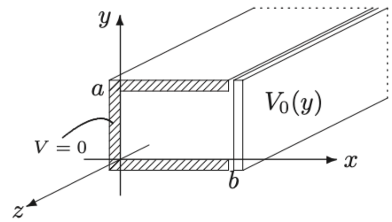
Problem 3.15

(a) $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = 0$, with boundary conditions



$$\left\{\begin{array}{ll} \text{(i)} & V(x,0)=0, \\ \text{(ii)} & V(x,a)=0, \\ \text{(iii)} & V(0,y)=0, \\ \text{(iv)} & V(b,y)=V_0(y). \end{array}\right\}$$

As in Ex. 3.4, separation of variables yields

$$V(x,y) = \left(Ae^{kx} + Be^{-kx}\right)(C\sin ky + D\cos ky).$$

Here (i)⇒ $D = 0$, (iii)⇒ $B = -A$, (ii)⇒ ka is an integer multiple of π:

$$V(x,y) = AC\left(e^{n\pi x/a} - e^{-n\pi x/a}\right)\sin(n\pi y/a) = (2AC)\sinh(n\pi x/a)\sin(n\pi y/a).$$

But $(2AC)$ is a constant, and the most general linear combination of separable solutions consistent with (i), (ii), (iii) is

$$\boxed{V(x,y) = \sum_{n=1}^{\infty} C_n \sinh(n\pi x/a)\sin(n\pi y/a).}$$

It remains to determine the coefficients C_n so as to fit boundary condition (iv):

$$\sum C_n \sinh(n\pi b/a)\sin(n\pi y/a) = V_0(y). \text{ Fourier's trick} \Rightarrow C_n \sinh(n\pi b/a) = \frac{2}{a}\int_0^a V_0(y)\sin(n\pi y/a)\,dy.$$

Therefore

$$\boxed{C_n = \frac{2}{a\sinh(n\pi b/a)}\int_0^a V_0(y)\sin(n\pi y/a)\,dy.}$$

(b) $$C_n = \frac{2}{a\sinh(n\pi b/a)}V_0\int_0^a \sin(n\pi y/a)\,dy = \frac{2V_0}{a\sinh(n\pi b/a)} \times \left\{\begin{array}{ll} 0, & \text{if } n \text{ is even,} \\ \frac{2a}{n\pi}, & \text{if } n \text{ is odd.} \end{array}\right\}$$

$$\boxed{V(x,y) = \frac{4V_0}{\pi}\sum_{n=1,3,5,\ldots}\frac{\sinh(n\pi x/a)\sin(n\pi y/a)}{n\sinh(n\pi b/a)}.}$$

Problem 3.16

Same format as Ex. 3.5, only the boundary conditions are:

$$\left\{\begin{array}{lll} \text{(i)} & V=0 & \text{when } x=0, \\ \text{(ii)} & V=0 & \text{when } x=a, \\ \text{(iii)} & V=0 & \text{when } y=0, \\ \text{(iv)} & V=0 & \text{when } y=a, \\ \text{(v)} & V=0 & \text{when } z=0, \\ \text{(vi)} & V=V_0 & \text{when } z=a. \end{array}\right\}$$

This time we want sinusoidal fuctions in x and y, exponential in z:

$$X(x) = A\sin(kx) + B\cos(kx), \quad Y(y) = C\sin(ly) + D\cos(ly), \quad Z(z) = Ee^{\sqrt{k^2+l^2}z} + Ge^{-\sqrt{k^2+l^2}z}.$$

(i)⇒ $B = 0$; (ii)⇒ $k = n\pi/a$; (iii)⇒ $D = 0$; (iv)⇒ $l = m\pi/a$; (v)⇒ $E + G = 0$. Therefore

$$Z(z) = 2E\sinh(\pi\sqrt{n^2+m^2}z/a).$$

Putting this all together, and combining the constants, we have:

$$V(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\sin(n\pi x/a)\sin(m\pi y/a)\sinh(\pi\sqrt{n^2+m^2}z/a).$$

It remains to evaluate the constants $C_{n,m}$, by imposing boundary condition (vi):

$$V_0 = \sum\sum\left[C_{n,m}\sinh(\pi\sqrt{n^2+m^2})\right]\sin(n\pi x/a)\sin(m\pi y/a).$$