Chapter 1

Introduction to Electrostatics

Electrostatics is the study of *time-independent* distributions of charges and fields.

1.1 Coulomb's Law

The foundation of electrostatics is **Coulomb's Law**, together with the **Superposition Principle** which we will discuss later.

Coulomb's Law

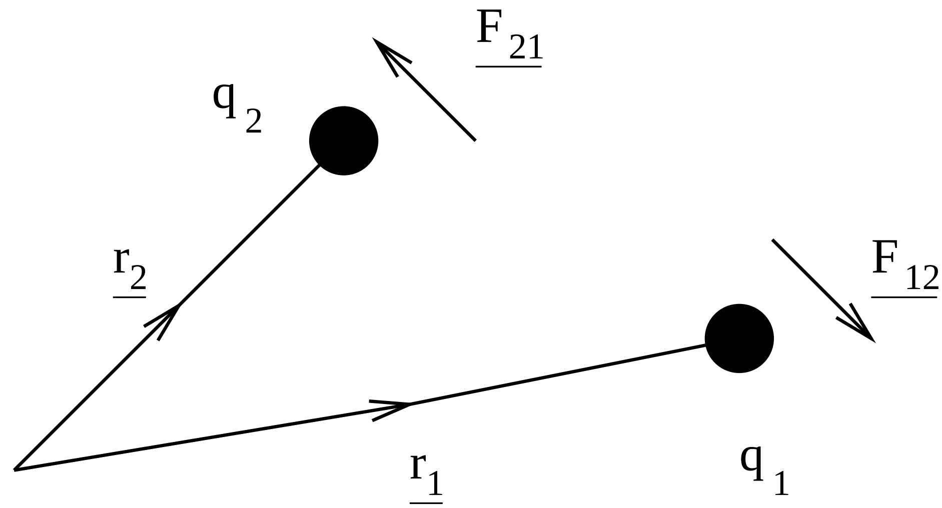
The force $\underline{F_{21}}$ on a particle of charge q_2 at $\underline{r_2}$ due to a particle of charge q_1 at $\underline{r_1}$ is given by

$$\underline{F_{21}} = kq_1q_2\frac{\underline{\hat{r}_{21}}}{|\underline{r_2} - \underline{r_1}|^2}$$

where

- $\underline{r_{21}} = \underline{r_2} - \underline{r_1}$
- $\underline{\hat{r}}$ is a unit vector in the direction of $\underline{r}$.

Coulomb's law is an **experimental observation**.



In SI units:

- $k = 1/4\pi\epsilon_0$ - the 4π is conventional.
- The charges q_1, q_2 are measured in **Coulombs** (C), and **defined** via the magnetic effects of currents.
- ϵ_0, the **Permitivity of Free Space** is also a **defined** quantity:

$$\epsilon_0 = 8.854\,187\,817\ldots \times 10^{-12}\,C^2 N^{-1} m^{-2}$$

There are two further observations that we can make:

- The forces on the two charges are **equal and opposite**, obeying Newton's third law: $\underline{F}_{12} = -\underline{F}_{21}$.
- The force is repulsive (attractive) for like (unlike) charges.

Electric Field: The electric field $\underline{E}$ at $\underline{r}$ is defined as the force acting on a unit test charge at that point. More strictly,

$$\underline{E}(\underline{r}) = \lim_{q \to 0} \frac{\underline{F}(\underline{r})}{q},$$

so that the electric field due to the test charge can be ignored.

1.2 The Superposition Principle and Extended Distributions

In the above we have looked at the fields due to **single, isolated point-like** charges. In this section, we will explore the second emperical ingredient necessary for our understanding of electrostatic fields, the **linear superposition principle**.

Linear Superposition Principle

> The resultant force on a test particle due to several charges is the **vector sum** of the forces due to the charges individually.

