HW assignment 6

Solution.

The kinetic energy is

$$T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{m}{2}\dot{z}^2$$

The constraint $\dot{\phi} = \omega$ is non-holonomic so I'll put it in explicitly

$$T = \frac{m}{2}(\dot{r}^2 + r^2\omega^2) + \frac{m}{2}\dot{z}^2$$

The potential energy is $V = mgz$ so the Lagrangian with constraint $z = \alpha\rho^2$ is

$$L = \frac{m}{2}(\dot{r}^2 + r^2\omega^2) + \frac{m}{2}\dot{z}^2 - mgz + \lambda(z - \alpha r^2)$$

Euler-Lagrange equations are

$$\begin{aligned}
\frac{\partial L}{\partial \lambda} &= z - \alpha r^2 = 0 &\Rightarrow\quad& z = \alpha r^2 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r} = \frac{\partial L}{\partial r} &= (m\omega^2 - 2\alpha\lambda)r &\Rightarrow\quad& \ddot{r} = (\omega^2 - 2\alpha\frac{\lambda}{m})r \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = m\ddot{z} = \frac{\partial L}{\partial z} &= (\lambda - mg) &\Rightarrow\quad& m\ddot{z} = \lambda - mg
\end{aligned}$$

From the last equation we see that λ is a progection of a normal force on the z axis.

If we use $z = \alpha r^2$ we get

$$\ddot{r} = (\omega^2 - 2\alpha\frac{\lambda}{m})r, \qquad \ddot{r}r + \dot{r}^2 = \frac{\lambda - mg}{2m\alpha}$$

Eliminating λ from the above equations, we get the second-order differential equation for $r(t)$

$$r\ddot{r}(1 + 4\alpha^2r^2) + 4\alpha^2r\dot{r}^2 + (2g\alpha - \omega^2)r = 0$$