Contents

Chapter 3

Boundary-value Problems in Curvilinear Coordinates

In the previous chapter, we saw how we could look for factorizable solutions to Laplace's Equation in Cartesian coordinates, and then construct the solution for more general boundary values using the completeness property of such factorized solutions. In this chapter we will employ analogous methods in *spherical polar* and *cylindrical* coordinate systems. In practice, the coordinate system that is appropriate depends on the *symmetry* or *geometry* of the problem.

3.1 Laplace's Equation in Spherical Polar Coordinates

We will denote our coordinates by (r, θ, φ), in terms of which Laplace's equation assumes the form

$$\nabla^2\Phi(r,\theta,\varphi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}\ . \tag{3.1.1}$$

We will now seek *factorizable* solutions of the form

$$\Phi(r,\theta,\varphi) = \frac{U(r)}{r}P(\theta)Q(\varphi)\ , \tag{3.1.2}$$

where the factor of $1/r$ is conventional. Substituting this into Laplace's equation, we have

$$\begin{aligned} &P(\theta)Q(\varphi)\frac{1}{r^2}\frac{d}{dr}\left[r^2\left(-\frac{1}{r^2}U(r) + \frac{1}{r}\frac{dU(r)}{dr}\right)\right] \\ &+ \quad \frac{U(r)Q(\varphi)}{r}\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP(\theta)}{d\theta}\right) + \frac{U(r)P(\theta)}{r}\frac{1}{r^2\sin^2\theta}\frac{d^2Q(\varphi)}{d\varphi^2} = 0\ , \end{aligned}$$

yielding

$$\frac{PQ}{r}\frac{d^2U}{dr^2}+\frac{UQ}{r^3\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\frac{UP}{r^3\sin^2\theta}\frac{d^2Q}{d\varphi^2}=0\ , \tag{3.1.3}$$

which we may write as

$$\frac{1}{Q}\frac{d^2Q}{d\varphi^2}+r^2\sin^2\theta\left[\frac{1}{U}\frac{d^2U}{dr^2}+\frac{1}{r^2\sin\theta}\frac{1}{P}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)\right]=0\ . \tag{3.1.4}$$

The first term is a function of φ alone, and the remaining term is a function of (r, θ) only. Thus they must be separately constant, and we may write

$$\frac{1}{Q}\frac{d^2Q}{d\varphi^2}=-m^2\ , \tag{3.1.5}$$

where m is a constant. Eq.(3.1.5) has solution

$$Q=e^{\pm im\varphi}\ . \tag{3.1.6}$$

We now observe that the solution must be periodic, with period 2π, in the azimuthal variable φ. Thus m must be a real **integer** number. Hence we may write Eq.(3.1.4) as

$$\frac{r^2}{U}\frac{d^2U}{dr^2}+\frac{1}{\sin\theta}\frac{1}{P}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)-\frac{m^2}{\sin^2\theta}=0\ . \tag{3.1.7}$$

We now observe that the first term is purely a function of r, whilst the remaining terms are purely a function of θ. Thus we may write

$$\frac{r^2}{U}\frac{d^2U}{dr^2}=l(l+1)\ , \tag{3.1.8}$$

where l is a constant – we will see the reason for expressing the constant in this way later. To solve this equation, we will take a trial solution

$$U(r)=r^\alpha, \tag{3.1.9}$$

yielding

$$\alpha(\alpha-1)=l(l+1) \tag{3.1.10}$$

with solutions $\alpha = l + 1, -l$. Thus we have

$$\boxed{U(r)=Ar^{l+1}+Br^{-l}\ , \qquad (3.1.11)}$$

or

$$\boxed{\frac{U(r)}{r} = Ar^l + Br^{-l-1}\ . \qquad (3.1.12)}$$

As we will see later on, l is integer. Thus, taking $l \geq 0$, we get for the r-dependent factor $U(r)/r$ all nonnegative integer powers of r from the r^l term and all negative integer powers of r from the r^{-l-1} term. Alternatively, taking a negative integer $l = -L-1$ (with nonnegative integer $L \geq 0$) we have $r^l \to r^{-L-1}$ and $r^{-l-1} \to r^L$, and we get again all integer powers of r as possible solutions. Hence, for definiteness, we may take $l \geq 0$.

The equation for the polar coordinate θ now assumes the form

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right] P = 0\ . \tag{3.1.13}$$

It is convenient to introduce the variable $x = \cos\theta$, with $-1 \leq x \leq 1$. Then

$$\frac{1}{\sin\theta}\frac{d}{d\theta} = -\frac{d}{dx} \tag{3.1.14}$$

and

$$\sin\theta\frac{d}{d\theta} = -(1-x^2)\frac{d}{dx}\ . \tag{3.1.15}$$

Thus we have

$$\boxed{\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right] P = 0\ . \qquad (3.1.16)}$$

This is the **Generalized Legendre Equation**, and is, once again, an equation of *Sturm-Liouville* type, with $p(x) = 1-x^2$, $q(x) = -m^2/(1-x^2)$, $\lambda = l(l+1)$, and $r(x) = 1$.

We will now seek solutions of this equation, first for the case $m = 0$, where the equation is known as the **Ordinary Legendre Equation**

$$\boxed{\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + l(l+1)P = 0\ . \qquad (3.1.17)}$$

We begin by noting that the solutions must be both **continuous** and **single-valued** in the region $-1 \le x \le 1$, corresponding to $0 \le \theta \le \pi$. We will obtain the solutions through **series substitution**, i.e. by trying a solution of the form

$$P = \sum_{n=0}^{\infty} c_n x^{\gamma+n} , \tag{3.1.18}$$

from which

$$\begin{aligned}
\frac{dP}{dx} &= \sum_{n=0}^{\infty} c_n(\gamma+n)x^{\gamma+n-1} \\
(1-x^2)\frac{dP}{dx} &= \sum_{n=0}^{\infty} c_n(\gamma+n)x^{\gamma+n-1} - \sum_{n=0}^{\infty} c_n(\gamma+n)x^{\gamma+n+1} , \\
\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] &= \sum_{n=0}^{\infty} c_n(\gamma+n)(\gamma+n-1)x^{\gamma+n-2} - \sum_{n=0}^{\infty} c_n(\gamma+n)(\gamma+n+1)x^{\gamma+n} .
\end{aligned}$$

Thus Legendre's equation becomes

$$\sum_{n=0}^{\infty} c_n(\gamma+n)(\gamma+n-1)x^{\gamma+n-2} + \sum_{n=0}^{\infty} c_n\left[l(l+1) - (\gamma+n)(\gamma+n+1)\right]x^{\gamma+n} = 0 . \tag{3.1.19}$$

The first sum here contains two terms, $x^{\gamma-2}$ and $x^{\gamma-1}$ (corresponding to $n = 0$ and $n = 1$, respectively) which are absent in the second sum. Writing them separately and shifting $n \to n+2$ for the remaining terms of the first sum, we have

$$\begin{aligned}
x^{\gamma-2}c_0\gamma(\gamma-1) + x^{\gamma-1}c_1(\gamma+1)\gamma &+ \sum_{n=0}^{\infty} c_n(\gamma+n+1)(\gamma+n+2)x^{\gamma+n} \\
&+ \sum_{n=0}^{\infty} c_n\left[l(l+1) - (\gamma+n)(\gamma+n+1)\right]x^{\gamma+n} = 0 .
\end{aligned} \tag{3.1.20}$$

As this equation must be valid $\forall x \in [-1, 1]$, we can equate the coefficients of the powers of x to zero. The lowest power of x is $x^{\gamma-2}$, and we use this equation, the **indicial equation**, to determine γ. Thus

- $x^{\gamma-2}$:

$$c_0\gamma(\gamma-1) = 0 \implies \gamma = 0 \ \text{ or } \ \gamma = 1 \tag{3.1.21}$$

- $x^{\gamma-1}$:

$$c_1(\gamma+1)\gamma = 0 \implies \begin{cases} \gamma = 0 : & c_1 \text{ undetermined} \\ \gamma = 1 : & c_1 = 0 \end{cases} \tag{3.1.22}$$

- $x^{\gamma+n}$, $n \geq 0$:

$$c_{n+2} = \frac{(\gamma+n)(\gamma+n+1) - l(l+1)}{(\gamma+n+1)(\gamma+n+2)}\, c_n \,. \tag{3.1.23}$$

Thus, specifying c_0, we get c_2, c_4, etc. from the recurrence relation. Note that the resulting power series involves only even powers of x for $\gamma = 0$ and only odd powers of x for $\gamma = 1$. In case of $\gamma = 0$, we may also specify c_1, and the recurrence relation will generate c_3, c_5, etc., i.e. a series involving odd powers of x. Thus, we can generate a series with odd powers of x in two ways.

First, we may take $\gamma = 1$ and start from c_0. This gives a series $c_0 x + c_2 x^3 + \ldots$, with

$$c_2 = \frac{\gamma(\gamma+1) - l(l+1)}{(\gamma+1)(\gamma+2)}\, c_0 \to \frac{2 - l(l+1)}{2\cdot 3}\, c_0 \,. \tag{3.1.24}$$

Second, we may take $\gamma = 0$ and start from c_1. This gives a series $c_1 x + c_3 x^3 + \ldots$, with

$$c_3 = \frac{(\gamma+1)(\gamma+2) - l(l+1)}{(\gamma+2)(\gamma+3)}\, c_1 \to \frac{2 - l(l+1)}{2\cdot 3}\, c_1 \,. \tag{3.1.25}$$

So, it looks like the two ways give us the same series, up to an overall factor determined by the coefficient of x^1.

Now we notice that the coefficient c_n accompanies the power term $x^{n+\gamma}$. Hence, denoting $k \equiv n + \gamma$, we may write

$$P_l(x) = \sum_{k=\gamma,\gamma+2,\gamma+4,\ldots} p_k x^k \tag{3.1.26}$$

($p_k = c_{k-\gamma}$) with summation over even or odd k (depending on γ) and the common recurrence relation

$$p^{(l)}_{k+2} = \frac{k(k+1) - l(l+1)}{(k+1)(k+2)}\, p^{(l)}_k = \frac{(k-l)(k+l+1)}{(k+1)(k+2)}\, p^{(l)}_k \,. \tag{3.1.27}$$

In particular,

$$p^{(l)}_2 = \frac{(0-l)(l+1)}{1\cdot 2}\, p^{(l)}_0 \;,\; p^{(l)}_3 = \frac{(1-l)(l+2)}{2\cdot 3}\, p^{(l)}_1 \;, \tag{3.1.28}$$

$$p^{(l)}_4 = \frac{(2-l)(l+3)}{3\cdot 4}\, p^{(l)}_2 \;,\; p^{(l)}_5 = \frac{(3-l)(l+4)}{4\cdot 5}\, p^{(l)}_4 \,, \tag{3.1.29}$$

etc., which gives $p^{(2)}_2/p^{(2)}_0 = -3$, $p^{(3)}_3/p^{(3)}_1 = -5/3$.

We have already noted that the solution must be valid for $x \in [-1, 1]$, and in particular at the end points $x = \pm 1$. Thus the series must be finite at $x = \pm 1$. To explore the convergence properties, we note that

$$p_{k+2}/p_k \longrightarrow 1 \text{ as } k \longrightarrow \infty \,, \tag{3.1.30}$$

and thus the series resembles a geometrical expansion $\sum x^{2k}$. This series diverges at $x = \pm 1$ unless the series terminates, i.e. unless $p_{k+2} = 0$ for some k. Thus our requirement for convergence is

$$k(k+1) - l(l+1) = 0 \text{ for some } k \ , \tag{3.1.31}$$

or

$$(k-l)(k+l+1) = 0 \text{ for some } k \ , \tag{3.1.32}$$

which holds for $k = l$ if l is nonnegative, and for $k = -l-1$ if l is negative. In the latter case, we write it as $l = -L-1$, and get $k = -l-1 = L \geq 0$ as a solution. Thus, as we have discussed already, we may take nonnegative l without a loss of generality.

Since k is an integer number, the Legendre series (3.1.26) becomes a polynomial only if l is an integer number. Then $p^{(l)}_{l+2} = 0 \cdot p^{(l)}_{l} = 0$, i.e., the highest power of x is x^l. We call the corresponding solutions

$$P_l(x) = \sum_{k=\gamma,\gamma+2,\gamma+4,\ldots}^{l} p_k x^k \tag{3.1.33}$$

the **Legendre Polynomials**. The overall factor in the definition of $P_l(x)$ is fixed by the requirement $P_l(1) = 1$. Using the recurrence relations (3.1.27) and imposing this condition, we find the first few Legendre Polynomials to be

$$\begin{aligned} P_0(x) &= 1 \ , \\ P_1(x) &= x \ , \\ P_2(x) &= \frac{1}{2}(3x^2 - 1) \ , \\ P_3(x) &= \frac{1}{2}(5x^3 - 3x) \ . \end{aligned}$$

3.1.1 Rodriques' Formula and Generating Function

We can write the Legendre polynomials in a more memorable form through **Rodrigues' Formula**:

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l. \tag{3.1.34}$$

Another useful way of determining the Legendre polynomials is through the **generating function**

$$g(t,x) \equiv (1 - 2xt + t^2)^{-1/2} = \sum_{l=0}^{\infty} P_l(x)\, t^l \ , \quad |t| < 1 \ . \tag{3.1.35}$$

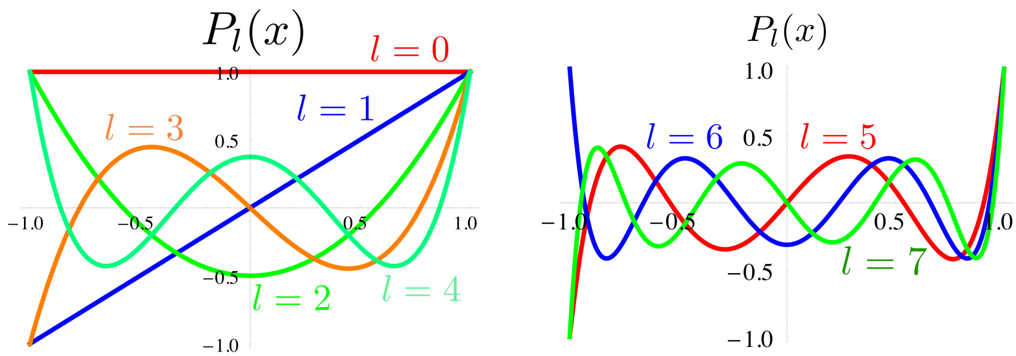


Figure 3.1: Legendre polynomials for $l = 0, 1, 2, 3, 4$ (left) and $l = 5, 6, 7$ (right).

One can use the generating function approach to get Legendre polynomials in *Mathematica*. In particular, the command

`Series[1/Sqrt[1 − 2tx + t^2], {t, 0, 10}]`

gives the first 11 Legendre polynomials $P_l(x)$ as factors accompanying t^l in

$$1 + xt + \frac{1}{2}\left(3x^2 - 1\right)t^2 + \frac{1}{2}x\left(5x^2 - 3\right)t^3 + \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)t^4$$
$$+\frac{1}{8}x\left(63x^4 - 70x^2 + 15\right)t^5 + \frac{1}{16}\left(231x^6 - 315x^4 + 105x^2 - 5\right)t^6$$
$$+\frac{1}{16}x\left(429x^6 - 693x^4 + 315x^2 - 35\right)t^7$$
$$+\frac{1}{128}\left(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35\right)t^8$$
$$+\frac{1}{128}x\left(12155x^8 - 25740x^6 + 18018x^4 - 4620x^2 + 315\right)t^9$$
$$+\frac{1}{256}\left(46189x^{10} - 109395x^8 + 90090x^6 - 30030x^4 + 3465x^2 - 63\right)t^{10} + O\left(t^{11}\right) \quad (3.1.36)$$

The first 8 Legendre polynomials are shown in Fig. 3.1. One can see that for large l, in the middle region, they are getting closer and closer to periodic oscillating functions jumping, however, to their ± 1 values at the end-points. In fact, we can check analytically that the generating function produces correct normalization of the Legendre polynomials for $x = 1$. We have

$$g(t, x = 1) = (1 - 2t + t^2)^{-1/2} = \frac{1}{1 - t} = \sum_{l=0}^{\infty} t^l \ , \quad (3.1.37)$$

hence $P_l(1) = 1$ for all l. Similarly, for $x = -1$, we have

$$g(t, x = -1) = (1 + 2t + t^2)^{-1/2} = \frac{1}{1 + t} = \sum_{l=0}^{\infty} (-1)^l t^l \ , \quad (3.1.38)$$

hence $P_l(-1) = (-1)^l$ for all l.