Example: We have N charges $q_i (i = 1, \ldots, N)$, situated at the points $\underline{r_i}$. The force on a test particle of charge q at the point $\underline{r}$ is given by

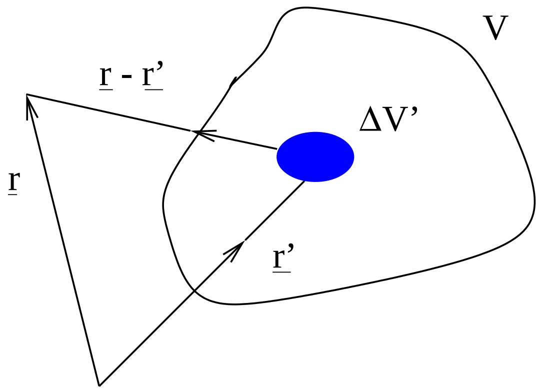
$$\underline{F}(\underline{r}) = kq \sum_{i=1}^{N} \frac{q_i(\underline{r} - \underline{r_i})}{\mid \underline{r} - \underline{r_i} \mid^3}$$

where $k = 1/4\pi\epsilon_0$ in SI units.
Thus the **electrostatic field** $\underline{E}(\underline{r})$ is

$$\underline{E}(\underline{r}) = k \sum_{i=1}^{N} \frac{q_i(\underline{r} - \underline{r_i})}{\mid \underline{r} - \underline{r_i} \mid^3}$$

1.2.1 Extended Charge Distributions

We will now apply the *linear superposition principle* to a **continuous** distribution of charge.
Consider a continuous distribution of charge density (*charge per unit volume*) $\rho(\underline{r}')$, confined to a volume V.



In order to use the superposition principle, we will divide the volume V into infinitessimal volume elements $\Delta V'$, centred at $\underline{r}'$. The charge occupying the volume element at $\underline{r}'$ is

$$dq = \rho(\underline{r}')dV'$$

Therefore, the electrostatic field at the point $\underline{r}$ due to the element of charge dq at $\underline{r}'$ is

$$\Delta\underline{E}(\underline{r}) = k\frac{\rho(\underline{r}')\,(\underline{r}-\underline{r}')\Delta V'}{\mid \underline{r}-\underline{r}' \mid^3}$$

where we take $\Delta\underline{E}(\underline{r}) \longrightarrow 0$ as $\underline{r} \longrightarrow \infty$. We now use the principle of linear superposition to write that the resultant field at $\underline{r}$ as a **sum** over the elements $\Delta V'$ in V

$$\underline{E}(\underline{r}) = k\sum_{\Delta V'}\frac{\rho(\underline{r}')\,(\underline{r}-\underline{r}')\Delta V'}{\mid \underline{r}-\underline{r}' \mid^3}$$

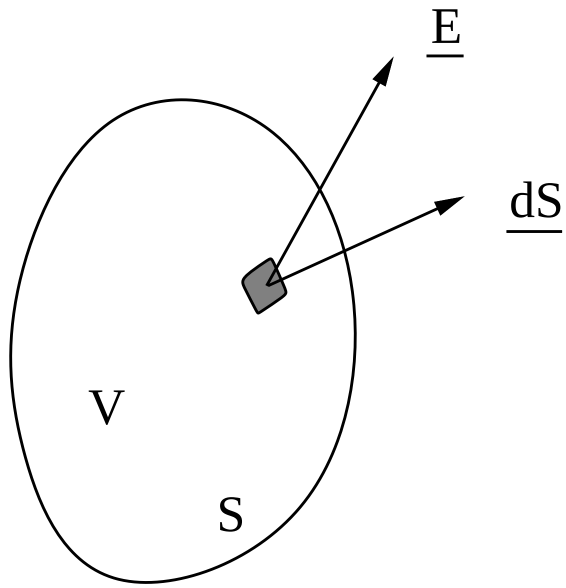
In the limit that $\Delta V'$ becomes infinitessimal, we have

$$\boxed{\underline{E}(\underline{r}) = k\int_V \frac{\rho(\underline{r}')(\underline{r}-\underline{r}')dV'}{\mid \underline{r}-\underline{r}' \mid^3}}$$

Much of the rest of this course is centred on methods for obtaining the electrostatic field, and we begin with one of the simplest - **Gauss' Law**.

