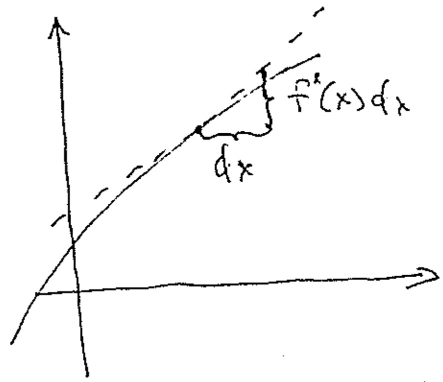
Differential calculus

Usual derivative: slope of the curve



$$df(x) = f'(x)\,dx$$

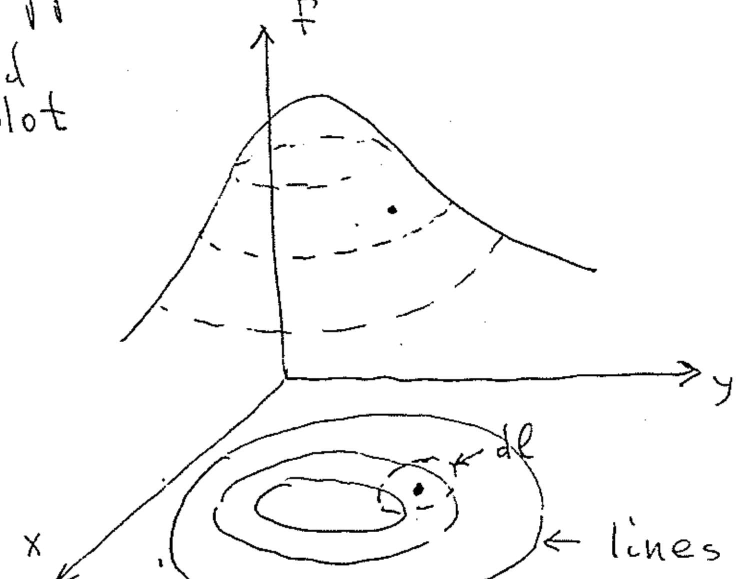
$$f(x+dx) - f(x) = f'(x)\,dx + O((dx)^2)$$

($O(dx)^2$ means, roughly speaking, $(dx)^2$ multiplied by some constant

Gradient

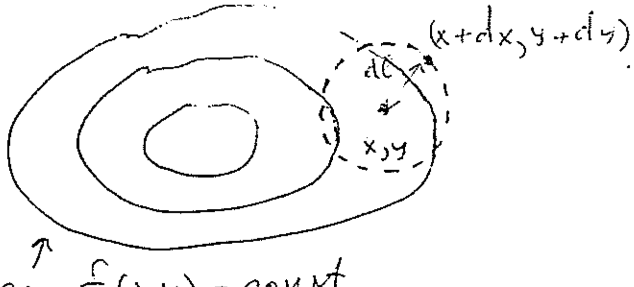
Suppose we have a function of two variables $f(x,y)$

3d plot



Let us look for the direction of the steepest ascent from the point (x,y). We draw a small circle with radius dl and check where $|f(x+dx,y+dy) - f(x,y)| = \max$.

$\sqrt{(dx)^2+(dy)^2} = dl$ fixed



$$f(x+dx,\, y+dy) \simeq$$

$$\simeq f(x,y) + \frac{\partial f(x,y)}{\partial x}\,dx + \frac{\partial f(x,y)}{\partial y}\,dy$$

$$\Rightarrow df = dx\,\frac{\partial f}{\partial x} + dy\,\frac{\partial f}{\partial y}$$

It is convenient to represent $df = dx \frac{\partial f}{\partial x} + dy \frac{\partial f}{\partial y}$ in the form of a dot product

$$df = \underbrace{\left(\frac{\partial f}{\partial x}\hat{e}_1 + \frac{\partial f}{\partial y}\hat{e}_2\right)}_{\equiv\ \vec{\nabla} f} \cdot \underbrace{(dx\,\hat{e}_1 + dy\,\hat{e}_2)}_{d\hat{e}} \Rightarrow$$

Definition: $\vec{\nabla} f(x,y) = \hat{e}_1 \frac{\partial f}{\partial x}(x,y) + \hat{e}_2 \frac{\partial f}{\partial y}(x,y)$ – „gradient" (vector)

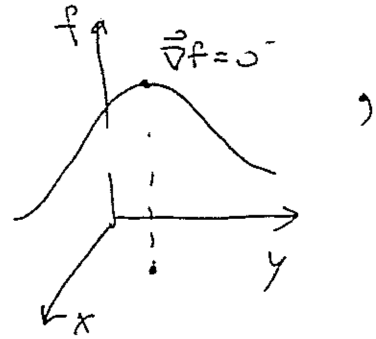
$$\Rightarrow df = \vec{\nabla} f \cdot d\vec{e} = |\vec{\nabla} f|\,|de|\cos\theta = |\vec{\nabla} f|\,de\cos\theta \qquad (*)$$

It is clear that $\theta = 0$ corresponds to the direction of the steepest ascent $\Rightarrow$

$\Rightarrow$ Direction of the steepest ascent in a given point (x,y), is given by the gradient $\vec{\nabla} f(x,y)$.

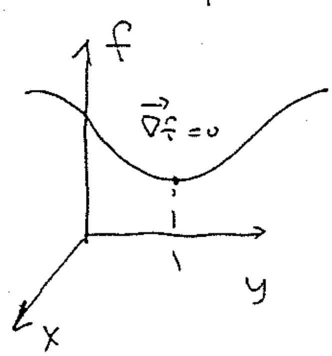
Moreover, in this direction $\frac{df}{de} = |\vec{\nabla} f| \Rightarrow$ the slope in the steepest ascent direction is given by the magnitude of the gradient

If $\vec{\nabla} f(x,y) = 0$ you have maximum

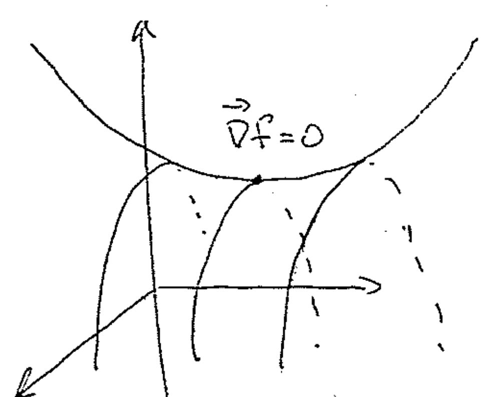


,

minimum



or a „saddle point"



As in the case of one variable, in order to distinguish between these three possibilities one should calculate second derivatives $\frac{\partial^2 f}{}$, $\frac{\partial^2 f}{}$ and $\frac{\partial^2 f}{}$

Similarly, one can define the gradient for a function of three variables $T(x,y,z)$

$$\vec{\nabla} T = \frac{\partial T(x,y,z)}{\partial x}\hat{e}_1 + \frac{\partial T(x,y,z)}{\partial y}\hat{e}_2 + \frac{\partial T(x,y,z)}{\partial z}\hat{e}_3$$

so

$$dT(x,y,z) = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz = \vec{\nabla}T \cdot \underbrace{d\vec{\ell}}_{\substack{\parallel \\ dx\,\hat{e}_1 + dy\,\hat{e}_2 + dz\,\hat{e}_3}}$$

Again, if $|d\vec{\ell}|$ is fixed,

$|dT| = |\vec{\nabla}T|\, d\ell \cos\theta$ $\Rightarrow$ gradient gives direction and magnitude of the steepest ascent of function $T(x,y,z)$ (a good example to visualize is a temperature in a room)

Let us calculate $\vec{\nabla}\frac{1}{r}$ $\qquad r = \sqrt{x^2+y^2+z^2}$

$$\frac{\partial}{\partial x}\frac{1}{r} = \frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2}$$

$$= -\frac{2x}{2\sqrt{x^2+y^2+z^2}} = -\frac{x}{r}$$

Similarly, $\frac{\partial}{\partial y}\frac{1}{r} = -\frac{y}{r}$ and $\frac{\partial \frac{1}{r}}{\partial z} = -\frac{z}{r}$

$$\Rightarrow \vec{\nabla}\frac{1}{r} = -\frac{x}{r}\hat{e}_1 - \frac{y}{r}\hat{e}_2 - \frac{z}{r}\hat{e}_3 = -\frac{1}{r^2}(x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3) = -\frac{\vec{r}}{r^3}$$

The operator $\vec{\nabla}$

$$\vec{\nabla}T = \frac{\partial T}{\partial x}\hat{e}_1 + \frac{\partial T}{\partial y}\hat{e}_2 + \frac{\partial T}{\partial z}\hat{e}_3 = \underbrace{\left(\hat{e}_1\frac{\partial}{\partial x} + \hat{e}_2\frac{\partial}{\partial y} + \hat{e}_3\frac{\partial}{\partial z}\right)}_{\vec{\nabla}} T(x,y,z)$$

$\vec{\nabla} \equiv \hat{e}_1\frac{\partial}{\partial x} + \hat{e}_2\frac{\partial}{\partial y} + \hat{e}_3\frac{\partial}{\partial z}$ – „vector operator"

(like $\frac{\partial}{\partial x}$ is a „scalar operator")

If we have „vector field"

$$\vec{v}(x,y,z) = v_1(x,y,z)\,\hat{e}_1 + v_2(x,y,z)\,\hat{e}_2 + v_3(x,y,z)\,\hat{e}_3$$

usual (scalar) functions

we can construct

$\vec{\nabla}\cdot\vec{v} \equiv$ „divergence"

and

$\vec{\nabla}\times\vec{v} \equiv$ „curl" $\qquad$ of the vector field $\vec{v}(x,y,z)$

The divergence

$$\nabla\cdot v = \left(\hat{e}_1 \frac{\partial}{\partial x} + \hat{e}_2 \frac{\partial}{\partial y} + \hat{e}_3 \frac{\partial}{\partial z}\right)\cdot\left(\hat{e}_1 v_1(x,y,z) + \hat{e}_2 v_2(x,y,z) + \hat{e}_3 v_3(x,y,z)\right) =$$

$$= \frac{\partial v_1(x,y,z)}{\partial x} + \frac{\partial v_2(x,y,z)}{\partial y} + \frac{\partial v_3(x,y,z)}{\partial z} = \frac{\partial}{\partial x_i} v_i$$

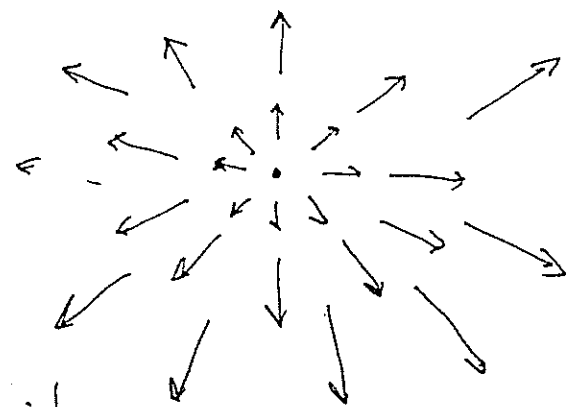
divergence is a scalar

$(x_1 \equiv x,\ x_2 \equiv y,\ x_3 \equiv z)$

Two examples

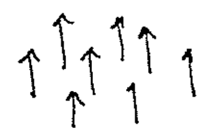
1. $v(x,y,z) = \vec{r} = \hat{e}_1 x + \hat{e}_2 y + \hat{e}_3 z$

$$\nabla\cdot v = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y + \frac{\partial}{\partial z} z = 3$$



Physical meaning: if, for example, $\vec{v}(x,y)$ is a field describing velocities of a liquid, then positive divergence corresponds to a source and negative to a sink.

