Contents

Chapter 1

Introduction to Electrostatics

Electrostatics is the study of *time-independent* distributions of charges and fields.

1.1 Coulomb's Law

The foundation of electrostatics is **Coulomb's Law**, together with the **Superposition Principle** which we will discuss later.

Coulomb's Law

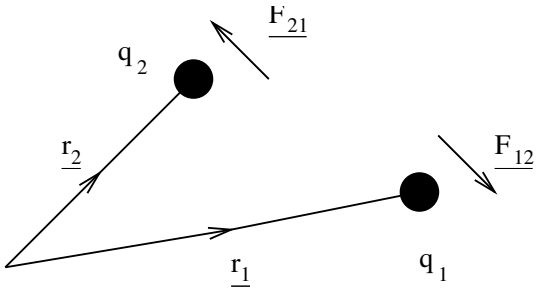
The force $\mathbf{F_{21}}$ on a particle of charge q_2 at $\mathbf{r_2}$ due to a particle of charge q_1 at $\mathbf{r_1}$ is given by

$$\mathbf{F_{21}} = kq_1q_2\frac{\hat{\mathbf{r}}_{\mathbf{21}}}{|\mathbf{r_2} - \mathbf{r_1}|^2} \ , \tag{1.1.1}$$

where

- $\mathbf{r_{21}} = \mathbf{r_2} - \mathbf{r_1}$
- $\hat{\mathbf{r}}$ is a unit vector in the direction of $\mathbf{r}$.

Coulomb's law is an **experimental observation**.



In SI units:

- $k = 1/4\pi\epsilon_0$ - the 4π is conventional.
- The charges q_1, q_2 are measured in **Coulombs** (C), and **defined** via the magnetic effects of currents (1 C = 1 A×1 s, where 1 A is 1 ampere. Also, charge of the proton is $1.602176487\times10^{-19}$ coulomb).
- ϵ_0, the **Permittivity of Free Space** is also a **defined** quantity:

$$\epsilon_0 = 8.854\,187\,817\cdots\times 10^{-12}\,\mathrm{C^2N^{-1}m^{-2}}\ . \tag{1.1.2}$$

Thus, one electron produces a field of approximately 1.44×10^{-9} V/m at a distance of 1 meter.
There are two further observations that we can make:

- The forces on the two charges are **equal and opposite**, obeying Newton's third law: $\mathbf{F_{12}} = -\mathbf{F_{21}}$.
- The force is repulsive (attractive) for like (unlike) charges.

Electric Field: The electric field **E** at **r** is defined as the force acting on a unit test charge at that point. More strictly,

$$\mathbf{E}(\mathbf{r}) = \lim_{q\to 0}\frac{\mathbf{F}(\mathbf{r})}{q}\ , \tag{1.1.3}$$

so that the electric field due to the test charge can be ignored.

1.2 The Superposition Principle and Extended Distributions

In the above we have looked at the fields due to **single, isolated point-like** charges. In this section, we will explore the second empirical ingredient necessary for our understanding

of electrostatic fields, the **linear superposition principle**.

Linear Superposition Principle

The resultant force on a test particle due to several charges is the **vector sum** of the forces due to the charges individually.

Example: We have N charges $q_i (i = 1, \ldots, N)$, situated at the points $\mathbf{r_i}$. The force on a test particle of charge q at the point $\mathbf{r}$ is given by

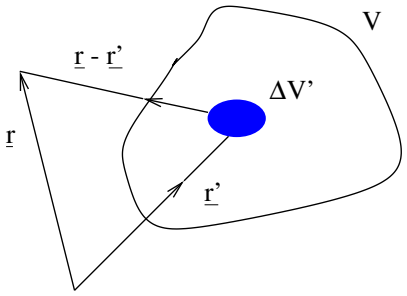
$$\mathbf{F}(\mathbf{r}) = kq \sum_{i=1}^{N} q_i \frac{\mathbf{r} - \mathbf{r_i}}{\mid \mathbf{r} - \mathbf{r_i} \mid^3} \ , \tag{1.2.1}$$

where $k = 1/4\pi\epsilon_0$ in SI units.
Thus the **electrostatic field** $\mathbf{E}(\mathbf{r})$ is

$$\mathbf{E}(\mathbf{r}) = k \sum_{i=1}^{N} q_i \frac{\mathbf{r} - \mathbf{r_i}}{\mid \mathbf{r} - \mathbf{r_i} \mid^3} \ . \tag{1.2.2}$$

1.2.1 Extended Charge Distributions

We will now apply the *linear superposition principle* to a **continuous** distribution of charge. Consider a continuous distribution of charge density (*charge per unit volume*) $\rho(\mathbf{r}')$, confined to a volume V.



In order to use the superposition principle, we will divide the volume V into infinitesimal volume elements $\Delta V'$, centered at $\mathbf{r}'$. The charge occupying the volume element at $\mathbf{r}'$ is

$$dq = \rho(\mathbf{r}')dV' \ . \tag{1.2.3}$$

Therefore, the electrostatic field at the point $\mathbf{r}$ due to the element of charge dq at $\mathbf{r}'$ is

$$\Delta\mathbf{E}(\mathbf{r}) = k\rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{\mid \mathbf{r} - \mathbf{r}' \mid^3} \Delta V' \ , \tag{1.2.4}$$

where we take $\Delta\mathbf{E}(\mathbf{r}) \longrightarrow 0$ as $\mathbf{r} \longrightarrow \infty$. We now use the principle of linear superposition to write that the resultant field at $\mathbf{r}$ as a **sum** over the elements $\Delta V'$ in V

$$\mathbf{E}(\mathbf{r}) = k \sum_{\Delta V'} \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{\mid \mathbf{r} - \mathbf{r}' \mid^3} \Delta V' \ . \tag{1.2.5}$$

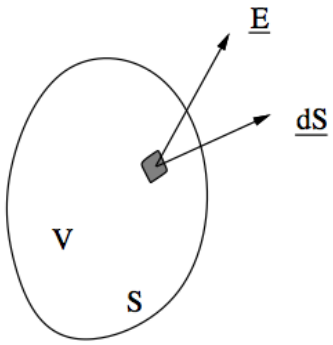
In the limit that $\Delta V'$ becomes infinitesimal, we have

$$\boxed{\mathbf{E}(\mathbf{r}) = k \int_V \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{\mid \mathbf{r} - \mathbf{r}' \mid^3} dV' \ .} \tag{1.2.6}$$

Much of the rest of this course is centered on methods for obtaining the electrostatic field, and we begin with one of the simplest – **Gauss' Law**.

1.3 Gauss' Law

Suppose that the charge density $\rho(\mathbf{r})$ is the sole source of the electrostatic field $\mathbf{E}(\mathbf{r})$. Gauss' Law relates the **flux** of $\mathbf{E}$ out of a closed surface S bounding a volume V to the **total charge** Q contained within V



Gauss' Law states that:

$$\int_S \mathbf{E} \cdot \mathbf{dS} = 4\pi k Q = \frac{Q}{\epsilon_0} \qquad \text{(in SI units)} \ , \tag{1.3.1}$$

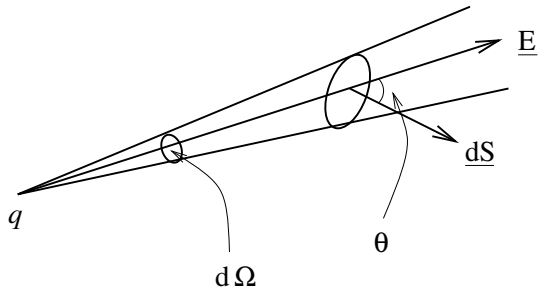
where

- $Q =$ **total charge** within S ,
- **dS** = **outward normal** to surface, having infinitesimal area dS .

Gauss' Law provides a powerful way to compute the electrostatic field for the case where there is spherical, or even cylindrical, symmetry. It will also form the starting point for our derivation of Laplace's equation later in the course.

1.3.1 Geometrical Interpretation of Gauss' Law

Consider a point charge q placed at the origin (not necessarily *inside* V), and the electrostatic flux across an area **dS**.

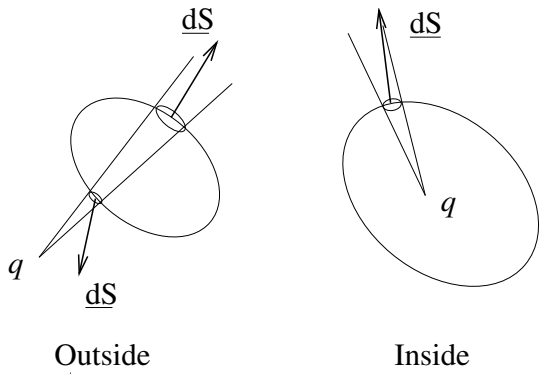


Then we have

$$\begin{aligned} \mathbf{E}\cdot\mathbf{dS} &= kq\,\frac{dS\,\cos\theta}{r^2} = kq\,\frac{r^2\,d\Omega}{r^2} \\ &= kq\,d\Omega \end{aligned}$$

where $d\Omega$ is the solid angle subtended by $d\mathbf{S}$ at the origin; $r^2 d\Omega$ is the *projection* of the surface element dS onto the sphere of radius r, and $d\Omega$ is its projection onto the unit sphere. Note that $\int_S d\Omega = 4\pi$ where S is a unit sphere, or any closed surface, enclosing the origin.

- If the charge q is **outside** the volume, then the total flux $\int_V \mathbf{E}\cdot\mathbf{dS}$ is zero; the contributions from two elements of surface area produced by the intersection of a cone with the surface **cancel**, see below.



