453 Midterm (17 points). 7 Oct pt 08, 14:50 – 16:15 p.m., rm 303

Problem 1 (5 points).

A wire loop of radius a and resistance R lies in the XY plane. There is a uniform magnetic field $\vec{B} = B\hat{z}$ filling the whole space. What total charge passes a given point in the loop when it is rotated by 90^o around the x axis?

Solution:

The induced current is

$$I(t) = \frac{1}{R}\mathcal{E}(t) = -\frac{1}{R}\frac{d\Phi}{dt}$$

so the total charge passing a given point has the form

$$Q = \int_0^\infty I(t)dt = -\frac{1}{R}\int_0^\infty \frac{d\Phi}{dt}dt = \frac{1}{R}(\Phi_{\text{initial}} - \Phi_{\text{final}}) = \frac{\pi a^2 B}{R}$$

Problem 2 (6 points).

A long cylindrical metal wire of radius a and length $l \gg a$ carries a uniform current of density J in the axial direction. The conductivity of the metal is σ.

(a) Find Poynting vector at the surface of the wire.

(b) Calculate the flux of energy through the surface of a cylinder.

Solution

From Ampere's law

$$\vec{B}(s) \stackrel{s\geq a}{=} \frac{\mu_0 I}{2\pi s}\hat{\phi} \quad \Rightarrow \quad \vec{B}(a) = \frac{\mu_0 I}{2\pi a}\hat{\phi} = \frac{\mu_0}{2}Ja\hat{\phi}$$

From Ohm's law

$$\vec{J} = \sigma\vec{E} \quad \Rightarrow \quad \vec{E} = \frac{1}{\sigma}J\hat{e}_3$$

$\Rightarrow$ the Poynting vector is

$$\vec{S}(a) = \frac{1}{\mu_0}\vec{E}(a)\times\vec{B}(a) = \frac{J^2 a}{2\sigma}\hat{e}_3\times\hat{\phi} = -\frac{J^2 a}{2\sigma}\hat{e}_r$$

The sign reflects the fact that the energy is pumped *into* the wire (to be dissipated like the ohmic heat).

Flux of the energy is

$$P = \int dA\,|\vec{S}| = 2\pi a l|\vec{S}| = \frac{\pi a^2 J^2}{\sigma}l$$

Check: ohmic heat

$$P = I^2 R = (J\pi a^2)^2\frac{l}{\sigma\pi a^2} = \frac{\pi a^2 J^2}{\sigma}l$$

Problem 3 (5 points).

A linearly polarized electromagnetic plane wave is normally incident on an infinitely large plane made from a perfect conductor. Find the charge and current densities induced on the conducting plane.

Extra credit - 3 points. Do the same for circularly polarized electromagnetic plane

Solution

Let us choose the z axis in the direction normal to the plane and the wave coming from above the plane with polarizqtion in x direction. The incident wave has the form ($\hat{e}_3 \times \hat{e}_+ = -i\hat{e}_+$)

$$\vec{E}_I \;=\; \hat{e}_1 E_0 e^{-i\omega t+ikz}, \quad \vec{B}_I \;=\; \frac{\hat{n}}{c}\times\vec{E} \;=\; -\frac{\hat{e}_3}{c}\times\vec{E} \;=\; -\hat{e}_2\frac{E_0}{c}e^{-i\omega t+ikz}$$

The boundary condition for the electric field is $E_{\parallel}^{\text{above}} = E_{\parallel}^{\text{below}} = 0$ so $\vec{E}_R = i\hat{e}_+ E_0$ and the reflected wave is

$$\vec{E}_R \;=\; -\hat{e}_1 E_0 e^{-i\omega t-ikz}, \qquad \vec{B}_R \;=\; \frac{\hat{e}_3}{c}\times\vec{E}_R\; e^{-i\omega t-ikz} \;=\; -\hat{e}_2\frac{E_0}{c}e^{-i\omega t-ikz}$$

The sum of the reflected and incident waves takes the form:

$$\vec{E} \;=\; 2\hat{e}_1 E_0 \sin kz\; e^{-i\omega t}, \quad \vec{B} \;=\; -2\hat{e}_2\frac{E_0}{c}\cos kz\; e^{-i\omega t}$$

Now, σ and K can be found from the remaining boundary conditions

$$\epsilon_1\vec{E}_{\perp}^{\text{above}} - \epsilon_2\vec{E}_{\perp}^{\text{below}} = \sigma, \quad \frac{1}{\mu_1}\vec{B}_{\perp}^{\text{above}} - \frac{1}{\mu_2}\vec{B}_{\perp}^{\text{below}} = \vec{K}\times\hat{e}_3$$

We get $\sigma = 0$ and

$$\mu_0\vec{K}\times\hat{e}_3 \;=\; -2\hat{e}_2\frac{E_0}{c}\; e^{-i\omega t} \;\Rightarrow\; \hat{e}_3\times(\vec{K}\times\hat{e}_3) = \;\vec{K} \;=\; -2\hat{e}_3\times\hat{e}_2\frac{E_0}{\mu_0 c}\; e^{-i\omega t} \;=\; 2\hat{e}_1\frac{E_0}{\mu_0 c}\; e^{-i\omega t}$$

. Choosing E_0 to be real and taking real part we get

$$\vec{K} \;=\; \frac{2}{\mu_0 c}\hat{e}_1 E_0\; \cos\omega t$$

Extra problem

Let us now consider plane wave linearly polarized in y direction. The incident and reflected waves are

$$\begin{aligned}
\vec{E}_I \;&=\; 2\hat{e}_2 E_0\; e^{ikz-i\omega t}, \quad \vec{B}_I \;=\; 2\hat{e}_1\frac{E_0}{c}\; e^{ikz-i\omega t} \\
\vec{E}_R \;&=\; -2\hat{e}_2 E_0\; e^{-ikz-i\omega t}, \quad \vec{B}_R \;=\; \frac{\hat{e}_3}{c}\times\vec{E}_R\; e^{-i\omega t-ikz} \;=\; 2\hat{e}_1\frac{E_0}{c}\; e^{-ikz-i\omega t}
\end{aligned}$$

and the sum of the incident and reflected waves takes the form:

$$\vec{E} \;=\; 2\hat{e}_2 E_0 \sin kz\; e^{-i\omega t}, \quad \vec{B} \;=\; -2\hat{e}_1\frac{E_0}{c}\cos kz\; e^{-i\omega t}$$

Same calculation yields $\sigma = 0$ and

$$\mu_0\vec{K}\times\hat{e}_3 \;=\; 2\hat{e}_1\frac{E_0}{c}\; e^{-i\omega t} \;\Rightarrow\; \hat{e}_3\times(\vec{K}\times\hat{e}_3) = \;\vec{K} \;=\; 2\hat{e}_3\times\hat{e}_1\frac{E_0}{\mu_0 c}\; e^{-i\omega t} \;=\; 2\hat{e}_2\frac{E_0}{\mu_0 c}\; e^{-i\omega t}$$

. The circularly polarized wave is a sum of x-polarized wave and y-polarized wave with phase shift $\pm\frac{\pi}{2}$ so

$$\vec{K} \;=\; \frac{2}{\mu_0 c}\hat{e}_1 E_0\; \cos\omega t + \frac{2}{\mu_0 c}\hat{e}_2 E_0\; \cos\left(\omega t \pm \frac{\pi}{2}\right)$$

In components

$$K_x \;=\; \frac{2}{\mu_0 c}E_0\; \cos\omega t, \quad K_y \;=\; \mp\frac{2}{\mu_0 c}E_0\; \sin\omega t$$