2. $v(x,y,z) = \hat{e}_3$ $\qquad \vec{\nabla}\cdot\vec{v} = 0$



no sources, no sinks.

Let us calculate

$$\vec{\nabla}\cdot\frac{\vec{r}}{r^3} = \frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial}{\partial y}\frac{y}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial}{\partial z}\frac{z}{(x^2+y^2+z^2)^{3/2}} =$$

$$= \left(\frac{1}{r^3} - \frac{3}{2}\cdot\frac{2x^2}{(x^2+y^2+z^2)^{5/2}}\right) + \left(\frac{1}{r^3} - \frac{3y^2}{r^5}\right) + \left(\frac{1}{r^3} - \frac{3z^2}{r^5}\right) = \frac{3}{r^3} - \frac{3(x^2+y^2+z^2)}{r^5} = 0$$

More careful analysis shows that $\vec{\nabla}\cdot\frac{\vec{r}}{r^3} = 4\pi\delta^{(3)}(\vec{r})$

„Dirac δ-function"

The curl

$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}$

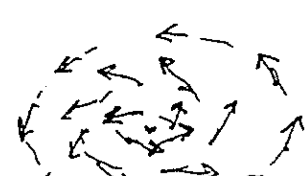
$\epsilon_{123} = 1$

$$(\vec{\nabla}\times\vec{v})_i = \epsilon_{ijk}\frac{\partial}{\partial x_j} v_k \qquad x_1 \equiv x,\ x_2 \equiv y,\ x_3 \equiv z$$

$$\Rightarrow \vec{\nabla}\times\vec{v} = \hat{e}_1\left(\frac{\partial}{\partial x_2} v_3 - \frac{\partial}{\partial x_3} v_2\right) + \hat{e}_2\left(\frac{\partial}{\partial x_3} v_1 - \frac{\partial}{\partial x_1} v_3\right) + \hat{e}_3\left(\frac{\partial}{\partial x_1} v_2 - \frac{\partial}{\partial x_2} v_1\right) =$$

$$= \hat{e}_1\left(\frac{\partial}{\partial y} v_3(x,y,z) - \frac{\partial}{\partial z} v_2(x,y,z)\right) + \hat{e}_2\left(\frac{\partial}{\partial z} v_1 - \frac{\partial}{\partial x} v_3\right) + \hat{e}_3\left(\frac{\partial}{\partial x} v_2 - \frac{\partial}{\partial y} v_1\right)$$

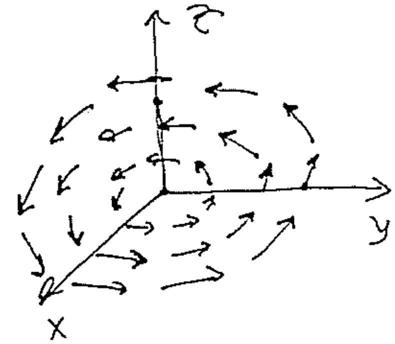
Picture of a curl: a whirlpool (if $\vec{v}(x,y)$ describes the field of the velocities of the liquid)



Example:

$\vec{V}(x,y) = -y\hat{e}_1 + x\hat{e}_2$

$|\vec{V}(x,y)| = r$



$\vec{\nabla}\times\vec{V} =$

$= \hat{e}_1\left(-\frac{\partial}{\partial z}x\right) + \hat{e}_2\left(\frac{\partial}{\partial z}(-y)\right) + \hat{e}_3\left(\frac{\partial}{\partial x}x - \frac{\partial}{\partial y}(-y)\right) = 2\hat{e}_3$

By the way, $\nabla\cdot v = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}x = \emptyset$ $\to$ no sinks or sources only curls

Another example: $\vec{\nabla}\times\frac{\vec{r}}{r^3}$

$\vec{\nabla}\times\frac{\vec{r}}{r^3} = \hat{e}_1\left(\frac{\partial}{\partial y}\frac{z}{(x^2+y^2+z^2)^{3/2}} - \frac{\partial}{\partial z}\frac{y}{(x^2+y^2+z^2)^{3/2}}\right) + \hat{e}_2\left(\frac{\partial}{\partial z}\frac{x}{r^3} - \frac{\partial}{\partial x}\frac{z}{r^3}\right) +$

$+\hat{e}_3\left(\frac{\partial}{\partial x}\frac{y}{r^3} - \frac{\partial}{\partial y}\frac{x}{r^3}\right) = \hat{e}_1\left(-\cancel{\frac{3yz}{r^5}} + \cancel{\frac{3yz}{r^5}}\right) + \emptyset\cdot\hat{e}_2 + \emptyset\cdot\hat{e}_3 = 0$

As for usual derivatives

$\vec{\nabla}(f+g) = \hat{e}_i\frac{\partial}{\partial x_i}(f+g) = \hat{e}_i\frac{\partial}{\partial x_i}f + \hat{e}_i\frac{\partial}{\partial x_i}g = \vec{\nabla}f + \vec{\nabla}g$

$\vec{\nabla}\cdot(\vec{A}+\vec{B}) = \frac{\partial}{\partial x_i}(A_i + B_i) = \frac{\partial}{\partial x_i}A_i + \frac{\partial}{\partial x_i}B_i = \vec{\nabla}\cdot\vec{A} + \vec{\nabla}\cdot\vec{B}$

$\vec{\nabla}\times(\vec{A}+\vec{B}) = \vec{\nabla}\times\vec{A} + \vec{\nabla}\times\vec{B}$

$\epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - m\leftrightarrow n$

Product rules

$\vec{\nabla}(fg) = e_i\frac{\partial}{\partial x_i}(fg) = e_i\left(\frac{\partial f}{\partial x_i}g + \frac{\partial g}{\partial x_i}f\right) = f\vec{\nabla}g + g\vec{\nabla}f$

$\vec{\nabla}(\vec{A}\cdot\vec{B}) = e_i\frac{\partial}{\partial x_i}(A_jB_j) = e_i\left\{\left(\frac{\partial}{\partial x_i}A_j\right)B_j + A_j\frac{\partial}{\partial x_i}B_j\right\} =$

$= e_iB_j\partial_iA_j + e_iA_j\partial_iB_j =$ $\qquad \partial_iA_j \equiv \frac{\partial}{\partial x_i}A_j$

$= e_iB_j(\partial_iA_j - \partial_jA_i + \partial_jA_i) + e_iA_j(\partial_iB_j - \partial_jB_i + \partial_jB_i) =$

$= e_iB_j(\partial_iA_j - \partial_jA_i) + e_i\vec{B}\cdot\vec{\nabla}A_i + e_iA_j(\partial_iB_j - \partial_jB_i) + e_i(\vec{A}\cdot\vec{\nabla})B_i$

$= e_iB_j(\partial_iA_j - i\leftrightarrow j) + e_iA_j(\partial_iB_j - i\leftrightarrow j) + (\vec{B}\cdot\vec{\nabla})\vec{A} + (\vec{A}\cdot\vec{\nabla})\vec{B}$

Let us prove that the first term is $\vec{B}\times(\vec{\nabla}\times\vec{A})$

$(\vec{B}\times(\vec{\nabla}\times\vec{A}))_i = \epsilon_{ijk}B_j(\vec{\nabla}\times\vec{A})_k = \epsilon_{ijk}B_j\epsilon_{klm}\partial_lA_m =$

$= \epsilon_{ijk}\epsilon_{lmk}B_j\partial_lA_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})B_j\partial_lA_m = B_m(\partial_iA_m - \partial_mA_i)$

$$\Rightarrow \vec{B}\times(\vec{\nabla}\times\vec{A}) = \hat{e}_i(\vec{B}\times(\vec{\nabla}\times\vec{A}))_i \doteq e_i B_m(\partial_i A_m - \partial_m A_i) = e_i B_j(\partial_i A_j - i\leftrightarrow j)$$

$A \leftrightarrow B$: $\quad \vec{A}\times(\vec{\nabla}\times\vec{B}) = e_i A_j(\partial_i B_j - i\leftrightarrow j)$

$$\Rightarrow \vec{\nabla}(\vec{A}\cdot\vec{B}) = \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) + (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A}$$

Similarly

$$\vec{\nabla}\cdot(f\vec{A}) = f(\vec{\nabla}\cdot\vec{A}) + \vec{A}\cdot(\vec{\nabla}f)$$

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$$

$$\vec{\nabla}\times(f\vec{A}) = f(\vec{\nabla}\times\vec{A}) - \vec{A}\times(\vec{\nabla}f)$$

$$\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{A}).$$

Second derivatives

$\vec{\nabla}\cdot(\vec{\nabla}T)$, $\vec{\nabla}\times(\vec{\nabla}T)$

$\vec{\nabla}(\vec{\nabla}\cdot\vec{v})$, $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v})$, $\vec{\nabla}\times(\vec{\nabla}\times\vec{v})$

1. $$\vec{\nabla}\cdot(\vec{\nabla}T) = \left(\hat{e}_1\frac{\partial}{\partial x} + \hat{e}_2\frac{\partial}{\partial y} + \hat{e}_3\frac{\partial}{\partial z}\right)\left(\hat{e}_1\frac{\partial T}{\partial x} + \hat{e}_2\frac{\partial T}{\partial y} + \hat{e}_3\frac{\partial T}{\partial z}\right) =$$

$$= \frac{\partial}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial}{\partial y}\frac{\partial T}{\partial y} + \frac{\partial}{\partial z}\frac{\partial T}{\partial z} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)}_{\text{Laplacian or } \nabla^2} T$$

$$\nabla^2 T = \left(\hat{e}_i\frac{\partial}{\partial x_i}\right)\left(\hat{e}_j\frac{\partial}{\partial x_j}\right)T = \delta_{ij}\frac{\partial^2 T}{\partial x_i\partial x_j} = \frac{\partial^2}{\partial x_i\partial x_i}T$$