1.3 Gauss' Law

Suppose that the charge density $\rho(\underline{r})$ is the sole source of the electrostatic field $\underline{E}(\underline{r})$. Gauss' Law relates the **flux** of $\underline{E}$ out of a closed surface S bounding a volume V to the **total charge** Q contained within V



Gauss' Law states that:

$$\int_S \underline{E} \cdot \underline{dS} = 4\pi k Q = \frac{Q}{\epsilon_0} \qquad \text{in SI units}$$

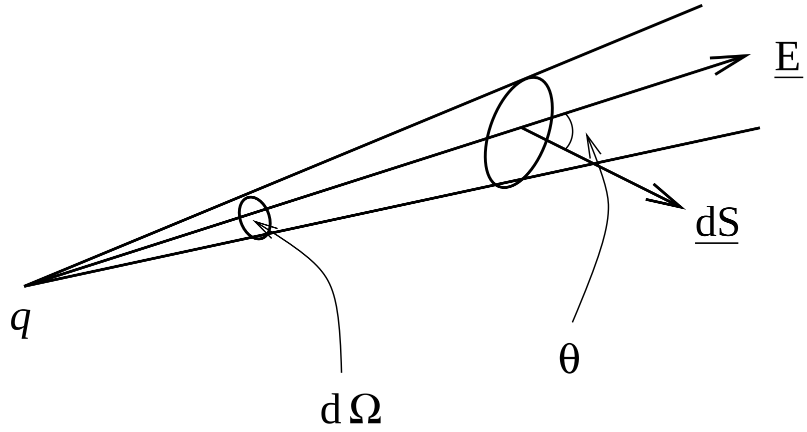
where

- Q = **total charge** within S
- $\underline{dS}$ = **outward normal** to surface, having infinitessimal area dS

Gauss' Law provides a powerful way to compute the electrostatic field for the case where there is spherical, or even cylindrical, symmetry. It will also form the starting point for our derivation of Laplace's equation later in the course.

1.3.1 Geometrical Interpretation of Gauss' Law

Consider a point charge q placed at the origin (not necessarily *inside* V), and the electrostatic flux across an area $\underline{dS}$.



