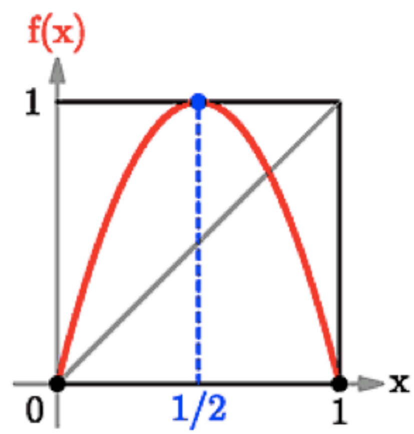


# Example of Lyapunov exponent

Even simple equations can produce complex behaviour when there is feedback in the system. For example, take a simple function like

$$f(x) = 4x(1-x)$$

The graph of this function is just an upside-down parabola passing through the $(x, y)$ pairs $(0,0)$, $(1/2,1)$, $(1,0)$, as shown in the following figure:




Graph of the function f

The number $4$ in the definition of the function is there so that the graph of the function fits neatly into the unit box; in other words, if we apply the function to any number in the unit interval $0 \le x \le 1$, then we get another number in the unit interval.

Now, pick a real number $x[0]$ between 0 and 1 (the "initial condition" or "initial point") and apply the function $f$ to this $x$ to get

$$x[1] = f(x[0]) = 4x[0](1 - x[0])$$

For example, suppose we choose $x[0] = 0.3$; then we get

$$x[1] = f(x[0]) = 4(0.3)(1 - 0.3) = 0.84.$$

Now apply "feedback" to $x[1]$ by applying the function again to get

$$x[2] = f(x[1]) = f(f(x[0]))$$

and keep repeating this process to get

$$x[3] = f(x[2]) = f(f(f(x[0])));$$

$$x[4] = f(x[3]) = f(f(f(f(x[0])))),$$

and so on.

So, for each number $n = 0, 1, 2, \ldots$, we put

$$x[n+1] = f(x[n]).$$

This process is called "iterating" the function $f$ with the initial condition (or "initial point") $x[0]$. The sequence of numbers

$$x[0], x[1], x[2], x[3], \ldots$$

is called the "orbit" of the initial condition $x[0]$.

If we add a small error to the initial point $x[0]$, e.g. we look at

$$u[0] = x[0] + 0.001$$

and then we look at the orbit $u[0], u[1], u[2], \ldots$ of $u[0]$, we might expect that the small error is not very important.

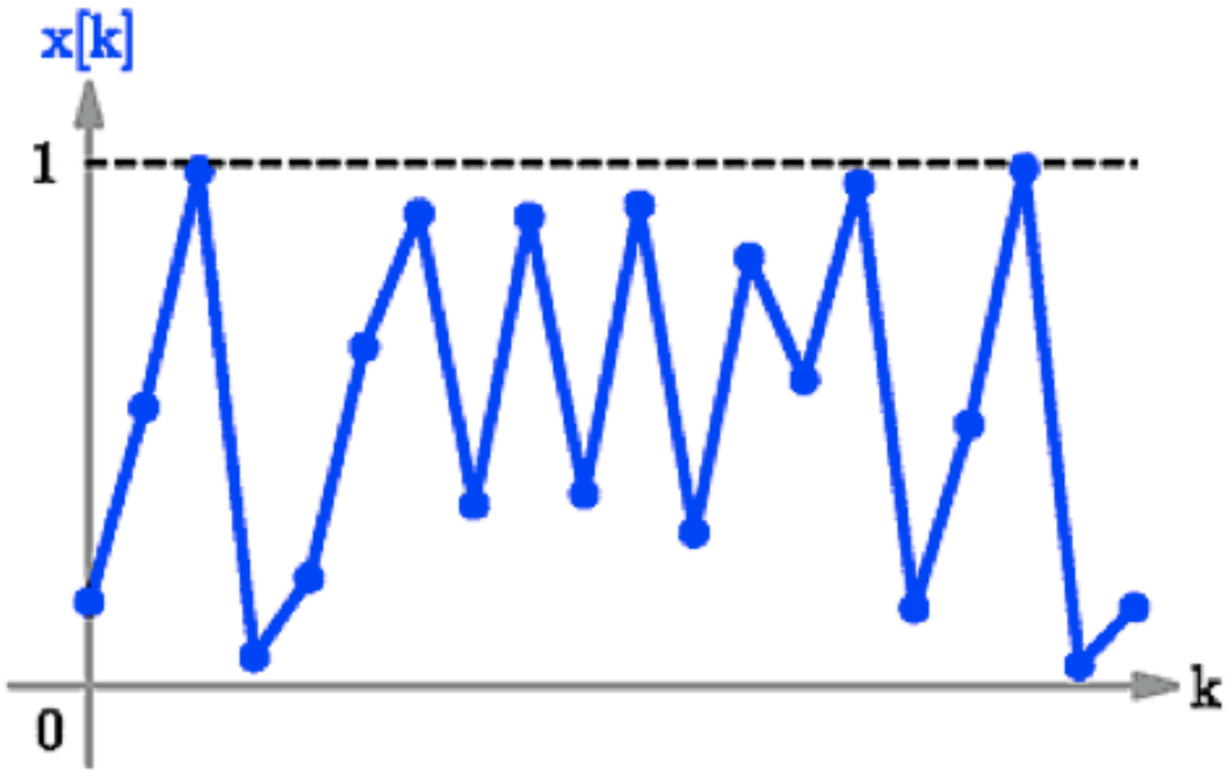
However, it turns out that a function like the one above tends to *amplify small errors* until they become large, so the difference or "error" between the two orbits, i.e. the sequence

