HW 3 solution

Problem 1 = Problem 4.9 except the proof of the last equation (4-25)

(a) Prove Eq. (4-22): $\left(\frac{\partial u}{\partial P}\right)_v = c_v\left(\frac{\partial T}{\partial P}\right)_v$

Solution:
Eq. (4-22) follows directly from Eq. (4-18)

(b) Prove Eq. (4-23): $\left(\frac{\partial h}{\partial v}\right)_P = c_P\left(\frac{\partial T}{\partial v}\right)_P$

Solution:

From Eq. (4-19) we get

$$h = u + Pv \qquad \Rightarrow \qquad \text{l.h.s.} = \left(\frac{\partial h}{\partial v}\right)_P = \left(\frac{\partial u}{\partial v}\right)_P + P = \left(\frac{\partial u}{\partial T}\right)_v\left(\frac{\partial T}{\partial v}\right)_P + \left(\frac{\partial u}{\partial v}\right)_T + P = c_v\left(\frac{\partial T}{\partial v}\right)_P + \left(\frac{\partial u}{\partial v}\right)_T + P$$

On the other hand, from Eq. (4-6)

$$c_P = c_v + \left[\left(\frac{\partial u}{\partial v}\right)_T + P\right]\left(\frac{\partial v}{\partial T}\right)_P$$

$$\Rightarrow \qquad \text{r.h.s.} = \left\{c_v + \left[\left(\frac{\partial u}{\partial v}\right)_T + P\right]\left(\frac{\partial v}{\partial T}\right)_P\right\}\left(\frac{\partial T}{\partial v}\right)_P = c_v\left(\frac{\partial T}{\partial v}\right)_P + \left(\frac{\partial u}{\partial v}\right)_T + P = \text{l.h.s.}$$

where I used "inverse" equation (2-43) $\left(\frac{\partial v}{\partial P}\right)_T = \left(\frac{\partial P}{\partial v}\right)_T^{-1}$

(c) Prove Eq. (4-24): $\delta q_T = c_P\left(\frac{\partial T}{\partial v}\right)_P dv_T + c_v\left(\frac{\partial T}{\partial P}\right)_v dP_T$

Solution #1:
Taking u as a function of T and v we get

$$du_T = u(T, v + dv) - u(T, v) = \left(\frac{\partial u}{\partial v}\right)_T dv_T$$

and therefore

$$\text{l.h.s.} = \delta q_T = du_T + P dv_T = \left(\frac{\partial u}{\partial v}\right)_T dv_T + P dv_T = \left[\left(\frac{\partial u}{\partial v}\right)_T + P\right] dv_T$$

On the other hand,

$$\text{r.h.s.} = c_P\left(\frac{\partial T}{\partial v}\right)_P dv_T + c_v\left(\frac{\partial T}{\partial P}\right)_v dP_T$$

Taking T as a function of v and P, for isothermal process we have a relation

$$0 = T(v + dv_T, P + dP_T) - T(v, P) = \left(\frac{\partial T}{\partial v}\right)_P dv_T + \left(\frac{\partial T}{\partial P}\right)_v dP_T \qquad \Rightarrow \quad dP_T = -\frac{\left(\frac{\partial T}{\partial v}\right)_P}{\left(\frac{\partial T}{\partial P}\right)_v} dv_T$$

and therefore

$$\text{r.h.s.} = c_P\left(\frac{\partial T}{\partial v}\right)_P dv_T + c_v\left(\frac{\partial T}{\partial P}\right)_v dP_T = \left[c_P\left(\frac{\partial T}{\partial v}\right)_P - c_v\left(\frac{\partial T}{\partial v}\right)_P\right] dv_T$$

$$= (c_P - c_v)\left(\frac{\partial T}{\partial v}\right)_P dv_T \stackrel{(4-9)}{=} \left[\left(\frac{\partial u}{\partial v}\right)_T + P\right]\left(\frac{\partial v}{\partial T}\right)_P\left(\frac{\partial T}{\partial v}\right)_P dv_T \stackrel{(2-43)}{=} \left[\left(\frac{\partial u}{\partial v}\right)_T + P\right] dv_T = \text{l.h.s.}$$

Solution #2

$$\delta q = du + P dv = \left[\left(\frac{\partial u}{\partial v}\right)_P + P\right] dv + \left(\frac{\partial u}{\partial P}\right)_v dP \stackrel{\text{part(a)}}{=} \left(\frac{\partial h}{\partial v}\right)_P dv + c_v\left(\frac{\partial T}{\partial P}\right)_v dP \stackrel{\text{part(b)}}{=} c_P\left(\frac{\partial T}{\partial v}\right)_P dv + c_v\left(\frac{\partial T}{\partial P}\right)_v dP$$