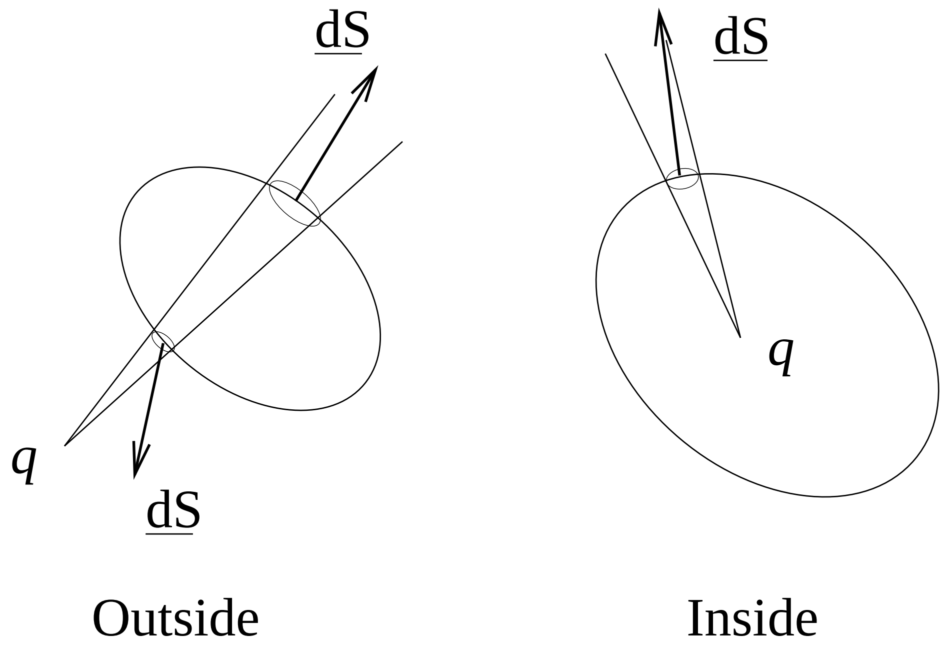
Then we have

$$\begin{aligned} \underline{E} \cdot \underline{dS} &= kq \frac{dS \ \cos\theta}{r^2} \\ &= kq\, d\Omega \end{aligned}$$

where $d\Omega$ is the solid angle subtended by $d\underline{S}$ at the origin; $d\Omega$ is the *projection* of the surface element dS onto the unit sphere. Note that $\int_S \, d\Omega = 4\pi$ where S is a unit sphere, or any closed surface, enclosing the origin.

- If the charge q is **outside** the volume, then the total flux $\int_V \underline{E} \cdot \underline{dS}$ is zero; the contributions from two elements of surface area produced by the intersection of a cone with the surface **cancel**, see below.

- If the charge q is **inside** the volume, the total flux $\int_V \underline{E} \cdot \underline{dS} = q/\epsilon_0$.



Though this provides an intuitive interpretation of Gauss' Law, we will now proceed to a more formal proof.

1.3.2 Proof of Gauss' Law

We will begin by proving Gauss' Law for a single, pointlike charge q at the origin.

Gauss' Law for a Single Charge

Our starting point is once again Coulomb's Law:

$$\underline{E}(\underline{r}) = kq\frac{\underline{r}}{r^3}$$

Lemma: For a single charge at the origin, $\underline{\nabla} \cdot \underline{E} = 0$ for $\underline{r} \neq 0$

Proof:

$$\begin{aligned} \underline{\nabla} \cdot \left(\frac{\underline{r}}{r^3}\right) &= \left(\underline{\nabla}\frac{1}{r^3}\right) \cdot \underline{r} + (\underline{\nabla} \cdot \underline{r})\frac{1}{r^3} \\ &= -\left(\frac{3\,\underline{r}}{r^5}\right) \cdot \underline{r} + \frac{3}{r^3} = 0 \qquad \text{when } \underline{r} \neq 0 \end{aligned}$$

Gauss' Law for a point charge is:

$$\int_S \underline{E} \cdot \underline{dS} = \begin{cases} 4\pi kq & \text{if the surface } S \text{ encloses the origin} \\ 0 & \text{otherwise} \end{cases}$$

Proof:

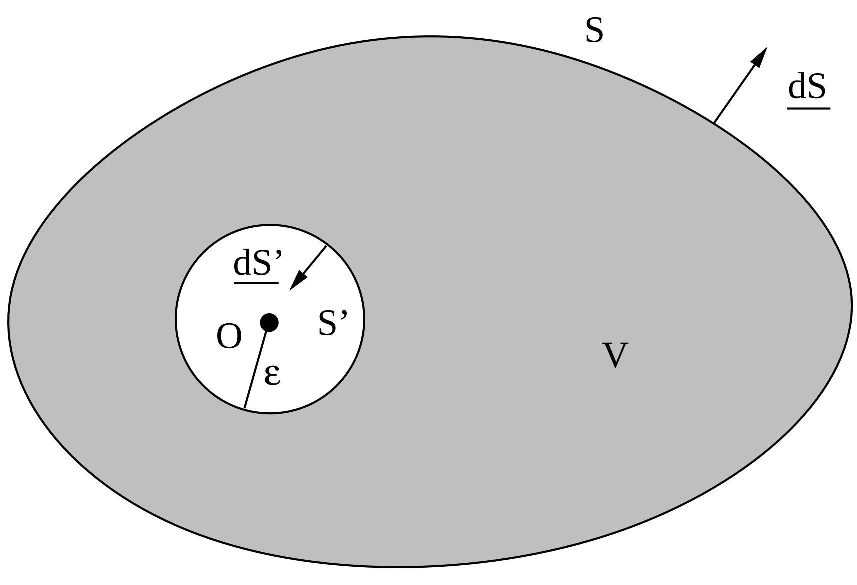
Origin outside V:

$\underline{E}(\underline{r})$ is continuously differentiable, and $\underline{\nabla} \cdot \underline{E} = 0$ everywhere within V. From the divergence theorem,

$$\int_S \underline{E} \cdot \underline{dS} = \int_V (\underline{\nabla} \cdot \underline{E})\, dV = 0 \qquad \text{if origin } \textbf{not within } V$$

Origin inside V:

$\underline{E}(\underline{r})$ is **undefined** at $\underline{r} = 0$. Therefore define V to be the region **between** the closed surfaces S' and S, where S' is a small sphere of radius ϵ centred at the origin:



Now in the region V, $\underline{\nabla} \cdot \underline{E} = 0$. Therefore, by the divergence theorem,

$$\int_V \underline{\nabla} \cdot \underline{E}\, dV = \int_S \underline{E} \cdot \underline{dS} + \int_{S'} \underline{E} \cdot \underline{dS} = 0$$

Introduce spherical polar coordinates (r, θ, ψ). Then on the sphere S' we have:

$$\underline{dS} = -\epsilon^2 \sin\theta\, d\theta\, d\psi\, \underline{e_r},$$

where the *outward* normal for S' points *towards* the origin. Therefore

$$\begin{aligned}\int_{S'} \underline{E} \cdot \underline{dS} &= \int_0^{2\pi} \int_0^{\pi} \left(kq \frac{\underline{e_r}}{r^2}\bigg|_{r=\epsilon} \right) \cdot (-\epsilon^2 \sin\theta \, d\theta \, d\psi \, \underline{e_r}) \\ &= -4\pi kq \qquad \text{independent of } \epsilon\end{aligned}$$

We now let $\epsilon \to 0$, so that $V \to$ total volume within S, and we have

$$\int_S \underline{E} \cdot \underline{dS} = 4\pi kq = \frac{q}{\epsilon_0} \qquad \text{in SI units}$$

so that the theorem is proved.

If the point charge is at the point $\underline{r_1}$, then we have

$$\underline{E}(\underline{r}) = kq \frac{\underline{r} - \underline{r_1}}{|\, \underline{r} - \underline{r_1} \,|^3}.$$

By changing variables to $\underline{\rho} = \underline{r} - \underline{r_1}$ it is easy to show

$$\int_S \underline{E} \cdot \underline{dS} = \begin{cases} 4\pi kq \;\; = q/\epsilon_0 \quad \text{in SI units} & \text{if } \underline{r_1} \in V \\ 0 & \text{otherwise} \end{cases}$$

Gauss' Law for Distribution of Point Charges

We can extend the proof of Gauss' Law for a single charge distribution to a set of N point charges $\{q_i\}$ at $\{\underline{r_i}\}$ using the linear-superposition principle:

$$\underline{E}(\underline{r}) = \sum_{i=1}^{N} \underline{E_i}(\underline{r})$$

where $E(\underline{r})$ is the total electrostatic field at the point $\underline{r}$, and $\underline{E_i}(\underline{r})$ is the electrostatic field at the point $\underline{r}$ due to the charge q_i at the point $\underline{r_i}$. Applying Gauss' Law for point charges proved above, we have

$$\int_S \underline{E_i} \cdot \underline{dS} = \begin{cases} 4\pi kq_i \;\; = q_i/\epsilon_0 \quad \text{in SI units} & \text{if } \underline{r_i} \in V \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$\begin{aligned}\int_S \underline{E} \cdot \underline{dS} &= \sum_i \int_S \underline{E_i} \cdot \underline{dS} = 4\pi k \sum_{i, \underline{r_i} \in V} q_i \\ &= 4\pi k Q = \frac{Q}{\epsilon_0} \qquad \text{(in SI units)}\end{aligned}$$