- If the charge q is **inside** the volume, the total flux $\int_V \mathbf{E}\cdot\mathbf{dS} = q/\epsilon_0$.

Though this provides a simple and intuitive proof of Gauss' Law, we will now proceed to a more formal discussion.

1.3.2 Gauss' Law and Divergence Theorem

We will begin by analyzing the Gauss' Law for a single, pointlike charge q at the origin.

Gauss' Law for a Single Charge

Our starting point is once again Coulomb's Law:

$$\mathbf{E}(\mathbf{r}) = kq\frac{\mathbf{r}}{r^3} \ . \tag{1.3.2}$$

Lemma: For a single charge at the origin, $\nabla \cdot \mathbf{E} = 0$ for $\mathbf{r} \neq 0$.
Proof:

$$\begin{aligned} \nabla \cdot \left(\frac{\mathbf{r}}{r^3}\right) &= \left(\nabla \frac{1}{r^3}\right) \cdot \mathbf{r} + (\nabla \cdot \mathbf{r})\frac{1}{r^3} \\ &= -\left(\frac{3\,\mathbf{r}}{r^5}\right) \cdot \mathbf{r} + \frac{3}{r^3} = 0 \qquad \text{when } \mathbf{r} \neq 0 \ . \end{aligned}$$

We used here

$$\frac{\partial}{\partial x}\frac{1}{(x^2+y^2+z^2)^{3/2}} = -\frac{3}{2}\frac{2x}{(x^2+y^2+z^2)^{5/2}} = -3\frac{x}{r^5} \ \text{ etc. for } \ y,\, z \ \Rightarrow \nabla\frac{1}{r^3} = -3\frac{\mathbf{r}}{r^5}$$

and

$$\nabla \cdot \mathbf{r} = \frac{\partial}{\partial x}\, x + \frac{\partial}{\partial y}\, y + \frac{\partial}{\partial z}\, z = 3 \ .$$

Gauss' Law for a point charge is:

$$\int_S \mathbf{E} \cdot \mathbf{dS} = \begin{cases} 4\pi kq & \text{if the surface } S \text{ encloses the origin} \\ 0 & \text{otherwise} \end{cases} \tag{1.3.3}$$

Proof:
Origin outside V:
$\mathbf{E}(\mathbf{r})$ is continuously differentiable, and $\nabla \cdot \mathbf{E} = 0$ everywhere within V. From the divergence theorem,

$$\int_S \mathbf{E} \cdot \mathbf{dS} = \int_V (\nabla \cdot \mathbf{E})\, dV = 0 \qquad \text{if origin } \textbf{not within } V \ . \tag{1.3.4}$$

Origin inside V:
$\mathbf{E}(\mathbf{r})$ is **undefined** at $\mathbf{r} = 0$. Therefore define V to be the region **between** the closed surfaces S' and S, where S' is a small sphere of radius ϵ centered at the origin:

Now in the region V, $\nabla \cdot \mathbf{E} = 0$. Therefore, by the divergence theorem,

$$\int_V \nabla \cdot \mathbf{E}\, dV = \int_S \mathbf{E} \cdot \mathbf{dS} + \int_{S'} \mathbf{E} \cdot \mathbf{dS} = 0 \tag{1.3.5}$$

Then on the sphere S' we have:

$$\mathbf{dS} = -\epsilon^2\, d\Omega\, \mathbf{e_r}\ , \tag{1.3.6}$$

where the *outward* normal for S' points *towards* the origin. Therefore

$$\int_{S'} \mathbf{E} \cdot \mathbf{dS} = \int \left(kq \frac{\mathbf{e_r}}{r^2}\Big|_{r=\epsilon} \right) \cdot (-\epsilon^2 d\Omega\, \mathbf{e_r}) = -4\pi kq \qquad \text{(independent of } \epsilon) \quad .$$

This calculation may be also done in a more detailed form, by introducing spherical polar coordinates (r, θ, ψ), in which

$$\mathbf{dS} = -\epsilon^2\, d\Omega\, \mathbf{e_r} = -\epsilon^2 \sin\theta\, d\theta\, d\psi\, \mathbf{e_r}\ , \tag{1.3.7}$$

so that

$$\begin{aligned} \int_{S'} \mathbf{E} \cdot \mathbf{dS} &= \int_0^{2\pi} \int_0^{\pi} \left(kq\, \frac{\mathbf{e_r}}{r^2}\Big|_{r=\epsilon} \right) \cdot (-\epsilon^2 \sin\theta\, d\theta\, d\psi\, \mathbf{e_r}) \\ &= -4\pi kq \qquad \text{(independent of } \epsilon) \quad . \end{aligned}$$

We now let $\epsilon \to 0$, so that $V \to$ total volume within S, and we have

$$\int_S \mathbf{E} \cdot \mathbf{dS} = 4\pi kq = \frac{q}{\epsilon_0} \qquad \text{in SI units}\ , \tag{1.3.8}$$

so that the theorem is proved.

This construction may be also used to established the form of $\nabla \cdot (\mathbf{r}/r^3)$ for $\mathbf{r} = 0$. Indeed, our derivation amounts to

$$\int_{S_\epsilon} \frac{\mathbf{e_r}}{r^2} \cdot \mathbf{dS} = 4\pi \tag{1.3.9}$$

for the surface of a sphere with an arbitrarily small radius ϵ. However, by divergence theorem

$$\int_{S_\epsilon} \frac{\mathbf{e_r}}{r^2} \cdot \mathbf{dS} = \int_{V_\epsilon} \nabla \cdot \frac{\mathbf{e_r}}{r^2}\, dV\ , \tag{1.3.10}$$

where V_ϵ is the volume surrounded by the arbitrarily small sphere. The fact that this volume integral equals to the same ϵ-independent constant 4π means that

$$\nabla \cdot \frac{\mathbf{e_r}}{r^2} = 4\pi\delta^3(\mathbf{r})\ . \tag{1.3.11}$$

If the point charge is at the point $\mathbf{r_1}$, then we have

$$\mathbf{E}(\mathbf{r}) = kq\frac{\mathbf{r}-\mathbf{r_1}}{\mid \mathbf{r}-\mathbf{r_1} \mid^3}. \tag{1.3.12}$$

By changing variables to $\mathbf{R} = \mathbf{r} - \mathbf{r_1}$ it is easy to show

$$\int_S \mathbf{E}\cdot\mathbf{dS} = \begin{cases} 4\pi kq \;\; = q/\epsilon_0 & \text{in SI units} \qquad \text{if } \mathbf{r_1} \in V \\ 0 & \qquad\qquad\qquad\quad \text{otherwise} \end{cases} \tag{1.3.13}$$

One may also write

$$\nabla_{\mathbf{r}} \cdot \frac{\mathbf{r}-\mathbf{r_1}}{\mid \mathbf{r}-\mathbf{r_1} \mid^3} = 4\pi\delta^3(\mathbf{r}-\mathbf{r_1}) \ . \tag{1.3.14}$$

Gauss' Law for Distribution of Point Charges

We can extend the proof of Gauss' Law for a single charge distribution to a set of N point charges $\{q_i\}$ at $\{\mathbf{r_i}\}$ using the linear-superposition principle:

$$\mathbf{E}(\mathbf{r}) = \sum_{i=1}^{N} \mathbf{E_i}(\mathbf{r}) \ , \tag{1.3.15}$$

where $\mathbf{E}(\mathbf{r})$ is the total electrostatic field at the point $\mathbf{r}$, and $\mathbf{E_i}(\mathbf{r})$ is the electrostatic field at the point $\mathbf{r}$ due to the charge q_i at the point $\mathbf{r_i}$. Applying Gauss' Law for point charges proved above, we have

$$\int_S \mathbf{E_i}\cdot\mathbf{dS} = \begin{cases} 4\pi kq_i \;\; = q_i/\epsilon_0 & \text{in SI units} \qquad \text{if } \mathbf{r_i} \in V \\ 0 & \qquad\qquad\qquad\quad \text{otherwise} \end{cases} \tag{1.3.16}$$

Hence

$$\begin{aligned} \int_S \mathbf{E}\cdot\mathbf{dS} &= \sum_i \int_S \mathbf{E_i}\cdot\mathbf{dS} = 4\pi k \sum_{i,\mathbf{r_i}\in V} q_i \\ &= 4\pi kQ = \frac{Q}{\epsilon_0} \qquad \text{(in SI units)} \end{aligned}$$

where Q is the sum of the charges contained within the volume V.

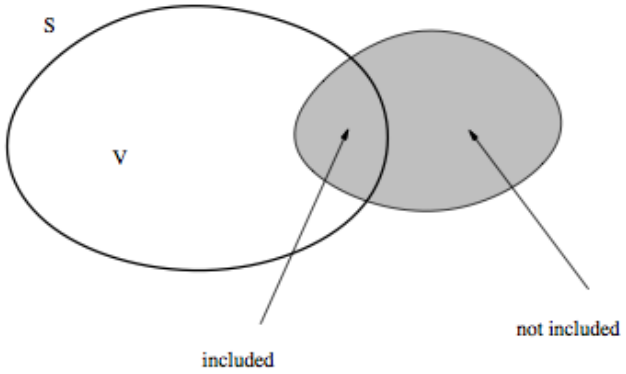
Gauss' Law for Continuous Distribution of Charge

This we prove by exact analogy with derivation of the *electrostatic field* for a continuous distribution: we divide up the volume V into elements of volume $\Delta V'$, centered at $\mathbf{r}'$, and

obtain

$$\begin{aligned}\int_S \mathbf{E}\cdot\mathbf{dS} &= 4\pi k \sum_{\Delta V'\in V} \rho(\mathbf{r}')\,\Delta V' \\ &\overset{\Delta V'\to 0}{\longrightarrow} 4\pi k \int_V \rho(\mathbf{r}')\,dV' = 4\pi k Q,\end{aligned}$$

where Q is the total charge contained within the volume V.



