Phys. 807 — Statistical Mechanics

Solution.

The velocities in generalized coordinates r, ϕ are

$$\begin{aligned}
\dot{x} &= \frac{d}{dt} r\cos\phi = \dot{r}\cos\phi - r\dot{\phi}\sin\phi \\
\dot{y} &= \frac{d}{dt} r\sin\phi = \dot{r}\sin\phi + r\dot{\phi}\cos\phi \\
\dot{z} &= \frac{d}{dt} r\cot\alpha = \dot{r}\cot\alpha
\end{aligned}$$

The kinetic energy is

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{r}^2\cot^2\alpha) = \frac{m}{2}(\frac{\dot{r}^2}{\sin^2\alpha} + r^2\dot{\phi}^2)$$

so the Lagrangian is

$$L = T - V = \frac{m}{2}(\frac{\dot{r}^2}{\sin^2\alpha} + r^2\dot{\phi}^2) - mgr\cot\alpha$$

The Lagrangian does not depend on ϕ so the first equation of motion is

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}} = \frac{d}{dt}(mr^2\dot{\phi}) = 0$$

so

$$mr^2\dot{\phi} = \text{const}$$

This is actually the z-component of the angular momentum:

$$L_z = x\dot{y} - y\dot{x} = \cos\phi(\dot{r}\sin\phi + r\dot{\phi}\cos\phi) - r\sin\phi(\dot{r}\cos\phi - r\dot{\phi}\sin\phi) = mr^2\dot{\phi}$$

The second of Euler-Lagrange equations is

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial\dot{r}} &= \frac{\partial L}{\partial r} \\
\frac{m\ddot{r}}{\sin^2\alpha} &= mr\dot{\phi}^2 - mg\cot\alpha
\end{aligned}$$

The second conserved quantity is the energy

$$E = T + V = \frac{m}{2}(\frac{\dot{r}^2}{\sin^2\alpha} + r^2\dot{\phi}^2) + mgr\cot\alpha$$

Check:

$$\frac{dE}{dt} = \frac{m}{\sin^2\alpha}\dot{r}\ddot{r} + mr^2\dot{\phi}\ddot{\phi} + m\dot{\phi}^2 r\dot{r} - mg\dot{r}\cot\alpha$$
$$= (mr\dot{\phi}^2 - mg\cot\alpha)\dot{r} + \dot{\phi}\frac{d}{dt}(mr^2\dot{\phi}) - mr\dot{r}\dot{\phi}^2 + m\dot{\phi}^2 r\dot{r} + mg\dot{r}\cot\alpha = 0$$

The conservation of energy can be rewritten as

$$\dot{r} = \sin\alpha\sqrt{\frac{2E}{m} - \frac{L_z^2}{r^2} - 2mgr\cot\alpha}$$

This ordinary differential equation can be integrated

$$\phi(t) - \phi_0 = \frac{1}{\sin\alpha}\int_{t_0}^{t} dt\ \frac{1}{\sqrt{\frac{2E}{m} - \frac{L_z^2}{r^2} - 2mgr\cot\alpha}}$$