3.1.2 Orthogonality and Normalization of Legendre Polynomials

We recall that the Legendre equation is of Sturm-Liouville type, with $p(x) = 1 - x^2$ and $r(x) = 1$. Since $p(x)$ vanishes for $x = \pm 1$, the solutions of the Legendre equation satisfy the boundary conditions

$$\left[p(x) \left(\psi_\lambda^* \frac{d\psi_{\lambda'}}{dx} - \psi_{\lambda'} \frac{d\psi_\lambda^*}{dx} \right) \right]_a^b = 0. \tag{3.1.39}$$

required by the orthogonality theorem, which states in this case that

$$[l(l-1) - l'(l'+1)] \int_{-1}^{1} dx\, P_l(x) P_{l'}(x) = 0 \Longrightarrow \int_{-1}^{1} dx\, P_l(x) P_{l'}(x) = 0,\ l \neq l', \tag{3.1.40}$$

i.e. the Legendre polynomials are *orthogonal* with the simplest possible weight $r(x) = 1$. To determine their normalization, we can use either Rodrigues' formula, or the generating function; we use the latter. From Eq.(3.1.35), we have

$$\begin{aligned} \int_{-1}^{1} dx\, [g(t,x)]^2 &= \int_{-1}^{1} dx \frac{1}{1 - 2xt + t^2} = \left\{ -\frac{1}{2t} \ln(1 - 2xt + t^2) \right\}_{-1}^{1} \\ &= -\frac{1}{2t} \ln \frac{(1-t)^2}{(1+t)^2} = 2 \sum_{l=0}^{\infty} \frac{t^{2l}}{2l+1}\ , \end{aligned}$$

where we have used the series expansion of $\ln(1 + t)$. However, we also have

$$\int_{-1}^{1} dx\, [g(t,x)]^2 \ = \ \sum_{l,l'=0}^{\infty} \int dx\, P_l(x) P_{l'}(x) t^{l+l'} = \sum_{l=0}^{\infty} t^{2l} \int_{-1}^{1} dx\, P_l(x)^2\ .$$

On the last step we used orthogonality of Legendre polynomials, i.e. the fact that only the $l' = l$ term is nonzero in the sum over l'. Equating the coefficients in these two expansions yields

$$2 \sum_{l=0}^{\infty} \frac{t^{2l}}{2l+1} \ = \ \sum_{l=0}^{\infty} t^{2l} \int_{-1}^{1} dx\, P_l(x)^2$$

and finally

$$\boxed{\int_{-1}^{1} dx\, P_l(x) P_l(x) = \frac{2}{2l+1}\ . \qquad (3.1.41)}$$

We can also combine the orthogonality property and normalization of the Legendre poynomials into one relation

$$\boxed{\int_{-1}^{1} dx\, P_l(x) P_{l'}(x) = \frac{2}{2l+1}\, \delta_{ll'} \ . \qquad (3.1.42)}$$

3.1.3 Recurrence Relations

Rodrigues' formula provides a means to obtain various **recurrence relations** between the Legendre Polynomials, for example:

$$\begin{aligned}
(l+1)P_{l+1}(x) - (2l+1)xP_l(x) + lP_{l-1}(x) &= 0 \\
\frac{d}{dx}P_{l+1}(x) - x\frac{dP_l(x)}{dx} - (l+1)P_l(x) &= 0 \\
(x^2-1)\frac{dP_l(x)}{dx} - lxP_l(x) + lP_{l-1}(x) &= 0 \\
\frac{d}{dx}P_{l+1}(x) - \frac{d}{dx}P_{l-1}(x)dx - (2l+1)P_l(x) &= 0.
\end{aligned}$$

Such recurrence relations would allow us to evaluate many of the integrals that will be encountered in the problems.

3.1.4 Completeness

Since the Legendre Polynomials form a complete set, we may write any function $f(x)$, $x \in [-1, 1]$ as

$$f(x) = \sum_{l=0}^{\infty} A_l P_l(x). \qquad (3.1.43)$$

We obtain the coefficients using the orthogonality relations

$$\begin{aligned}
\int_{-1}^{1} dx\, f(x) P_l(x) &= \sum_{l'=0}^{\infty} A_l \int_{-1}^{1} dx\, P_l(x) P_{l'}(x) \\
&= A_l \frac{2}{2l+1} \ ,
\end{aligned}$$

whence

$$A_l = \frac{2l+1}{2} \int_{-1}^{1} dx\, f(x) P_l(x) \ . \qquad (3.1.44)$$

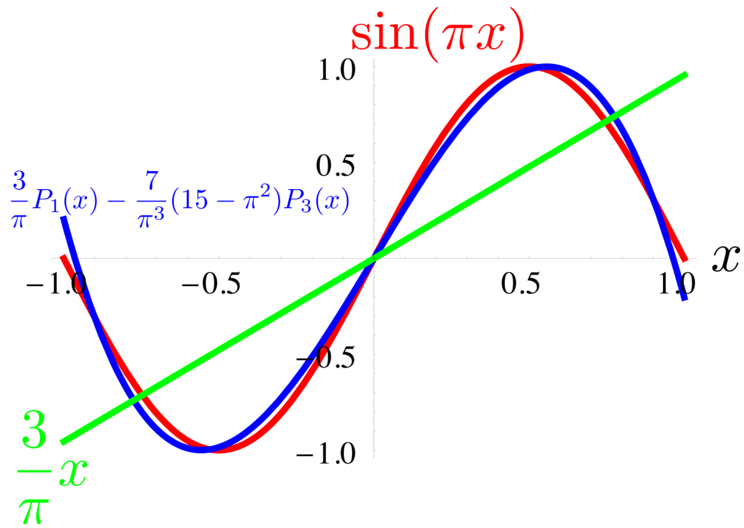


Figure 3.2: Approximation of $\sin(\pi x)$ by one and two Legendre polynomials.

Examples of Legendre expansion

To give an illustration of the expansion in Legendre polynomials, let us consider the function $f(x) = \sin(\pi x)$. It is an odd function of x, hence only $l =$odd contribute. We have

$$A_l = \frac{2l+1}{2}\int_{-1}^{1} dx\, \sin(\pi x)\, P_l(x)\ , \tag{3.1.45}$$

or

$$A_1 = \frac{3}{2}\int_{-1}^{1} dx\, x\, \sin(\pi x) = \frac{3}{2}\times\frac{2}{\pi} = \frac{3}{\pi} \tag{3.1.46}$$

and

$$A_3 = \frac{7}{2}\int_{-1}^{1} dx\, \frac{1}{2}\left(5x^3 - 3x\right)\sin(\pi x) = \frac{7}{2}\times\frac{2}{\pi^3}(\pi^2 - 15) = -\frac{7}{\pi^3}(15-\pi^2)\ . \tag{3.1.47}$$

The first term $3x/\pi$ of the Legendre expansion

$$\sin(\pi x) = \frac{3}{\pi}\, P_1(x) - \frac{7}{\pi^3}(15-\pi^2)\, P_3(x) + \ldots \tag{3.1.48}$$

looks rather remote from the expanded function (see Fig. 3.2). However, after adding the $l = 3$ term, we get an approximation that reproduces $\sin(\pi x)$ very closely.

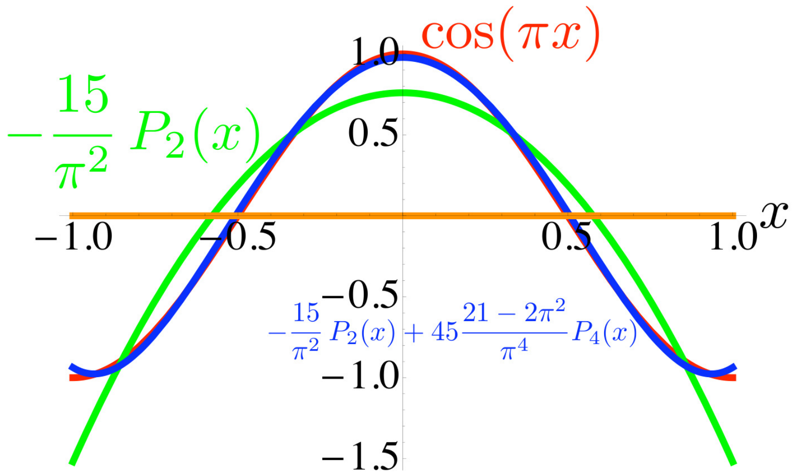


Figure 3.3: Approximation of $\cos(\pi x)$ by one, two and three Legendre polynomials.

For comparison, let us also construct the Legendre expansion for the function $f(x) = \cos(\pi x)$. It is an even function of x, hence only l =even contribute. Now we have

$$A_l = \frac{2l+1}{2} \int_{-1}^{1} dx \, \cos(\pi x) \, P_l(x) \ , \tag{3.1.49}$$

or

$$A_0 = \frac{1}{2} \int_{-1}^{1} dx \, \cos(\pi x) = \frac{1}{2} \times 0 = 0 \ . \tag{3.1.50}$$

The explanation of this outcome is simple: we integrate $\cos(\pi x)$ over its full periodicity region, as a result, the integral vanishes. For the next two coefficients we have

$$A_2 = \frac{5}{2} \int_{-1}^{1} dx \, \frac{5}{2} \left(3x^2 - 1\right) \cos(\pi x) = \frac{5}{2} \times \left(-\frac{6}{\pi^2}\right) = -\frac{15}{\pi^2} \tag{3.1.51}$$

and

$$A_4 = \frac{9}{2} \int_{-1}^{1} dx \, \frac{1}{8} \left(35x^4 - 30x^2 + 3\right) \cos(\pi x) = \frac{9}{2} \times \frac{10}{\pi^4}(21 - 2\pi^2) = \frac{45}{\pi^4}(21 - 2\pi^2) \ . \tag{3.1.52}$$

The first term of the Legendre expansion

$$\cos(\pi x) = 0 - \frac{15}{\pi^2} P_2(x) + \frac{45}{\pi^4}(21 - 2\pi^2) \, P_4(x) + \dots \tag{3.1.53}$$

equals zero now, and is very remote from the expanded function (see Fig. 3.3). The second term $-\frac{15}{\pi^2} P_2(x)$, however, is rather close to it, and after adding the third term we get the approximation curve which is almost indistinguishable from the curve for $\cos(\pi x)$.

Legendre expansion of antisymmetric step function

Consider the step-function $f(x)$ defined by

$$f(x) = \begin{cases} 1 & , \ 0 < x \le 1 \\ -1 & , \ -1 \le x < 0 \end{cases} \tag{3.1.54}$$

Then we have

$$\begin{aligned} A_l &= \frac{2l+1}{2} \int_{-1}^{1} dx f(x) P_l(x) \\ &= \frac{2l+1}{2} \left\{ \int_0^1 dx\, P_l(x) - \int_{-1}^{0} dx\, P_l(x) \right\} \\ &= \frac{2l+1}{2} \int_0^1 dx \left\{ P_l(x) - P_l(-x) \right\}. \end{aligned}$$

Thus we see that A_l is non-zero only for l odd:

$$A_l = \begin{cases} (2l+1) \int_0^1 dx\, P_l(x) & : l \text{ odd} \\ 0 & : l \text{ even} \end{cases} \tag{3.1.55}$$

Now by the last of our recurrence relations

$$\begin{aligned} A_l &= \int_0^1 dx \left\{ \frac{d}{dx} P_{l+1}(x) - \frac{d}{dx} P_{l-1}(x) \right\} \\ &= P_{l+1}(1) - P_{l+1}(0) - P_{l-1}(1) + P_{l-1}(0) \\ &= P_{l-1}(0) - P_{l+1}(0) \end{aligned}$$

where we have used the normalization condition $P_l(1) = 1$. But we have (from Rodrigues's formula, with a little work)

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l. \tag{3.1.56}$$

For even l

$$P_l(0) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (-1)^{l/2} x^l \left. \frac{l!}{((l/2)!)^2} \right|_{x=0} = \frac{1}{2^l} (-1)^{l/2} \frac{l!}{((l/2)!)^2} \tag{3.1.57}$$

$$P_l(0) = \begin{cases} \frac{1}{2^l} (-1)^{l/2} \frac{l!}{((l/2)!)^2} & : l \text{ even} \\ 0 & : l \text{ odd} \end{cases} . \tag{3.1.58}$$

Thus $P_0(0) = 1$, $P_2(0) = -\frac{1}{2^2} \frac{2!}{(1!)^2} = -\frac{1}{2}$,

$$P_4(0) = -\frac{1}{2^4} \frac{4!}{(2!)^2} P_2(0) = \frac{1}{2} \frac{3 \cdot 4}{2^2} = -\frac{3}{4} P_2(0) = \frac{3}{8} \ ,$$

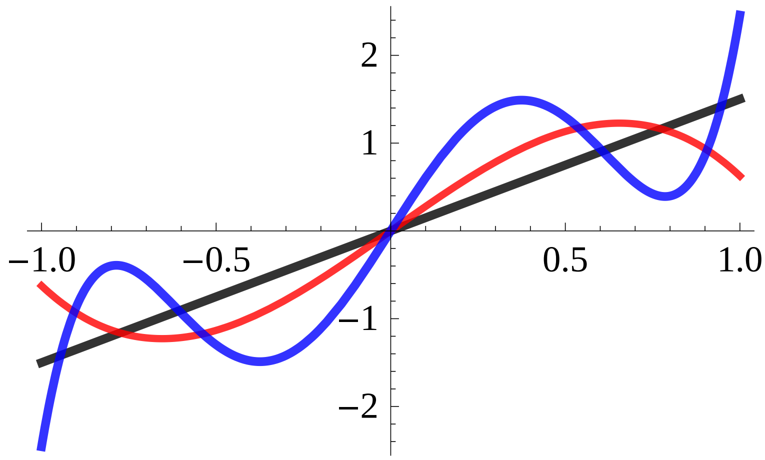


Figure 3.4: Approximation of step function by one, two and three Legendre polynomials.

$$P_6(0) = -\frac{1}{2^2}\frac{6\cdot 5}{3^2}P_4(0) = -\frac{5}{6}P_4(0) = -\frac{5}{16}, \text{etc.}$$

Since $P_{l-1}(0) - P_{l+1}(0)$, we have $A_1 = P_0(0) - P_2(0) = \frac{3}{2}$, $A_3 = P_2(0) - P_4(0) = -\frac{7}{8}$, $A_5 = P_4(0) - P_6(0) = -\frac{11}{16}$, etc.

Finally, we obtain

$$f(x) = \frac{3}{2}P_1(x) - \frac{7}{8}P_3(x) + \frac{11}{16}P_5(x) + \dots \tag{3.1.59}$$

3.2 Boundary-Value Problems with Azimuthal Symmetry

We may now write our general solution for the boundary-value problem in spherical coordinates with azimuthal symmetry, i.e. no φ dependence, as

$$\boxed{\Phi(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-l-1}\right)P_l(\cos\theta), \qquad (3.2.1)}$$

where the coefficients A_l and B_l are determined from the boundary conditions.

Example:

Consider the case of a sphere, of radius a, with no charge inside but potential $V(\theta)$ specified on the surface.