We can also define

$$\nabla^2\vec{v} = \hat{e}_1\nabla^2 v_1 + \hat{e}_2\nabla^2 v_2 + \hat{e}_3\nabla^2 v_3$$

2. $\vec{\nabla}\times(\vec{\nabla}T) = 0$

Proof: $(\vec{\nabla}\times(\vec{\nabla}T))_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}(\vec{\nabla}T)_k = \epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}T = 0$ because $\epsilon_{ijk} = -\epsilon_{ikj}$

3. $\vec{\nabla}(\vec{\nabla}\cdot\vec{v})$ – no special name

4. $$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v}) = \frac{\partial}{\partial x_i}(\vec{\nabla}\times\vec{v})_i = \frac{\partial}{\partial x_i}\epsilon_{ijk}\frac{\partial}{\partial x_j}v_k = \epsilon_{ijk}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}v_k = 0$$

5. $\vec{\nabla}\times(\vec{\nabla}\times\vec{v}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{v}) - \nabla^2\vec{v}$

Proof: $$(\vec{\nabla}\times(\vec{\nabla}\times\vec{v}))_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}(\vec{\nabla}\times\vec{v})_k = \epsilon_{ijk}\frac{\partial}{\partial x_j}\left(\epsilon_{klm}\frac{\partial}{\partial x}v_m\right) =$$

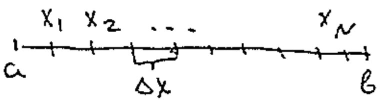
$$= \epsilon_{ijk}\epsilon_{lmk}\frac{\partial^2 v_m}{\partial x_j\partial x_l} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j\partial_l v_m = \partial_i\partial_m v_m - \partial_m\partial_m v_i$$

Integral calculus

Reminder: ordinary integration. Fundamental theorem:

$$\int_a^b dx\, \frac{df(x)}{dx} = f(b) - f(a) \iff \int_a^b F(x)\,dx = f(b) - f(a) \quad \text{if} \quad F(x) = f'(x)$$

Sketch of the proof:

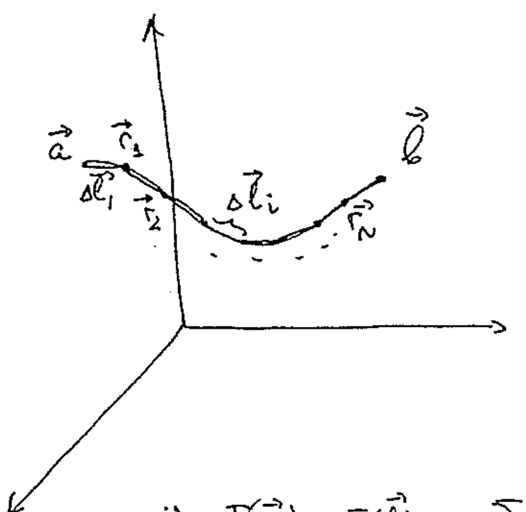


$$f(b) - f(a) = \underbrace{f(b) - f(x_N)}_{\simeq \Delta x\, f'(b)} + \underbrace{f(x_N) - f(x_{N-1})}_{\simeq \Delta x\, f'(x_N)} + \dots + \underbrace{f(x_1) - f(a)}_{\simeq \Delta x\, f'(a)} \simeq \sum_{n=1}^{N} \Delta x\, f'(x_n)$$

At $\Delta x \to 0$ we get $\sum_{n=1}^{N} \Delta x\, f'(x_n) \to \int_a^b dx\, f'(x) \Rightarrow f(b) - f(a) = \int_a^b dx\, f'(x)$

In general, the integral of a derivative over some interval is given by the value of the functions at the boundaries ($\iff$ end points of the interval).

The fundamental theorem for gradients



$$T(\vec{a}) - T(\vec{b}) = T(\vec{a}) - T(\vec{r}_1) + T(\vec{r}_1) - T(\vec{r}_2) + \dots + T(\vec{r}_N) - T(\vec{b})$$

$$T(\vec{a}) - T(\vec{r}_1) \simeq \Delta\vec{l}_1 \cdot \vec{\nabla} T(\vec{a})$$

$$T(\vec{r}_1) - T(\vec{r}_2) \simeq \Delta\vec{l}_2 \cdot \vec{\nabla} T(\vec{r}_1)$$

$$\vdots$$

$$T(\vec{r}_N) - T(\vec{b}) \simeq \Delta\vec{l}_N \cdot \vec{\nabla} T(\vec{r}_N)$$

$$\Rightarrow T(\vec{a}) - T(\vec{b}) \simeq \sum \Delta\vec{l}_k \cdot \vec{\nabla} T(\vec{r}_k)$$

$$|\Delta l| \to 0 \quad \Rightarrow \quad T(\vec{a}) - T(\vec{b}) = \int_b^a d\vec{l} \cdot (\vec{\nabla} T)$$

Corollary: $\int_b^a (\vec{\nabla} T) \cdot d\vec{l}$ is independent of path taken from a to b

$$\iff \oint \vec{\nabla} T \cdot d\vec{l} = 0$$

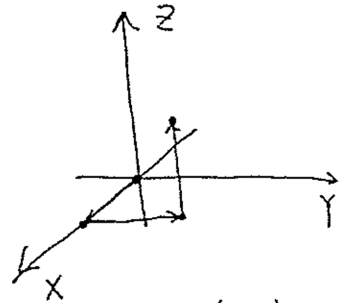
(Recall conservative forces and potential energy in mechanics).

Example: $T = x^2 + 4xy + 2yz^3$

$$\vec{\nabla} T = \frac{\partial T}{\partial x}\hat{e}_1 + \frac{\partial T}{\partial y}\hat{e}_2 + \frac{\partial T}{\partial z}\hat{e}_3 = (2x+4y)\hat{e}_1 + (4x+2z^3)\hat{e}_2 + 6yz^2\hat{e}_3$$

$$\int_{(0,0,0)}^{(1,1,1)} \vec{\nabla} T \cdot d\vec{\ell} = ?$$

Let us pick the path $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1)$



$$\int_{(0,0,0)}^{(1,1,1)} \vec{\nabla} T \cdot d\vec{\ell} = \int_{(0,0,0)}^{(1,0,0)} \vec{\nabla} T \cdot d\vec{\ell} + \int_{(1,0,0)}^{(1,1,0)} \vec{\nabla} T \cdot d\ell + \int_{(1,1,0)}^{(1,1,1)} \vec{\nabla} T \cdot d\ell$$

$$\int_{(0,0,0)}^{(1,0,0)} \vec{\nabla} T \cdot d\vec{\ell} = ? \qquad d\vec{\ell} = dx\,\hat{e}_1 \qquad \vec{\nabla} T \cdot \hat{e}_1 = 2x + 4y \quad \Rightarrow \int_{(0,0,0)}^{(1,0,0)} (2x + \underset{=0}{\cancel{4y}})\,dx =$$

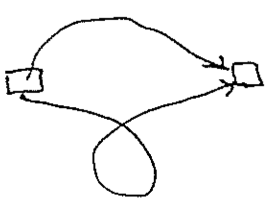
$$\int_{(1,0,0)}^{(1,1,0)} \vec{\nabla} T \cdot d\vec{\ell} = \int_{(1,0,0)}^{(1,1,0)} \vec{\nabla} T \cdot \hat{e}_2\, dy = \int_{(1,0,0)}^{(1,1,0)} (4\underset{=1}{x} + \cancel{2z^3})\,dy = 4\int_0^1 dy = 4$$

$$\int_{(1,1,0)}^{(1,1,1)} \vec{\nabla} T \cdot d\vec{\ell} = \int_{(1,1,0)}^{(1,1,1)} \vec{\nabla} T \cdot \hat{e}_3\, dz = \int_{(1,1,0)}^{(1,1,1)} 6yz^2\,dz = 6\int_0^1 z^2\,dz = 2$$

$$\Rightarrow \int_{(0,0,0)}^{(1,1,1)} \vec{\nabla} T \cdot d\vec{\ell} = 1 + 4 + 2 = 7.$$

Check: $T(1,1,1) - T(0,0,0) = 1 + 4 + 2 = 7.$

It is worth noting that for the arbitrary vector functions $\vec{v}(x,y,z)$ the integral $\int_a^b \vec{v} \cdot d\vec{\ell}$ depends on the path (Example: work done by the force of a friction on a plane)



different works

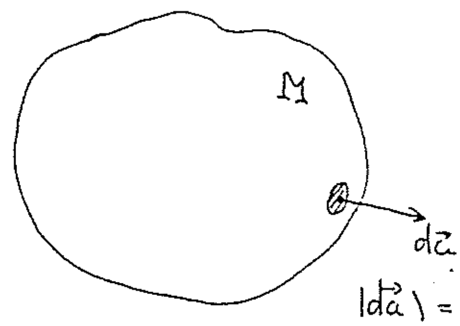
In this case, the integral $\oint \vec{v} \cdot d\vec{\ell} \neq 0.$

Fundamental theorem for divergences

Gauss' theorem
Green's theorem

$$\int_{M} (\vec{\nabla}\cdot\vec{v})\, d\tau = \oint_{\partial M} \vec{v}\cdot d\vec{a}$$

M (volume) $\quad$ ∂M (surface)



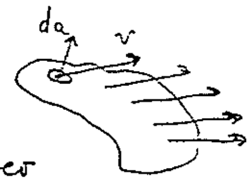
A convenient notation:
$\partial M \stackrel{\text{def}}{\equiv}$ surface of volume M

$|d\vec{a}|$ = area of the element of the surface

Direction of $d\vec{a}$ is normal to the surface (and out of the volume)

$\int_{\text{surface}} \vec{v}\cdot d\vec{a}$ is called the flux of $\vec{v}$ through the surface

Geometrical interpretation



$\vec{v}$ - velocity of some liquid

$\vec{v}\cdot d\vec{a}$ = amount of liquid which passes through da per unit time

$\int_{\text{surface}} \vec{v}\cdot d\vec{a}$ = amount of liquid which passes through the surface per unit time

Recall geometrical interpretation of the divergence: strength of the faucet (source) in a given point.

