HW 9 solution.

The generalized coordinates are α and β. The angle between axes is $\beta - \alpha$ so the kinetic energy is

$$T = \frac{m}{2}l^2\dot{\alpha}^2 + \frac{m}{2}l^2[\dot{\alpha}^2 + \dot{\beta}^2 + 2\dot{\alpha}\dot{\beta}\cos(\alpha-\beta)]$$

and the potential

$$V = -mgl(2\cos\alpha + \cos\beta) + \frac{\kappa}{2}(\alpha - \beta + \pi)^2$$

The Lagrangian is

$$L = \frac{m}{2}l^2\dot{\alpha}^2 + \frac{m}{2}l^2[\dot{\alpha}^2 + \dot{\beta}^2 + 2\dot{\alpha}\dot{\beta}\cos(\alpha-\beta)] + mgl(2\cos\alpha + \cos\beta) - \frac{\kappa}{2}(\alpha - \beta + \pi)^2$$

The derivatives of the Lagrangian are

$$\frac{\partial L}{\partial \dot{\alpha}} = ml^2[2\dot{\alpha} + \dot{\beta}\cos(\alpha-\beta)], \qquad \frac{\partial L}{\partial \alpha} = -ml^2\dot{\alpha}\dot{\beta}\sin(\alpha-\beta) - 2mgl\sin\alpha - \kappa(\alpha-\beta+\pi)$$
$$\frac{\partial L}{\partial \dot{\beta}} = ml^2[\dot{\alpha}\cos(\alpha-\beta) + \dot{\beta}], \qquad \frac{\partial L}{\partial \beta} = ml^2\dot{\alpha}\dot{\beta}\sin(\alpha-\beta) - mgl\sin\beta + \kappa(\alpha-\beta+\pi)$$

so the equations of motion read

$$2\ddot{\alpha} + \ddot{\beta}\cos(\alpha-\beta) - \dot{\beta}(\dot{\alpha}-\dot{\beta})\sin(\alpha-\beta) = -\dot{\alpha}\dot{\beta}\sin(\alpha-\beta) - \frac{2g}{l}\sin\alpha - \frac{\kappa}{ml^2}(\alpha-\beta+\pi)$$
$$\ddot{\alpha}\cos(\alpha-\beta) + \ddot{\beta} - \dot{\alpha}(\dot{\alpha}-\dot{\beta})\sin(\alpha-\beta) = \dot{\alpha}\dot{\beta}\sin(\alpha-\beta) - \frac{g}{l}\sin\beta + \frac{\kappa}{ml^2}(\alpha-\beta+\pi)$$

or

$$2\ddot{\alpha} + \ddot{\beta}\cos(\alpha-\beta) + \dot{\beta}^2\sin(\alpha-\beta) + \frac{2g}{l}\sin\alpha + \frac{\kappa}{ml^2}(\alpha-\beta+\pi) = 0$$
$$\ddot{\alpha}\cos(\alpha-\beta) + \ddot{\beta} - \dot{\alpha}^2\sin(\alpha-\beta) + \frac{g}{l}\sin\beta - \frac{\kappa}{ml^2}(\alpha-\beta+\pi) = 0$$