Since there are no charges inside the sphere, the potential Φ **inside** must be regular everywhere. Thus $B_l = 0\ \forall l$, and we may write the solution as

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)\ . \tag{3.2.2}$$

Imposing the boundary conditions at $r = a$ yields

$$V(\theta) = \sum_{l=0}^{\infty} A_l\, a^l P_l(\cos\theta), \tag{3.2.3}$$

so that, using the normalization condition on the Legendre polynomials, we have

$$A_l = \frac{2l+1}{2a^l}\int_0^{\pi} d\theta\,\sin\theta\, V(\theta) P_l(\cos\theta). \tag{3.2.4}$$

Suppose now that we require the solution **outside** the sphere. Then the solution must be finite as $r \to \infty$, and thus

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} B_l r^{-l-1} P_l(\cos\theta) \tag{3.2.5}$$

with

$$V(\theta) = \sum_{l=0}^{\infty} B_l a^{-l-1} P_l(\cos\theta), \tag{3.2.6}$$

so that

$$B_l = \frac{2l+1}{2} a^{l+1}\int_0^{\pi} d\theta\,\sin\theta\, V(\theta) P_l(\cos\theta). \tag{3.2.7}$$

Let us now go back to the problem in Section 2.4.2:

$$V(\theta) = \begin{cases} V & : 0 \le \theta \le \pi/2 \\ -V & : \pi/2 \le \theta \le \pi \end{cases} \tag{3.2.8}$$

Then we have

$$\begin{aligned} B_l &= \frac{2l+1}{2} a^{l+1} V \left\{ \int_0^{\pi/2} P_l(\cos\theta)\sin\theta d\theta - \int_{-\pi/2}^{\pi} P_l(\cos\theta)\sin\theta d\theta \right\} \\ &= \frac{2l+1}{2} a^{l+1} V \left\{ \int_0^{1} dx P_l(x) - \int_{-1}^{0} dx P_l(x) \right\} \\ &= \frac{2l+1}{2} a^{l+1} V \left\{ \int_{-1}^{1} dx f(x) P_l(x) \right\} \end{aligned}$$

where

$$f(x) = \begin{cases} 1 & 0 < x \le 1 \\ -1 & -1 \le x < 0 \end{cases} \tag{3.2.9}$$

This is just the expression we evaluated in Section 3.1.4, and thus we have:

$$B_l = \begin{cases} V a^{l+1} (-\frac{1}{2})^{\frac{l-1}{2}} \frac{(l-2)!!(2l+1)}{2(\frac{l+1}{2})!} & l \text{ odd} \\ 0 & l \text{ even} \end{cases} \tag{3.2.10}$$

so that

$$\Phi(r,\theta) = V \left\{ \frac{3}{2}\frac{a^2}{r^2} P_1(\cos\theta) - \frac{7}{8}\frac{a^4}{r^4} P_3(\cos\theta) + \frac{11}{16}\frac{a^6}{r^6} P_5(\cos\theta) + \ldots \right\}. \tag{3.2.11}$$

Recall that in Section 2.4.2 we obtained

$$\begin{aligned} \Phi(r,\theta,\varphi) &= \frac{3Va^2}{2r^2} \left\{ \cos\theta - \frac{7a^2}{12r^2}\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right) + \mathcal{O}\left(\frac{a^4}{r^4}\right) \right\} \\ &= V \left\{ \frac{3}{2}\frac{a^2}{r^2} P_1(\cos\theta) - \frac{7}{8}\frac{a^4}{r^4} P_3(\cos\theta) + \ldots \right\}, \end{aligned}$$

which is precisely the first two terms in the expansion of Eq.(3.2.11).
The crucial observation in such problems is that the series expansion

$$\Phi(r,\theta) = \sum_l \left(A_l\, r^l + B_l\, r^{-l-1} \right) P_l(\cos\theta) \tag{3.2.12}$$

is unique. Thus it is possible to determine the coefficients A_l and B_l from a knowledge of the solution in some limited domain. As an illustration, we recall that we obtained a closed solution to the above problem above the north pole, i.e. at $\theta = 0$:

$$\Phi(z = r, \theta = 0) = V \left\{ 1 - \frac{r^2 - a^2}{r\sqrt{r^2 + a^2}} \right\}. \tag{3.2.13}$$

We can use the Taylor expansion to express this as a series in a/r. It may be obtained by *Mathematica*, if one looks for a Series expansion of the function

$$f(x) = 1 - \frac{1 - x^2}{\sqrt{1 + x^2}} \ . \tag{3.2.14}$$

The result is

$$f(x) = 1 - \frac{1 - x^2}{\sqrt{1 + x^2}} = \frac{3}{2}\, x^2 - \frac{7}{8}\, x^4 + \frac{11}{16}\, x^6 - \frac{75}{128}\, x^8 + \frac{133}{256}\, x^{10} + \ldots \ . \tag{3.2.15}$$

The same result may be obtained analytically using the binomial expansion:

$$\Phi(z=r,\theta=0) = V\left\{1-(1-a^2/r^2)\sum_{j=0}^{\infty}\frac{\Gamma(\frac{1}{2})}{\Gamma(j+1)\Gamma(\frac{1}{2}-j)}\left(\frac{a}{r}\right)^{2j}\right\}. \tag{3.2.16}$$

If we use the property

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z} \tag{3.2.17}$$

and note that $\Gamma(1/2)=\sqrt{\pi}$, we obtain, after a little manipulation (*exercise*),

$$\Phi(r,\theta=0) = \frac{V}{\sqrt{\pi}}\sum_{j=1}^{\infty}(-1)^{j-1}\frac{(2j-\frac{1}{2})\Gamma(j-\frac{1}{2})}{j!}\left(\frac{a}{r}\right)^{2j}. \tag{3.2.18}$$

We now compare this series solution with Eq.(3.2.12), evaluated at $\theta=0$, and observe that only terms with $l=2j-1$ enter, and that

$$B_{2j-1} = \frac{V}{\sqrt{\pi}}(-1)^{j-1}\frac{(2j-\frac{1}{2})\Gamma(j-\frac{1}{2})}{j!}a^{2j}. \tag{3.2.19}$$

Let us try the first couple of terms

$$\begin{aligned} j=1: \quad B_1 &= \tfrac{V}{\sqrt{\pi}}(-1)^0\tfrac{(3/2)\Gamma(1/2)}{1!}a^2 = 3Va^2/2 \\ j=2: \quad B_3 &= \tfrac{V}{\sqrt{\pi}}(-1)^1\tfrac{(5/2)\Gamma(3/2)}{2!}a^4 = -\tfrac{7}{8}Va^4, \end{aligned} \tag{3.2.20}$$

and once again we reproduce the expression Eq.(3.2.11).

3.2.1 Expansion of $\frac{1}{|\mathbf{x}-\mathbf{x}'|}$

We conclude this section by looking at the expansion of this critical quantity that occurs in the construction of the Green's function. We begin by observing that the result can depend only on r, r' and γ, the angle between $\mathbf{x}$ and $\mathbf{x}'$. We may thus simplify the problem by choosing the azimuthal direction (z axis) along the $\mathbf{x}'$ axis. The problem then displays manifest azimuthal symmetry, and we may write

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty}\left[A_l(r')r^l + B_l(r')r^{-l-1}\right]P_l(\cos\gamma) \tag{3.2.21}$$

We now consider the case where $\mathbf{x}$ lies parallel to $\mathbf{x}'$, when $\cos\gamma=1$. Then the l.h.s. of Eq.(3.2.21) becomes

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{|r-r'|}. \tag{3.2.22}$$

There are two cases:

$$r > r' \quad : \quad \frac{1}{|r-r'|} = \frac{1}{r-r'} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l = \sum_{l=0}^{\infty}\frac{r'^l}{r^{l+1}}$$

$$r < r' \quad : \quad \frac{1}{|r-r'|} = \frac{1}{r'-r} = \frac{1}{r'}\sum_{l=0}^{\infty}\left(\frac{r}{r'}\right)^l = \sum_{l=0}^{\infty}\frac{r^l}{(r')^{l+1}}$$

Let us introduce $r_> = \max(r, r')$ and $r_< = \min(r, r')$. Then we may write

$$\frac{1}{|r-r'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}} \tag{3.2.23}$$

and, comparing with Eq.(3.2.21), we have

$$\boxed{\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma) \quad (3.2.24)}$$

3.3 Solution of the Generalized Legendre Equation

Let us now consider the case where we no longer assume azimuthal symmetry. Then we are concerned with solutions of the *Generalized Legendre Equation*,

$$\frac{d}{dx}\left[(1-x^2)\frac{dP(x)}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P(x) = 0. \tag{3.3.1}$$

We can obtain a series solution in an analogous way to that of the *ordinary Legendre equation*. For solutions to be finite at $x = \pm 1$, corresponding to $\theta = 0, \pi$, we require that l must be a positive integer or zero, **and** that m takes the values

$$m = -l, -l+1, \ldots, l-1, l. \tag{3.3.2}$$

Recall that we already know that m must be an integer by the requirement that the azimuthal function $Q(\varphi)$ be single-valued.

For the case where m is *positive*, we can write the solutions $P_l^m(x)$ as

$$P_l^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_l(x) \tag{3.3.3}$$

or for both positive and negative m by adopting Rodrigues' formula:

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l. \tag{3.3.4}$$

Note that Eq.(3.3.1) depends only on m^2. Thus we have that $P_l^{-m}(x)$ must be proportional to $P_l^m(x)$, and in fact

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x). \tag{3.3.5}$$

Eq. (3.3.1) is an equation of Sturm-Liouville class, with eigenvalues $l(l+1)$. We can apply the orthogonality theorem *at fixed* m, and we have

$$\int_{-1}^{1} dx P_{l'}^m(x) P_l^m(x) = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{ll'}. \tag{3.3.6}$$

3.4 Spherical Harmonics

We began by looking at separable solutions in spherical polar coordinates, and writing

$$\Phi(r,\theta,\varphi) = \frac{1}{r} U(r) P(\theta) Q(\varphi). \tag{3.4.1}$$

Then we derived the equation

$$\frac{PQ}{r}\frac{d^2U}{dr^2} + \frac{UQ}{r^3 \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right) + \frac{UP}{r^3 \sin^2\theta}\frac{d^2Q}{d\varphi^2} = 0 \ , \tag{3.4.2}$$

which may be also written as

$$\frac{r^2}{U}\frac{d^2U}{dr^2} + \frac{1}{PQ}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) + \frac{1}{\sin^2\theta}\frac{d^2}{d\varphi^2}\right] PQ = 0 \ , \tag{3.4.3}$$

Since $U(r)$ satisfies Eq. (3.1.8)

$$\frac{r^2}{U}\frac{d^2U}{dr^2} = l(l+1) \ , \tag{3.4.4}$$

the angular part $Y(\theta,\varphi) \equiv P(\theta)Q(\varphi)$ satisfies

$$-\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) + \frac{1}{\sin^2\theta}\frac{d^2}{d\varphi^2}\right] Y \equiv -\nabla_\Omega^2 Y = l(l+1) Y \ . \tag{3.4.5}$$

Thus, it is convenient to combine the angular functions into solutions on the unit sphere:

$$Y_{lm}(\theta,\varphi) = \sqrt{\frac{(l-m)!(2l+1)}{4\pi(l+m)!}} P_l^m(\cos\theta) e^{im\varphi} \ . \tag{3.4.6}$$

The spherical harmonics (3.4.6) satisfy the equation

$$-\nabla_\Omega^2 Y_{lm}(\theta,\varphi) = l(l+1) Y_{lm}(\theta,\varphi) \tag{3.4.7}$$

or, in explicit form

$$\left[-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)-\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]Y_{lm}(\theta,\varphi)=l(l+1)Y_{lm}(\theta,\varphi)\ . \tag{3.4.8}$$

(A person familiar with quantum mechanics may recognize the expression in square brackets on lhs of this equation as a square of operator of anglular momentum L^2.)
Using the relation between $P_l^{-m}(\cos\theta)$ and $P_l^m(\cos\theta)$ we have

$$Y_{l,-m}(\theta,\varphi)=(-1)^m Y^*_{lm}(\theta,\varphi) \tag{3.4.9}$$

and the normalization condition is

$$\int_0^{2\pi}d\varphi\int_0^{\pi}Y^*_{lm}(\theta,\varphi)Y_{l'm'}(\theta,\varphi)\,\sin\theta\,d\theta\ =\delta_{ll'}\delta_{mm'}, \tag{3.4.10}$$

i.e.

$$\int d\Omega\,Y_{lm}(\theta,\varphi)Y^*_{l'm'}(\theta,\varphi)=\delta_{mm'}\delta_{ll'}. \tag{3.4.11}$$

For the case $m=0$, the solution clearly reduces to the Legendre polynomial, up to some normalization:

$$Y_{l0}(\theta,\varphi)=\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)\ . \tag{3.4.12}$$

3.4.1 Completeness

Any arbitrary function $g(\theta,\varphi)$ defined on $0\le\theta\le\pi$, $0\le\varphi\le 2\pi$ may be expressed in terms of Y_{lm}:

$$g(\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_{lm}Y_{lm}(\theta,\varphi)\ , \tag{3.4.13}$$

where

$$\begin{aligned}A_{lm}\ &=\ \int_0^{\pi}d\theta\,\sin\theta\int_0^{2\pi}g(\theta,\varphi)Y^*_{lm}(\theta,\varphi)\,d\varphi\\ &=\ \int d\Omega\,Y^*_{lm}(\theta,\varphi)g(\theta,\varphi)\ .\end{aligned}$$

3.4.2 General Solution

We can now write the general solution of the Laplace boundary value problem as

$$\Phi(r,\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_{lm}r^l+B_{lm}r^{-l-1}\right]Y_{lm}(\theta,\varphi)\ . \tag{3.4.14}$$

3.4.3 Addition Theorem for Spherical Harmonics

Consider two vectors $\mathbf{x}, \mathbf{x}'$, with coordinates (r, θ, φ) and (r', θ', φ') respectively. Let γ be the angle between $\mathbf{x}$ and $\mathbf{x}'$, so that

$$\cos\gamma = \frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}||\mathbf{x}'|} = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi') \ . \tag{3.4.15}$$

Then we have

$$\boxed{P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\varphi')Y_{lm}(\theta,\varphi) \ . \quad (3.4.16)}$$

This is proved in Jackson, but is more easily proved using group theory. Note that we can rewrite this in the form

$$P_l(\cos\gamma) = P_l(\cos\theta)P_l(\cos\theta') + 2\sum_{m=1}^{l}\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta)P_l^m(\cos\theta')\cos m(\varphi-\varphi') \tag{3.4.17}$$

Example

An important application is to the expansion of $\frac{1}{|\mathbf{x}-\mathbf{x}|}$, discussed in section 3.2.1:

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma). \tag{3.4.18}$$

Using the addition theorem, we can rewrite this as

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y^*_{lm}(\theta',\varphi')Y_{lm}(\theta,\varphi). \tag{3.4.19}$$

Superficially, this looks like a much more complicated expression, since we have introduced an additional sum over m. But it is now a sum over terms that factorize into a function of (θ, φ) and a function of (θ', φ'), and thus much more useful.

3.5 Expansion of Green Function in Spherical Polar Coordinates

The solutions found by separation of variables constituted complete sets of orthogonal functions satisfying the appropriate boundary conditions. This means that any function, and in particular the Green function, satisfying the same boundary conditions can be expanded as a series of these orthogonal functions.

3.5.1 Reminder: Green Functions

Green's theorem tells us that the potential $\Phi(\mathbf{x})$ related to the charge density $\rho(\mathbf{x}')$ by

$$\nabla'^2\Phi(\mathbf{x}') = -\rho(\mathbf{x}')/\epsilon_0. \tag{3.5.1}$$

can be written as

$$\boxed{\begin{aligned}\Phi(\mathbf{x}) = &\frac{1}{4\pi\epsilon_0}\int_V d^3x'\, G(\mathbf{x},\mathbf{x}')\rho(\mathbf{x}') \\ &+\frac{1}{4\pi}\int_{S=\partial V} dS'\,\left\{G(\mathbf{x},\mathbf{x}')\frac{\partial\Phi(\mathbf{x}')}{\partial n'} - \Phi(\mathbf{x}')\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n'}\right\} \quad (3.5.2)\end{aligned}}$$

The function $G(\mathbf{x},\mathbf{x}')$ is said to be a **Green function** for the problem, it is a function satisfying

$$\nabla'^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta^{(3)}(\mathbf{x}-\mathbf{x}'). \tag{3.5.3}$$

In general, it has the form

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} + F(\mathbf{x},\mathbf{x}'), \tag{3.5.4}$$

where $F(\mathbf{x},\mathbf{x}')$ is a solution of Laplace's equation

$$\nabla'^2 F(\mathbf{x},\mathbf{x}') = 0. \tag{3.5.5}$$

The utility of this representation is that we can choose $G(\mathbf{x},\mathbf{x}')$ so that the surface integral depends only on the prescribed boundary values of Φ (Dirichlet) or $\partial\Phi/\partial n'$ (Neumann).
In Dirichlet problem, the value of $\Phi(\mathbf{x}')$ is specified on the surface, and therefore it is natural to impose that the Green function $G_D(\mathbf{x},\mathbf{x}')$ satisfy

$$G_D(\mathbf{x},\mathbf{x}') = 0 \quad \text{for } \mathbf{x}' \text{ on } S, \tag{3.5.6}$$

and thus

$$\begin{aligned}\Phi(\mathbf{x}) =&\frac{1}{4\pi\epsilon_0}\int_V d^3x'\, G_D(\mathbf{x},\mathbf{x}')\rho(\mathbf{x}') - \frac{1}{4\pi}\int_S dS'\,\Phi(\mathbf{x}')\frac{\partial G_D(\mathbf{x},\mathbf{x}')}{\partial n'} \\ \equiv& \Phi_\rho(\mathbf{x}) + \Phi_S(\mathbf{x}) \ .\end{aligned} \tag{3.5.7}$$

Thus the surface integral only involves $\Phi(\mathbf{x}')$, and not the unknown $\partial\Phi(\mathbf{x}')/\partial n'$.
Recall that $G_D(\mathbf{x},\mathbf{x}')$ corresponds to the potential at point $\mathbf{x}$ of a point charge located at $\mathbf{x}'$, subject to the condition that the potential vanishes on the surface S. Thus, the first term, $\Phi_\rho(\mathbf{x})$, corresponds to the potential of the charge distribution $\rho(\mathbf{x}')$ in the presence of a grounded conducting surface S.

3.5.2 Green function for the Sphere in Spherical Harmonics

We have already seen that expansion of $1/|\mathbf{x}-\mathbf{x}'|$ may be written in terms of spherical harmonics, viz.