For the (incompressible) fluid, per unit time

$$\int_{M} (\text{all faucets in the volume}) = \oint_{\partial M} (\text{flux out through the surface})$$

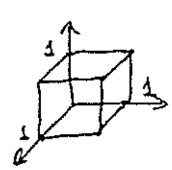
Example. check the divergence theorem for

$$\vec{v}(x,y,z) = y^2\hat{e}_1 + (2xy+z^2)\hat{e}_2 + 2yz\,\hat{e}_3$$

for the unit cube

$$\int_{M} (\vec{\nabla}\cdot\vec{v})\, d\tau$$

$d\tau \equiv dx\,dy\,dz$ = element of the volume



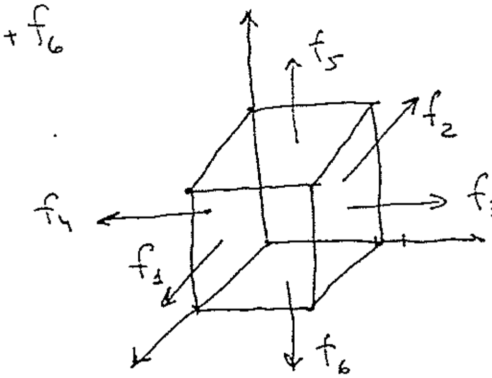
$$\vec{\nabla}\cdot\vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = \emptyset + 2x + 2y = 2(x+y)$$

$$\int_{M} 2(x+y)\,dx\,dy\,dz = 2\int_0^1 dx \int_0^1 dy \int_0^1 dz\,(x+y) = 2\underbrace{\int_0^1 x\,dx}_{1/2}\underbrace{\int_0^1 dy}_{1}\underbrace{\int_0^1 dz}_{1} + 2\underbrace{\int_0^1 dx}_{1}\underbrace{\int_0^1 y\,dy}_{1/2}\underbrace{\int_0^1 dz}_{1}$$

$= 2$

$\Rightarrow$ l.h.s. $= \int (\vec{\nabla}\cdot\vec{v})\, d\tau = 2$

r.h.s. = total flux $= f_1 + f_2 + f_3 + f_4 + f_5 + f_6$



f_1:

$$\int_{\partial M_1} \vec{v}\cdot d\vec{a} = ? \qquad d\vec{a} = dy\, dz\, \hat{e}_1$$

$$\int_{\partial M_1} \vec{v}\cdot\hat{e}_1\, dy\, dz = \int_{\partial M_1} y^2 dy\, dz =$$

$$= \int_0^1 dy \int_0^1 dz\, y^2 = \int_0^1 y^2 dy \int_0^1 dz = \frac{1}{3}\cdot 1 = \frac{1}{3} \quad \Rightarrow f_1 = \frac{1}{3}$$

f_2:

$$\int_{\partial M_2} \vec{v}\cdot d\vec{a} = \int_{\partial M_2} \vec{v}\cdot(-\hat{e}_1)\, dy\, dz = -\int_0^1 dy \int_0^1 dz\; y^2 = -\frac{1}{3} \Rightarrow f_2 = -\frac{1}{3}$$

f_3:

$$\int_{\partial M_3} \vec{v}\cdot d\vec{a} = \int_{\partial M_3} \vec{v}\cdot\hat{e}_2\, dx\, dz = \int_{\partial M_3} (2x\overset{1}{\overset{\shortparallel}{y}} + z^2)\, dx\, dz = \int_0^1 dx \int_0^1 dz\, (2x + z^2) = 2\int_0^1 x\, dx \int_0^1 dz$$
$$+ \int_0^1 dx \int_0^1 z^2 dz = 1 + \frac{1}{3} = \frac{4}{3}$$

f_4:

$$\int_{\partial M_4} \vec{v}\cdot d\vec{a} = \int_{\partial M_4} \vec{v}\cdot(-\hat{e}_2)\, dx\, dz = -\int_{\partial M_4} (2x\overset{\emptyset}{\overset{\shortparallel}{y}} + z^2)\, dx\, dz = -\int_0^1 dx \int_0^1 dz\, z^2 = -\frac{1}{3}$$

f_5:

$$\int_{\partial M_5} \vec{v}\cdot d\vec{a} = \int_{\partial M_5} \vec{v}\cdot\hat{e}_3\, dx\, dy = \int_{\partial M_5} 2y\overset{1}{\overset{\shortparallel}{z}}\, dx\, dy = 2\int_0^1 dx \int_0^1 y\, dy = 1$$

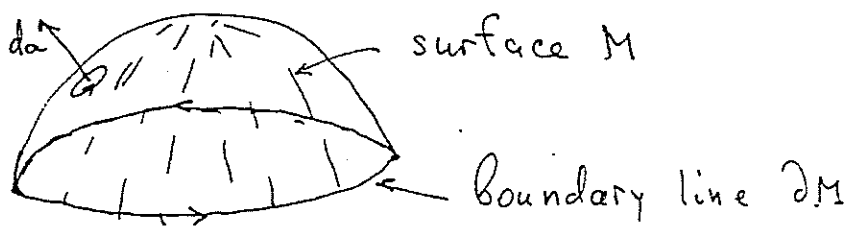
f_6:

$$\int_{\partial M_6} \vec{v}\cdot d\vec{a} = \int_{\partial M_6} \vec{v}\cdot(-\hat{e}_3)\, dx\, dy = -\int_{\partial M_6} 2y\underset{\emptyset}{\underset{\shortparallel}{z}}\, dx\, dy = \emptyset$$

$$\Rightarrow f_1 + f_2 + f_3 + f_4 + f_5 + f_6 = \oint_{\partial M} \vec{v}\cdot d\vec{a} = \frac{1}{3} - \frac{1}{3} + \frac{4}{3} - \frac{1}{3} + 1 + \emptyset = 2 \Rightarrow$$

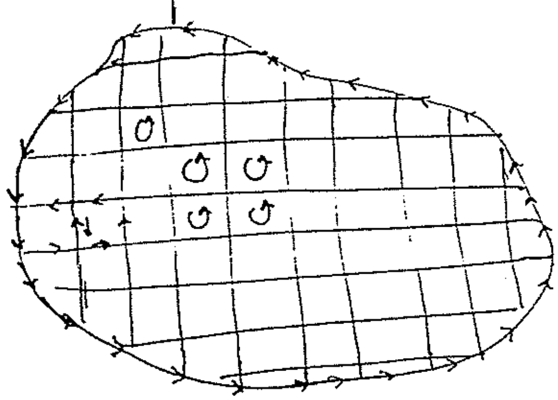
l.h.s = r.h.s.

The fundamental theorem for curls



$$\int_M (\vec\nabla\times\vec v)\cdot d\vec a = \oint_{\partial M} \vec v\cdot d\vec l \qquad \text{Stokes' theorem}$$

Geometrical interpretation



For one „□"

$$\int_{\square} (\vec\nabla\times\vec v)\cdot d\vec a \simeq (\vec\nabla\times\vec v)\cdot\hat e_3\, dx\, dy$$

$$\simeq \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) dx\, dy$$

$$\oint_{\square\,(x,y)\to(x+dx,\,y+dy)} \vec v\cdot d\vec l \approx \begin{array}{l} v_1\left(x+\frac{dx}{2}, y\right) + \\ v_2\left(x+dx, y+\frac{dy}{2}\right) \\ - v_1\left(x+\frac{dx}{2}, y+dy\right) dx \\ - v_2\left(x, y+\frac{dy}{2}\right) dy = \end{array}$$

$$= dx\left(v_1\left(x+\frac{dx}{2}, y\right) - v_1\left(x+\frac{dx}{2}, y+dy\right)\right) +$$
$$+ dy\left(v_2\left(x+dx, y+\frac{dy}{2}\right) - v_2\left(x, y+\frac{dy}{2}\right)\right)$$
$$= -dx\,dy\,\frac{\partial v_1}{\partial y}(x,y) + dx\,dy\,\frac{\partial v_2}{\partial x}(x,y)$$

Corollary:

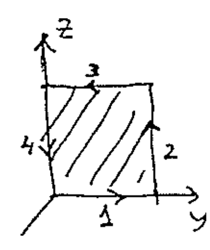
$\int_{\text{surface } M} (\vec\nabla\times\vec v)\cdot d\vec a$ depends only on boundary line ∂M $\Big\}\Rightarrow$ for a closed surface $\oint_M (\vec\nabla\times\vec v)\cdot da = 0$

$$\Rightarrow \oint (\vec\nabla\times\vec v)\cdot d\vec a = \emptyset$$

Example: $\vec v(x,y,z) = (2xz + 3y^2)\,\hat e_2 + 4yz^2\,\hat e_3$

$$\vec\nabla\times\vec v = (4z^2 - 2x)\,\hat e_1 + 2z\,\hat e_3$$

Let us check Stokes' theorem for M =



$$\text{l.h.s.} = \int_M dy\,dz\;\hat{e}_1\cdot(\vec{\nabla}\times\vec{v}) = \int_0^1 dy\int_0^1 dz\,(4z^2-2\underbrace{x}_{0}) = 4\int_0^1 dy\int_0^1 z^2 dz = \frac{4}{3}$$

$$\text{r.h.s.} = \int_{(1)}\vec{v}\cdot d\vec{l} + \int_{(2)}\vec{v}\cdot d\vec{l} + \int_{(3)}\vec{v}\cdot d\vec{l} + \int_{(4)}\vec{v}\cdot d\vec{l}$$

$$\int_{(1)}\vec{v}\cdot d\vec{l} = \int_0^1 dy\,v_2(0,y,0) = \int_0^1 dy\,(2\underbrace{xz}_{\emptyset}+3y^2) = 3\int_0^1 y^2 dy = 1$$

$$\int_{(2)}\vec{v}\cdot d\vec{l} = \int_0^1 dz\,v_3(0,1,z) = \int_0^1 dz\,4\underbrace{y}_{1}z^2 = 4\int_0^1 z^2 dz = \frac{4}{3}$$

$$\int_{(3)}\vec{v}\cdot d\vec{l} = -\int_0^1 dy\,v_2(0,y,1) = -\int_0^1 dy\,3y^2 = -1$$

$$\int_{(4)}\vec{v}\cdot d\vec{l} = -\int_0^1 dz\,v_3(0,0,z) = \emptyset$$

$$\Rightarrow \text{r.h.s.} = \oint\vec{v}\cdot d\vec{l} = 1+\frac{4}{3}-1+\emptyset = \frac{4}{3} \quad\Rightarrow\quad \text{r.h.s.} = \text{l.h.s.}$$

Relations between the fundamental theorems.