$$u[0] - x[0],\ u[1] - x[1],\ u[2] - x[2],\ \ldots u[k] - x[k],\ \ldots$$

tends to grow until it is as large as the numbers $x[k]$ themselves.

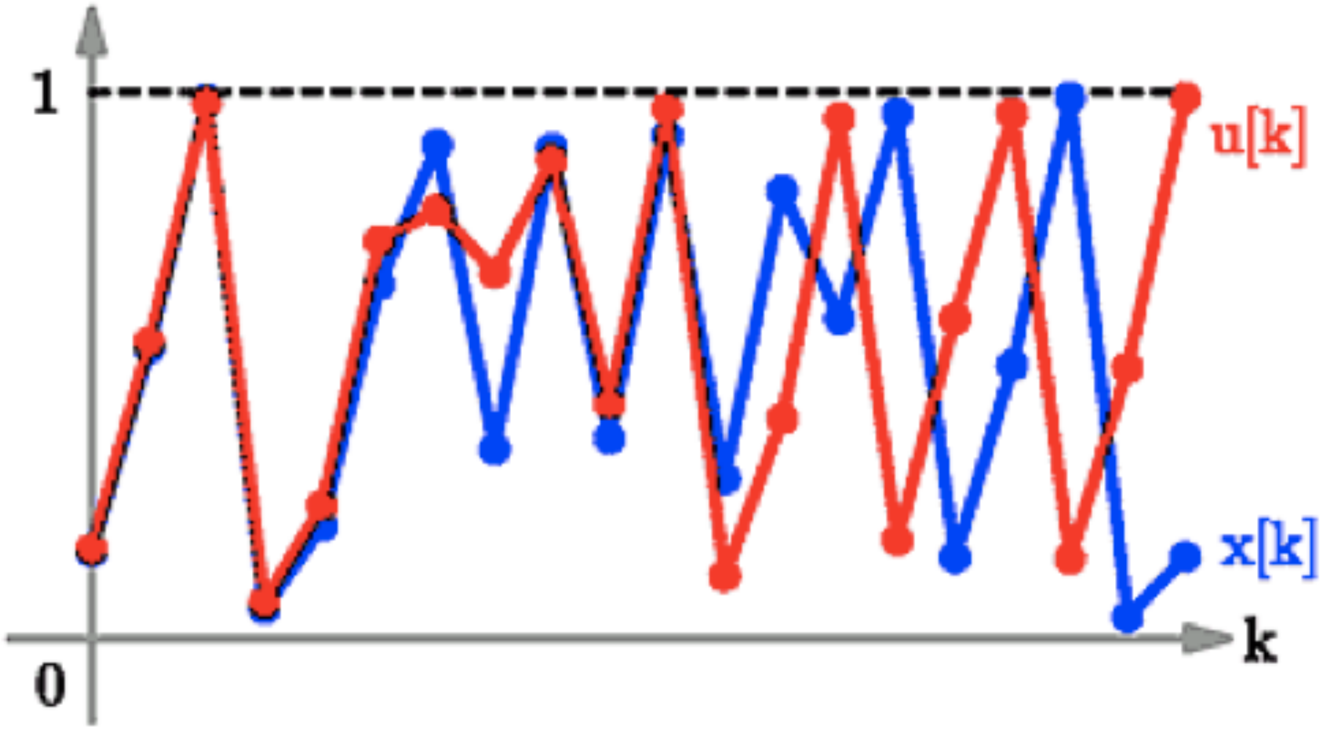
The following diagrams illustrate this phenomenon: first, we show the orbit of an initial point in blue. We plot the points $x[k]$ (for $k = 0, 1, 2$ etc.) against the step-number $k$: this is called a "time-series".