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi). \tag{3.5.8}$$

Suppose we wish to construct the Dirichlet Green function for the *outside* of a sphere of radius a. To get $G(\mathbf{x},\mathbf{x}') = 0$ for $\mathbf{x}'$ on the sphere, we use the *method of images*, and obtain

$$\begin{aligned} G(\mathbf{x},\mathbf{x}') &= \frac{1}{|\mathbf{x}'-\mathbf{x}|} - \frac{a}{r|\mathbf{x}'-\mathbf{x}a^2/r^2|} \\ &= 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)\left\{\frac{r_<^l}{r_>^{l+1}} - \frac{a}{r}\frac{(a^2/r)^l}{r'^{l+1}}\right\}, \end{aligned}$$

where we note that, for the *image charge*, $r_> = r', r_< = a^2/r$, since the image charge is *always inside* the sphere. Then we have

$$G(\mathbf{x},\mathbf{x}') = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)\left\{\frac{r_<^l}{r_>^{l+1}} - \frac{1}{a}\left(\frac{a^2}{rr'}\right)^{l+1}\right\}. \tag{3.5.9}$$

We have thus accomplished our goal of expressing the Green function as an expansion over orthogonal functions. There are some important observations we can make by looking at the radial part

$$\left\{\frac{r_<^l}{r_>^{l+1}} - \frac{1}{a}\left(\frac{a^2}{rr'}\right)^{l+1}\right\} = \begin{cases} \frac{1}{r'^{l+1}}\left[r^l - \frac{a^{2l+1}}{r^{l+1}}\right] & r < r' \\ \frac{1}{r^{l+1}}\left[r'^l - \frac{a^{2l+1}}{r'^{l+1}}\right] & r > r' \end{cases}. \tag{3.5.10}$$

- The radial part manifestly vanishes at $r = a$ and $r' = a$.
- It is symmetric under $r \leftrightarrow r'$.
- The solution is a linear combination of the solutions of Laplace's equation, regarded as a function of r' for fixed r, but a different linear combination for $r' > r$ and $r' < r$. We will see how this property arises below.

3.5.3 General construction of Green function in spherical coordinates

In spherical polars, the Green function satisfies

$$\nabla'^2 G(\mathbf{x}, \mathbf{x}') = -\frac{4\pi}{r'^2}\delta(r - r')\delta(\varphi - \varphi')\delta(\cos\theta - \cos\theta'), \tag{3.5.11}$$

where

$$\nabla'^2 = \frac{1}{r'^2}\frac{\partial}{\partial r'}\left(r'^2\frac{\partial}{\partial r'}\right) + \frac{1}{r'^2\sin\theta'}\frac{\partial}{\partial\theta'}\left(\sin\theta'\frac{\partial}{\partial\theta'}\right) + \frac{1}{r'^2\sin^2\theta'}\frac{\partial^2}{\partial\varphi'^2}\ , \tag{3.5.12}$$

or using the ${\nabla'_\Omega}^2$ notation

$$\nabla'^2 = \frac{1}{r'^2}\left[\frac{\partial}{\partial r'}\left(r'^2\frac{\partial}{\partial r'}\right) + {\nabla'_\Omega}^2\right]\ , \tag{3.5.13}$$

where

$${\nabla_\Omega}^2 \equiv \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) + \frac{1}{\sin^2\theta}\frac{d^2}{d\varphi^2}\ . \tag{3.5.14}$$

We will consider the case where we require the Green function over the full angular range $0 \le \theta' \le \pi$, $0 \le \varphi' \le 2\pi$. Thus we can expand the Green function, as a function of the *primed* variables with the unprimed variables fixed, in spherical harmonics:

$$G(\mathbf{x}, \mathbf{x}') = \sum_{l',m'} F_{l'm'}(r, \theta, \varphi; r')Y_{l'm'}(\theta', \varphi'). \tag{3.5.15}$$

Substituting this into the inhomogeneous equation we have

$$\begin{aligned}\frac{1}{r'^2}\sum_{l',m'}\left\{\frac{\partial}{\partial r'}\left[r'^2\frac{\partial F_{l'm'}}{\partial r'}\right]Y^*_{l'm'}(\theta', \varphi') + F_{l'm'}{\nabla'_\Omega}^2 Y^*_{l'm'}(\theta', \varphi')\right\}\\ = -\frac{4\pi}{r'^2}\delta(r - r')\delta(\varphi - \varphi')\delta(\cos\theta - \cos\theta').\end{aligned} \tag{3.5.16}$$

Now the spherical harmonics are solutions of Laplace's equation on the unit sphere, and, from Eq.(3.1.4), satisfy

$${\nabla'_\Omega}^2 Y^*_{l'm'}(\theta', \varphi') + l'(l'+1)Y^*_{l'm'}(\theta', \varphi') = 0. \tag{3.5.17}$$

Thus our Green function equation becomes (after canceling $1/r'2$)

$$\begin{aligned}\sum_{l'm'}\left\{\frac{\partial}{\partial r'}\left[r'^2\frac{\partial F_{l'm'}}{\partial r'}\right] - F_{l'm'}l'(l'+1)\right\}Y^*_{l'm'}(\theta', \varphi')\\ = -4\pi\delta(r - r')\delta(\varphi - \varphi')\delta(\cos\theta - \cos\theta').\end{aligned} \tag{3.5.18}$$

We now multiply by $Y_{lm}(\theta', \varphi')$, and use the orthogonality properties of the spherical harmonics:

$$\begin{aligned}
&\frac{\partial}{\partial r'}\left[r'^2 \frac{\partial F_{lm}}{\partial r'}\right] - \frac{F_{lm}}{r'^2} l(l+1) \\
&= -4\pi \int d\Omega' \, Y_{lm}(\theta', \varphi')\delta(r - r')\delta(\varphi - \varphi')\delta(\cos\theta - \cos\theta') \\
&= -4\pi \, Y_{lm}(\theta, \varphi)\delta(r - r'). \qquad (3.5.19)
\end{aligned}$$

Thus, the angular dependence of F_{lm} is given by $Y_{lm}(\theta, \varphi)$, and we may write

$$F_{lm}(r, \theta, \varphi; r') = g_l(r, r')Y_{lm}(\theta, \varphi) \qquad (3.5.20)$$

where $g_l(r, r')$ satisfies

$$\frac{d}{dr'}\left(r'^2 \frac{d}{dr'} g_l(r, r')\right) - l(l+1)g_l(r, r') = -4\pi\delta(r - r'). \qquad (3.5.21)$$

This is just the radial part of Laplace's equation. To proceed further, we must specifiy boundary conditions.

3.5.4 Green Function for the Sturm-Liouville Equation

We wish to determine the Green function to the equation

$$\frac{d}{dx'}\left[p(x')\frac{dg(x, x')}{dx'}\right] + q(x')g(x, x') = -4\pi\delta(x - x'), \qquad (3.5.22)$$

defined on the interval $x' \in [a, b]$, with homogeneous boundary conditions at a and b. Note that we regard x as some arbitrary, fixed parameter.
The Green function must possess the following properties:

1. For $x' \neq x$, $g(x, x')$ satisfies the homogeneous equation, i.e. the Sturm-Liouville equation with no source on the r.h.s..

2. $g(x, x')$ satisfies the homogeneous boundary condition at $x' = a$ and $x' = b$, e.g. $g(x, a) = 0$ and $g(x, b) = 0$.

3. $g(x, x')$ must be continuous at $x' = x$. This is subtle; otherwise dg/dx' would contain a δ-function, and d^2g/dx'^2 would contain the *derivative* of a δ-function at $x' = x$, which is more singular than the r.h.s. of the equation.

To see what happens at $x' = x$, we integrate the equation from $x - \epsilon$ to $x + \epsilon$:

$$\int_{x-\epsilon}^{x+\epsilon} dx' \left\{ \frac{d}{dx'} \left[p(x') \frac{dg(x,x')}{dx'} \right] + q(x')g(x,x') \right\} = -4\pi \int_{x-\epsilon}^{x+\epsilon} dx' \, \delta(x - x'), \tag{3.5.23}$$

leading to

$$\left[p(x') \frac{dg(x,x')}{dx'} \right]_{x-\epsilon}^{x+\epsilon} + \int_{x-\epsilon}^{x+\epsilon} dx' \, q(x')g(x,x') = -4\pi. \tag{3.5.24}$$

Both $q(x')$ and $g(x,x')$ are finite (and even continuous) at $x' = x$, and therefore we have

$$\lim_{\epsilon \to 0} \int_{x-\epsilon}^{x+\epsilon} dx' \, q(x')g(x,x') = 0, \tag{3.5.25}$$

and we may write

$$\lim_{\epsilon \to 0} \left[p(x') \frac{dg(x,x')}{dx'} \right]_{x-\epsilon}^{x+\epsilon} = -4\pi. \tag{3.5.26}$$

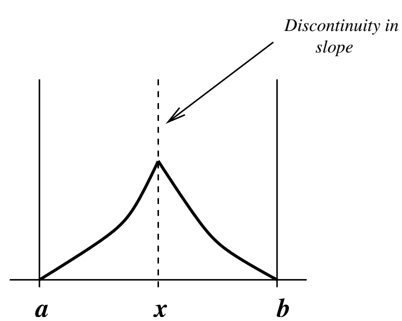
The function $p(x')$ is also continuous at $x' = x$, and thus

$$p(x) \lim_{\epsilon \to 0} \left\{ \frac{dg(x, x' = x + \epsilon)}{dx'} - \frac{dg(x, x' = x - \epsilon)}{dx'} \right\} = -4\pi \ , \tag{3.5.27}$$

which we write as

$$\delta \left[\frac{dg(x,x')}{dx'} \right]_{x'=x} = -\frac{4\pi}{p(x)} \, , \tag{3.5.28}$$

i.e. there is a discontinuity in the slope of the Green function of magnitude $4\pi/p(x)$ at $x' = x$.



Thus we will write our Green function as

- $a \leq x' \leq x$:

$$g(x,x') = C_1(x) y_1(x'), \tag{3.5.29}$$

  where $y_1(x')$ is a solution of the homogeneous equation satisfying the appropriate boundary condition at $x' = a$.

- $x \le x' \le b$:

$$g(x,x') = C_2(x) y_2(x') \tag{3.5.30}$$

where $y_2(x')$ is a solution of the homogeneous equation satisfying the appropriate boundary condition at $x' = b$.

We now impose the conditions on the Green function at $x' = x$

- $g(x,x')$ continuous at $x' = x$:

$$C_1(x) y_1(x) - C_2(x) y_2(x) = 0 \tag{3.5.31}$$

- Discontinuity in slope is $-4\pi/p(x)$:

$$C_2(x) y_2'(x) - C_1(x) y_1'(x) = -\frac{4\pi}{p(x)} \tag{3.5.32}$$

From Eq.(3.5.31), we have

$$C_2(x) = \frac{C_1(x) y_1(x)}{y_2(x)}. \tag{3.5.33}$$

Substituting into Eq.(3.5.32), we find

$$\begin{aligned}
\frac{C_1(x) y_1(x) y_2'(x)}{y_2(x)} - C_1(x) y_1'(x) &= -\frac{4\pi}{p(x)} \\
\Rightarrow C_1(x) &= -\frac{4\pi}{p(x)} \frac{y_2(x)}{W[y_1(x), y_2(x)]}, \\
\Rightarrow C_2(x) &= -\frac{4\pi}{p(x)} \frac{y_1(x)}{W[y_1(x), y_2(x)]},
\end{aligned}$$

where the W is the **Wronskian** (named after a Polish-French mathematician J. Wroński),

$$W[y_1(x), y_2(x)] = y_1(x) y_2'(x) - y_2(x) y_1'(x). \tag{3.5.34}$$

Note that this method only works if y_1 and y_2 are *linearly independent*, since otherwise the Wronskian vanishes.

Thus our general form for the Green function is

$$g(x,x') = \begin{cases} -\dfrac{4\pi}{p(x)} \dfrac{y_2(x) y_1(x')}{W[y_1(x), y_2(x)]}, & a \le x' \le x \;; \\ -\dfrac{4\pi}{p(x)} \dfrac{y_1(x) y_2(x')}{W[y_1(x), y_2(x)]}, & x \le x' \le b \;. \end{cases} \tag{3.5.35}$$

So, the Green function in the regions $x' < x$ and $x' > x$ is given by two different, linearly independent solutions of the homogeneous equation.

One can show that the combination $p(x)W[y_1(x), y_2(x)]$ in fact does not depend on x, if $y_1(x), y_2(x)$ are two independent solutions of the Sturm-Liouville equation

$$\frac{d}{dx}\left[p(x)y'(x)\right] + q(x)y(x) = 0 \ . \tag{3.5.36}$$

Indeed, let us write this equation for $y_1(x)$ and multiply it by $y_2(x)$,

$$y_2(x)\ \left[\frac{d}{dx}\left[p(x)y_1'(x)\right] + q(x)y_1(x)\right] = 0 \ , \tag{3.5.37}$$

and then write the equation for $y_2(x)$ and multiply it by $y_1(x)$,

$$y_1(x)\ \left[\frac{d}{dx}\left[p(x)y_2'(x)\right] + q(x)y_2(x)\right] = 0 \ . \tag{3.5.38}$$

The difference of these two expressions gives

$$y_2(x)\frac{d}{dx}\left[p(x)y_1'(x)\right] - y_1(x)\frac{d}{dx}\left[p(x)y_2'(x)\right] = 0 \ . \tag{3.5.39}$$

Integrating this equation over x from some fixed point c in the interval $[a, b]$ to some point z inside the same interval gives, after using integration by parts,

$$\left[y_2(x)p(x)y_1'(x) - y_1(x)p(x)y_2'(x)\right]_c^z - \int_c^z \left[y_2'(x)p(x)y_1'(x) - y_1'(x)p(x)y_2'(x)\right]dx = 0 \ . \tag{3.5.40}$$

Since the integrand of the x-integral above vanishes, we have

$$y_2(z)p(z)y_1'(z) - y_1(z)p(z)y_2'(z) = y_2(c)p(c)y_1'(c) - y_1(c)p(c)y_2'(c) \ , \tag{3.5.41}$$

or

$$-p(z)W[y_1(z), y_2(z)] = -p(c)W[y_1(c), y_2(c)] \ , \tag{3.5.42}$$

i.e., the combination $p(z)W[y_1(z), y_2(z)]$ is a constant independent of z. As a result, we may write $g(x, x')$ as

$$g(x, x') = \begin{cases} C\, y_2(x)y_1(x') \ , & a \le x' \le x \ ; \\ C\, y_1(x)y_2(x') \ , & x \le x' \le b \ . \end{cases} \tag{3.5.43}$$

where $C \equiv -4\pi/\{p(x)W[y_1(x), y_2(x)]\}$ is a constant. Even more compact form is

$$g(x, x') = C\, y_1(x_<)\, y_2(x_>) \ , \tag{3.5.44}$$

where $x_<$ ($x_>$) is smaller (larger) of x, x'.

3.5.5 Dirichlet Green Function between Spheres at $r = a$ and $r = b$

We require $g_l(r, r')$ subject to the boundary conditions $g_l(r, a) = g_l(r, b) = 0$.