Gradient theorem: $$\int_a^b \vec{\nabla}T\cdot d\vec{l} = T(b)-T(a) \qquad (1)$$

Divergence $$\int_{\text{volume } M}(\vec{\nabla}\cdot\vec{v})\,d\tau = \oint_{\partial M}\vec{v}\cdot d\vec{a} \qquad (2)$$

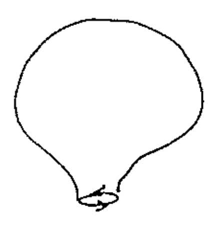
Curl $$\int_{\text{surface } M}(\vec{\nabla}\times\vec{v})\cdot d\vec{a} = \int_{\partial M}\vec{v}\cdot d\vec{l} \qquad (3)$$

I. Combine (1) and (3)

$$a=b \overset{(1)}{\Rightarrow} \oint\vec{\nabla}T\cdot d\vec{l} = 0 \overset{(3)}{\Longrightarrow} 0 = \int_{\text{surface}}(\vec{\nabla}\times(\vec{\nabla}T))\cdot d\vec{a} \Rightarrow$$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}T) = \emptyset$$

II. Combine (2) and (3)

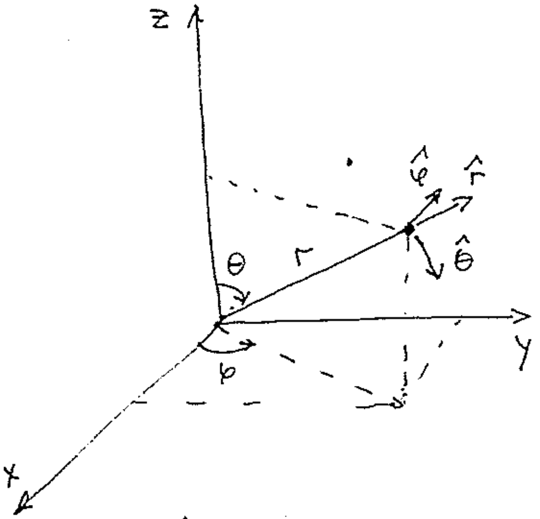


$$\int_{\text{surface } M}(\vec{\nabla}\times\vec{v})\cdot d\vec{a} = \oint_{\partial M}\vec{v}\cdot d\vec{l} \quad\Rightarrow\quad \oint_{\text{closed surface}}(\vec{\nabla}\times\vec{v})\cdot d\vec{a} = 0$$

$$\Rightarrow 0 = \oint_M(\vec{\nabla}\times\vec{v})\cdot da = \int_{\text{volume}}(\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v}))\,d\tau \quad\Rightarrow\quad \vec{\nabla}\cdot(\vec{\nabla}\times\vec{v}) = \emptyset$$

Curvilinear coordinates.

I. Spherical polar coordinates



$x = r \sin\theta \cos\varphi$
$y = r \sin\theta \sin\varphi$
$z = r \cos\theta$

Inverse formulas

$r = \sqrt{x^2 + y^2 + z^2}$

$\theta = \arccos \frac{z}{\sqrt{x^2+y^2+z^2}}$

$\varphi = \arctan \frac{y}{x}$

$\hat{r} \equiv$ unit vector in the direction of increase of r
$\hat{\theta} \equiv$ — " — " — " — " — " — " — " — θ
$\hat{\varphi} \equiv$ — " — " — " — " — " — " — " — φ

Any vector $\vec{A}$ can be expressed as

$\vec{A} = A_r \hat{r} + A_\theta \hat{\theta} + A_\varphi \hat{\varphi}$ (similarly to $\vec{A} = A_x \hat{e}_1 + A_y \hat{e}_2 + A_z \hat{e}_3$)

NB: Unlike $\hat{e}_1, \hat{e}_2$, and $\hat{e}_3$ the vectors $\hat{r}, \hat{\theta}$, and $\hat{\varphi}$ depend on the point.

Element of the volume in polar coordinates

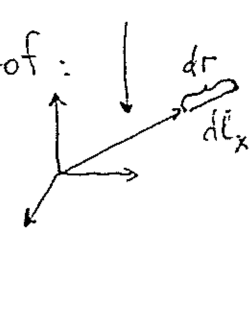
In Cartesian coordinates, $d\vec{l} = dx\, \hat{e}_1 + dy\, \hat{e}_2 + dz\, \hat{e}_3$ ← infinitesimal displacement

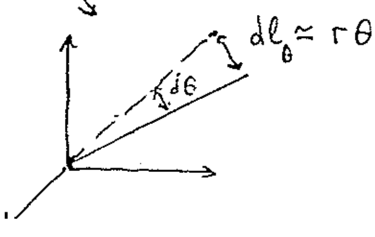
$d\tau = dx\,dy\,dz$ ← infinitesimal element of volume

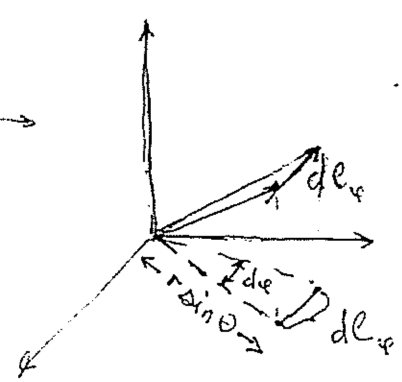
In polar coordinates

$d\vec{l} = dr\, \hat{r} + r\, d\theta\, \hat{\theta} + r \sin\theta\, d\theta\, \hat{\varphi}$

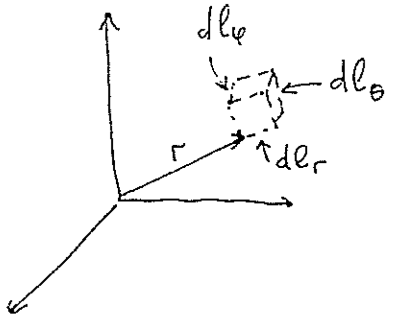
Proof:



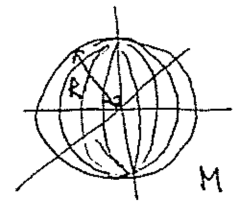




Element of the volume is $dl_r \, dl_\theta \, dl_\varphi = r^2 \sin\theta \, dr \, d\theta \, d\varphi$



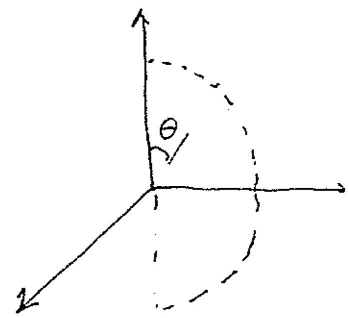
Exersize: volume of a sphere of radius R



$$\text{volume} = \int_M d\tau = \int_0^R dr \int_0^\pi d\theta \int_0^{2\pi} d\varphi \; r^2 \sin\theta =$$

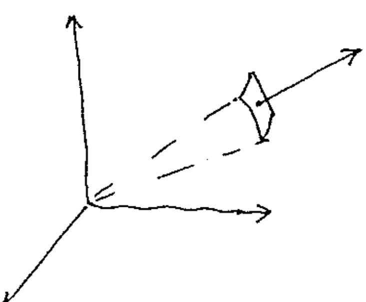
$$= \int_0^R r^2 dr \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\varphi = \frac{R^3}{3} \cdot 2 \cdot 2\pi = \frac{4}{3}\pi R^3$$

Note that θ varies from 0 to π (and φ from 0 to 2π)



Element of the surface area $d\vec{a}$ depends on the orientation of the surface.

For the surface of a sphere, r is constant so



$$|d\vec{a}| = dl_\theta \, dl_\varphi = r^2 \sin\theta \, d\theta \, d\varphi$$

and

$$d\vec{a} \uparrow\uparrow \vec{r} \quad \Rightarrow \quad d\vec{a} = r^2 \sin\theta \, d\theta \, d\varphi \, \hat{r}$$

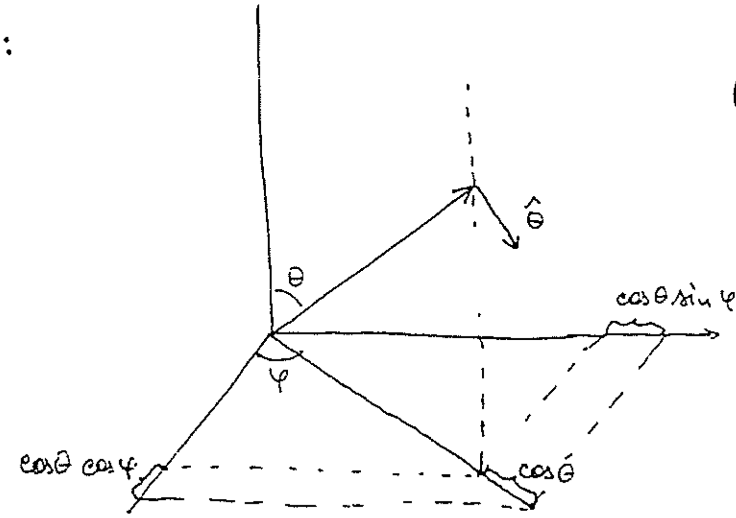
Check: the area of the surface of the sphere is

$$A = \int_{\partial M} |d\vec{a}| = \int_{\partial M} dl_\theta \, dl_\varphi = \int_0^\pi d\theta \int_0^{2\pi} d\varphi \; R^2 \sin\theta = R^2 \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\varphi = 4\pi R^2$$

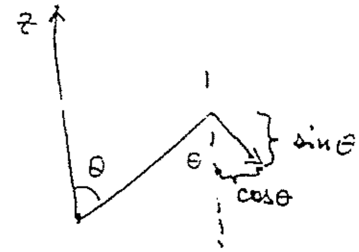
Relations between $\hat{r}, \hat{\theta}, \hat{\varphi}$ and Cartesian unit vectors

$$\hat{r} = \frac{\vec{r}}{r} = \frac{x}{r}\hat{e}_1 + \frac{y}{r}\hat{e}_2 + \frac{z}{r}\hat{e}_3 = \sin\theta\cos\varphi \, \hat{e}_1 + \sin\theta \sin\varphi \, \hat{e}_2 + \cos\theta \, \hat{e}_3$$

$\hat{\theta}$:



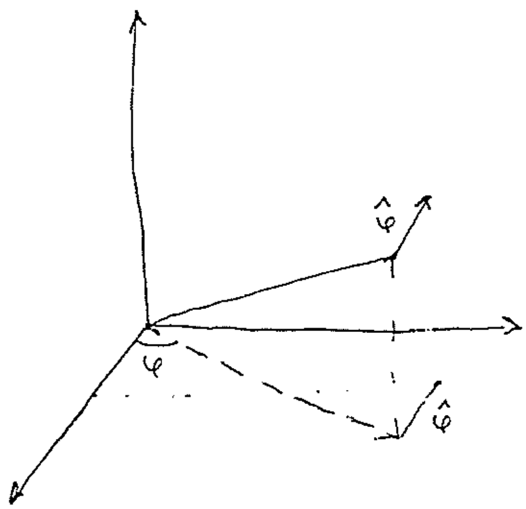
$$(\hat{\theta})_3 = -\cos\left(\tfrac{\pi}{2}-\theta\right) = -\sin\theta$$



$$(\hat{\theta})_1 = \cos\theta\cos\varphi$$
$$(\hat{\theta})_2 = \cos\theta\sin\varphi \quad \Rightarrow$$