# Orbit of an initial point



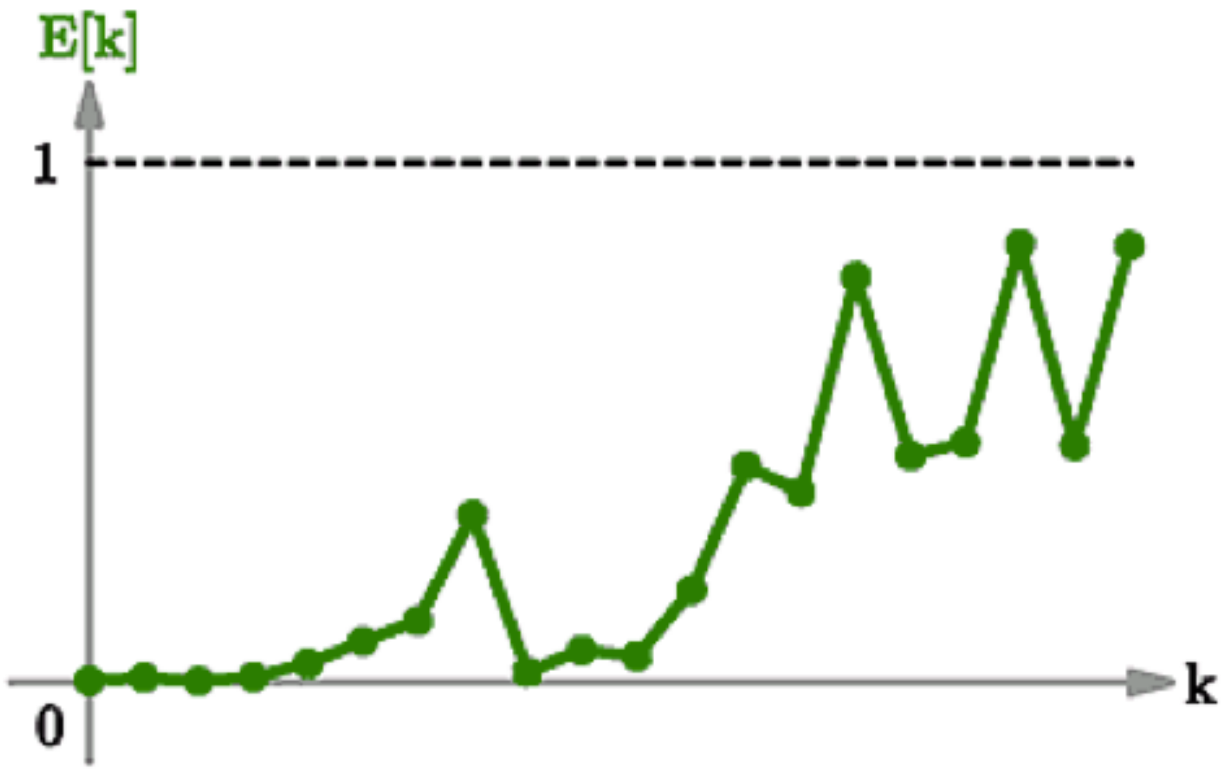

Orbit of an initial point

# Orbit of two initially close points




Orbits of two initially close-by points

# Error between the orbits




Error between the orbits

In fact, no matter how tiny we make the initial error, in most cases it will tend to be amplified until it grows large. This behaviour, where small errors tend to become amplified until they are as large as the "signal" itself, is called *sensitive dependence on initial conditions*.

Another popular name for this phenomenon is the "*Butterfly Effect*", so-called because the idea is rather like the tiny flapping of a butterfly's wings becoming amplified until it causes a hurricane on the other side of the world.

This is because the weather is sensitive to small errors in the same way that the function above is, so that - no matter how accurately we try to measure the weather right now (the temperature, humidity, etc.) - inevitably small errors in our measurements are amplified until they make our long-term weather forecasts unreliable.

This effect is *one* of the hallmarks of "chaos": chaotic systems all have this feature (but not all sensitive systems are chaotic!) The other hallmarks of chaos are to do with periodic orbits ("cycles", where there is a repeating pattern in the orbit $x[0], x[1], \ldots$) and "mixing" behaviour (like the kneading of dough, where orbits tend to get thoroughly "mixed around" throughout all the possible values).

# Growth of small errors

## Growth of small errors

Remember that we begin with a small initial error. Let's call this $E\,[0]$. So, we begin with