1.3.3 Applications of Gauss' Law

Gauss' Law provides a powerful method of determining the electrostatic field where we have **spherical** or **cylindrical** symmetry.

Spherical Symmetry

Suppose we have a **spherically symmetric** distribution of charge $\rho = \rho(r)$, where $r = |\mathbf{r}|$. Then the electrostatic field will depend only on r, and therefore must be in the radial direction.
Choose a spherical surface S of radius r, centered on the center of the charge distribution. Then we have that

$$\int_S \mathbf{E}(\mathbf{r})\cdot\mathbf{dS} = \int_S E(r)\,\mathbf{e_r}\cdot\mathbf{dS} = \int_\Omega E(r)\,r^2\,d\Omega = 4\pi E(r)\,r^2. \tag{1.3.17}$$

But by Gauss' Law, we have

$$\int_S \mathbf{E}\cdot\mathbf{dS} = 4\pi k Q(r), \tag{1.3.18}$$

where $Q(r) = \int_V \rho(r')\,dV$ is the *total charge* contained within the sphere of radius r.

Thus we have

$$\mathbf{E}(\mathbf{r}) = \frac{kQ(r)}{r^2}\,\mathbf{e_r} = \frac{Q(r)}{4\pi\epsilon_0 r^2}\,\mathbf{e_r} \qquad \text{in SI units.} \tag{1.3.19}$$

Note that **outside** a spherically symmetric charge distribution, the field is the **same** as if we had a **point-like** charge $Q(r)$ at the origin.

Example: Consider a thin spherical shell of charge Q. We can say immediately:

- **Outside** the shell, the electrostatic field is the same as that of the equivalent point charge Q at its center:

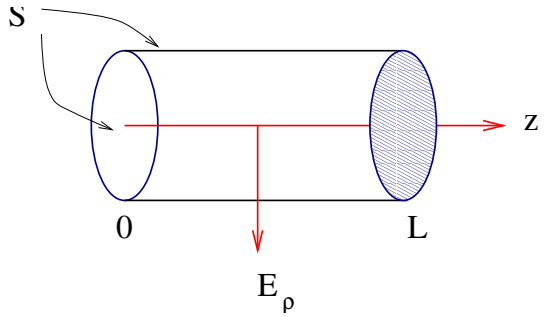
$$\mathbf{E}(\mathbf{r}) = \frac{kQ}{r^2}\mathbf{e_r} \tag{1.3.20}$$

- **Inside** the shell, the field is **zero**.

Cylindrical Symmetry

Suppose we have an infinitely long, **cylindrically symmetric** distribution of charge, with the axis of symmetry along the z axis. Introduce cylindrical coordinates (ρ, φ, z).

Consider an *element* of length L, and radius ρ, containing a charge $Q(\rho, L)$:



The field will depend solely on ρ, and therefore must be in the $\mathbf{e}_\rho$ direction, $\mathbf{E}(\mathbf{r}) = E(\rho)\mathbf{e}_\rho$. Applying Gauss' Law to the cylinder we have

$$\int_S \mathbf{E}\cdot\mathbf{dS} = 4\pi k Q(\rho, L) \tag{1.3.21}$$

Now on the "end-caps", $z = 0$ and $z = L$, $\mathbf{E}\cdot\mathbf{dS} = 0$, and therefore

$$\int_S \mathbf{E}\cdot\mathbf{dS} = \int_S E(\rho)\mathbf{e}_\rho\cdot\mathbf{dS} = E(\rho)\int_S dS = E(\rho)2\pi\rho L. \tag{1.3.22}$$

Thus

$$E(\rho) = \frac{2kQ(\rho, L)}{\rho L} = \frac{2Q(\rho, L)}{4\pi\epsilon_0 \rho L} \qquad \text{in SI units} \tag{1.3.23}$$

Example: Infinitely long, thin rod carrying charge λ per unit length. Thus, $Q(\rho, L) = \lambda L$ and we have

$$E(\rho) = \frac{\lambda}{2\pi\epsilon_0 \rho}. \tag{1.3.24}$$

We expect the treatment of the rod as *infinitely long* to be a good approximation for a rod of finite length providing

$$w < \rho \ll l \tag{1.3.25}$$

where w and l and the *width* and the *length* of the rod respectively.

1.3.4 Maxwell's First Equation (ME1)

Our starting point is Gauss' Law:

$$\int_S \mathbf{E} \cdot \mathbf{dS} = 4\pi k \int_V \rho(\mathbf{r}')\, dV' \tag{1.3.1}$$

where $\rho(\mathbf{r}')$ is the charge density. By the *divergence theorem*, we have

$$\int_S \mathbf{E} \cdot \mathbf{dS} = \int_V \nabla \cdot \mathbf{E}\, dV', \tag{1.3.2}$$

and thus

$$\int_V \{\nabla \cdot \mathbf{E} - 4\pi k\rho\}\, dV' = 0. \tag{1.3.3}$$

This applies for *any* volume V, and therefore the integrand itself must vanish:

$$\nabla \cdot \mathbf{E} = 4\pi k\rho = \frac{\rho}{\epsilon_0}. \tag{1.3.4}$$

This is **Maxwell's First Equation** (ME1). ME1 is essentially an expression of Gauss' law in differential form.

1.4 The Scalar Potential

1.4.1 Maxwell's Second Equation for Electrostatics

The mathematical basis for the Maxwell's First Equation is given by the relation

$$\nabla \cdot \frac{\mathbf{e_r}}{r^2} = 4\pi\delta^3(\mathbf{r}) \tag{1.4.1}$$

satisfied by the $\mathbf{e_r}/r^2$ factor governing the coordinate dependence of the electric field generated by a unit point charge located at the origin.
Another important equation satisfied by this factor is

$$\nabla \times \frac{\mathbf{e_r}}{r^2} = 0 \ . \tag{1.4.2}$$

Indeed (shifting $\mathbf{r} \to \mathbf{r} - \mathbf{r}'$ for generality),

$$\begin{aligned} \nabla \times \left\{ \frac{\mathbf{r}-\mathbf{r}'}{\mid \mathbf{r}-\mathbf{r}' \mid^3} \right\} =& \nabla \left(\frac{1}{\mid \mathbf{r}-\mathbf{r}' \mid^3} \right) \times (\mathbf{r}-\mathbf{r}') + \frac{1}{\mid \mathbf{r}-\mathbf{r}' \mid^3} \nabla \times (\mathbf{r}-\mathbf{r}') \\ =& \frac{-3(\mathbf{r}-\mathbf{r}')}{\mid \mathbf{r}-\mathbf{r}' \mid^5} \times (\mathbf{r}-\mathbf{r}') + 0 = 0 \ , \end{aligned} \tag{1.4.3}$$

since $\mathbf{C} \times \mathbf{C} = 0$ for any vector $\mathbf{C}$, and $\nabla \times \mathbf{r} = 0$.
Thus, one has

$$\nabla \times \mathbf{E}_q(\mathbf{r}) = 0 \tag{1.4.4}$$

for a field $\mathbf{E}_q(\mathbf{r})$ of a point charge. Due to the linear superposition principle, one would have

$$\nabla \times \mathbf{E}(\mathbf{r}) = 0 \tag{1.4.5}$$

for electric field generated by any system of static charges, i.e., $\nabla \times \mathbf{E}(\mathbf{r}) = 0$ *for any electrostatic field*. This is **Maxwell's Second Equation** (ME2) for electrostatics.

1.4.2 Introducing Scalar Potential

It can be shown that any vector field $\mathbf{E}(\mathbf{r})$ that is given by a gradient of a scalar function, call it $-\Phi(\mathbf{r})$, has the property $\nabla \times \mathbf{E}(\mathbf{r}) = 0$. Indeed, consider a vector $\mathbf{C} \equiv \nabla \times (\nabla\Phi(\mathbf{r}))$. Its x-component is given by

$$C_x = \frac{\partial}{\partial y}\left(\frac{\partial \Phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \Phi}{\partial y}\right) = 0 \ , \tag{1.4.6}$$

and similarly for two other components. Thus,

$$\nabla \times (\nabla \Phi(\mathbf{r})) = 0 \ . \tag{1.4.7}$$

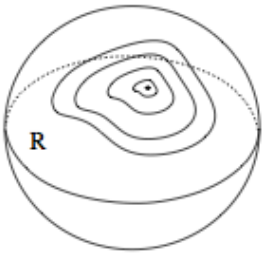
In other words, fields that can be written as $\mathbf{E}(\mathbf{r}) = -\nabla\Phi(\mathbf{r})$, satisfy $\nabla \times \mathbf{E}(\mathbf{r}) = 0$. But it may also be shown that if a field $\mathbf{E}(\mathbf{r})$ satisfies $\nabla \times \mathbf{E}(\mathbf{r}) = 0$ then it can be written as $\mathbf{E}(\mathbf{r}) = -\nabla\Phi(\mathbf{r})$, i.e., *any electrostatic field* $\mathbf{E}$ *may be obtained from the scalar potential* Φ. ME1 has provided us with a differential equation to describe the electric field, $\mathbf{E}(\mathbf{r})$, but it would be easier were we able to work with a *scalar* quantity. The **scalar potential** provides a means of so doing.