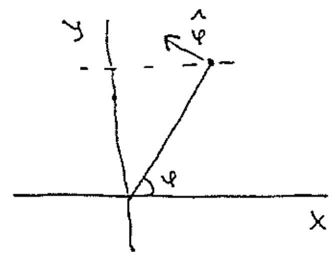
$$\Rightarrow \hat{\theta} = \cos\theta\cos\varphi\,\hat{e}_1 + \cos\theta\sin\varphi\,\hat{e}_2 - \sin\theta\,\hat{e}_3$$

$\hat{\varphi}$:



$$(\hat{\varphi})_3 = \phi$$

In the xy plane:



$$(\hat{\varphi})_1 = -\sin\varphi \qquad (\hat{\varphi})_2 = \cos\varphi$$

$$\Rightarrow \quad \hat{\varphi} = -\sin\varphi\,\hat{e}_1 + \cos\varphi\,\hat{e}_2$$

Summary:

$$\hat{r} = \hat{e}_1\sin\theta\cos\varphi + \hat{e}_2\sin\theta\sin\varphi + \hat{e}_3\cos\theta$$
$$\hat{\theta} = \hat{e}_1\cos\theta\cos\varphi + \hat{e}_2\cos\theta\sin\varphi - \hat{e}_3\sin\theta$$
$$\hat{\varphi} = -\hat{e}_1\sin\varphi + \hat{e}_2\cos\varphi$$

It is easy to check that $\hat{r}^2 = \hat{\theta}^2 = \hat{\varphi}^2 = 1$ and $\hat{r}\cdot\hat{\theta} = \hat{\theta}\cdot\hat{\varphi} = \hat{r}\cdot\hat{\varphi} = \phi$

Gradient in polar coordinates

$$\vec{\nabla}T = \frac{\partial T}{\partial x}\hat{e}_1 + \frac{\partial T}{\partial y}\hat{e}_2 + \frac{\partial T}{\partial z}\hat{e}_3 = \left(\frac{\partial T}{\partial r}\frac{\partial r}{\partial x}\hat{e}_1 + \frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial x}\hat{e}_1 + \frac{\partial T}{\partial \varphi}\frac{\partial \varphi}{\partial x}\hat{e}_1\right) +$$
$$+\left(\frac{\partial T}{\partial r}\frac{\partial r}{\partial y}\hat{e}_2 + \frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial y}\hat{e}_2 + \frac{\partial T}{\partial \varphi}\frac{\partial \varphi}{\partial y}\hat{e}_2\right) + \left(\frac{\partial T}{\partial r}\frac{\partial r}{\partial z}\hat{e}_3 + \frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial z}\hat{e}_3 + \frac{\partial T}{\partial \varphi}\frac{\partial \varphi}{\partial z}\hat{e}_3\right) =$$
$$= \frac{\partial T}{\partial r}\left(\frac{\partial r}{\partial x}\hat{e}_1 + \frac{\partial r}{\partial y}\hat{e}_2 + \frac{\partial r}{\partial z}\hat{e}_3\right) + \frac{\partial T}{\partial \theta}\left(\frac{\partial \theta}{\partial x}\hat{e}_1 + \frac{\partial \theta}{\partial y}\hat{e}_2 + \frac{\partial \theta}{\partial z}\hat{e}_3\right) + \frac{\partial T}{\partial \varphi}\left(\frac{\partial \varphi}{\partial x}\hat{e}_1 + \frac{\partial \varphi}{\partial y}\hat{e}_2 + \frac{\partial \varphi}{\partial z}\hat{e}_3\right)$$

We need 9 partial derivatives

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r} = \sin\theta\cos\varphi$$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta\sin\varphi$ and $\frac{\partial r}{\partial z} = \frac{z}{r} = \cos\varphi$

$$\theta = \arctan\frac{\sqrt{x^2+y^2}}{z} \Rightarrow$$

$$\frac{\partial\theta}{\partial x} = \frac{\partial}{\partial x}\arctan\frac{\sqrt{x^2+y^2}}{z} = \frac{1}{1+\frac{x^2+y^2}{z^2}}\,\frac{1}{z}\,\frac{\partial}{\partial x}\sqrt{x^2+y^2} = \frac{z^{\cancel{2}}}{x^2+y^2+z^2}\,\frac{1}{\cancel{z}}\,\frac{x}{\sqrt{x^2+y^2}} = \frac{z}{r^2}\frac{x}{\sqrt{x^2+y^2}}$$

$$= \frac{1}{r}\cos\theta\cos\varphi$$

Similarly, $\frac{\partial\theta}{\partial y} = \frac{1}{r}\frac{z}{r}\frac{y}{\sqrt{x^2+y^2}} = \frac{1}{r}\cos\theta\sin\varphi$

$$\frac{\partial\theta}{\partial z} = \frac{\partial}{\partial z}\arctan\frac{\sqrt{x^2+y^2}}{z} = \frac{1}{1+\frac{x^2+y^2}{z^2}}\sqrt{x^2+y^2}\,\underbrace{\frac{\partial}{\partial z}\frac{1}{z}}_{-1/z^2} = -\frac{\sqrt{x^2+y^2}}{x^2+y^2+z^2} = -\frac{1}{r}\sin\theta$$

Finally, $\varphi = \arctan y/x \Rightarrow$

$$\frac{\partial\varphi}{\partial z} = 0 \qquad \frac{\partial\varphi}{\partial x} = \frac{\partial}{\partial x}\arctan\frac{y}{x} = \frac{1}{1+\frac{y^2}{x^2}}\,y\frac{d}{dx}\frac{1}{x} = -\frac{y/x^2}{1+y^2/x^2} = -\frac{y}{x^2+y^2} = -\frac{\sin\varphi}{r\sin\theta}$$

$$\frac{\partial\varphi}{\partial y} = \frac{\partial}{\partial y}\arctan\frac{y}{x} = \frac{1}{1+y^2/x^2}\,\frac{d}{dy}\frac{y}{x} = \frac{1}{x(1+y^2/x^2)} = \frac{x}{x^2+y^2} = \frac{x}{\sqrt{x^2+y^2}}\,\frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}\frac{\cos\varphi}{\sin\theta}$$

Now we can assemble $\vec{\nabla}T$ in polar coordinates

$$\vec{\nabla}T = \frac{\partial T}{\partial r}(\underbrace{\hat{e}_1\sin\theta\cos\varphi + \hat{e}_2\sin\theta\sin\varphi + e_3\cos\varphi}_{\hat{r}}) + \frac{1}{r}\frac{\partial T}{\partial\theta}(\underbrace{\hat{e}_1\cos\theta\cos\varphi + \hat{e}_2\cos\theta\sin\varphi - \hat{e}_3\sin\theta}_{\hat{\theta}})$$

$$+ \frac{\partial T/\partial\varphi}{r\sin\theta}(\underbrace{-\hat{e}_1\sin\varphi + \hat{e}_2\cos\varphi}_{\hat{\varphi}}) = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial\varphi}\hat{\varphi}$$

Similarly one can obtain

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi}$$

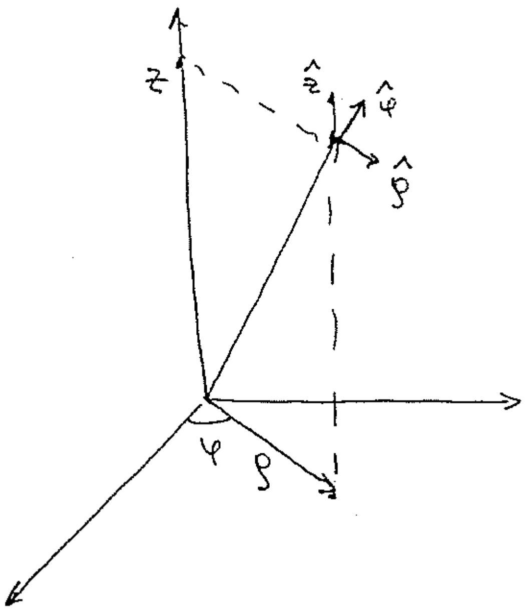
$$\vec{\nabla}\times\vec{v} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta\, v_\varphi) - \frac{\partial v_\theta}{\partial\varphi}\right) + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\varphi} - \frac{\partial}{\partial r}(r v_\varphi)\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right)\hat{\varphi}$$

where $\vec{v} = v_1(x,y,z)\hat{e}_1 + v_2(x,y,z)\hat{e}_2 + v_3(x,y,z)\hat{e}_3 = v_r(r,\theta,\varphi)\hat{r} + v_\theta(r,\theta,\varphi)\hat{\theta} + v_\varphi(r,\theta,\varphi)\hat{\varphi}$

Laplacian in spherical polar coordinates:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\varphi^2}$$

Cylindrical coordinates

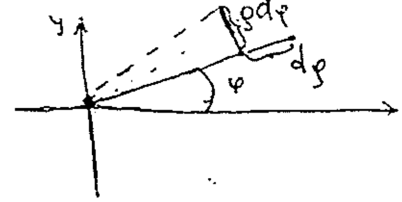


$$x = \rho\cos\varphi \qquad \rho = \sqrt{x^2+y^2}$$
$$y = \rho\sin\varphi \quad \Longleftrightarrow \quad \varphi = \arctan\frac{y}{x}$$
$$z = z \qquad z = z$$

Infinitesimal displacements

$$dl_\rho = d\rho \qquad dl_\varphi = \rho\, d\varphi \qquad dl_z = dz$$

xy plane:

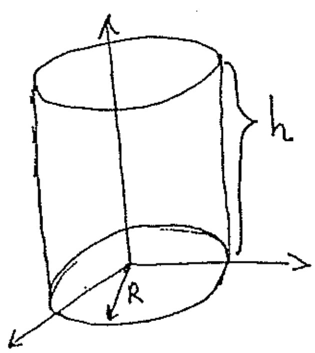


Unit vectors: $\hat{z} = \hat{e}_3$, $\hat{\rho} = \frac{x}{\sqrt{x^2+y^2}}\hat{e}_1 + \frac{y}{\sqrt{x^2+y^2}}\hat{e}_2 = \hat{e}_1\cos\varphi + \hat{e}_2\sin\varphi$

$\hat{\varphi} = -\hat{e}_1\sin\varphi + \hat{e}_2\cos\varphi$ (same as in polar coordinates)