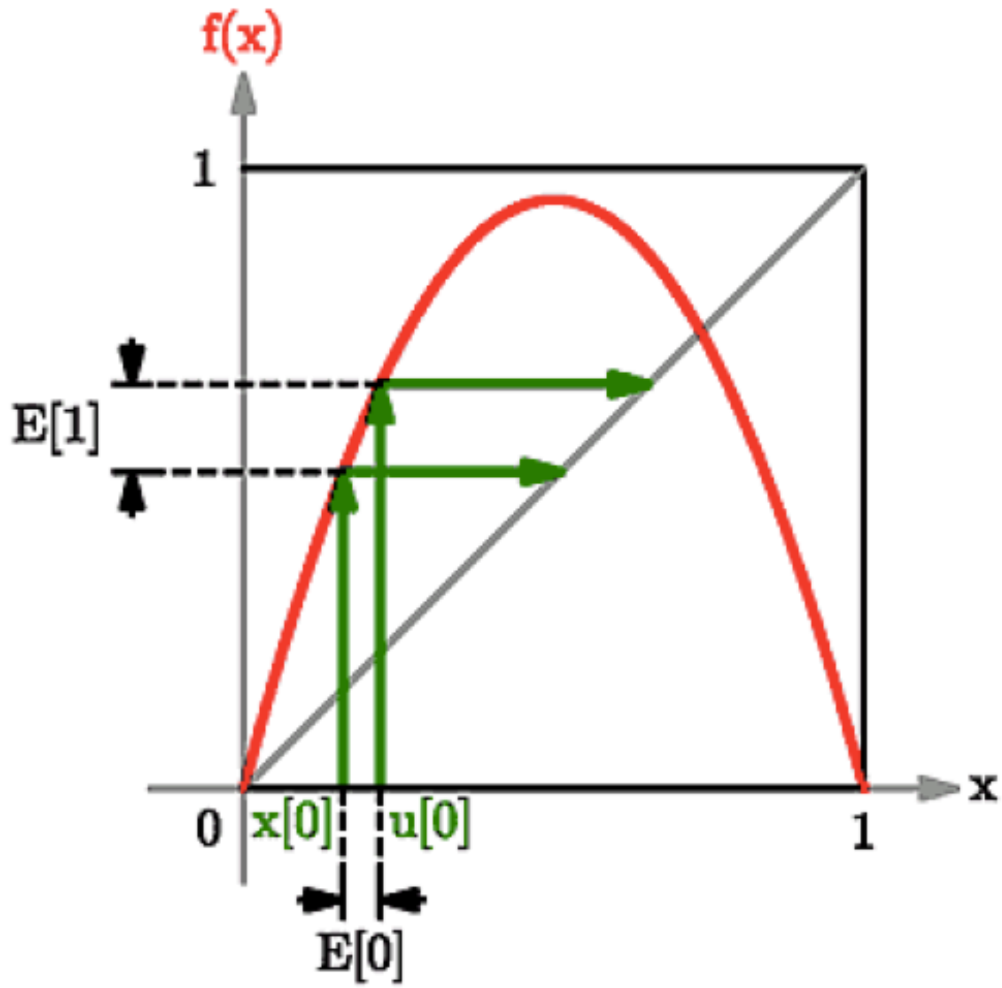
$$u\,[0] = x\,[0] + E\,[0]$$

i.e. initial condition $x\,[0]$ and error $E\,[0]$. Then we compare the "exact" orbit $x[0],\ x[1],\ x[2]$,... against the orbit with the error $u\,[0]\,,\ u\,[1]\,,\ u\,[2]$ ,... and see how the difference between them grows.

In other words, we look at the sequence of errors:

$$\begin{aligned} E\,[0] &= u\,[0] - x\,[0]\,, \\ E\,[1] &= u\,[1] - x\,[1]\,, \\ E\,[2] &= u\,[2] - x\,[2]\,, \\ \ldots \end{aligned}$$

# Growth of an initial error




Growth of an initial error

To see how much the error is amplified at each iteration step $k$, we compute

$$\left|\frac{E[k+1]}{E[k]}\right|$$

i.e. we see how big the next error $E[k+1]$ is, in comparison with the current error $E[k]$. (For example, if the next error were twice as big as the current one, then the expression above would have the value $2$.)

After $n$ steps, the initial error $E[0]$ has been amplified by a factor of:

$$\left|\frac{E[n]}{E[0]}\right| = \left|\frac{E[n]}{E[n-1]}\right| \cdot \left|\frac{E[n-1]}{E[n-2]}\right| \cdot \cdots \cdot \left|\frac{E[1]}{E[0]}\right|$$

Now, suppose we had instead just looked at the simple linear function

Now, suppose we had instead just looked at the simple linear function

$$g(x) = cx,$$

where $c$ is some constant greater than 1. Then any error is simply magnified by $c$ at each iteration:

$$\begin{aligned} g(x+E) &= c(x+E) \\ &= cx + cE \\ &= g(x) + cE; \\ g(g(x)+cE) &= c(g(x)+cE) \\ &= cg(x) + \left(c^2\right)E \\ &= g(g(x)) + \left(c^2\right)E, \end{aligned}$$

so that after $n$ steps, the initial error has been magnified by

$$\left|\frac{E[n]}{E[0]}\right| = c^n$$

(Note that if $c$ were less than 1, the error would actually be decreased rather than increased.)

If we want to find out what the value of $c$ is for such a linear function $g$, we have

$$\ln(c) = \frac{1}{n}\ln\left(\left|\frac{E[n]}{E[0]}\right|\right)$$

This is the idea behind the Lyapunov exponent: we take a *general* function $f$ and use this same formula to see how much small errors tend to be magnified. We take more and more iterates (larger and larger $n$) and calculate the corresponding values for $c$. If, as we let $n$ grow larger and larger, these values settle down to a constant, then this suggests that the error tends to grow on average like $c^n$.