Scalar Potential

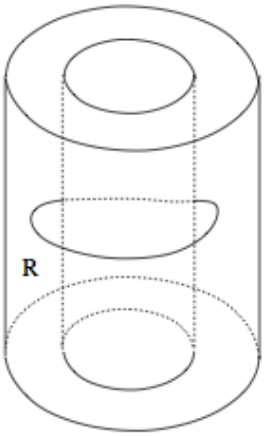
- Given a vector field $\mathbf{A}(\mathbf{r})$, under what conditions can we write $\mathbf{A}$ as the **gradient** of a **scalar** field Φ, viz. $\mathbf{A}(\mathbf{r}) = -\nabla\,\Phi(\mathbf{r})$, where the minus sign is conventional?
- What can we say about the *uniqueness* of $\Phi(\mathbf{r})$?

Definition: A **simply connected** region R is a region where every **closed** curve in R can be shrunk continuously to a point whilst remaining entirely in R.

Examples:



The inside of a sphere is **simply connected**



The region between two cylinders is **not** simply connected: it's **doubly connected**

1.4.3 Theorems on Scalar Potentials

Let $\mathbf{A}(\mathbf{r})$ be a continuously differentiable vector field defined in a simply connected region R. Then the following three statements are equivalent, i.e. **any one implies the other two**:

1. $\nabla \times \mathbf{A}(\mathbf{r}) = 0$ for all points $\mathbf{r} \in R$
2. (a) $\oint_C \mathbf{A}(\mathbf{r}') \cdot \mathbf{dr}' = 0$, where C is any **closed** curve in R

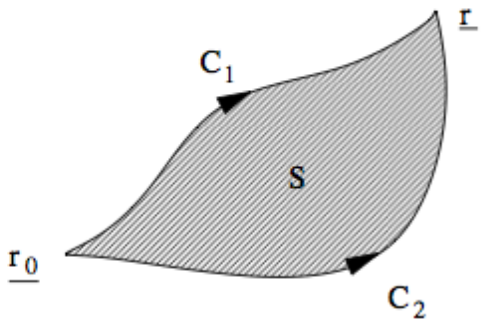
   (b) $\Phi(\mathbf{r}) \equiv -\int_{\mathbf{r_0}}^{\mathbf{r}} \mathbf{A}(\mathbf{r}') \cdot \mathbf{dr}'$ does not depend on the path between $\mathbf{r_0}$ and $\mathbf{r}$.
3. $\mathbf{A}(\mathbf{r})$ can be written as the **gradient** of a **scalar potential** $\Phi(\mathbf{r})$

$$\mathbf{A}(\mathbf{r}) = -\nabla \Phi(\mathbf{r}) \quad \text{with} \quad \Phi(\mathbf{r}) = -\int_{\mathbf{r_0}}^{\mathbf{r}} \mathbf{A}(\mathbf{r}') \cdot \mathbf{dr}' \tag{1.4.8}$$

where $\mathbf{r_0}$ is some **arbitrary fixed point** in R.

Proof that (1) implies (2):

Let $\nabla \times \mathbf{A}(\mathbf{r}) = 0$ in R, and consider any two curves, C_1 and C_2 from the point $\mathbf{r}_0$ to the point $\mathbf{r}$ in R. Introduce the *closed* curve $C = C_1 - C_2$, and let S be a surface spanning C.



Apply Stokes' theorem:

$$\oint_C \mathbf{A}(\mathbf{r}') \cdot \mathbf{dr}' = \int_S \nabla \times \mathbf{A} \cdot \mathbf{dS} = 0 \tag{1.4.9}$$

since $\nabla \times \mathbf{A} = 0$ everywhere. Note that we use $\mathbf{r}'$ as integration variable to distinguish it from the **end-points** of C_1 and C_2, $\mathbf{r}_0$ and $\mathbf{r}$.

Thus we have:

$$\nabla \times \mathbf{A} = 0 \Rightarrow \oint_C \mathbf{A}(\mathbf{r}') \cdot \mathbf{dr}' = 0 \tag{1.4.10}$$

for any curve C in R, and the first part of the proof is done.
For the second part of the proof, we observe

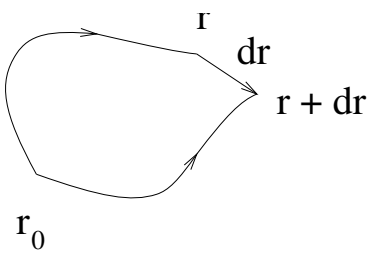
$$\int_{C_1} \mathbf{A}(\mathbf{r}') \cdot \mathbf{dr}' - \int_{C_2} \mathbf{A}(\mathbf{r}') \cdot \mathbf{dr}' = \oint_C \mathbf{A}(\mathbf{r}') \cdot \mathbf{dr}' = 0. \tag{1.4.11}$$

Thus the **scalar potential** $\Phi(\mathbf{r})$ of the vector field $\mathbf{A}(\mathbf{r})$ defined by

$$\Phi(\mathbf{r}) = -\int_{\mathbf{r_0}}^{\mathbf{r}} \mathbf{A}(\mathbf{r}') \cdot \mathbf{dr}' \tag{1.4.12}$$

is **independent of the path of integration** joining $\mathbf{r}_0$ and $\mathbf{r}$.
Proof that (2) implies (3)
Consider two neighbouring points $\mathbf{r}$ and $\mathbf{r} + \mathbf{dr}$. Define the scalar potential as before:

$$\Phi(\mathbf{r}) = -\int_{\mathbf{r_0}}^{\mathbf{r}} \mathbf{A}(\mathbf{r}') \cdot \mathbf{dr}' \tag{1.4.13}$$



Now define the quantity $\delta\Phi(\mathbf{r})$:

$$\begin{aligned}
\delta\Phi(\mathbf{r}) &= \Phi(\mathbf{r}+\mathbf{dr}) - \Phi(\mathbf{r}) \\
&= \left\{ -\int_{\mathbf{r_0}}^{\mathbf{r}+\mathbf{dr}} \mathbf{A}(\mathbf{r}')\cdot\mathbf{dr}' + \int_{\mathbf{r_0}}^{\mathbf{r}} \mathbf{A}(\mathbf{r}')\cdot\mathbf{dr}' \right\} \quad \text{(by definition)} \\
&= -\left\{ \int_{\mathbf{r_0}}^{\mathbf{r}+\mathbf{dr}} \mathbf{A}(\mathbf{r}')\cdot\mathbf{dr}' + \int_{\mathbf{r}}^{\mathbf{r}_0} \mathbf{A}(\mathbf{r}')\cdot\mathbf{dr}' \right\} \quad \text{(swapped limits on 2nd } \textstyle\int\text{)} \\
&= -\left\{ \int_{\mathbf{r}}^{\mathbf{r}_0} \mathbf{A}(\mathbf{r}')\cdot\mathbf{dr}' + \int_{\mathbf{r_0}}^{\mathbf{r}+\mathbf{dr}} \mathbf{A}(\mathbf{r}')\cdot\mathbf{dr}' \right\} \text{(interchanged the integrals)} \\
&= -\int_{\mathbf{r}}^{\mathbf{r}+\mathbf{dr}} \mathbf{A}(\mathbf{r}')\cdot\mathbf{dr}' \quad \text{(Combined the integrals using path-independence of integral)} \\
&= -\left[\int_{\mathbf{r}}^{\mathbf{r}+\mathbf{dr}} \mathbf{A}(\mathbf{r})\cdot\mathbf{dr}' \right] = -\left[\mathbf{A}(\mathbf{r})\cdot\mathbf{r}' \right]_{\mathbf{r}}^{\mathbf{r}+\mathbf{dr}} \quad \text{(for } \textit{infinitesimal}\ \mathbf{dr}\text{)} \\
&= \mathbf{A}(\mathbf{r})\cdot\{ -(\mathbf{r}+\mathbf{dr}) + \mathbf{r}\}
\end{aligned}$$

$$\text{So} \quad \delta\Phi(\mathbf{r}) = -\mathbf{A}(\mathbf{r})\cdot\mathbf{dr} \tag{1.4.14}$$

To perform the integral, we used path independence and integrated along the infinitesimal straight line between $\mathbf{r}$ and $\mathbf{r}+\mathbf{dr}$ along which $\mathbf{A}(\mathbf{r}')$ is **constant** up to effects of $O(\mathbf{dr})$.

But, by Taylor's theorem, we also have

$$\delta\Phi(\mathbf{r}) = \sum_i \frac{\partial\Phi(\mathbf{r})}{\partial x_i}\, dx_i = \nabla\Phi(\mathbf{r})\cdot\mathbf{dr} \tag{1.4.15}$$

Since equations (1.4.14) and (1.4.15) hold for an arbitrary infinitesimal $\mathbf{dr}$, we have

$$\mathbf{A}(\mathbf{r}) = -\nabla\Phi(\mathbf{r})\ . \tag{1.4.16}$$

Thus we have shown that **path independence** implies the existence of a **scalar potential** Φ for the vector field $\mathbf{A}$.

Proof that (3) implies (1)

$$\mathbf{A} \;=\; -\nabla\,\Phi \quad \Rightarrow \quad \nabla \times \mathbf{A} \;=\; -\nabla \times (\nabla\,\Phi) \equiv 0 \tag{1.4.17}$$

because, as we established earlier, $\nabla \times\ (\nabla\ \Phi)$ is identically zero (i.e. it's zero for *any* scalar field Φ).

1.4.4 Terminology

Such a vector field is called

- **Irrotational:** $\nabla \times \mathbf{A}(\mathbf{r}) \;=\; 0 \quad \Leftrightarrow \quad \oint_C \mathbf{A}(\mathbf{r}') \cdot \mathbf{dr}' \;=\; 0$

  Sometimes, the notation "curl $\mathbf{A}(\mathbf{r})$" is used for $\nabla \times \mathbf{A}(\mathbf{r})$. Also, if you look in older textbooks, you will sometimes see "rot" rather than "curl".