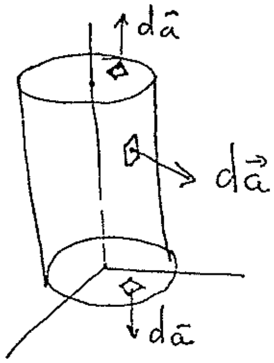
Element of the volume $\quad d\tau = dl_\rho\, dl_\varphi\, dl_z = \rho\, d\rho\, d\varphi\, dz$

Example: volume of a cylinder



$$\text{volume} = \int \rho\, d\rho\, d\varphi\, dz =$$
$$= \int_0^R \rho\, d\rho \int_0^{2\pi} d\varphi \int_0^h dz = \frac{R^2}{2}\, 2\pi h = \pi R^2 h$$

Element of the surface of the cylinder



Side surface:

$d\rho = 0 \Rightarrow |d\vec{a}| = dl_\varphi\, dl_z = \rho\, d\varphi\, dz$

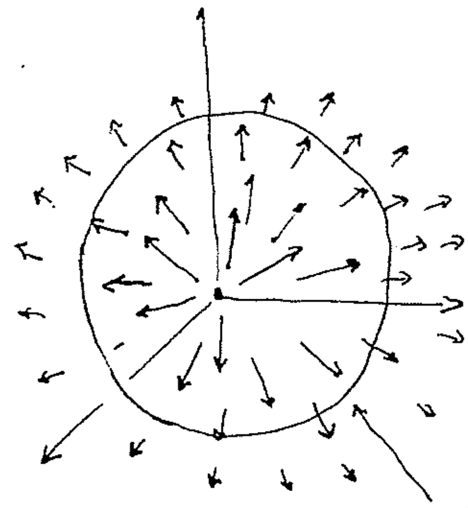
$d\vec{a} \uparrow\uparrow \vec{\rho} \Rightarrow d\vec{a} = \rho\, d\varphi\, dz\, \hat{\rho}$

Upper base $\quad dz = 0 \Rightarrow d\vec{a} = \rho\, d\rho\, d\varphi\, \hat{e}_3$

Lower base $\quad d\vec{a} = -\hat{e}_3\, \rho\, d\rho\, d\varphi$

The Dirac δ-function

Warm-up exersize: check of Gauss' theorem for $\vec{v}(\vec{r}) = \frac{\vec{r}}{r^3}$



$$\int_M \vec{\nabla}\cdot\vec{v}\, d\tau = \oint_{\partial M} \vec{v}\cdot d\vec{a}$$

Gaussian sphere M (radius R)

We know that $\vec{\nabla}\cdot\vec{v} = 0$ everywhere except possibly $\vec{r} = 0$

(One more check: in polar coordinates $\vec{v} = \frac{1}{r^2}\hat{r} \Rightarrow$
$\Rightarrow \vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) = \frac{1}{r^2}\frac{\partial}{\partial r}(\cancel{r^2}\frac{1}{\cancel{r^2}}) = 0$ everywhere except $r = 0$).

On the other hand

$$\oint_{\partial M} \vec{v}\cdot d\vec{a} = \oint_{\partial M} \vec{v}\cdot\hat{r}\, R^2 \sin\theta\, d\theta\, d\varphi = \oint_{\partial M} \frac{1}{\cancel{R^2}}\hat{r}\cdot\hat{r}\,\cancel{R^2}\sin\theta\, d\theta\, d\varphi = \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\varphi = 4\pi$$

$\Rightarrow \int_M d\tau\, \vec{\nabla}\cdot\vec{v} = 4\pi$, but $\vec{\nabla}\cdot\vec{v} = 0$ everywhere except $\vec{r} = 0$

How can it be?

Roughly speaking, $\vec{\nabla}\cdot\vec{v}\,\big|_{\vec{r}=0} = \infty$, so (value of the function $\vec{\nabla}\cdot\vec{v}$)(support of the function $\vec{\nabla}\cdot\vec{v}$) $= \infty \cdot 0 = 4\pi$.

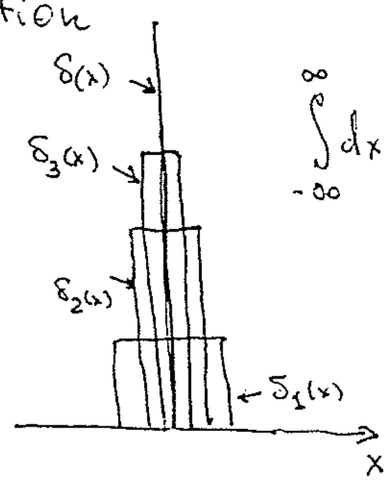
Mathematically, $\vec{\nabla}\cdot\vec{v}$ is the example of so-called Dirac δ-function: $\vec{\nabla}\cdot\vec{v} = 4\pi\,\delta^{(3)}(\vec{r})$ where $\delta^3(\vec{r})$ is 0 everywhere except $\vec{r} = 0$ where it is ∞ and $\int d\vec{\tau}\,\delta^3(\vec{r}) =$

For simplicity, consider at first the δ-function of one variable.

The one-dimensional Dirac δ-function

Consider

$$\delta_n(x) = \begin{cases} 0 & \text{if } |x| > \frac{1}{2n} \\ n & \text{if } |x| < \frac{1}{2n} \end{cases}$$



$$\int_{-\infty}^{\infty} dx\, \delta_n(x) = 1$$

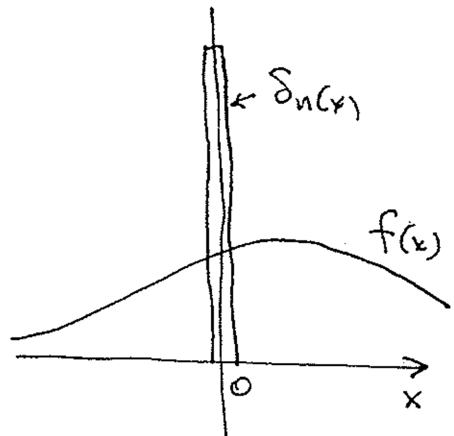
Definition

$$\delta(x) = \lim_{n\to\infty} \delta_n(x)$$

$$\int_{-\infty}^{\infty} dx\, \delta(x) = 1$$

Main property:

$$(*) \quad \int_{-\infty}^{\infty} dx\, f(x)\,\delta(x) = f(0)$$

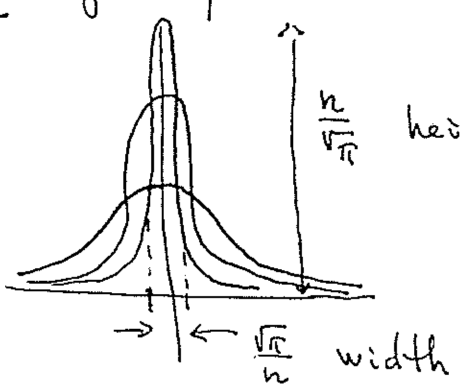


Proof:
$$\int_{-\infty}^{\infty} dx\, f(x)\,\delta(x) = \lim_{n\to\infty} \int_{-\infty}^{\infty} dx\, \delta_n(x)\, f(x) =$$
$$= \lim_{n\to\infty} n \int_{-1/2n}^{1/2n} dx\, f(x) = \lim_{n\to\infty} n \left(\frac{1}{n} f(0) + O\left(\frac{1}{n^2}\right)\right) = f(0)$$

In mathematics, the δ-function is defined as a „distribution" satisfying the condition (*).

Instead of limiting sequence $\delta_n(x)$ we could have chosen

$$\tilde{\delta}_n(x) = \frac{n}{\sqrt{\pi}} e^{-x^2 n^2}$$



$$\int_{-\infty}^{\infty} dx\, \frac{n}{\sqrt{\pi}} e^{-x^2 n^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dy\, e^{-y^2} = 1$$

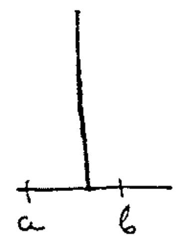
The limit at $n\to\infty$ is the same $\lim_{n\to\infty} \tilde{\delta}_n(x) = \delta(x)$

$$\underbrace{\int_{-\infty}^{\infty} dx\, \delta(x) = 1 \quad \text{and} \quad \delta(x) = 0 \text{ everywhere except } x = 0}$$

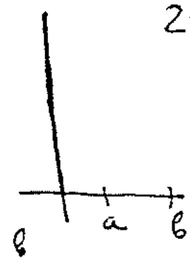
This property defines δ-function independently of the limiting sequence $\delta_n(x)$

Properties of the δ-function

I. $\int_a^b dx\, \delta(x) = \begin{cases} 1 & \text{if } b>0>a \\ 0 & \text{otherwise} \end{cases}$



$\int_a^b dx\, \delta(x) = \int_{-\infty}^{\infty} dx\, \delta(x) = 1$

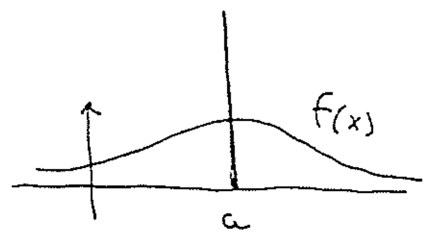


$\int_a^b dx\, \delta(x) = 0$

$\Downarrow$

$\int_a^b dx\, \delta(x) f(x) = \begin{cases} f(0) & \text{if } b>0>a \\ 0 & \text{otherwise} \end{cases}$

II. $\int_{-\infty}^{\infty} dx\, \delta(x-a)\, f(x) = f(a)$



Proof is trivial: $\int_{-\infty}^{\infty} dx\, \delta(x-a)\, f(x) = \{\text{shift } x \to x+a\}$

$= \int_{-\infty}^{\infty} dx\, \delta(x)\, f(x+a) = f(a)$

Corollary: $\int_a^b dx\, \delta(x-c)\, f(x) = \begin{cases} f(c) & \text{if } b>c>a \\ 0 & \text{otherwise} \end{cases}$

III. $\delta(x)$ is an even function of x: $\delta(x) = \delta(-x)$

This is easily seen from the limiting sequence because $\delta_n(x)$ are even functions; however, it can be proved independently of the form of $\delta_n(x)$

$$\left.\begin{aligned} &\int_{-\infty}^{\infty} dx\, \delta(-x)\, f(x) = \{\text{change of variable } x \to -x\} = \int_{-\infty}^{\infty} dx\, \delta(x)\, f(-x) = f(0) \\ &\int_{-\infty}^{\infty} dx\, \delta(x)\, f(x) = f(0) \end{aligned}\right\} \Rightarrow$$

$\Rightarrow \delta(x) = \delta(-x)$

(If $\int_{-\infty}^{\infty} dx\, f(x)\, g_1(x) = \int_{-\infty}^{\infty} dx\, f(x)\, g_2(x)$ for any function $f(x)$, the functions $g_1(x)$ and $g_2(x)$ are identical. By definition, the generalized functions ($\equiv$ distributions) D_1 and D_2 are said to be identical if $\int_{-\infty}^{\infty} dx\, f(x)\, D_1(x) = \int_{-\infty}^{\infty} dx\, f(x)\, D_2(x)$ for any $f(x)$.)