Taking the logarithm turns a product into a sum:

$$\begin{aligned}\frac{1}{n}\ln\left(\left|\frac{E[n]}{E[0]}\right|\right) &= \frac{1}{n}\ln\left(\left|\frac{E[n]}{E[n-1]}\right|\cdot\left|\frac{E[n-1]}{E[n-2]}\right|\cdot\dots\cdot\left|\frac{E[1]}{E[0]}\right|\right)\\ &= \left(\frac{1}{n}\right)\sum_{k=1}^{k=n}\ln\left(\left|\frac{E[k]}{E[k-1]}\right|\right)\end{aligned}$$

The Lyapunov Exponent is defined to be the limiting value of the above quantity as the initial error $E[0]$ is made ever smaller-and-smaller, and the number of iterations $n$ is sent to infinity.

Notice that

$$E[k] = f(x[k-1] + E[k-1]) - f(x[k-1])$$

so that

$$\frac{E[k]}{E[k-1]} = \frac{f(x[k-1] + E[k-1]) - f(x[k-1])}{E[k-1]}$$

Now, suppose that the function $f$ is smooth (i.e. we can differentiate it to get $f\prime(x)$). In this case, you should notice that the expression on the right-hand side looks familiar! Let's rewrite it like this (put $h = E[k-1]$ and $x = x[k-1]$ ):

$$\frac{E[k]}{E[k-1]} = \frac{f(x+h) - f(x)}{h}$$

If we let $h$ go to zero, then this is just the expression for the derivative! So, as $h$ tends to zero we have:

$$\frac{E[k]}{E[k-1]} \rightarrow f\prime(x[k-1])$$

Thus we have

$$\left(\frac{1}{n}\right)\ln\left(\left|\frac{E[n]}{E[0]}\right|\right) = \left(\frac{1}{n}\right)\sum_{k=1}^{k=n}\ln\left(\left|\frac{E[k]}{E[k-1]}\right|\right)$$
$$= \frac{1}{n}\sum_{k=1}^{k=n}\ln\left(\left|f'(x[k-1])\right|\right)$$

Finally, letting $n$ go to infinity (i.e. looking at more and more iterates), we obtain a formula for the Lyapunov Exponent of a differentiable function $f$:

$$\lambda(x[0]) = \lim_{n\to\infty}\left(\frac{1}{n}\right)\sum_{k=1}^{k=n}\ln\left(\left|f'(x[k-1])\right|\right).$$

If this number is positive, then this means that small errors in the initial condition $x[0]$ will tend to be magnified, and the system is showing sensitivity. (Remember that sensitivity is one of the three hallmarks of chaos.) If this number is negative, then small errors tend to be reduced, and the system is showing stability.

So, the Lyapunov exponent gives us a way to measure the stability of our system.

# Order and chaos in the same system

Even some very simple systems are capable of showing *both* orderly and chaotic behaviour.

These systems may make transitions (called "bifurcations") from one type of behaviour to another, as some parameter is varied, rather like tuning-in to different radio stations by turning the dial on a radio.