- **Conservative:** e.g. if $\mathbf{A}$ = force, then Φ is potential energy and total energy is conserved (see later).

- The field $\Phi(\mathbf{r})$ is the **scalar potential** for the vector field $\mathbf{A}(\mathbf{r})$.

1.4.5 Uniqueness

$\Phi(\mathbf{r})$ is **uniquely determined** up to a **constant**.

Proof:
Let Φ and Ψ be scalar potentials obtained by **different** choices of $\mathbf{r_0}$. Then

$$\nabla\,\Phi \;-\; \nabla\,\Psi \;=\; \mathbf{A} \;-\; \mathbf{A} \;=\; 0\ . \tag{1.4.18}$$

Therefore

$$\nabla\,(\Psi \;-\; \Phi) \;=\; 0\ . \tag{1.4.19}$$

Integration of this equation wrt any of x, y, or z gives

$$\Psi \;-\; \Phi \;=\; \text{constant}\ . \tag{1.4.20}$$

Therefore

$$\Psi \;=\; \Phi \;+\; \text{constant}\ . \tag{1.4.21}$$

The **absolute value** of a scalar potential has no meaning, only **potential differences** are significant.

1.4.6 Existence of Scalar Potential for Electrostatic Field

After the digression on subject of the scalar potentials, it is time to show that the electrostatic field is, indeed, irrotational.
The central result of this chapter was the expression for the electrostatic field due to a continuous charge distribution

$$\mathbf{E}(\mathbf{r}) = k \int_V \frac{\rho(\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{\mid \mathbf{r}-\mathbf{r}' \mid^3} \, dV' \ . \tag{1.4.22}$$

Thus we have

$$\begin{aligned} \nabla \times \mathbf{E}(\mathbf{r}) &= \int \nabla \times \left\{ \frac{\rho(\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{\mid \mathbf{r}-\mathbf{r}' \mid^3} \right\} dV' \\ &= \int_V \rho(\mathbf{r}') \left\{ \nabla \left(\frac{1}{\mid \mathbf{r}-\mathbf{r}' \mid^3} \right) \times (\mathbf{r}-\mathbf{r}') + \frac{1}{\mid \mathbf{r}-\mathbf{r}' \mid^3} \nabla \times (\mathbf{r}-\mathbf{r}') \right\} dV' \\ &= \int \rho(\mathbf{r}') \left\{ \frac{-3(\mathbf{r}-\mathbf{r}')}{\mid \mathbf{r}-\mathbf{r}' \mid^5} \times (\mathbf{r}-\mathbf{r}') \ + \ 0 \right\} dV' \\ &= 0 \ , \end{aligned} \tag{1.4.23}$$

where the derivatives act only on the *unprimed* indices.

The electrostatic field $\mathbf{E}(\mathbf{r})$ can be written in terms of a scalar potential

$$\mathbf{E}(\mathbf{r}) = \ -\nabla\Phi(\mathbf{r})$$

1.4.7 Methods for finding Scalar Potentials

We have shown that the scalar potential $\Phi(\mathbf{r})$ for an irrotational vector field $\mathbf{A}(\mathbf{r})$ can be constructed via

$$\Phi(\mathbf{r}) \ = \ - \int_{\mathbf{r_0}}^{\mathbf{r}} \mathbf{A}(\mathbf{r}') \cdot \mathbf{dr}' \tag{1.4.24}$$

for some suitably chosen $\mathbf{r_0}$ and any path which joins $\mathbf{r_0}$ and $\mathbf{r}$. Sensible choices for $\mathbf{r_0}$ are often $\mathbf{r_0} = 0$ or $\mathbf{r_0} = \infty$.

We have also shown that the line integral is *independent* of the path of integration between the endpoints. Therefore, a convenient way of evaluating such integrals is to integrate along a **straight line** between the points $\mathbf{r}_0$ and $\mathbf{r}$. Choosing $\mathbf{r}_0 = 0$, we can write this integral in

parametric form as follows:

$$\begin{aligned} \mathbf{r}' &= \lambda\,\mathbf{r} \quad \text{where} \quad \{0 \le \lambda \le 1\} \\ \text{so} \quad \mathbf{dr}' &= \mathbf{r}\,d\lambda \qquad \text{and therefore} \\ \Phi(\mathbf{r}) &= -\int_{\lambda=0}^{\lambda=1} \mathbf{A}(\lambda\,\mathbf{r})\cdot(\mathbf{r}\,d\lambda) \end{aligned}$$

Example:
Let $\mathbf{A}(\mathbf{r}) = (\mathbf{a}\cdot\mathbf{r})\,\mathbf{a}$ where $\mathbf{a}$ is a constant vector.
It is easy to show that $\nabla\times((\mathbf{a}\cdot\mathbf{r})\,\mathbf{a}) = 0$. Indeed,

$$(\nabla\times((\mathbf{a}\cdot\mathbf{r})\,\mathbf{a}))_x = \partial_y(\mathbf{a}\cdot\mathbf{r})a_z - \partial_z(\mathbf{a}\cdot\mathbf{r})a_y = a_y a_z - a_z a_y = 0\ , \tag{1.4.25}$$

etc. Or, in vector form, using $\nabla(\mathbf{a}\cdot\mathbf{r}) = \mathbf{a}$

$$\nabla\times((\mathbf{a}\cdot\mathbf{r})\,\mathbf{a}) = \mathbf{a}\times\mathbf{a} = 0\ . \tag{1.4.26}$$

Thus

$$\begin{aligned} \Phi(\mathbf{r}) &= -\int_0^{\mathbf{r}} \mathbf{A}(\mathbf{r}')\cdot\mathbf{dr}' \\ &= -\int_0^{\mathbf{r}} \Big((\mathbf{a}\cdot\mathbf{r}')\,\mathbf{a}\Big)\cdot\mathbf{dr}' \\ &= -\int_0^{1} \Big((\mathbf{a}\cdot\lambda\,\mathbf{r})\,\mathbf{a}\Big)\cdot(\mathbf{r}\,d\lambda) \\ &= -(\mathbf{a}\cdot\mathbf{r})^2\int_0^1 \lambda\,d\lambda \\ &= -\frac{1}{2}(\mathbf{a}\cdot\mathbf{r})^2 \end{aligned}$$

Of course, this is all rather artifical. What we really want to do is to obtain Φ and $\mathbf{A}$ *from first principles.*

1.4.8 Singular Fields

We have seen that, for the case of a point-charge at the origin, the electric field is *singular* at $\mathbf{r} = 0$. In such cases, it is not possible to obtain the corresponding scalar potential at $\mathbf{r}$ by integration along a path from the origin. All is not lost – remember that the starting point for our path is arbitrary, and often it is convenient to take it at infinity.

Example: Electric field due to point charge at $\mathbf{r} = \mathbf{0}$: $\mathbf{E}(\mathbf{r}) = kq\mathbf{r}/r^3$, so that $\mathbf{E}(\mathbf{r} = \mathbf{0})$ is singular, and hence undefined. As in the proof of Gauss' law, our region R must exclude an infinitesimal sphere centered at $\mathbf{r} = 0$.
Here we choose a path from $r_0 = \infty$, yielding

$$\begin{aligned} \Phi(\mathbf{r}) &= -\int_\infty^{\mathbf{r}} \mathbf{E}(\mathbf{r}') \cdot \mathbf{dr}' = -\int_\infty^1 \mathbf{E}(\lambda\mathbf{r}) \cdot d\lambda\, \mathbf{r} \\ &= -kq \int_\infty^1 \frac{\lambda\, \mathbf{r}\cdot\mathbf{r}}{\lambda^3 r^3}\, d\lambda = -kq \int_\infty^1 \frac{d\lambda}{\lambda^2}\frac{r^2}{r^3} \\ &= kq\,\frac{1}{r}\ . \end{aligned} \tag{1.4.27}$$

Thus we have the famous $1/r$ potential due to a point charge.
Since one electron produces a field of 1.438×10^{-9} V/m at a distance of 1 meter, the electrostatic potential energy of two electrons $e^2/(4\pi\epsilon_0 r)$ separated by 1 meter equals to 1.438×10^{-9} electonvolts.
The electric field of a point charge is given by $\mathbf{E}(\mathbf{r}) = kq\,\mathbf{e_r}/r^2$, hence, one should have

$$\nabla\left(\frac{1}{r}\right) = -\frac{\mathbf{e_r}}{r^2}\ , \tag{1.4.28}$$

and

$$\nabla_{\mathbf{r}}\left(\frac{1}{|\ \mathbf{r}-\mathbf{r}'\ |}\right) = -\frac{\mathbf{r}-\mathbf{r}'}{|\ \mathbf{r}-\mathbf{r}'\ |^3}\ , \tag{1.4.29}$$

which may be easily checked by a direct differentiation. Recalling Eq. (1.3.14),

$$\nabla_{\mathbf{r}}\cdot\frac{\mathbf{r}-\mathbf{r_1}}{|\ \mathbf{r}-\mathbf{r_1}\ |^3} = 4\pi\delta^3(\mathbf{r}-\mathbf{r_1})\ , \tag{1.4.30}$$

we conclude that

$$\nabla^2_{\mathbf{r}}\left(\frac{1}{|\ \mathbf{r}-\mathbf{r}'\ |}\right) = -4\pi\delta^3(\mathbf{r}-\mathbf{r_1})\ . \tag{1.4.31}$$

Because of the *linearity* of the gradient operation, we can impose the linear superposition principle on the potential, and hence obtain an expression for the potential due to an extended charge distribution:

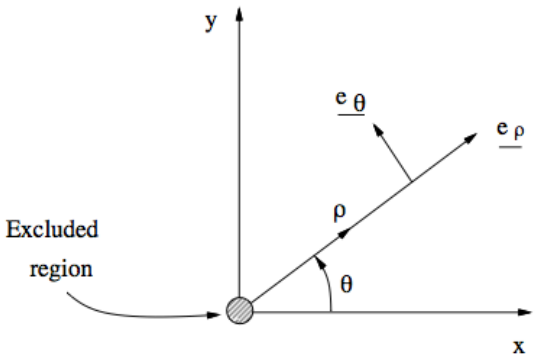
$$\boxed{\Phi(\mathbf{r}) = k\int_V \frac{\rho(\mathbf{r}')}{|\ \mathbf{r}-\mathbf{r}'\ |}\, dV'} \tag{1.4.32}$$

1.4.9 Multiply-connected Regions

In this case, $\nabla \times \mathbf{A} = 0$ does not imply the existence of a scalar potential function.
Example: Work using **cylindrical coordinates** (ρ, φ, z). A vector field $\mathbf{A}$, with

$$A_\rho = A_z = 0, \quad A_\varphi = \frac{a}{\rho} \tag{1.4.33}$$

where a is a constant, is defined outside an infinitesimal cylinder about the z-axis, where A_φ is **singular**. This region is **doubly connected** (cf. example above where we exclude an infinitesimal sphere).



Then we have $\nabla \times \mathbf{A} = 0$. Indeed,

$$\begin{aligned}
\nabla \times \mathbf{A} =& \mathbf{e}_\rho \left(\frac{1}{\rho} \frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z} \right) + \mathbf{e}_\varphi \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) + \mathbf{e}_z \frac{1}{\rho} \left(\frac{\partial}{\partial \rho}(\rho A_\varphi) - \frac{\partial A_\rho}{\partial \varphi} \right) \\
=& -\mathbf{e}_\rho \frac{\partial A_\varphi}{\partial z} + \mathbf{e}_\varphi \cdot 0 + \mathbf{e}_z \frac{1}{\rho} \frac{\partial}{\partial \rho}(\rho A_\varphi) \\
=& 0 + 0 + \mathbf{e}_z \frac{1}{\rho} \frac{\partial}{\partial \rho}(a) = 0 \ .
\end{aligned} \tag{1.4.34}$$

But, for a circular path C of radius ρ enclosing the z-axis, we have

$$\oint_C \mathbf{A} \cdot \mathbf{dr} = \int_0^{2\pi} A_\varphi \, \rho \, d\varphi = 2\pi a \neq 0 \ . \tag{1.4.35}$$

In this case, the "potential" would depend on the choice of path, and in particular the *winding number* - the number of times that a path wraps around the z-axis.
Examples: Vortices in superconductors, Cosmic strings...
Note also, that the value of the integral does not depend on the radius ρ of the curve, i.e., it is the same for any circle. This also means that the integral equals $2\pi a$ for any curve encircling the origin (one time).

Another corollary is that if one takes a combined contour made of two circles C_1 and C_2 with radii ρ_1, ρ_2 passed in opposite directions, the line integral for the combined contour is zero, which is consistent with the fact that $\nabla \times \mathbf{A} = 0$ everywhere on the surface between these two circles.

1.4.10 Conservative Forces and Physical Interpretation of Potential

To see how the name *conservative field* arises, consider a vector field $\mathbf{F}(\mathbf{r})$ corresponding to the only force acting on some test particle of mass m. The **work done** by the force in going around a closed curve C is

$$W = \oint_C \mathbf{F}(\mathbf{r}) \cdot \mathbf{dr} \ . \tag{1.4.36}$$

For a *conservative* force, $\nabla \times \mathbf{F} \ = \ 0$, the earlier theorems tell us:

- The total work done by the force in moving the particle around a closed curve is zero.
- We can write the force in terms of a scalar potential

$$\mathbf{F}(\mathbf{r}) = -\nabla U(\mathbf{r}) \ , \tag{1.4.37}$$

  where the minus sign is conventional (see later).

We will now show that for a conservative force, the **total energy** is **constant** in time.

Proof

The particle moves under the influence of Newton's Second Law:

$$m\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}). \tag{1.4.38}$$

Consider a small displacement $\mathbf{dr}$ taking time dt along the path followed by the particle. Then we have

$$m\ddot{\mathbf{r}} \cdot \mathbf{dr} = \mathbf{F}(\mathbf{r}) \cdot \mathbf{dr} = -\nabla U(\mathbf{r}) \cdot \mathbf{dr}. \tag{1.4.39}$$

Integrating this expression along the path from $\mathbf{r_A}$ at time $t = t_A$ to $\mathbf{r_B}$ at time $t = t_B$ yields

$$m \int_{\mathbf{r_A}}^{\mathbf{r_B}} \ddot{\mathbf{r}} \cdot \mathbf{dr} = -\int_{\mathbf{r_A}}^{\mathbf{r_B}} \nabla U(\mathbf{r}) \cdot \mathbf{dr}. \tag{1.4.40}$$

We can simplify the left-hand side of equation (1.4.40) by writing $\mathbf{dr} = (\mathbf{dr}/dt)dt = \dot{\mathbf{r}}dt$ to obtain

$$m \int_{\mathbf{r_A}}^{\mathbf{r_B}} \ddot{\mathbf{r}} \cdot \mathbf{dr} = m \int_{t_A}^{t_B} \ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} \, dt = m \int_{t_A}^{t_B} \frac{1}{2} \left(\frac{d}{dt} \dot{\mathbf{r}}^2 \right) \, dt = \frac{1}{2} m [v_B^2 - v_A^2], \tag{1.4.41}$$

where v_A and v_B are the magnitudes of the velocities at the points labelled by A and B respectively. The limits of integration over t correspond to time moments when the particle was at $\mathbf{r_A}$ (time $t = t_A$) and $\mathbf{r_B}$ (time $t = t_B$).

To integrate the right-hand side of equation (1.4.40), we appeal to Taylor's theorem to note that

$$\nabla U(\mathbf{r}) \cdot \mathbf{dr} = \sum_i \frac{\partial U}{\partial x_i} dx_i = dU \tag{1.4.42}$$

is the change is U when we move from $\mathbf{r}$ to $\mathbf{r} + \mathbf{dr}$. Thus we have

$$-\int_{\mathbf{r_A}}^{\mathbf{r_B}} \nabla U(\mathbf{r}) \cdot \mathbf{dr} = -\int_{\mathbf{r_A}}^{\mathbf{r_B}} dU = U_A - U_B \tag{1.4.43}$$

where U_A and U_B are the values of the potential U at $\mathbf{r_A}$ and $\mathbf{r_B}$, respectively.
Thus we have that

$$\frac{1}{2}mv_A^2 + U_A = \frac{1}{2}mv_B^2 + U_B \ . \tag{1.4.44}$$

- The first term on both sides we recognize as the **kinetic energy**
- The second term we identify as the **potential energy**

The **Total Energy**

$$E = \frac{1}{2}mv^2 + U \tag{1.4.45}$$

is **conserved**, i.e. *constant in time.*

We have seen that the existence of a scalar potential is associated with the *irrotational* or *conservative* nature of a vector field. Where the vector field corresponds to a *force*, we have a neat physical motivation for the name: a force is conservative if the work done in going around a closed path is zero, and if a particle moves solely under the influence of that force, then the energy is conserved.

Physical Interpretation of $\Phi(\mathbf{r})$

In electrostatics, the force $\mathbf{F}$ acting on a charge q due to an electrostatic field $\mathbf{E}$ is $\mathbf{F}(\mathbf{r}) = q\mathbf{E}(\mathbf{r})$. Now $\mathbf{E}(\mathbf{r}) = -\nabla\Phi(\mathbf{r})$ so that

$$\mathbf{F}(\mathbf{r}) = -\nabla(q\Phi(\mathbf{r})). \tag{1.4.46}$$

We have seen that the (conservative) force acting on a particle is minus the gradient of its potential energy: $\mathbf{F}(\mathbf{r}) = -\nabla U(\mathbf{r})$.

The **potential energy** $U(\mathbf{r})$ of a charge q situated at $\mathbf{r}$ in an electrostatic potential $\Phi(\mathbf{r})$ is

$$U(\mathbf{r}) = q\Phi(\mathbf{r}). \tag{1.4.47}$$

1.4.11 Potential Energy of Charge Distribution

Consider two charges q_1, q_2 located at $\mathbf{r}_1$ and $\mathbf{r}_2$. Choosing the potential produced by a particular charge q_j as vanishing at infinity, i.e. in the form

$$\Phi_j(\mathbf{r}) = \frac{q_j}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_j|} \ , \tag{1.4.48}$$

and using $U = q\Phi(\mathbf{r})$, we may write the potential energy due to electrostatic interaction of two charges as $q_1\Phi_2(\mathbf{r}_1)$ or $q_2\Phi_1(\mathbf{r}_2)$, which gives the same result

$$U_{12} = \frac{q_1 q_2}{4\pi\epsilon_0|\mathbf{r}_1-\mathbf{r}_2|} \ . \tag{1.4.49}$$

Take now a system of N point charges q_i located at $\mathbf{r}_i$, $i = 1, \ldots, N$. For any pair of charges q_i, q_j the potential energy U_{ij} of their electrostatic interaction is given by

$$U_{ij} = \frac{q_i q_j}{4\pi\epsilon_0|\mathbf{r}_i-\mathbf{r}_j|} \ . \tag{1.4.50}$$

Thus, we may write

$$U = \sum_{\text{all pairs } i,j} U_{ij} \ . \tag{1.4.51}$$

Evidently $j \neq i$. Then

$$U = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\frac{q_i q_j}{|\mathbf{r}_i-\mathbf{r}_j|} \ , \tag{1.4.52}$$

where the factor $1/2$ compensates for the fact that doing independent summation over i and j, we count each pair twice.