IV. $\delta(kx) = \frac{1}{|k|}\delta(x)$

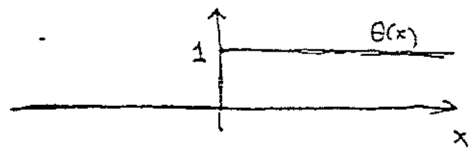
$$\int_{-\infty}^{\infty} dx\, \delta(kx)\, f(x) = \{\text{change of variables } x \to \tfrac{x}{k}\} = \frac{1}{k}\int_{-\infty}^{\infty} dx\, \delta(x)\, f(\tfrac{x}{k}) = \frac{1}{k} f(0)$$

$$\left.\begin{array}{l} \int_{-\infty}^{\infty} dx\, \delta(kx)\, f(x) = \frac{1}{k} f(0) \\ \int_{-\infty}^{\infty} dx\, \frac{1}{k}\delta(x)\, f(x) = \frac{1}{k} f(0) \end{array}\right\} \Rightarrow \delta(kx) = \frac{1}{k}\delta(x) \quad \text{if } k>0$$

For negative k, $\delta(kx) = \delta(|k|x)$ since δ is an even function $\Rightarrow$
$\Rightarrow$ in general, $\delta(kx) = \frac{1}{|k|}\delta(x)$.

V. Define the „step function"

$$\theta(x) = \begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x \leq 0 \end{cases}$$



Property: $\delta(x) = \theta'(x)$

Proof: $\int_{-\infty}^{\infty} dx\, \theta'(x) f(x) = \{\text{integration by parts}\} = f(x)\theta(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx\, f'(x)\theta(x)$

$$\left.\begin{array}{l} = f(\infty) - \int_{0}^{\infty} dx\, f'(x) = \cancel{f(\infty)} - \cancel{f(\infty)} + f(0) = f(0) \\ \int_{-\infty}^{\infty} dx\, \delta(x)\, f(x) = f(0) \end{array}\right\} \Rightarrow \delta(x) = \theta'(x)$$

VI $\quad x\,\delta'(x) = -\delta(x) \qquad \delta'(x) \equiv \frac{d}{dx}\delta(x)$

Proof: $\int_{-\infty}^{\infty} dx\, x \frac{d}{dx}\delta(x) = \{\text{integration by parts}\} = \cancel{x\,\delta(x) f(x)\Big|_{-\infty}^{\infty}} -$
$- \int_{-\infty}^{\infty} dx\, \delta(x) \frac{d}{dx}(x f(x)) = -\int_{-\infty}^{\infty} dx\, \delta(x)\,(x f'(x) + f(x)) = -(x f'(x) + f(x))\Big|_{x=0} = -f(0)$

Compare to $\int_{-\infty}^{\infty} dx\, (-\delta(x)) f(x) = -f(0) \Rightarrow x\frac{d}{dx}\delta(x) = -\delta(x)$

The three-dimensional δ-function

Definition: $\delta^{(3)}(\vec{r}) = \delta(x)\delta(y)\delta(z)$

Main property: $\int d\tau\, \delta^{(3)}(\vec{r})\, f(\vec{r}) = f(0)$

Proof: $\int d\tau\, \delta^{(3)}(\vec{r}) f(\vec{r}) = \int dx\,dy\,dz\, \delta(x)\delta(y)\delta(z)\, f(x,y,z) = \int dx\,dy\, \delta(x)\delta(y) \int dz$
$\delta(z) f(x,y,z) = \int dx\,dy\, \delta(x)\delta(y)\, f(x,y,0) = \int dx\, \delta(x) \int dy\, \delta(y) f(x,y,0) = \int dx\, \delta(x) f(x,0,0) =$
$= f(0,0,0)$

Similarly to the one-dimensional case

$$\int d\tau\, \delta^{(3)}(\vec{r}-\vec{a})\, f(\vec{r}) = f(\vec{a})$$

Proof: shift $\vec{r} \to \vec{r}+\vec{a}$ in the integration variables

$$\int d\tau\, \delta^{(3)}(\vec{r}-\vec{a})\, f(\vec{r}) = \int d\tau\, \delta^{(3)}(\vec{r})\, f(\vec{r}+\vec{a}) = f(\vec{a})).$$

Now we can return to the Gauss' theorem for $\vec{v}(\vec{r}) = \frac{\vec{r}}{r^3}$

The divergence $\frac{1}{4\pi}\vec{\nabla}\cdot\vec{v}(\vec{r})$ is a scalar function $f(\vec{r})$ satisfying the conditions

$$\left.\begin{array}{l} f(\vec{r}) = 0 \quad \text{if } \vec{r} \neq 0 \\ \int d\tau\, f(\vec{r}) = 1 \end{array}\right\} \quad f(\vec{r}) = \delta^{(3)}(\vec{r}) \quad \Rightarrow \quad \vec{\nabla}\cdot\frac{\vec{r}}{r^3} = 4\pi\delta^{(3)}(\vec{r})$$

Similarly, $\vec{\nabla}\cdot\dfrac{\vec{r}-\vec{a}}{|\vec{r}-\vec{a}|^3} = 4\pi\,\delta^{(3)}(\vec{r}-\vec{a})$

Another useful formula:

$$\nabla^2\frac{1}{|\vec{r}-\vec{a}|} = -4\pi\,\delta^{(3)}(\vec{r}-\vec{a})$$

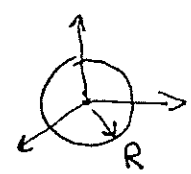
We have proved that $\vec{\nabla}\frac{1}{|\vec{r}-\vec{a}|} = \hat{e}_1\frac{\partial}{\partial x}\frac{1}{\sqrt{(x-a_1)^2+(y-a_2)^2+(z-a_3)^2}} +$

$$+\hat{e}_2\frac{\partial}{\partial y}\frac{1}{\sqrt{\dots}} + \hat{e}_3\frac{\partial}{\partial z}\frac{1}{\sqrt{\dots}} = -\hat{e}_1\frac{x-a_1}{(\sqrt{\dots})^3} - \hat{e}_2\frac{y-}{(\sqrt{\dots})^3} - \hat{e}_3\frac{z-a_3}{(\sqrt{\dots})^3} =$$

$$= -\frac{\vec{r}-\vec{a}}{|\vec{r}-\vec{a}|^3} \quad \Rightarrow$$

$$\nabla^2\frac{1}{|\vec{r}-\vec{a}|} = -\vec{\nabla}\cdot\frac{\vec{r}-\vec{a}}{|\vec{r}-\vec{a}|^3} = -4\pi\,\delta^{(3)}(\vec{r}-\vec{a})$$

Exersize: $\int_M d\tau\,(r^2+2)\,\vec{\nabla}\cdot\frac{\vec{r}}{r^3}$ where M = sphere with radius R



First method:

$$\vec{\nabla}\cdot\frac{\vec{r}}{r^3} = 4\pi\delta^{(3)}(\vec{r}) \quad \Rightarrow \quad \int_M d\tau\,(r^2+2)\,4\pi\,\delta^{(3)}(\vec{r}) = 8\pi$$

Second method: integration by parts

$$\vec{\nabla}\cdot(f\vec{A}) = \frac{\partial}{\partial x_i}(fA_i) = \frac{\partial f}{\partial x_i}A_i + f\frac{\partial A_i}{\partial x_i} = \vec{A}\cdot(\vec{\nabla}f) + f(\vec{\nabla}\cdot\vec{A})$$

$$\Rightarrow \int_M d\tau\, f(\vec\nabla\cdot\vec A) = \int_M d\tau\, \vec\nabla\cdot(f\vec A) - \int_M \vec A\cdot\nabla f\, d\tau = \int_{\partial M} d\vec a\cdot\vec A f - \int_M d\tau\, \vec A\cdot\vec\nabla f$$

(This is the 3-dimensional analog of the formula
$$\int_a^b dx\, f(x)\, g'(x) = f(x)g(x)\Big|_a^b - \int_a^b dx\, g(x)\, f'(x)\ )$$

Using this formula for $f(\vec r) = r^2+2$ and $\vec A(\vec r) = \frac{\vec r}{r^3}$ we get

$$\int_M d\tau\, (r^2+2)\,\vec\nabla\cdot\frac{\vec r}{r^3} = \int_{\partial M} (R^2+2)\frac{\vec r}{R^3}\cdot d\vec a - \int_M d\tau\, \frac{\vec r}{r^3}\cdot\vec\nabla(r^2+\cancel{2}) =$$

$$\nabla r^2 = \hat e_1\frac{\partial r^2}{\partial x} + \hat e_2\frac{\partial r^2}{\partial y} + \hat e_3\frac{\partial r^2}{\partial z} = 2\vec r$$

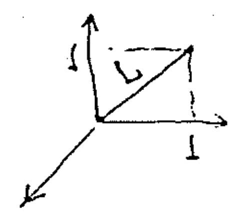
$$= \int_{\partial M} (R^2+2)\frac{\hat r}{R^2}\cdot\hat r\, R^2\sin\theta\, d\theta\, d\varphi - \int_M d\tau\, \frac{\vec r}{r^3}\cdot 2\vec r = (R^2+2)\int_0^\pi d\theta\sin\theta\int_0^{2\pi} d\varphi - 2\int_M \frac{d\tau}{r}$$

$$= 4\pi(R^2+2) - 2\int_0^R \frac{r^{\cancel{2}}}{\cancel{r}}\, dr\int_0^\pi d\theta\sin\theta\int_0^{2\pi} d\varphi = 4\pi(R^2+2) - 8\pi\frac{R^2}{2} = 8\pi \Rightarrow$$

$\Rightarrow$ same result.

About the problem 1.33 from HW 1:

Let us calculate $\int_L \vec v\cdot d\vec l$ for the path L =

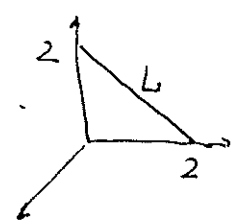


$dl = \hat e_2\, dy + \hat e_3\, dz$

$$\Rightarrow \int_L \vec v\cdot d\vec l = \int_L v_2\, dy + \int_L v_3\, dz = \int_0^1 dy\, v_2(0,y,y) + \int_0^1 dz\, v_3(0,z,z) =$$

$$= \int_0^1 dy\,(2y^2) + \int_0^1 dz\cdot 0 = 1$$

The integral along the path



is calculated similarly.