1. $a \le r' \le r$: The solution $y_1(r')$ of the homogeneous equation must satisfy $y_1(a) = 0$. Now the general solution is of the form

$$y_1(r') = A_1 r'^l + B_1 r'^{-l-1}, \tag{3.5.45}$$

and thus we have

$$A_1 a^l + B_1 a^{-l-1} = 0 \;\Rightarrow\; B_1 = -A_1 a^{2l+1} \tag{3.5.46}$$

yielding

$$y_1(r') = A_1 \left[r'^l - \frac{a^{2l+1}}{r'^{l+1}} \right]. \tag{3.5.47}$$

2. $r \le r' \le b$: Then the solution $y_2(r')$ of the homogeneous equation must satisfy $y_2(b) = 0$, and thus we have

$$A_2 b^l + B_2 b^{-l-1} = 0 \;\Rightarrow\; A_2 = -B_2 b^{-2l-1} \tag{3.5.48}$$

yielding

$$y_2(r') = B_2 \left[\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}} \right] \tag{3.5.49}$$

We now construct the Wronskian

$$\begin{aligned} W[y_1(r), y_2(r)] &= y_1(r) y_2'(r) - y_2(r) y_1'(r) && (3.5.50) \\ &= A_1 \left[r^l - \frac{a^{2l+1}}{r^{l+1}} \right] B_2 \left[-(l+1)\frac{1}{r^{l+2}} - l\frac{r^{l-1}}{b^{2l+1}} \right] && (3.5.51) \\ &- B_2 \left[\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}} \right] A_1 \left[l r^{l-1} + (l+1)\frac{a^{2l+1}}{r^{l+2}} \right] && (3.5.52) \\ &= -A_1 B_2 \frac{2l+1}{r^2} \left[1 - \frac{a^{2l+1}}{b^{2l+1}} \right]. && (3.5.53) \end{aligned}$$

One can see that r^{2l-1} and $1/r^{2l+3}$ terms canceled, and only $1/r^2$ terms remained. Noting that $p(r) = r^2$, we observe that, once again, the product of $p(r)$ and Wronskian $W[y_1(r), y_2(r)]$

is independent of the evaluation point, and we have general solution

$$g_l(r,r') = \begin{cases} -4\pi \dfrac{A_1\left(r'^l - \dfrac{a^{2l+1}}{r'^{l+1}}\right) B_2\left(\dfrac{1}{r^{l+1}} - \dfrac{r^l}{b^{2l+1}}\right)}{-A_1 B_2(2l+1)\left(1-\dfrac{a^{2l+1}}{b^{2l+1}}\right)}; & a \le r' \le r \le b \\ -4\pi \dfrac{A_1\left(r^l - \dfrac{a^{2l+1}}{r^{l+1}}\right) B_2\left(\dfrac{1}{r'^{l+1}} - \dfrac{r'^l}{b^{2l+1}}\right)}{-A_1 B_2(2l+1)\left(1-\dfrac{a^{2l+1}}{b^{2l+1}}\right)}; & a \le r \le r' \le b \end{cases}, \tag{3.5.54}$$

which we may write in the more compact form

$$g_l(r,r') = \frac{4\pi}{2l+1}\left(1-\frac{a^{2l+1}}{b^{2l+1}}\right)^{-1}\left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right), \tag{3.5.55}$$

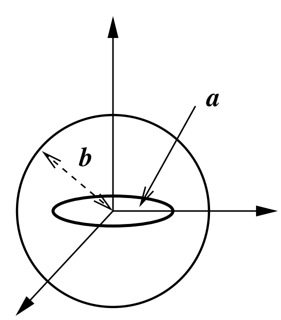
and hence

$$G(\mathbf{x},\mathbf{x}') = \sum_{l,m} g_l(r,r') Y_{lm}^*(\theta',\varphi') Y_{lm}(\theta,\varphi). \tag{3.5.56}$$

Note that it is also possible to recover this result using the *method of images*, but in this case an infinite number of image charges are required.

Example:

Consider the potential inside a grounded, conducting sphere of radius b, due to a uniform ring of charge of radius $a < b$, and total charge Q, lying in the plane through the equator, and centered at the center of the sphere.



We can obtain the Green function by taking the $a \to 0$ limit of Eq.(3.5.56):

$$G(\mathbf{x},\mathbf{x}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1} r_<^l \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right) Y_{lm}^*(\theta',\varphi') Y_{lm}(\theta,\varphi)\ . \tag{3.5.57}$$

The potential is then given by

$$\begin{aligned} \Phi(\mathbf{x}) =& \frac{1}{4\pi\epsilon_0}\int_V' d^3x'\, G(\mathbf{x},\mathbf{x}')\rho(\mathbf{x}') - \frac{1}{4\pi}\int_{S=\partial V} dS'\, \Phi(\mathbf{x}') \frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n'} \\ \equiv& \Phi_\rho(\mathbf{x}) + \Phi_S(\mathbf{x})\ . \end{aligned} \tag{3.5.58}$$

In our case, the surface integral $\Phi_S(\mathbf{x})$ vanishes, because the potential vanishes there. However, it is instructive to calculate this contribution for a situation when the potential on this spherical surface is given by some nontrivial function $V(\theta', \varphi')$. We can write it as an expansion over spherical harmonics:

$$V(\theta', \varphi') = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} V_{lm} Y_{lm}(\theta', \varphi') \ . \tag{3.5.59}$$

The coefficients V_{lm} can be obtained using the orthogonality relation for spherical harmonics (3.4.11),

$$\begin{aligned} V_{lm} &= \int_0^{\pi} d\theta' \sin\theta' \int_0^{2\pi} V(\theta', \varphi') Y_{lm}^*(\theta', \varphi') \\ &= \int d\Omega' \, Y_{lm}^*(\theta', \varphi') V(\theta', \varphi') \ . \end{aligned}$$

To calculate the derivative $\partial G(\mathbf{x}, \mathbf{x}')/\partial n'$ of the Green function on the surface S (i.e., for $r' = b$, when r' takes the largest possible value), we should take $r_< = r$ and $r_> = r'$ in Eq. (3.5.57):

$$G(\mathbf{x}, \mathbf{x}')|_{r' \to b} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} r^l \left(\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}} \right) Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi) \ . \tag{3.5.60}$$

Then

$$-\frac{1}{4\pi} \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'} \bigg|_S = -\frac{1}{4\pi} \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial r'} \bigg|_{r'=b} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{r^l}{b^{l+2}} Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi) \ . \tag{3.5.61}$$

Using $dS' = b^2 d\Omega'$ and the orthogonality of spherical harmonics, we obtain

$$\Phi_S(\mathbf{x}) \equiv -\frac{1}{4\pi} \int_{S=\partial V} dS' \, \Phi(\mathbf{x}') \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{r^l}{b^l} V_{lm} \, Y_{lm}(\theta, \varphi) \ . \tag{3.5.62}$$

We see that, for $r = b$, the potential $\Phi_S(\mathbf{x})$ is given by $V(\theta, \varphi)$. For $r < b$, the $Y_{lm}(\theta, \varphi)$ spherical component of $\Phi_S(\mathbf{x})$ is accompanied by the $(r/b)^l$ factor, i.e., to get the potential one should expand $V(\theta, \varphi)$ in spherical harmonics and attach the $(r/b)^l$ factor to each $Y_{lm}(\theta, \varphi)$. The result for $\Phi_S(\mathbf{x})$ depends only on the boundary conditions function $V(\theta, \varphi)$: it is universal for any charge distribution inside S.

Let us now return to our original problem of finding potential of a uniformly charged ring inside a grounded sphere. The (linear) charge density in this case is given by

$$\rho(\mathbf{x}') = \frac{Q}{2\pi a^2} \delta(r' - a) \delta(\cos\theta') \ . \tag{3.5.63}$$

Exercise: verify that the total charge is indeed Q.
Thus the potential is

$$\begin{aligned}\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int d\varphi' d(\cos\theta') dr'\, r'^2 \frac{Q}{2\pi a^2}\delta(r'-a)\delta(\cos\theta') \\ \times \quad 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1} Y^*_{lm}(\theta',\varphi') Y_{lm}(\theta,\varphi)\, r_<^l \left\{ \frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}} \right\} \ .\end{aligned}$$

In this case we have azimuthal symmetry, and the only non-vanishing integrals arise from the terms with $m = 0$, for which

$$Y_{l0}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta). \tag{3.5.64}$$

Thus we have

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\int dr' r'^2 \frac{Q}{a^2}\delta(r'-a) \sum_{l=0}^{\infty} P_l(0) P_l(\cos\theta)\, r_<^l \left\{ \frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}} \right\} \\ &= \frac{Q}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{2^n n!}\, r_<^{2n} \left\{ \frac{1}{r_>^{2n+1}} - \frac{r_>^{2n}}{b^{4n+1}} \right\} P_{2n}(\cos\theta) \ ,\end{aligned}$$

where we have used

$$\begin{aligned}P_{2n+1}(0) &= 0 \\ P_{2n}(0) &= \frac{(-1)^n (2n-1)!!}{2^n n!} \ .\end{aligned}$$

Here, $r_< = \min(r,a)$, $r_> = \max(r,a)$; hence, the radial dependence is given by $r^{2n}(1/a^{2n+1} - a^{2n}/b^{4n+1})$ for $r < a$ and by $a^{2n}(1/r^{2n+1} - r^{2n}/b^{4n+1})$ for $a < r < b$.
It is instructive to write this result as a function of ratios r/b, r/a and a/b:

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{Q}{4\pi\epsilon_0 a} \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{2^n n!} P_{2n}(\cos\theta) \\ &\times \left[\left(\frac{r}{a}\right)^{2n} \left\{ 1 - \left(\frac{a}{b}\right)^{4n+1} \right\} \theta(0 \le r \le a) + \left(\frac{a}{r}\right)^{2n+1} \left\{ 1 - \left(\frac{r}{b}\right)^{4n+1} \right\} \theta(a \le r \le b) \right] \ .\end{aligned} \tag{3.5.65}$$

The first terms of the expansion are given by

$$\begin{aligned}\Phi(\mathbf{x}) = \left[\frac{Q}{4\pi\epsilon_0}\right] \Bigg\{ &\left[\left(\frac{1}{a} - \frac{1}{b}\right) \theta(0 \le r \le a) + \frac{a}{r}\left\{1 - \frac{r}{b}\right\} \theta(a \le r \le b) \right] \\ &- \frac{1}{2a} P_2(\cos\theta) \left[\left(\frac{r}{a}\right)^2 \left\{ 1 - \left(\frac{a}{b}\right)^5 \right\} \theta(0 \le r \le a) + \left(\frac{a}{r}\right)^3 \left\{ 1 - \left(\frac{r}{b}\right)^5 \right\} \theta(a \le r \le b) \right] \\ &+ \frac{3}{8a} P_4(\cos\theta) \left[\left(\frac{r}{a}\right)^4 \left\{ 1 - \left(\frac{a}{b}\right)^9 \right\} \theta(0 \le r \le a) + \left(\frac{a}{r}\right)^5 \left\{ 1 - \left(\frac{r}{b}\right)^9 \right\} \theta(a \le r \le b) \right] \\ &+ \ldots \Bigg\} \ .\end{aligned}$$

Let us examine this result.
At the origin, i.e. for $r = 0$, only the $n = 0$ works, and we have $\Phi(r = 0) = (Q/4\pi\epsilon_0)[1/a - 1/b]$. Here, the $Q/4\pi\epsilon_0 a$ term is generated by the ring: total charge Q, all located distance a from the origin. The induced charge on the sphere should be $-Q$, since there is no field outside the sphere. This charge (all located distance b from the origin) produces potential $-Q/4\pi\epsilon_0 b$ at the origin.
Returning to the general case of $r \neq 0$, we note that in the absence of the sphere, i.e. for $b = \infty$, we have

$$\begin{aligned}\Phi(\mathbf{x})|_{b=\infty} &= \frac{Q}{4\pi\epsilon_0 a}\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{2^n n!}P_{2n}(\cos\theta) \\ &\times \left[\left(\frac{r}{a}\right)^{2n}\theta(0\le r\le a)+\left(\frac{a}{r}\right)^{2n+1}\theta(a\le r\le\infty)\right]. \end{aligned} \tag{3.5.66}$$

Naturally, this potential is non-zero for $r = b$. Our experience with image charges suggests that to get the total zero potential on the sphere of radius b, we should add, for any infinitesimal charge δq on the a-ring, an image charge of size $-(b/a)\delta q$ located distance b^2/a from the origin. Thus, we need to consider the potential of a ring located at distance b^2/a from the origin in the same plane as the original ring, and with total charge $-Qb/a$. This potential is obtained by changing $a \to b^2/a$ and $Q \to -Qb/a$ in the equation above, which gives

$$\begin{aligned}\Phi^{\text{image}}(\mathbf{x}) = &- \frac{Qb/a}{4\pi\epsilon_0(b^2/a)}\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{2^n n!}P_{2n}(\cos\theta) \\ &\times \left[\left(\frac{r}{b^2/a}\right)^{2n}\theta(0\le r\le b^2/a)+\left(\frac{b^2/a}{r}\right)^{2n+1}\theta(b^2/a\le r\le\infty)\right]. \end{aligned}$$

Since the image ring is outside the b-sphere, i.e. $r < b^2/a$, working inside the sphere we deal with the first contribution

$$\Phi^{\text{image}}(\mathbf{x})|_{r\le b} = - \frac{Q}{4\pi\epsilon_0 b}\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{2^n n!}P_{2n}(\cos\theta)\left(\frac{ar}{b^2}\right)^{2n}$$

only. It may be rewritten as a sum of $a < r$ and $r > a$ terms (each of which comes from the same functional form $(ar/b^2)^{2n}/b = (r/a)^{2n}(a/b)^{4n+1}/a = (a/r)^{2n+1}(r/b)^{4n+1}$):

$$\begin{aligned}\Phi^{\text{image}}(\mathbf{x})|_{r\le b} = &- \frac{Q}{4\pi\epsilon_0 a}\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{2^n n!}P_{2n}(\cos\theta) \\ &\times \left[\left(\frac{r}{a}\right)^{2n}\left(\frac{a}{b}\right)^{4n+1}\theta(0\le r\le a)+\left(\frac{a}{r}\right)^{2n+1}\left(\frac{r}{b}\right)^{4n+1}\theta(a\le r\le b)\right]. \end{aligned} \tag{3.5.67}$$

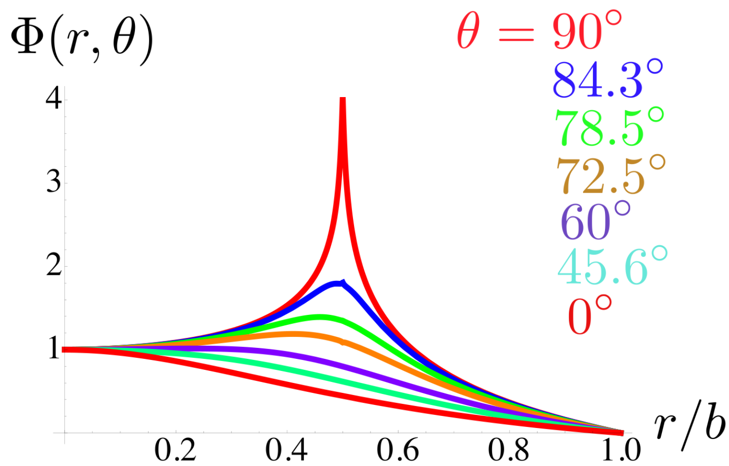


Figure 3.5: Potential $\Phi(r,\theta)$ (in units of $Q/4\pi\epsilon_0 b$) calculated as a sum of Legendre polynomials P_{2n} up to $n = 100$ in case of $a = b/2$ as a function of the radial variable r/b for the values of the polar angle θ corresponding to $\cos\theta = 0,\ 0.1,\ 0.2,\ 0.3,\ 0.5, 0.6,\ 1$.

One can see that the sum of the potential of the ring (3.5.66) and its image (3.5.67) produces the potential (3.5.65) of the ring inside a grounded sphere.

It is also interesting to study the r-dependence at different angles. Take, for example, $a = b/2$. Then, at the origin we have $\Phi(r = 0) = Q/4\pi\epsilon_0 b$ for all angles. For $r = b$, the potential vanishes for all angles. The ring is located at $r = a$ and $\theta = \pi/2$. One can see that for $\theta = 90°$ the potential curve has a cusp for $r = a$. When one slightly deviates from the ring plane, the potential still reflects the existence of the ring at $r = a$. However, when $\theta < 75°$ the curves show no bumps in the $r \sim a$ region.

In fact, the potential of the disk in the absence of the sphere may be easily obtained directly. First, consider the potential on the z-axis. Then all the points on the ring are located at a distance $\sqrt{a^2 + z^2}$ from the point z on the z-axis, i.e.

$$\Phi|_{b=\infty}(z) = \frac{Q}{4\pi\epsilon_0\sqrt{a^2 + z^2}}\ . \tag{3.5.68}$$

Using Taylor expansion

$$(1 + \alpha)^{-1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n(2n-1)!!}{2^n n!}\,\alpha^n \tag{3.5.69}$$

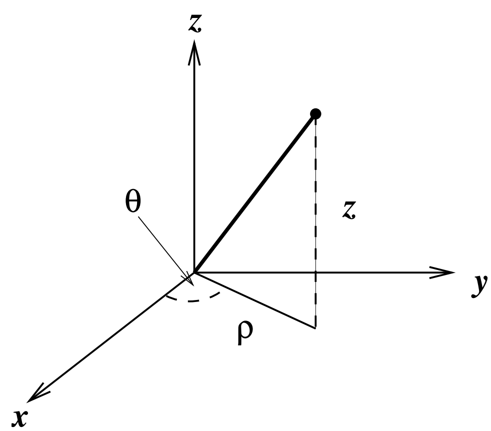
valid for $|\alpha| < 1$, we obtain

$$\Phi|_{b=\infty}(z) = \frac{Q}{4\pi\epsilon_0 a}\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{2^n n!}\left[\left(\frac{z}{a}\right)^{2n}\theta(0\le z\le a) + \left(\frac{a}{z}\right)^{2n+1}\theta(a\le z\le\infty)\right]. \tag{3.5.70}$$

As we know, to get solution outside the z-axis, we should change $z \to r$ and add the Legendre polynomial $P_{2n}(\cos\theta)$ corresponding to r^{2n} and $1/r^{2n+1}$ powers. This gives Eq. (3.5.66) . However, the number of problems solvable by the method of images is very limited, while the Green function method allows one to calculate (at least numerically) for any charge distribution inside a sphere, if the potential on the sphere is given.