This result may be also obtained from the following construction. Note that for the case where Φ vanishes at infinity, the potential energy $U(\mathbf{r})$ is the work done, W, in bringing the charge q from infinity to the point $\mathbf{r}$. We will now consider the work done in assembling a set of point charges q_i at $\mathbf{r}_i$, $i = 1, \ldots, N$.

We do this by bringing each charge i in turn, one at a time, to position $\mathbf{r}_i$, and then fixing it in position. The work done in bringing charge i is

$$W_i = \frac{q_i}{4\pi\epsilon_0}\sum_{j=1}^{i-1}\frac{q_j}{|\mathbf{r}_i-\mathbf{r}_j|} \tag{1.4.53}$$

and thus the total work done in assembling the charges is

$$W = \frac{1}{4\pi\epsilon_0}\sum_{i=2}^{N}\sum_{j=1}^{i-1}\frac{q_i q_j}{|\mathbf{r}_i-\mathbf{r}_j|} = U \ , \tag{1.4.54}$$

where U is the potential energy of the system. In this sum we have $j < i$, so each pair is counted only once. We can write this in a more symmetric form as

$$U = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} \ , \tag{1.4.55}$$

where we do not include the **self-energy** term, $i = j$. An extra factor $1/2$ appears because each pair is now counted twice.
We can generalize this to a *continuous* charge distribution in the usual way, viz

$$U = \frac{1}{8\pi\epsilon_0} \int d^3r \, d^3r' \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}, \tag{1.4.56}$$

and we now use Eq.(1.4.32) to write

$$U = \frac{1}{2} \int \rho(\mathbf{r})\Phi(\mathbf{r}) dV \ , \tag{1.4.57}$$

analogous to Eq.(1.4.47).
We can also interpret the potential energy in terms of the *electric field*, by using **ME1**

$$\begin{aligned} U &= \frac{\epsilon_0}{2} \int dV \, \nabla \cdot \mathbf{E}(\mathbf{r})\Phi(\mathbf{r}) \\ &= -\frac{\epsilon_0}{2} \int dV \, \mathbf{E}(\mathbf{r}) \cdot \nabla\Phi(\mathbf{r}) \quad (\textit{Integration by parts}) \\ &= \frac{\epsilon_0}{2} \int dV \, |\mathbf{E}|^2. \end{aligned} \tag{1.4.58}$$

We now identify the integrand as the **energy density**

$$u(\mathbf{r}) = \frac{\epsilon_0}{2} |\mathbf{E}(\mathbf{r})|^2. \tag{1.4.59}$$

1.5 Laplace's and Poisson's Equation

We are now ready to derive a differential equation for the potential. Our starting point is **Maxwell's First Equation** (ME1), derived earlier:

$$\nabla \cdot \mathbf{E} = 4\pi k\rho = \frac{\rho}{\epsilon_0}. \tag{1.5.1}$$

We now make use of the irrotational nature of $\mathbf{E}(\mathbf{r})$ to write $\mathbf{E} = -\nabla\Phi(\mathbf{r})$. Thus ME1 becomes

$$\nabla^2\Phi(\mathbf{r}) = -4\pi k\rho(\mathbf{r}) = -\rho(\mathbf{r})/\epsilon_0 \quad \text{in SI units} \tag{1.5.2}$$

where $\nabla^2\Phi(\mathbf{r}) \equiv \nabla\cdot(\nabla\Phi(\mathbf{r})) \equiv \sum_i \partial^2\Phi(\mathbf{r})/\partial x_i^2$.

- This equation is **Poisson's Equation**.
  $\rho(\mathbf{r})$ is the **source** for the electrostatic potential $\Phi(\mathbf{r})$.
- If we have that the **source** $\rho(\mathbf{r}) \equiv 0$ everywhere, then this equation becomes

$$\nabla^2\Phi = 0. \tag{1.5.3}$$

  This is **Laplace's Equation**.

These are two of the most important equations in physics. They, or close variants, occur in:

- Electromagnetism, as above
- Gravitation, with $k \to -G$, ρ the *mass density*, and Φ the gravitational potential
- Fluid dynamics, for the *irrotational* flow of a fluid.

1.5.1 Uniqueness of Solutions of Laplace's and Poisson's Equation

Laplace's and Poisson's equations are *linear, second order, partial differential equations*; to determine a solution we have also to specify **boundary conditions**.
Example: One-dimensional problem

$$\frac{d^2\Phi(x)}{dx^2} = \lambda \tag{1.5.4}$$

for $x \in [0, L]$, where λ is a constant. This has solution

$$\Phi(x) = \frac{1}{2}\lambda x^2 + Ax + B \tag{1.5.5}$$

where A, B are constants. To determine these constants, we might specify the values of $\Phi(x=0)$ and $\Phi(x=L)$, i.e. the values on the boundary.

Consider the solution of Poisson's Equation within a finite volume V, bounded by a *closed* surface S. Boundary conditions are classified as:

- **Dirichlet** boundary conditions, where we require

$$\Phi(\mathbf{r}) = f(\mathbf{r}) \qquad \text{on surface } S, \tag{1.5.6}$$

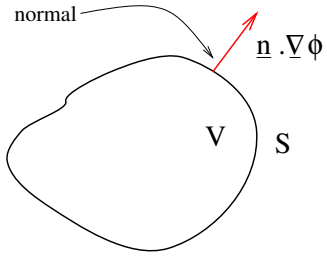
  i.e. we specify the *value* of $\Phi(\mathbf{r})$ on the boundary.
  Example: Electrostatic potential outside a *conductor*, with Φ specified on the boundaries.

- **Neumann** boundary conditions, where we require

$$\mathbf{n} \cdot \nabla\Phi(\mathbf{r}) = \frac{\partial\Phi}{\partial n} = g(\mathbf{r}) \qquad \text{on surface } S, \tag{1.5.7}$$

  where $\mathbf{n}$ is a unit vector normal to the surface S, i.e. we specify the *normal derivative* of Φ on the boundary.

Example: Electrostatic potential inside S, with *charge* on S specified on the boundaries.



We will proceed to show that the solutions of Laplace's and Poisson's equations are **unique**, up to a constant (*Neumann*), if subject to either of the above boundary conditions.

Green's First Identity and Green's Theorem

We begin with a couple of identities that will be useful both in this proof and later.
Let ψ_1 and ψ_2 be two continuously differentiable, arbitrary scalar fields defined in a volume V bounded by a closed surface S. Introduce the vector field $\mathbf{A}(\mathbf{r}) = \psi_1\nabla\psi_2$.
From the divergence theorem, we have

$$\int_V \nabla \cdot \mathbf{A}\, dV = \int_S \mathbf{A} \cdot \mathbf{n}\, dS \ , \tag{1.5.8}$$

where $\mathbf{n}$ is the unit *outward* normal to the surface S.
We now apply the vector identity

$$\nabla \cdot \mathbf{A} = \psi_1\nabla^2\psi_2 + \nabla\psi_1 \cdot \nabla\psi_2 \ , \tag{1.5.9}$$

to obtain

$$\int_V \left(\psi_1 \nabla^2 \psi_2 + \nabla\psi_1 \cdot \nabla\psi_2\right)\, dV = \int \psi_1 \nabla\psi_2 \cdot \mathbf{n}\, dS\ . \tag{1.5.10}$$

This is known as **Green's first identity**.
If we write down Eq.(1.5.10) with ψ_1 and ψ_2 interchanged,

$$\int_V \left(\psi_2 \nabla^2 \psi_1 + \nabla\psi_2 \cdot \nabla\psi_1\right)\, dV = \int \psi_2 \nabla\psi_1 \cdot \mathbf{n}\, dS\ , \tag{1.5.11}$$

and take the difference of the two equations, we obtain

$$\int_V \left(\psi_1 \nabla^2 \psi_2 - \psi_2 \nabla^2 \psi_1\right)\, dV = \int_S \left(\psi_1 \nabla\psi_2 - \psi_2 \nabla\psi_1\right) \cdot \mathbf{n}\, dS\ . \tag{1.5.12}$$

This identity is **Green's Theorem**.

1.5.2 Proof of Uniqueness of Solutions of Laplace's and Poisson's Equations

We now proceed to the formal proof. Let $\Phi_1(\mathbf{r})$ and $\Phi_2(\mathbf{r})$ be solutions of Poisson's equation $\nabla^2\Phi_i = -\rho/\epsilon_0$ inside a volume V bounded by surface S, satisfying **either**:

1. Dirichlet boundary conditions

$$\Phi_i(\mathbf{r}) = f(\mathbf{r}) \qquad \text{for } \mathbf{r} \text{ on surface } S\ ; \tag{1.5.13}$$

2. Neumann boundary conditions

$$\mathbf{n} \cdot \nabla\Phi_i(\mathbf{r}) = g(\mathbf{r}) \qquad \text{for } \mathbf{r} \text{ on surface } S\ ; \tag{1.5.14}$$

where $f(\mathbf{r})$ and $g(\mathbf{r})$ are continuous functions defined on the surface S.
Consider the function

$$\psi(\mathbf{r}) = \Phi_1(\mathbf{r}) - \Phi_2(\mathbf{r}). \tag{1.5.15}$$

Then ψ satisfies *Laplace's* equation:

$$\nabla^2\psi(\mathbf{r}) = 0 \qquad \text{in } V \tag{1.5.16}$$

with **either**

1. $\psi(\mathbf{r}) = 0$ for $\mathbf{r}$ on surface S – **Dirichlet**.