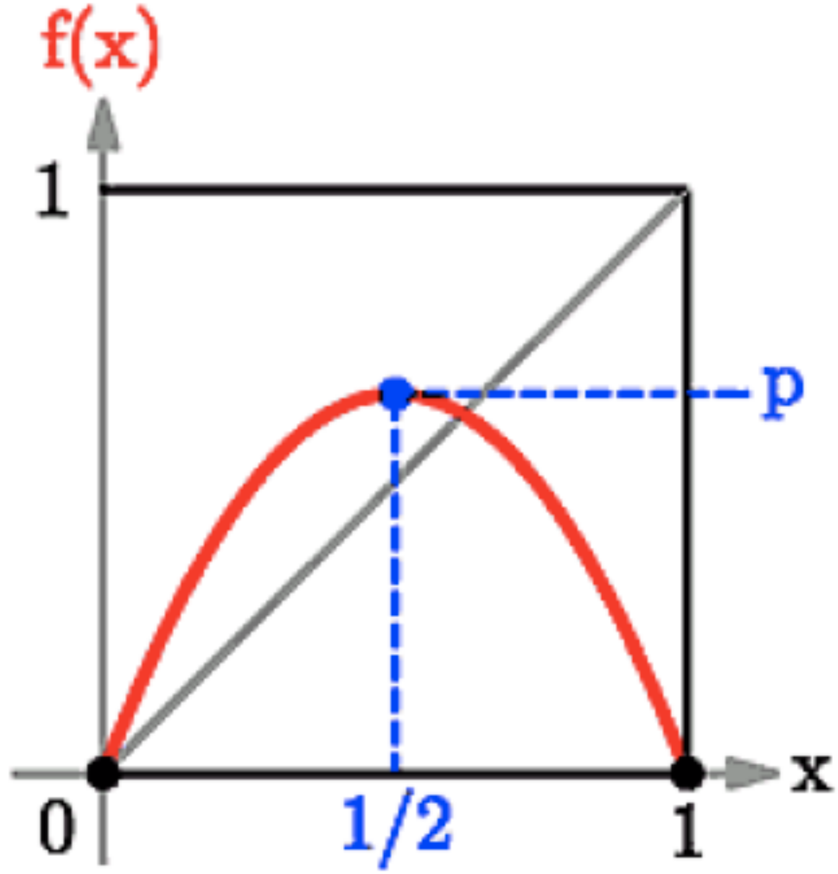
For example, remember our function

$$f(x) = 4x(1-x)$$

whose graph was a parabola through $(x, y) = (0, 0)$, $(1/2, 1)$, and $(1, 0)$. We can add a parameter $p$ to let us move the height of the parabola up and down:

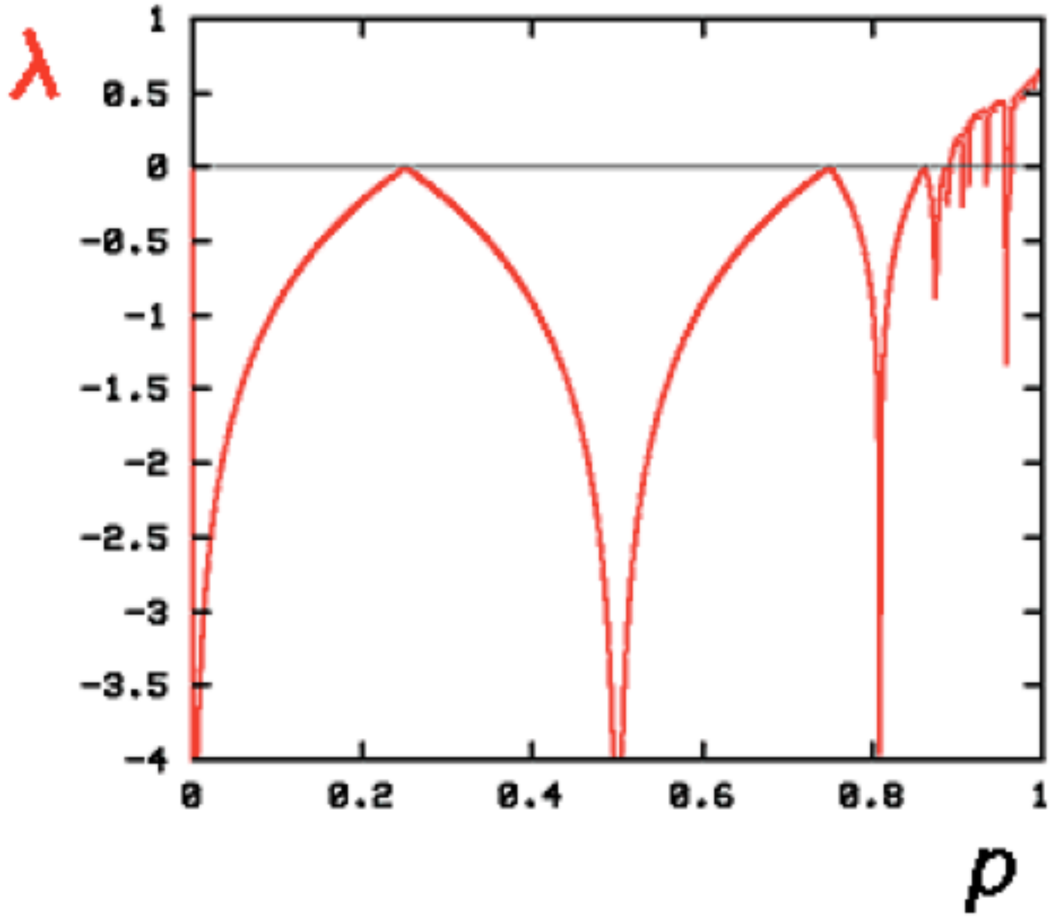
$$f_p(x) = (4p)\, x\, (1-x)$$

# Graph of function with a parameter p




Graph of function with a parameter p

# Lyapunov exp versus parameter



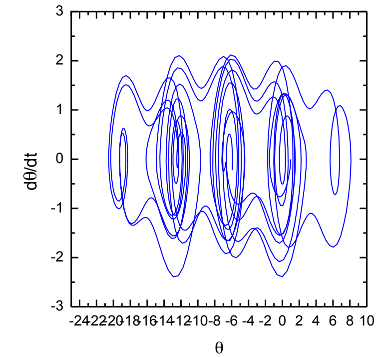

Lyapunov exponent vs. parameter

Notice the large negative spikes pointing downward. In reality, these spikes in the graph go off to minus infinity, showing that the behaviour of the function is incredibly stable at those parameter values. Remember that we have calculated an estimate of the *logarithm* of the error-multiplier: if this value is very large and negative, then the error-multiplier is very small, which indicates that errors tend to vanish away and the orbit of the system is super-stable.