3.6 Laplace's Equation in Cylindrical Polar Coordinates

We will denote the coordinates by (s, φ, z)



In terms of these coordinates, Laplace's equation assumes the form

$$\nabla^2\Phi(s,\varphi,z) = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial\Phi}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2\Phi}{\partial\varphi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0. \tag{3.6.1}$$

As before, we look for separable solutions of the form

$$\Phi(s,\varphi,z) = R(s)T(\varphi)Z(z), \tag{3.6.2}$$

so that Laplace's equation becomes

$$TZ\frac{1}{s}\frac{d}{ds}\left(s\frac{dR}{ds}\right) + RZ\frac{1}{s^2}\frac{d^2T}{d\varphi^2} + RT\frac{d^2Z}{dz^2} = 0, \tag{3.6.3}$$

which we may rewrite as

$$\frac{1}{sR}\frac{d}{ds}\left(s\frac{dR}{ds}\right) + \frac{1}{s^2 T}\frac{d^2 T}{d\varphi^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = 0. \tag{3.6.4}$$

The third term is a function of z alone, whilst the others are a function of s and φ alone. Thus we may write

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = k^2 \tag{3.6.5}$$

where k is a (not necessarily real) constant, with solution

$$Z(z) = e^{\pm kz}. \tag{3.6.6}$$

Thus we may now rewrite Laplace's equation as

$$\frac{s}{R}\frac{d}{ds}\left(s\frac{dR}{ds}\right) + \frac{1}{T}\frac{d^2 T}{d\varphi^2} + k^2 s^2 = 0, \tag{3.6.7}$$

and so for the angular term we have

$$\frac{1}{T}\frac{d^2 T}{d\varphi^2} = -\nu^2 \tag{3.6.8}$$

with solution

$$T(\varphi) = e^{\pm i\nu\varphi}. \tag{3.6.9}$$

For the solution to be single valued at $\varphi = 0$ and 2π, ν must be an **integer**.
Finally, the radial equation is

$$\frac{s}{R}\frac{d}{ds}\left(s\frac{dR}{ds}\right) - \nu^2 + k^2 s^2 = 0. \tag{3.6.10}$$

We can eliminate the constant k by the substitution $x = ks$, yielding

$$\frac{x}{R}\frac{d}{dx}\left(x\frac{dR}{dx}\right) - \nu^2 + x^2 = 0 \tag{3.6.11}$$

which we write as

$$\boxed{\frac{d^2 R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right) R = 0 \quad (3.6.12)}$$

This is the **Bessel Equation**.
As in the case of the Legendre equation, we find a solution by **series substitution**

$$R(x) = \sum_{n=0}^{\infty} c_n x^{\gamma+n} : \quad c_0 \neq 0 \tag{3.6.13}$$

ASIDE: why do we have to introduce the power x^γ, rather than just looking for a solution in terms of a Taylor expansion about $x = 0$? The reason is that there is a *regular singular point* at $x = 0$, i.e. the coefficients of R'' and R' in the Bessel equation written as

$$x^2 \frac{d^2R}{dx^2} + x\frac{dR}{dx} + \left(x^2 - \nu^2\right) R = 0 \tag{3.6.14}$$

vanish for $x = 0$, and therefore the solution can have a singularity there. In the case of the Legendre equation, there are regular singular points at $x = \pm 1$.
From Eq.(3.6.13), we have

$$\begin{aligned} \frac{dR}{dx} &= \sum_{n=0}^{\infty} c_n(\gamma+n)x^{\gamma+n-1} \\ \frac{d^2R}{dx^2} &= \sum_{n=0}^{\infty} c_n(\gamma+n)(\gamma+n-1)x^{\gamma+n-2}, \end{aligned}$$

and substituting into the Bessel equation we have

$$\sum_{n=0}^{\infty} c_n(\gamma+n)(\gamma+n-1)x^{\gamma+n-2} + \sum_{n=0}^{\infty} c_n(\gamma+n)x^{\gamma+n-2} + \sum_{n=0}^{\infty} c_n x^{\gamma+n} - \nu^2 \sum_{n=0}^{\infty} c_n x^{\gamma+n-2} = 0 \ , \tag{3.6.15}$$

or

$$\sum_{n=0}^{\infty} c_n[(\gamma+n)^2 - \nu^2]x^{\gamma+n-2} + \sum_{n=0}^{\infty} c_n x^{\gamma+n} = 0 \ . \tag{3.6.16}$$

The lowest power of x is $x^{\gamma-2}$, and equating the coefficients of this to zero gives the indicial equation which determines γ.

- $x^{\gamma-2}$:

$$c_0[\gamma^2 - \nu^2] = 0 \Rightarrow \gamma = \pm\nu, \quad \text{since } c_0 \neq 0. \tag{3.6.17}$$

- $x^{\gamma-1}$:

$$\begin{aligned} 0 &= c_1[(\gamma+1)^2 - \nu^2] \\ &= c_1(\gamma^2 + 2\gamma + 1 - \nu^2) \\ &= c_1(2\gamma+1) \quad \text{since } \gamma^2 = \nu^2 \\ \Rightarrow c_1 &= 0 \quad \text{since } \nu \text{ is an integer.} \end{aligned}$$

To proceed further, we rewrite Eq. (3.6.16) as

$$\sum_{n=-2}^{\infty} c_{n+2}[(\gamma+n+2)^2-\nu^2]x^{\gamma+n} + \sum_{n=0}^{\infty} c_n x^{\gamma+n} = 0 \ . \tag{3.6.18}$$

- $x^{n+\gamma}, n \geq 0$:

$$\begin{aligned} c_{n+2}[(\gamma+n+2)^2-\nu^2]+c_n &= 0 \\ \Rightarrow c_{n+2}[(\gamma+n+2)^2-\nu^2] &= -c_n \end{aligned}$$

Using in the last line $\nu^2 = \gamma^2$, we have

$$c_{n+2}[(\gamma+n+2)^2-\gamma^2] = -c_n$$

or

$$c_{n+2} = -\frac{c_n}{(n+2)(n+2+2\gamma)} \ .$$

As in the case of Legendre's equation, the recurrence relation connects either odd or even values of n. However, we have seen that $c_1 = 0$. Thus $c_n = 0$ *for all odd* n. Therefore, let us make the substitution $n = 2j$, and write the recurrence relation as

$$\begin{aligned} c_{2j+2} &= -\frac{c_{2j}}{4(j+1)(j+1+\gamma)}, \quad j = 0,1,2,\ldots \\ c_{2j+1} &= \ 0 \ . \end{aligned}$$

Rewriting

$$c_{2j} = -\frac{c_{2j-2}}{4j(j+\gamma)}, \quad j = 1,2,\ldots \ ,$$

we can now iterate this recurrence relation to obtain

$$\begin{aligned} c_{2j} &= \left(-\frac{1}{4}\right)^j \frac{c_0}{[j(j-1)\ldots 1]\,[(j+\gamma)(j-1+\gamma)\ldots(1+\gamma)]} \\ &= (-1)^j \left(\frac{1}{2}\right)^{2j} \frac{\gamma!}{j!(\gamma+j)!}\, c_0 = (-1)^j \left(\frac{1}{2}\right)^{2j} \frac{\Gamma(\gamma+1)}{\Gamma(j+1)\Gamma(\gamma+j+1)}\, c_0. \end{aligned} \tag{3.6.19}$$

We switched to the Gamma-function notations since γ is not necessarily positive. The Gamma-function representation allows also to define Bessel functions for non-integer γ. For further convenience, we choose

$$c_0 = \frac{1}{2^\gamma \Gamma(\gamma+1)}, \tag{3.6.20}$$

so that the solutions may be written as

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j+1)\Gamma(\nu+j+1)} \left(\frac{x}{2}\right)^{2j} \tag{3.6.21}$$

$$J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j+1)\Gamma(j-\nu+1)} \left(\frac{x}{2}\right)^{2j}. \tag{3.6.22}$$

These are the **Bessel Functions of the first kind of order** $\pm\nu$. Particular examples:

$$J_0(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(j!)^2} \left(\frac{x^2}{4}\right)^j, \tag{3.6.23}$$

$$J_1(x) = \frac{x}{2} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+1)!} \left(\frac{x^2}{4}\right)^j. \tag{3.6.24}$$

It is easy to see that $J_0'(x) = -J_1(x)$, i.e.. maxima or minima of $J_0(x)$ are located at zeros of $J_1(x)$. Also, $(xJ_1(x))' = xJ_0(x)$. These are particular examples of recurrence relations between Bessel functions.

Let us now list some general observations about Bessel functions:

- The series producing Bessel functions converge for all finite x
- If ν is *not* an integer, the solutions are *linearly independent.*
- If ν *is* an integer, they are *linearly dependent,* and in particular

$$J_{-m}(x) = (-1)^m J_m(x). \tag{3.6.25}$$

Proof: This is a consequence of the properties of the gamma function $\Gamma(z)$, which has singularities for $z = 0$ and for z a negative integer - recall the earlier relation

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{3.6.26}$$

We have

$$J_{-m}(x) = \left(\frac{x}{2}\right)^{-m} \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j+1)\Gamma(j-m+1)} \left(\frac{x}{2}\right)^{2j}. \tag{3.6.27}$$

Now $\Gamma(j-m+1) \longrightarrow \infty$ as argument approaches 0 or a negative integer. Thus only those terms in the sum for which $j - m + 1 \geq 1$ contribute, and we can write

$$\begin{aligned}
J_{-m}(x) &= \left(\frac{x}{2}\right)^{-m} \sum_{j=m}^{\infty} \frac{(-1)^j}{\Gamma(j+1)\Gamma(j-m+1)} \left(\frac{x}{2}\right)^{2j} \\
&= \left(\frac{x}{2}\right)^{-m} \left(\frac{x}{2}\right)^{2m} \sum_{l=0}^{\infty} \frac{(-1)^{l+m}}{\Gamma(l+1)\Gamma(l+m+1)} \left(\frac{x}{2}\right)^{2l} \\
&= \left(\frac{x}{2}\right)^{m} (-1)^m \sum_{l=0}^{\infty} \frac{(-1)^{l}}{\Gamma(l+1)\Gamma(l+m+1)} \left(\frac{x}{2}\right)^{2l} \\
&= (-1)^m J_m(x)
\end{aligned}$$

Because of the linear dependence of $J_{-m}(x)$ on $J_m(x)$, we introduce a second, linearly independent function

$$N_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi} \tag{3.6.28}$$

known as the **Neumann Function** or the **Bessel Function of the second kind**. Conventionally, we choose as our linearly independent functions $J_\nu(x)$ and $N_\nu(x)$ even if ν is not an integer.

Bessel Function of the Third Kind

These are just another pair of linearly independent solutions of the Bessel equation:

$$\begin{aligned}
H_\nu^{(1)}(x) &= J_\nu(x) + iN_\nu(x) \; , \\
H_\nu^{(2)}(x) &= J_\nu(x) - iN_\nu(x) \; .
\end{aligned}$$

These are also known as **Hankel Functions**. Their utility is that they have a more straightforward integral representation than $J_\nu(x)$ and $N_\nu(x)$.

3.6.1 Recurrence Relations

The sets of solutions of the Bessel equation are collectively known as **cylinder functions**, and satisfy recurrence relations in the same manner as the Legendre polynomials, e.g.

$$\begin{aligned}
\Omega_{\nu-1}(x) + \Omega_{\nu+1}(x) &= \frac{2\nu}{x}\Omega_\nu(x) \\
\Omega_{\nu-1}(x) - \Omega_{\nu+1}(x) &= 2\frac{d\Omega_\nu(x)}{dx}
\end{aligned}$$

3.6.2 Limiting Behaviour of Solutions

In the limit $x \ll 1$, we have

$$\begin{aligned}
J_\nu(x) &\to \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu \\
N_\nu(x) &\to \begin{cases} \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right)+\gamma_E+\ldots\right] & \nu=0 \\ -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu & \nu\neq 0 \end{cases}
\end{aligned}$$

where ν is real and non-negative, and $\gamma_E = 0.5772\ldots$ is the *Euler-Mascheroni constant.* Note that, when constructing solutions of the boundary-value problem, only $J_\nu(x)$ is regular as $x \to 0$.

In the limit $x \gg 1, \nu$, we have

$$\begin{aligned}
J_\nu(x) &\to \sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right) \\
N_\nu(x) &\to \sqrt{\frac{2}{\pi x}}\sin\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right).
\end{aligned} \tag{3.6.29}$$

The transition between these limiting forms occurs at $x \sim \nu$.

3.6.3 Roots of the Bessel functions

From the limiting forms (3.6.29), we see that each Bessel function has an infinite number of roots, which we denote $x_{\nu n}, n = 1, 2, 3, \ldots$ where

$$J_\nu(x_{\nu n}) = 0, \quad \text{for } n = 1, 2, 3, \ldots. \tag{3.6.30}$$

In particular, we have

$$\begin{aligned}
\nu = 0 : x_{0n} &= 2.405, 5.520, 8.654, \ldots \\
\nu = 1 : x_{1n} &= 3.832, 7.016, 10.173, \ldots \\
\nu = 2 : x_{2n} &= 5.136, 8.417, 11.620, \ldots
\end{aligned}$$

3.6.4 Ortogonality of the Bessel Functions

The roots of the Bessel function $J_\nu(x)$ are crucial when we consider its *orthogonality properties*, which take a rather unexpected form. We introduce the functions

$$\sqrt{s}J_\nu(x_{\nu n}s/a), n = 1, 2, 3, \ldots \tag{3.6.31}$$

and will now show that, for fixed $\nu \geq 0$, these functions, identified by n, form an orthogonal set on $0 \leq s \leq a$.

Proof

Recall the Bessel equation:

$$x\frac{d}{dx}\left(x\frac{dR}{dx}\right) + (x^2 - \nu^2)R = 0 \; . \tag{3.6.32}$$

Let us make the change of variable $x \to x_{\nu n}s/a$. Then

$$s\frac{d}{ds}\left(s\frac{dJ_\nu(x_{\nu n}s/a)}{ds}\right) + (x_{\nu n}s/a)^2 - \nu^2)J_\nu(x_{\nu n}s/a) = 0 \; , \tag{3.6.33}$$

or

$$\frac{1}{s}\frac{d}{ds}\left[s\frac{d}{ds}J_\nu(x_{\nu n}s/a)\right] + \left(\frac{x_{\nu n}^2}{a^2} - \frac{\nu^2}{s^2}\right)J_\nu(x_{\nu n}s/a) = 0 \; . \tag{3.6.34}$$

We now rewrite this as

$$\frac{d}{ds}\left[s\frac{dJ_\nu}{ds}\right] - \frac{\nu^2}{s}J_\nu = -\frac{x_{\nu n}^2}{a^2}sJ_\nu. \tag{3.6.35}$$

This is the Sturm-Liouville equation, with

$$\begin{aligned} p(x) &= s, \\ q(x) &= -\nu^2/s, \\ r(x) &= s, \\ \lambda &= x_{\nu n}^2/a^2. \end{aligned}$$

Thus we have

$$(x_{\nu n}^2 - x_{\nu n'}^2)\int_0^a ds\, sJ_\nu(x_{\nu n'}s/a)J_\nu(x_{\nu n}s/a) = 0 \tag{3.6.36}$$

providing

$$\left[s\left\{J_\nu(x_{\nu n'}s/a)\frac{d}{ds}J_\nu(x_{\nu n}s/a) - J_\nu(x_{\nu n}s/a)\frac{d}{ds}J_\nu(x_{\nu n'}s/a)\right\}\right]_0^a = 0. \tag{3.6.37}$$

At the upper limit, $s = a$, this expression vanishes since $x_{\nu n}$ and $x_{\nu n'}$ are roots of the Bessel function, and at the lower limit, $s = 0$, the expression vanishes because of the factor of s. Thus we have

$$\int_0^a ds\, sJ_\nu\left(\frac{x_{\nu n}s}{a}\right)J_\nu\left(\frac{x_{\nu n'}s}{a}\right) = 0,\; n \neq n' \tag{3.6.38}$$

The integral can be evaluated for $n' = n$, with the result

$$\boxed{\int_0^a ds\, sJ_\nu\left(\frac{x_{\nu n}s}{a}\right)J_\nu\left(\frac{x_{\nu n'}s}{a}\right) = \frac{a^2}{2}\left[J_{\nu+1}(x_{\nu n})\right]^2\delta_{nn'}. \qquad (3.6.39)}$$

3.6.5 Completeness

We now assume that the Bessel functions satisfy the completeness relation, and therefore we can expand *any* function on $0 \le s \le a$ as

$$f(s) = \sum_{n=1}^{\infty} A_{\nu n} J_\nu(x_{\nu n} s/a) \tag{3.6.40}$$

where

$$A_{\nu n} = \frac{2}{a^2 J_{\nu+1}^2(x_{\nu n})} \int_0^a ds\, s f(s) J_\nu\left(\frac{x_{\nu n} s}{a}\right). \tag{3.6.41}$$

This is a **Fourier-Bessel series**. This expansion is particularly useful for the case where $f(a) = 0$, e.g. the Dirichlet problem, since each term in the expansion satisfies the boundary conditions. An alternative set of basis functions is provided by

$$\sqrt{s} J_\nu\left(\frac{y_{\nu n} s}{a}\right), \tag{3.6.42}$$

where the $y_{\nu n}$ are the roots of $dJ_\nu/dx = 0$, because this set still satisfies the condition of Eq.(3.6.37). This choice is often more appropriate for the Neumann problem.