2. $\mathbf{n} \cdot \nabla\psi(\mathbf{r}) = 0$ for $\mathbf{r}$ on surface S – **Neumann**

We now apply *Green's first identity* for the case $\psi_1 = \psi_2 = \psi$,

$$\int_V \left(\psi\nabla^2\psi + \nabla\psi \cdot \nabla\psi\right)\, dV = \int \psi\nabla\psi \cdot \mathbf{n}\, dS\ , \tag{1.5.17}$$

and obtain

$$\begin{aligned} \int_V |\nabla\psi|^2\, dV &= \int_V (\psi\nabla^2\psi + |\nabla\psi|^2)\, dV \qquad (\text{since } \nabla^2\psi = 0 \text{ in } V) \\ &= \int_S \psi\nabla\psi \cdot \mathbf{n}\, dS \qquad (\text{from Eq.(1.5.17)}) \\ &= 0\ , \end{aligned} \tag{1.5.18}$$

since either $\psi(\mathbf{r}) = 0$ or $\nabla\psi \cdot \mathbf{n} = 0$ on surface S. Now $|\nabla\psi(\mathbf{r})|^2$ is **positive definite**, i.e.

$$|\nabla\psi(\mathbf{r})|^2 \geq 0 \tag{1.5.19}$$

for all $\mathbf{r} \in V$. Therefore, using equation (1.5.18), we have that $\nabla\psi(\mathbf{r}) = 0$ everywhere in V, and thus

$$\psi(\mathbf{r}) = \text{constant} \tag{1.5.20}$$

for all $\mathbf{r} \in V$.
Thus we have

- **Dirichlet Problem**: $\psi(\mathbf{r})$ is continuous at surface S, and $\psi(\mathbf{r}) = 0$ on the surface. Therefore $\psi(\mathbf{r}) = 0$ everywhere, and solution is **unique**.
- **Neumann Problem**: $\nabla\psi(\mathbf{r}) \cdot \mathbf{n} = 0$ on the surface S, and the constant undetermined. Solution is **unique up to an additive constant**.

Some observations on the proof:

- We can specify **either** Dirichlet or Neumann boundary conditions at each point on the boundary, but not **both**. To specify both is inconsistest, since the solution is then overdetermined.
- However, we can specify either Dirichlet **or** Neumann boundary conditions on **different** parts of the surface.

- The uniqueness property means we can use any method we wish to obtain the solution – if it satisfies the correct boundary conditions, and is a solution of the equation, then it is the correct solution. A good example: *Method of Images*, to be covered in the next chapter.

1.5.3 Uniqueness Theorem in an Infinite Region

We need a slight refinement of the proof if the region is infinite, i.e. if S contains a "surface at infinity". If the potential is due to a **localized** charge distribution, then it contains a part that falls off as $(Q/r)/(4\pi\epsilon_0)$ when $r \to \infty$, where Q is total charge of the distribution. This part vanishes at infinity. But recall that the potential Φ is defined up to an additive constant, so we can also add a constant term. So, a necessary condition for uniqueness is that one should specify the value Φ_∞ of $\Phi(\mathbf{r})$ at infinity. However, we should also specify the rate at which $\Phi(\mathbf{r})$ approaches Φ_∞ when r tends to infinity.
A sufficient condition for uniqueness of solutions of Poisson's equation is that the potential $\Phi(\mathbf{r})$ approaches Φ_∞ as const$/r$. So, if we assume now that there are two solutions $\Phi_1(\mathbf{r})$ and $\Phi_2(\mathbf{r})$, having the same Φ_∞ , their difference $\psi = \Phi_1 - \Phi_2$ vanishes at least as $O(1/r)$ when $r \to \infty$.
Let us show that then the surface integral vanishes at infinity.

Consider a **sphere**, radius r, area $S = 4\pi r^2$. Suppose

$$\psi = \Phi_1 - \Phi_2 = O(1/r) \quad \text{as} \quad r \to \infty\,, \qquad \text{so that} \qquad \nabla\psi = O(1/r^2)\,.$$

Then $$\int_S \psi\,\nabla\psi\cdot\mathbf{dS} = O(1/r)$$

which vanishes as $r \to \infty$. Thus, the volume integral of $|\nabla\psi|^2$ vanishes, hence $\nabla\psi = 0$ and $\psi(\mathbf{r})$ is constant in the whole space. Since this constant becomes zero at infinity, we conclude that $\psi(\mathbf{r}) = 0$ everywhere, i.e. the solution for $\Phi(\mathbf{r})$ is unique.
Sometimes a **uniform field** is specified at infinity. For example, if the uniform field $\mathbf{E}$ is in the z direction, then

$$\Phi(\mathbf{r}) = K - E\,z \tag{1.5.21}$$

where K is a constant. In this case, the uniqueness theorem holds because the 'two' solutions must satisfy the boundary condition

$$\Phi(\mathbf{r}) + E\,z \to K + O(1/r) \tag{1.5.22}$$

as $r \to \infty$.

The next couple of chapters of this course will be concerned with solving such *boundary-value* problems. We will conclude this chapter by discussing the boundary conditions to impose on our solutions, and in particular the boundary conditions at a conductor.

1.6 Boundary Conditions at a Conductor

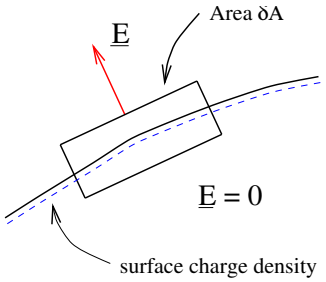
- In a conductor, electrons are able to move freely so as to set up a charge distribution.
- In the presence of an external electrostatic field, a charge distribution is generated under the influence of this field, and itself gives rise to an electrostatic field.
- Once equilibrium is attained (about 10^{-18} secs. for a good conductor), there are no current flows, and thus *the electric field* $\mathbf{E}$ *is zero throughout the body of a conductor.*
- If the electric field vanishes in a conductor, the potential must be constant. This provides the defining property of a conductor, namely that the boundary of a conductor is an equipotential surface.

> On the boundary of a conductor, $\Phi(\mathbf{r}) = \text{const.}$

- Conventionally, we take $\Phi = 0$ for a *grounded* conductor.
- The electric field at the surface of a conductor is **normal** to the surface; a tangential field would give rise to a charge flow along the surface.

1.6.1 Surface Charge Density at a Conductor

Within a conductor, the electrostatic field $\mathbf{E}$ must be zero. However, the field is zero because of an induced charge density sufficient to annul the external field.
Now ME1 tells us that $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$, where ρ is the charge density. Thus if $\mathbf{E}$ is zero within the conductor, the charge density must be zero. So where does the induced charge density reside?
The charge density is confined solely to the surface of the conductor
We can compute this *surface charge density* using Gauss' Law.



Consider applying Gauss' Law to the infinitesimal "pill-box" of height δh and area δA, as shown. Within the conductor, $\mathbf{E} = 0$, and at the surface of the conductor $\mathbf{E}$ is *normal* to the surface.

Therefore we have

$$\mathbf{E} \cdot \mathbf{n}\, \delta A = \delta A\, \sigma/\epsilon_0 \tag{1.6.1}$$

where σ is the surface charge density, and $\mathbf{n}$ is the outward normal to the surface of the conductor.

Thus we have that the *surface charge density* is proportional to the **discontinuity** in the normal electrostatic field at the conductor.

$$\boxed{\mathbf{E} \cdot \mathbf{n} = \sigma/\epsilon_0}$$

Note: the **surface charge density** discussed here is different to a **sheet** of charge of density σ per unit area discussed earlier in the course. The latter may best be viewed as a charge distribution in an insulator, i.e. a fixed charge distribution. Unfortunately, the two terms are often confused in the literature.

1.6.2 Capacitance and Potential Energy of Conductors

Consider now a set of N isolated conductors, with charge q_i, $i = 1 \ldots N$, and with no external electric field. Then each conductor is an equipotential Φ_i, and the charges reside on the surface of the conductor.

Thus the potential energy of this system is

$$U = \frac{1}{2} \int dV\, \rho(\mathbf{r}) \Phi(\mathbf{r}) = \sum_i \frac{1}{2}\, q_i\, \Phi_i. \tag{1.6.2}$$

The potentials Φ_i and the charges q_i are not independent. In particular, for a given set of charges q_i the potentials are determined by the solutions of the field equations. Because of the linearity of the field equations, the relationship between the Φ's and the q's must be linear, i.e.

$$\Phi_i = \sum_{j=1}^{N} P_{ij} q_j, \tag{1.6.3}$$

which in matrix form may be written as

$$\overrightarrow{\Phi} = \hat{P}\vec{q}. \tag{1.6.4}$$

We can invert this equation to obtain

$$q_i = \sum_{j=1}^{N} C_{ij}\Phi_j \tag{1.6.5}$$

where, formally, $\hat{C} = \hat{P}^{-1}$.
The diagonal elements of this matrix C_{ii} are the **capacitances**, whilst the off-diagonal elements $C_{ij}, i \neq j$ are the **coefficients of induction**. We can use Eq.(1.6.5) to write the *potential energy* of a system of conductors in terms either of the potentials or charges alone:

$$U = \frac{1}{2}\sum_{ij} \Phi_i C_{ij} \Phi_j = \frac{1}{2}\sum_{ij} q_i C_{ij}^{-1} q_j \tag{1.6.6}$$