Between the spikes, the graph touches the $p$-axis (so that the Lyapunov exponent is zero): this indicates that these parameters give an orbit which is on the edge of stable and unstable behaviour. It is here that sudden changes (bifurcations) in the behaviour of orbits take place.

Notice that the graph stays below zero until it reaches a point near the right hand side of the plot: this is where sensitivity first appears. As the graph rises above the axis, this means that the Lyapunov exponent is positive so that errors tend to be magnified. Even so, in this right-hand region there are still spikes where super-stable orbits appear.

# Chaotic structure in phase space




1. Limit cycles: ellipse-like figures with frequencies greater then $\omega_0$
2. Strange attractors: well-defined, yet complicated semi-periodic behavior. Those are highly sensitive to initial conditions. Even after millions of observations, the motion remains *attracted* to those paths
3. Predictable attractors: well-defined, yet fairy simple periodic behaviors that not particularly sensitive to initial conditions
4. Chaotic paths: regions of phase space that appear as filled-in bands rather then lines

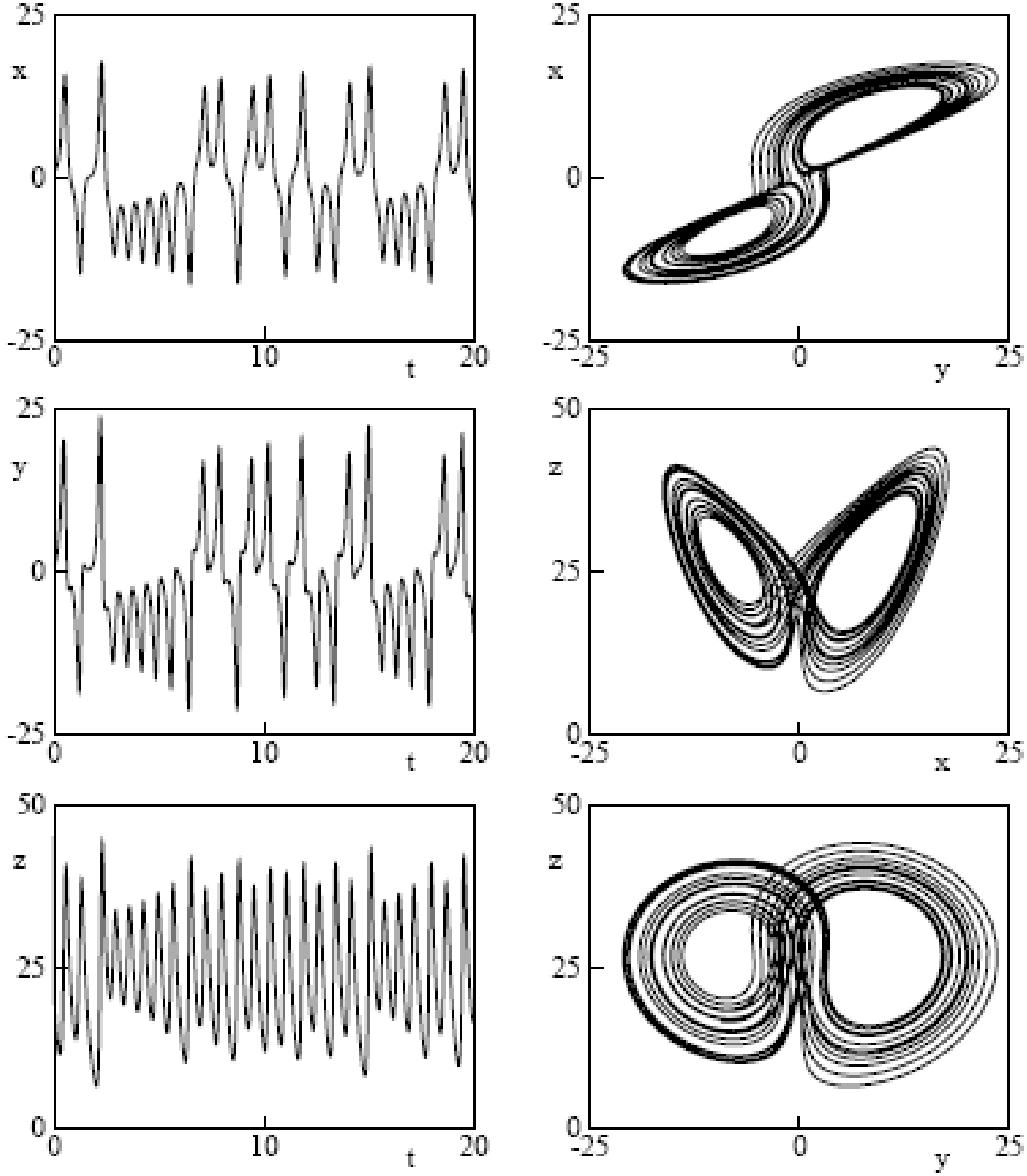
# The Lorenz Model & the *butterfly effect*