3.6.6 Modified Bessel Functions

Note that if we have chosen a separation constant such that the solution in the z-variable was

$$Z(z) = e^{\pm ikz}, \tag{3.6.43}$$

then the equation for $R(s)$ would have been

$$\frac{d^2R}{ds^2} + \frac{1}{s}\frac{dR}{ds} - \left(k^2 + \frac{\nu^2}{s^2}\right) R = 0, \tag{3.6.44}$$

which, after our usual substitution $x = ks$, becomes

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{d^2R}{dx} - \left(1 + \frac{\nu^2}{x^2}\right) R = 0. \tag{3.6.45}$$

with solutions

$$\begin{aligned} I_\nu(x) &= i^{-\nu} J_\nu(ix), \\ K_\nu(x) &= \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix). \end{aligned}$$

These, like I_ν and N_ν, are real functions of a real variable x, with limiting forms:

$x \ll 1$

$$
\begin{aligned}
I_\nu(x) &\rightarrow \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu \\
K_\nu(x) &\rightarrow \begin{cases} -\left[\ln\left(\frac{x}{2}\right) + \gamma_E + \dots\right] & \nu = 0 \\ \frac{\Gamma(\nu)}{2}\left(\frac{2}{x}\right)^\nu & \nu \neq 0 \end{cases}
\end{aligned}
$$

$x \gg 1, \nu$

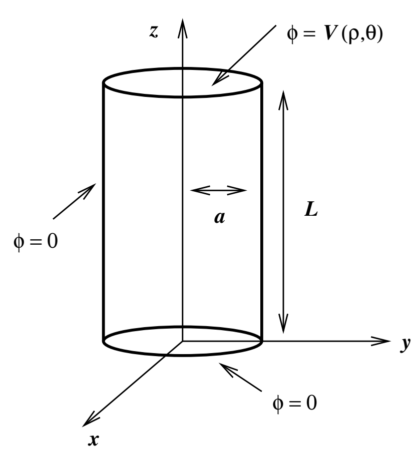
$$
\begin{aligned}
I_\nu(x) &\rightarrow \frac{1}{\sqrt{2\pi x}} e^x \left[1 + \mathcal{O}\left(\frac{1}{x}\right)\right] \\
K_\nu(x) &\rightarrow \sqrt{\frac{\pi}{2x}} e^{-x} \left[1 + \mathcal{O}\left(\frac{1}{x}\right)\right].
\end{aligned}
$$

Note again that only $I_\nu(x)$ is regular as $x \rightarrow 0$.

3.7 Boundary-value Problems in Cylindrical Coordinates

Consider the solution of the boundary-value problem in a cylinder of radius a, and length L, subject to the boundary conditions

$$
\begin{aligned}
\Phi(s, \varphi, 0) &= 0 \\
\Phi(a, \varphi, z) &= 0;\ 0 \leq z \leq L \\
\Phi(s, \varphi, L) &= V(s, \varphi)
\end{aligned}
$$



We look for separable solutions of the form

$$\Phi(s,\varphi,z) = R(s)T(\varphi)Z(z). \tag{3.7.1}$$

The angular factor has the form

$$T_m(\varphi) = A\sin m\varphi + B\cos m\varphi \tag{3.7.2}$$

where m is an integer greater than or equal to zero. The z factor is of the form

$$Z(z) = \sinh kz \tag{3.7.3}$$

where k is the separation constant, and we have imposed the boundary condition $Z(0) = 0$. Finally, the radial component is of the form

$$R_m(s) = C_m J_m(ks) + D_m N_m(ks). \tag{3.7.4}$$

Since there are no charges in the region $s \leq a$, the solution must be regular there, and in particular must be finite at $s = 0$. Thus we have $D_m = 0$. Furthermore, R must vanish at $s = a$, and thus

$$J_m(ka) = 0 \tag{3.7.5}$$

and hence the values of k are

$$k_{mn} = x_{mn}/a,\ n = 1, 2, 3, \ldots \tag{3.7.6}$$

where x_{mn} is the n-th root of $J_m(x) = 0$. Thus our general solution may be written

$$\begin{aligned} \Phi(s,\varphi,z) &= \sum_{n=1}^{\infty} \frac{B_{0n}}{2} J_0(k_{0n}s)\sinh(k_{0n}z) \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m(k_{mn}s)\sinh(k_{mn}z)\left[A_{mn}\sin m\varphi + B_{mn}\cos m\varphi\right]. \end{aligned} \tag{3.7.7}$$

We now impose the boundary condition at $z = L$:

$$\begin{aligned} V(s,\varphi) &= \sum_{n=1}^{\infty} \frac{B_{0n}}{2} J_0(k_{0n}s)\sinh(k_{0n}L) \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m(k_{mn}s)\sinh(k_{mn}L)\left[A_{mn}\sin m\varphi + B_{mn}\cos m\varphi\right]. \end{aligned}$$

This is a **Fourier series** in φ and a **Fourier-Bessel series** in s. We apply the orthogonality conditions, e.g., for A_{mn}:

$$\begin{aligned}
&\int_0^a ds\, s \int_0^{2\pi} d\varphi\, V(s,\varphi) J_{m'}(k_{m'n'}s)\sin m'\varphi = \\
&\qquad \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \sinh(k_{mn}L)\left\{\int_0^a ds\, s J_m(k_{mn}s) J_{m'}(k_{m'n'}s)\right\} \\
&\qquad \times\left\{A_{mn}\int_0^{2\pi} d\varphi\, \sin m\varphi \sin m'\varphi + B_{mn}\int_0^{2\pi} d\varphi\, \cos m\varphi \sin m'\varphi\right\} \\
&= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \sinh(k_{mn}L) A_{mn}\left\{\frac{a^2}{2}[J_{m+1}(x_{mn})]^2 \delta_{nn'}\right\}\{\pi\delta_{mm'}\}
\end{aligned}$$

and thus

$$\begin{aligned}
A_{mn} &= \frac{2}{\pi a^2 \sinh(k_{mn}L)[J_{m+1}(x_{mn})]^2}\int_0^a ds\, s\int_0^{2\pi} d\varphi\, V(s,\varphi) J_m(k_{mn}s)\sin m\varphi \\
B_{mn} &= \frac{2}{\pi a^2 \sinh(k_{mn}L)[J_{m+1}(x_{mn})]^2}\int_0^a ds\, s\int_0^{2\pi} d\varphi\, V(s,\varphi) J_m(k_{mn}s)\cos m\varphi
\end{aligned}$$

This form of the Fourier-Bessel seires is appropriate for problems confined to a finite region of s. Suppose, however, that we are interested in the solution for all $0 \le s \le \infty$.

Example

Determine $\Phi(s,\varphi,z)$ for the upper half-space $z \ge 0$, with $\Phi(s,\varphi,0) = V(s,\varphi)$, and Φ finite as $z \to \infty$. Then the separable solutions are of the form

$$e^{-kz}[A\sin m\varphi + B\cos m\varphi] J_m(ks)\ , \tag{3.7.8}$$

but there is now no restriction on the value of k other than it be positive (to ensure that Φ is finite as $z \to \infty$). Thus the sum over *discrete* values of k becomes an integral over k, and our general solution is

$$\begin{aligned}
\Phi(s,\varphi,z) &= \int_0^\infty dk\, e^{-kz}\,\frac{B_0(k)}{2} J_0(ks) \\
&+ \sum_{m=1}^{\infty}\int_0^\infty dk\, e^{-kz}\{A_m(k)\sin m\varphi + B_m(k)\cos m\varphi)\} J_m(ks).
\end{aligned}\tag{3.7.9}$$

We still have a Fourier series in φ, but the Fourier-Bessel series has evolved to a **Bessel transform**.

Imposing the boundary conditions at $z = 0$, we have

$$V(s,\varphi) = \int_0^\infty dk\, \frac{B_0(k)}{2} J_0(ks) \tag{3.7.10}$$

$$+ \sum_{m=1}^{\infty} \int_0^\infty dk\, \{A_m(k) \sin m\varphi + B_m(k) \cos m\varphi\}\, J_m(ks) \tag{3.7.11}$$

and we can invert the Fourier series to obtain

$$\begin{aligned} \frac{1}{\pi} \int_0^{2\pi} d\varphi'\, V(s,\varphi) \sin m\varphi' &= \int_0^\infty dk\, A_m(k) J_m(ks) \\ \frac{1}{\pi} \int_0^{2\pi} d\varphi'\, V(s,\varphi) \cos m\varphi' &= \int_0^\infty dk\, B_m(k) J_m(ks) \ . \end{aligned} \tag{3.7.12}$$

The integral of some function $A(k)$ with a Bessel function $J_m(ks)$

$$\mathcal{H}_m(s) = \int_0^\infty dk\, A(k) J_m(ks) \tag{3.7.13}$$

is called the **Hankel transforms** $\mathcal{H}_m(s)$. We can invert it using the completeness relation

$$\int_0^\infty ds\, s J_m(ks) J_m(k's) = \frac{1}{k} \delta(k' - k) \ . \tag{3.7.14}$$

Using it, one has

$$A(k) = k \int_0^\infty ds\, s J_m(ks) \mathcal{H}_m(s) \ . \tag{3.7.15}$$

In our case, applying (3.7.14) to the first line of Eq.(3.7.12), we have

$$\begin{aligned} \frac{1}{\pi} \int_0^\infty ds'\, s' \int_0^{2\pi} d\varphi'\, V(s',\varphi') \sin m\varphi' J_m(ks') &= \int_0^\infty ds'\, s' \int_0^\infty dk' A_m(k') J_m(k's') J_m(ks') \\ &= \int_0^\infty dk'\, A_m(k') \frac{1}{k} \delta(k - k') \\ &= \frac{1}{k} A_m(k), \end{aligned}$$

and thus we have

$$\begin{aligned} A_m(k) &= \frac{k}{\pi} \int_0^\infty ds'\, s' \int_0^{2\pi} d\varphi\, V(s',\varphi') \sin m\varphi' J_m(ks') \\ B_m(k) &= \frac{k}{\pi} \int_0^\infty ds'\, s' \int_0^{2\pi} d\varphi'\, V(s',\varphi') \cos m\varphi' J_m(ks') \end{aligned}$$

Substituting these coefficients into the expression (3.7.10) for $\Phi(s,\varphi,z)$, we have

$$\Phi(s,\varphi,z) = \int_0^\infty s'\,ds' \int_0^{2\pi} d\varphi'\; V(s',\varphi')\,\frac{1}{\pi}\int_0^\infty k\,dk\,e^{-kz}\left\{\frac{1}{2}J_0(ks')J_0(ks) + \sum_{m=1}^{\infty}\cos m(\varphi-\varphi')J_m(ks')J_m(ks)\right\} \;. \tag{3.7.16}$$

Rewriting this result as

$$\Phi(s,\varphi,z) = \int_0^\infty s'\,ds' \int_0^{2\pi} d\varphi'\; B(s',\varphi';s,\varphi,z)\; V(s',\varphi') \tag{3.7.17}$$

where

$$B(s',\varphi';s,\varphi,z) = \frac{1}{\pi}\int_0^\infty k\,dk\,e^{-kz}\left\{\frac{1}{2}J_0(ks')J_0(ks) + \sum_{m=1}^{\infty}\cos m(\varphi-\varphi')J_m(ks')J_m(ks)\right\} \;, \tag{3.7.18}$$

and treating $B(s',\varphi';s,\varphi,z)$ as

$$B(s',\varphi';s,\varphi,z) = \frac{\partial}{\partial z'}G(s',\varphi',z';s,\varphi,z)\bigg|_{z'=0} \;, \tag{3.7.19}$$

(recall that $\partial/\partial n' = -\partial/\partial z'$ for the $z'>0$ region) where

$$G(s',\varphi',z';s,\varphi,z) = \frac{1}{\pi}\int_0^\infty dk\,e^{-k(z-z')}\left\{\frac{1}{2}J_0(ks')J_0(ks) + \sum_{m=1}^{\infty}\cos m(\varphi-\varphi')J_m(ks')J_m(ks)\right\} \;, \tag{3.7.20}$$

we realize that we have obtained a Green function type representation of the solution of our problem.

3.7.1 General Solution of Green Function in Cylindrical Polars

So, let us find the Green function in cylindrical coordinates using a general method based on solving the equation

$$\nabla'^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}') \;. \tag{3.7.21}$$

As boundary conditions, let us take the simplest case of zero potential at infinity, i.e., we will find Green function in unbounded space.

To express the r.h.s. in terms of cylindrical coordinates, we recall that

$$\delta[g(x)] = \sum_i \frac{1}{|g'(x_i)|}\delta(x_i) \tag{3.7.22}$$

where x_i are the roots of $g(x) = 0$. Thus, in three dimensions, we have

$$\delta(\mathbf{x} - \mathbf{x}') = \left|\frac{\partial(x, y, z)}{\partial(s, \varphi, z)}\right|^{-1} \delta(s - s')\delta(\varphi - \varphi')\delta(z - z') = \frac{1}{s'}\delta(s - s')\delta(\varphi - \varphi')\delta(z - z'). \tag{3.7.23}$$

Hence the Green function satisfies

$$\nabla'^2 G(\mathbf{x} - \mathbf{x}') = -\frac{4\pi}{s'}\delta(s - s')\delta(\varphi - \varphi')\delta(z - z') \tag{3.7.24}$$

where

$$\nabla'^2 = \frac{1}{s'}\frac{\partial}{\partial s'}\left(s'\frac{\partial}{\partial s'}\right) + \frac{1}{s'^2}\frac{\partial^2}{\partial\varphi'^2} + \frac{\partial^2}{\partial z'^2}. \tag{3.7.25}$$

Note that in the following we will treat the *unprimed* indices as fixed parameters.
We will now specialise to the case where we wish to obtain the Green function in a volume V encompassing the full angular range $0 \le \varphi \le 2\pi$. Then any solution can be expressed as a Fourier series in φ',

$$G(\mathbf{x}, \mathbf{x}') = G(s, \varphi, z; s', \varphi', z') = \sum_{m'=-\infty}^{\infty} F_{m'}(s, \varphi, z; s', z')e^{-im'\varphi'}. \tag{3.7.26}$$

Substituting this into Eq.(3.7.24), we have

$$\sum_{m'=-\infty}^{\infty} e^{-im'\varphi'}\left\{\frac{1}{s'}\frac{\partial}{\partial s'}\left[s'\frac{\partial}{\partial s'}F_{m'}(s, \varphi, z; s', z')\right] - m'^2\frac{1}{s'^2}F_{m'}(s, \varphi, z; s', z')\right.$$
$$\left.+\frac{\partial^2}{\partial z'^2}F_{m'}(s, \varphi, z; s', z')\right\} = -\frac{4\pi}{s'}\delta(s - s')\delta(\varphi - \varphi')\delta(z - z').$$

We now use the orthogonality properties of the $\exp im\varphi$ to obtain

$$\sum_{m'=-\infty}^{\infty}\int_0^{2\pi} d\varphi'\, e^{i(m-m')\varphi'}\left\{\frac{1}{s'}\frac{\partial}{\partial s'}\left[s'\frac{\partial}{\partial s'}F_{m'}(s, \varphi, z; s', z')\right] - m'^2\frac{1}{s'^2}F_{m'}(s, \varphi, z; s', z')\right.$$
$$\left.+\frac{\partial^2}{\partial z'^2}F_{m'}(s, \varphi, z; s', z')\right\} = -\frac{4\pi}{s'}\delta(s - s')\delta(z - z')\int_0^{2\pi} d\varphi' e^{im\varphi'}\delta(\varphi - \varphi'),$$

yielding

$$\frac{1}{s'}\frac{\partial}{\partial s'}\left[s'\frac{\partial F_m}{\partial s'}\right] - \frac{m^2}{s'^2}F_m + \frac{\partial^2 F_m}{\partial z'^2} = -\frac{2}{s'}\delta(s - s')\delta(z - z')e^{im\varphi}. \tag{3.7.27}$$

Thus we have explicitly exhibited the φ dependence, and can write

$$F_m(s,\varphi,z;s',z') = f_m(s,z;s',z')e^{im\varphi}, \tag{3.7.28}$$

where f_m obeys the P.D.E.