- In 1962 Lorenz was looking for a simple model for weather predictions and simplified the heat-transport equations to the three equations

$$\frac{dx}{dt} = 10(y - x)$$
$$\frac{dy}{dt} = -xz + 28x - y$$
$$\frac{dz}{dt} = xy - \frac{8}{3}z$$

- The solution of these simple nonlinear equations gave the complicated behavior that has led to the modern interest in chaos

# Example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0
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t
20
50
z
25
0
-25
0
x
25
50
z
25
0
0
10
t
20
50
z
25
0
-25
0
y
25

# Summary

- The simple systems can exhibit complex behavior
- Chaotic systems exhibit extreme sensitivity to initial conditions.

# Practice

⇨ Duffing Oscillator

$$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} - \frac{1}{2}x(1 - x^2) = f\cos(\omega t)$$

⇨ Write a program to solve the Duffing model. Is there a parametric region in $(\alpha, f, \omega)$ where the system is chaotic

# Fourier Analysis of Nonlinear Oscillations

- The traditional tool for decomposing both periodic and non-periodic motions into an infinite number of harmonic functions
- It has the distinguishing characteristic of generating periodic approximations

# Fourier series

For a periodic function

$$y(t+T) = y(t)$$

one may write

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos(n\omega t) + b_n \sin(n\omega t) \right), \qquad \omega = \frac{2\pi}{T}$$

The Fourier series is a “best fit” in the least square sense of data fitting

A general function may contain infinite number of components. In practice a good approximation is possible with about 10 harmonics

# Coefficients:

the coefficients are determined by the standard technique for orthogonal function expansion

$$a_n = \frac{2}{T}\int_0^T \cos(n\omega t)\, y(t)\, dt,$$

$$b_n = \frac{2}{T}\int_0^T \sin(n\omega t)\, y(t)\, dt,$$

$$\omega = \frac{2\pi}{T}$$

# Fourier transform

The right tool for non-periodic functions

$$y(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} Y(\omega) e^{i\omega t} d\omega$$

and the inverse transform is

$$Y(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} y(t) e^{-i\omega t} dt$$

a plot of $\left|Y(\omega)\right|^2$ versus $\omega$ is called the power spectrum

# Spectral function

If $y(t)$ represent the response of some system as a function of time, $Y(\omega)$ is a spectral function that measures the amount of frequency $\omega$ making up this response

# Methods to calculate Fourier transform

- ⇨ Analytically
- ⇨ Direct numerical integration
- ⇨ Discrete Fourier transform
  (for functions that are known only for a finite number of times $t_k$
- ⇨ Fast Fourier transform (FFT)

# Discrete Fourier transform

- Assume that a function y(t) is sampled at a discrete number of N+1 points, and these times are evenly spaced
- Let T is the time period for the sampling:
  a function y(t) is periodic with T, y(t+T)=y(t)
- The largest frequency for this time interval is
  $\omega_1 = 2\pi / T$ and $\omega_n = n\omega_1 = n2\pi / T = n2\pi / (Nh)$

# Discrete Fourier transform

- The discrete Fourier transform, after applying a trapezoid rule

$$Y(\omega_n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega_n t} y(t) dt = \frac{h}{\sqrt{2\pi}} \sum_{k=1}^{N} e^{-i\frac{2\pi kn}{N}} y_k$$

$$y(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega_n t} Y(\omega) d\omega = \frac{\sqrt{2\pi}}{hN} \sum_{n=1}^{N} e^{i\frac{2\pi nt}{hN}} Y(\omega_n)$$

# DFT in terms of separate real and imaginary parts

$$e^{ix} = \cos(x) + i\sin(x)$$

$$\begin{aligned}
Y(\omega_n) \quad &= \quad \frac{h}{\sqrt{2\pi}} \sum_{k=1}^{N} \; [(\cos(2\pi kn/N)\,\mathrm{Re}(y_k) \\
&+ \quad \sin(2\pi kn/N)\,\mathrm{Im}(y_k)) \\
&+ \quad i(\cos(2\pi kn/N)\,\mathrm{Im}(y_k) \\
&- \quad \sin(2\pi kn/N)\,\mathrm{Re}(y_k))]
\end{aligned}$$

# Practice for the simple pendulum

Solve the simple pendulum for harmonic motion, beats, and chaotic motion (the dissipation and driven forces are included)

- Decompose your numerical solutions into a Fourier series. Evaluate contribution from the first 10 terms
- Evaluate the power spectrum from your numerical solutions