$$\frac{1}{s'}\frac{\partial}{\partial s'}\left[s'\frac{\partial f_m}{\partial m'}\right] - \frac{m^2}{s'^2}f_m + \frac{\partial^2 f_m}{\partial z'^2} = -\frac{2}{s'}\delta(s-s')\delta(z-z'). \tag{3.7.29}$$

Thus from our original P.D.E. in *three* variables we now have a two-variable P.D.E..
To proceed further, we must say something about the boundary conditions, or at the very least specify the volume V. We will assume that is covers $-\infty \leq z \leq \infty$, and then write f_m in the Fourier representation

$$f_m(s,z;s',z') \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} dk' e^{-ik'z'}\tilde{f}_m(s,z;s',k').$$

Substituting this representation into Eq.(3.7.29), we have

$$\frac{1}{2\pi}\int_{\infty}^{\infty} dk'\, e^{-ik'z'}\left\{\frac{1}{s'}\frac{\partial}{\partial s'}\left[s'\frac{\partial f_m}{\partial s'}\right] - \frac{m^2}{s'^2}f_m - k'^2 f_m\right\} = -\frac{2}{s'}\delta(s-s')\delta(z-z')\ . \tag{3.7.30}$$

Now, we multiply both sides with $\exp ikz'$ and integrate them over z'. Using the orthogonality properties of the $\exp ikz$ exponentials gives

$$\frac{1}{s'}\frac{\partial}{\partial s'}\left[s'\frac{\partial \tilde{f}_m}{\partial s'}\right] - \frac{m^2}{s'^2}\tilde{f}_m - k^2\tilde{f}_m = -\frac{2}{s'}\delta(s-s')e^{ikz}\ , \tag{3.7.31}$$

where $\tilde{f}_m = \tilde{f}_m(s,z;s',k)$. We have now exhibited the z dependence of the function, and may write

$$\tilde{f}_m(s,z;s',k) = \frac{1}{2\pi}e^{ikz}g_m(s,s';k), \tag{3.7.32}$$

giving

$$\frac{1}{s'}\frac{\partial}{\partial s'}\left[s'\frac{\partial g_m}{\partial s'}\right] - \left(\frac{m^2}{s'^2} + k^2\right)g_m = -\frac{4\pi}{s'}\delta(s-s'). \tag{3.7.33}$$

This is just a **one-dimensional** Green function equation, which we may write in a more familiar form by substituting

$$\begin{aligned} x &= |k|s \\ x' &= |k|s', \end{aligned}$$

yielding

$$\frac{\partial}{\partial x'}\left[x'\frac{\partial g_m(x,x')}{\partial x'}\right] - x'\left(1+\frac{m^2}{x'^2}\right) g_m(x,x') = -4\pi\delta(x-x'). \tag{3.7.34}$$

This is just the modified Bessel equation, with inhomogeneous source. As we noted earlier, the modified Bessel equation (like the Legendre equation) is of Sturm-Liouville type:

$$\frac{d}{dx'}\left[p(x')\frac{dg(x,x')}{dx'}\right] + q(x')g(x,x') = -4\pi\delta(x-x') \tag{3.7.35}$$

with

$$\begin{aligned} p(x') &= x' \\ q(x') &= -x'\left(1+\frac{m^2}{x'^2}\right). \end{aligned}$$

Thus we have finally reduced the problem to the solution of the Green function for the Sturm-Liouville equation.

3.7.2 Green Function for Modified Bessel Equation

Thus we have to analyze the modified Bessel equation with δ-function source

$$\frac{d}{dx'}\left[x'\frac{dg(x,x')}{dx'}\right] - x'\left[1+\frac{m^2}{x'^2}\right] g(x,x') = -4\pi\delta(x-x'). \tag{3.7.36}$$

A pair of linearly independent solutions is provided by the modified Bessel functions $I_m(x')$ and $K_m(x')$. Let us now consider the case where we require the solution over all space, i.e. $x' \in [0,\infty]$. The solution must be *finite* at $x = 0$, and thus

$$y_1(x') = I_m(x'). \tag{3.7.37}$$

If we further require that the solution be finite as $x' \to \infty$, then we have

$$y_2(x') = K_m(x'), \tag{3.7.38}$$

which we can see from the limiting behaviour quoted earlier. In this case, the Wronskian is (see Jackson)

$$W[I_m(x), K_m(x)] = -\frac{1}{x} \tag{3.7.39}$$

(note that $p(x) = x$ in this case, hence $p(x)W[I_m(x), K_m(x)] = -1 = \text{const}$, in agreement with the general result discussed above) and thus our general solution for the Green function is

$$g_m(x, x') = \begin{cases} -\dfrac{4\pi}{x}\dfrac{K_m(x)I_m(x')}{-1/x} & 0 \le x' \le x \\ -\dfrac{4\pi}{x}\dfrac{K_m(x')I_m(x)}{-1/x} & x \le x' \le \infty \end{cases}, \tag{3.7.40}$$

which we may express as

$$\boxed{g_m(x, x') = 4\pi I_m(x_<)K_m(x_>) \qquad (3.7.41)}$$

where $x_< = \min(x, x')$ and $x_> = \max(x, x')$.

Reconstruction of the Full Green Function

We reconstruct the full Green function in four steps:

1.

$$\begin{aligned} \tilde{f}_m(s, z; s', k) &= g_m(s, s'; k)e^{ikz}/2\pi \\ &= 2I_m(|k|s_<)K_m(|k|s_>)e^{ikz} \end{aligned} \tag{3.7.42}$$

2.

$$\begin{aligned} f_m(s, z; s', z') &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-ikz'}\tilde{f}_m(s, z; s', k) \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} dk\, e^{ik(z-z')}I_m(|k|s_<)K_m(|k|s_>) \end{aligned} \tag{3.7.43}$$

3.

$$F_m(s, \varphi, z; s', z') = f_m(s, z; s', z')e^{im\varphi} \tag{3.7.44}$$

4.

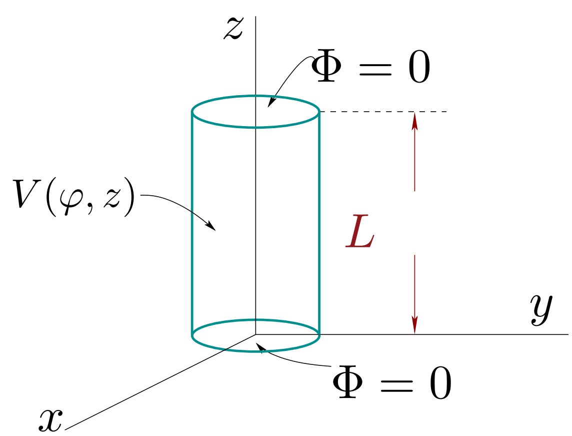
$$\boxed{G(\mathbf{x}, \mathbf{x}') = \frac{1}{\pi}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi')}\int_{-\infty}^{\infty} dk\, e^{ik(z-z')}I_m(|k|s_<)K_m(|k|s_>). \qquad (3.7.45)}$$

Since we have evaluated the Green function with boundary conditions at infinity, this last expression is just the expansion of $|\mathbf{x} - \mathbf{x}'|^{-1}$ in cylindrical polar coordinates.

Example

Consider the solution of the boundary-value problem in a cylinder of radius a, and length L, subject to the boundary conditions

$$\begin{aligned}\Phi(s,\varphi,0) &= 0\\ \Phi(s,\varphi,L) &= 0\\ \Phi(a,\varphi,z) &= V(\varphi,z);\ 0\le z\le L\end{aligned}$$



We look for separable solutions of the form

$$\Phi(s,\varphi,z)=R(s)T(\varphi)Z(z). \tag{3.7.46}$$

The angular factor has the form

$$T_m(\varphi)=A\sin m\varphi+B\cos m\varphi \tag{3.7.47}$$

where m is an integer greater than or equal to zero. Since the potential vanishes both for $z=0$ and $z=L$, it makes sense to take the z factor in a sine form

$$Z(z)=\sin(kz) \tag{3.7.48}$$

where k is the separation constant, and we have imposed the boundary condition $Z(0)=0$. Imposing the boundary condition $Z(L)=0$, we conclude that k should be given by

$$k_n=n\frac{\pi}{L}\quad,\quad n=1,2,3,\dots\ . \tag{3.7.49}$$

Then, for the radial component $R(s)$ we will have the modified Bessel equation

$$\frac{1}{s}\frac{\partial}{\partial s}\left[s\frac{\partial R_m(s,k)}{\partial s}\right]-\left(\frac{m^2}{s^2}+k^2\right)R_m(s,k)=0\ , \tag{3.7.50}$$

and thus

$$R_m(s,k) = C_m I_m(ks) + D_m K_m(ks). \tag{3.7.51}$$

Since there are no charges in the region $s \le a$, the solution must be regular there, and in particular must be finite at $s = 0$. Thus we have $D_m = 0$. As a result, our general solution may be written as

$$\begin{aligned}\Phi(s,\varphi,z) &= \sum_{n=1}^{\infty} \frac{B_{0n}}{2} I_0(k_n s)\sin(k_n z) \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} I_m(k_n s)\sin(k_n z)\,[A_{mn}\sin m\varphi + B_{mn}\cos m\varphi]. \end{aligned} \tag{3.7.52}$$

We now impose the boundary condition at $s = a$:

$$\begin{aligned}V(\varphi,z) &= \sum_{n=1}^{\infty} \frac{B_n}{2} J_0(k_n a)\sin(k_n z) \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} I_m(k_n a)\sin(k_n z)\,[A_{mn}\sin m\varphi + B_{mn}\cos m\varphi]. \end{aligned}$$

This is a **Fourier series** in both φ and z. We apply the orthogonality conditions, e.g., for A_{mn}:

$$\begin{aligned} &\int_0^L dz \int_0^{2\pi} d\varphi\, V(\varphi,z)\sin(m'\varphi)\sin(n'\pi z/L) = \\ &\int_0^L dz \int_0^{2\pi} d\varphi\, \sin(m'\varphi)\sin(n'\pi z/L)\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} I_m(n\pi a/L)\sin(n\pi z/L)\,[A_{mn}\sin m\varphi + B_{mn}\cos m\varphi]\right] \\ &= \frac{L}{2}\pi \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} I_m(n\pi a/L) A_{mn}\,\{\delta_{nn'}\}\,\{\pi\delta_{mm'}\} \\ &= \frac{L}{2}\pi I_{m'}(n'\pi a/L) A_{m'n'} \end{aligned}$$

and thus

$$\begin{aligned} A_{mn} &= \frac{2}{\pi L I_m(n\pi a/L)} \int_0^L dz \int_0^{2\pi} d\varphi\, V(\varphi,z)\sin(m\varphi)\sin(n\pi z/L) \\ B_{mn} &= \frac{2}{\pi L I_m(n\pi a/L)} \int_0^L dz \int_0^{2\pi} d\varphi\, V(\varphi,z)\cos(m\varphi)\sin(n\pi z/L) \end{aligned}$$

The solution of the boundary-value problem is given by

$$\begin{aligned}\Phi(s,\varphi,z) =& \frac{1}{\pi L}\sum_{n=1}^{\infty}\frac{I_0(n\pi s/L)}{I_0(n\pi a/L)}\sin(n\pi z/L)\\ &\times\int_0^L dz'\int_0^{2\pi} d\varphi'\, V(\varphi',z')\sin(n\pi z'/L)\\ &+\frac{2}{\pi L}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{I_m(n\pi s/L)}{I_m(n\pi a/L)}\sin(n\pi z/L)\\ &\times\int_0^L dz'\int_0^{2\pi} d\varphi'\, V(\varphi',z')\sin(n\pi z'/L)\cos m(\varphi-\varphi')\end{aligned}$$

3.8 Expansion of Green Function in terms of Eigenfunctions

A closely related method to those discussed above is the expansion of the Green function in terms of the *eigenfunctions* of some related problem. Consider the solution of

$$\nabla^2\varphi(\mathbf{x}) + [f(\mathbf{x})+\lambda]\varphi(\mathbf{x}) = 0, \tag{3.8.1}$$

in a volume V bounded by a surface S, subject to φ satisfying certain *homogeneous* boundary conditions for $x \in S$. In general, consistent solutions can be obtained only for certain (possibly continuous) values of λ, which we will denote λ_n, the **eigenvalues**. The corresponding solutions, the **eigenfunctions**, we will denote $\varphi_n(\mathbf{x})$. The eigenvalue equation is then

$$\nabla^2\varphi_n + [f(\mathbf{x})+\lambda_n]\varphi_n = 0. \tag{3.8.2}$$

The eigenfunctions form a complete, orthogonal set of functions (the proof of orthogonality follows that for the Sturm-Liouville equation), and we will assume that they are normalized:

$$\int d^3x\, \varphi_m^*\varphi_n = \delta_{mn}. \tag{3.8.3}$$

Then *any* function satisfying the *same homogeneous boundary conditions* may be expanded as a series in the eigenfunctions. Consider in particular a Green function, satisfying

$$\nabla'^2 G(\mathbf{x},\mathbf{x}') + [f(\mathbf{x}')+\lambda]G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}') \tag{3.8.4}$$

where λ is, in general, not an eigenvalue. The corresponding eigenfunction expansion is

$$G(\mathbf{x},\mathbf{x}') = \sum_n a_n(\mathbf{x})\varphi_n(\mathbf{x}'), \tag{3.8.5}$$

and, inserting in Eq.(3.8.4), we obtain

$$\sum_n a_n(\mathbf{x})\{\nabla'^2\varphi_n(\mathbf{x}') + f(\mathbf{x}')\varphi_n(\mathbf{x}') + \lambda\varphi_n(\mathbf{x}')\} = -4\pi\delta(\mathbf{x}-\mathbf{x}'). \tag{3.8.6}$$

We now use that φ_n is an eigenfunction of Eq.(3.8.2) with eigenvalue λ_n, and obtain

$$\sum_n a_n(\mathbf{x})[\lambda - \lambda_n]\varphi_n(\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}'). \tag{3.8.7}$$

Using the orthonormal property of the eigenfunctions, we obtain

$$a_n(\mathbf{x}) = 4\pi\frac{\varphi_n^*(\mathbf{x})}{\lambda_n - \lambda} \tag{3.8.8}$$

and hence

$$\boxed{G(\mathbf{x},\mathbf{x}') = 4\pi\sum_n \frac{\varphi_n^*(\mathbf{x})\varphi_n(\mathbf{x}')}{\lambda_n - \lambda}}$$

This is often referred to as the **spectral representation** of the Green function.

Example: Green function in free space

Let us now specialise to Poisson's equation, i.e. we set $f(\mathbf{x}) = 0$ and $\lambda = 0$ in Eq.(3.8.4). We will begin by considering the solution in free space, for which the most closely related eigenvalue equation is the wave equation

$$(\nabla^2 + k^2)\varphi_{\mathbf{k}}(\mathbf{x}) = 0 \tag{3.8.9}$$

where k^2 is the (continuous) eigenvalue, and the corresponding normalized eigenfunction is

$$\varphi_{\mathbf{k}}(\mathbf{x}) = \left(\frac{1}{2\pi}\right)^{3/2} e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{3.8.10}$$

with normalization

$$\int d^3x\, \varphi_{\mathbf{k}'}^*(\mathbf{x})\varphi_{\mathbf{k}}(\mathbf{x}) = \delta(\mathbf{k}-\mathbf{k}'). \tag{3.8.11}$$

Then the expression for the Green function is

$$G(\mathbf{x},\mathbf{x}') = 4\pi\int d^3k \frac{e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})}}{k^2}\left(\frac{1}{2\pi}\right)^3 \tag{3.8.12}$$

which we observe may be written as

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{2\pi^2}\int d^3k \frac{e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})}}{k^2}. \tag{3.8.13}$$

Example: Dirichlet Green function inside a rectangular box

We define the surface of the box to be the planes $x = 0, a$, $y = 0, b$, and $z = 0, c$. The most closely related eigenvalue problem is

$$\nabla^2 \varphi + k_{lmn}^2 \varphi_{lmn} = 0, \tag{3.8.14}$$

where the eigenvalues and normalized eigenfunctions are

$$\begin{aligned} k_{lmn}^2 &= \pi^2 \left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} \right) \\ \varphi_{lmn}(\mathbf{x}) &= \sqrt{\frac{8}{abc}} \sin \frac{l\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c}. \end{aligned}$$

Thus we can immediately write down the Green function as

$$G(\mathbf{x}, \mathbf{x}') = \frac{32}{\pi abc} \sum_{l,m,n} \frac{\sin \frac{l\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} \sin \frac{l\pi x'}{a} \sin \frac{m\pi y'}{b} \sin \frac{n\pi z'}{c}}{\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}}. \tag{3.8.15}$$