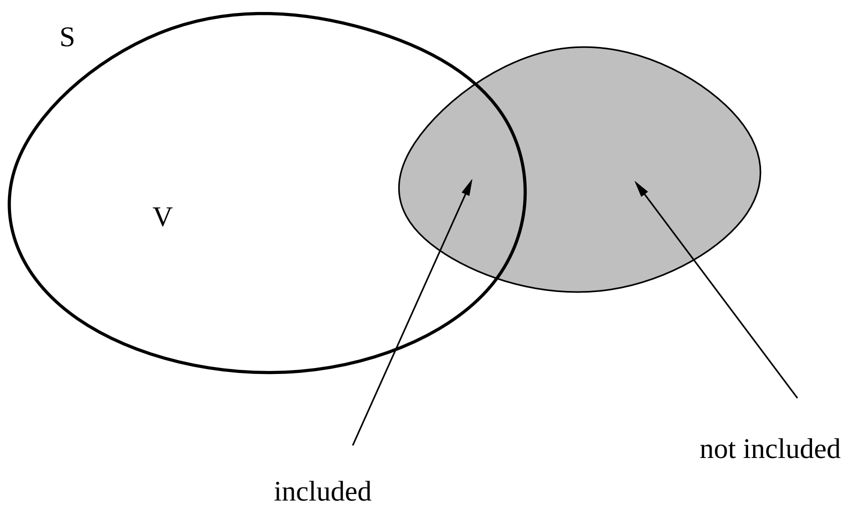
where Q is the sum of the charges contained within the volume V.

Gauss' Law for Continuous Distribution of Charge

This we prove by exact analogy with derivation of the *electrostatic field* for a continuous distribution: we divide up the volume V into elements of volume $\Delta V'$, centred at $\underline{r}'$, and obtain

$$\begin{aligned}\int_S \underline{E}\cdot\underline{dS} &= 4\pi k \sum_{\Delta V'\in V} \rho(\underline{r}')\,\Delta V' \\ &\stackrel{\Delta V'\to 0}{\longrightarrow} 4\pi k \int_V \rho(\underline{r}')\,dV' = 4\pi k Q,\end{aligned}$$

where Q is the total charge contained within the volume V.



1.3.3 Applications of Gauss' Law

Gauss' Law provides a powerful method of determining the electrostatic field where we have **symmetrical** or **cylindrical** symmetry.

Spherical Symmetry

Suppose we have a **spherically symmetric** distribution of charge - or mass - $\rho = \rho(r)$, where $r = |\underline{r}|$. Then the electrostatic field will depend only on r, and therefore must be in the radial direction.

Choose a spherical surface S of radius r, centred on the centre of the charge distribution. Then we have that

$$\int_S \underline{E}(\underline{r}) \cdot \underline{dS} = \int_S E(r)\, \underline{e_r} \cdot \underline{dS} = \int_\Omega E(r)\, r^2\, d\Omega = 4\pi E(r)\, r^2.$$

But by Gauss' Law, we have

$$\int_S \underline{E} \cdot \underline{dS} = 4\pi k Q(r),$$

where $Q(r) = \int_V \rho(r')\, dV$ is the *total charge* contained within the sphere of radius r.

Thus we have

$$\underline{E}(\underline{r}) = \frac{kQ(r)}{r^2}\, \underline{e_r} = \frac{Q(r)}{4\pi\epsilon_0 r^2}\, \underline{e_r} \qquad \text{in SI units.}$$

Note that **outside** a spherically symmetric charge distribution, the field is the **same** as if we had a **point-like** charge $Q(r)$ at the origin.

Example: Consider a thin spherical shell of charge Q. We can say immediately:

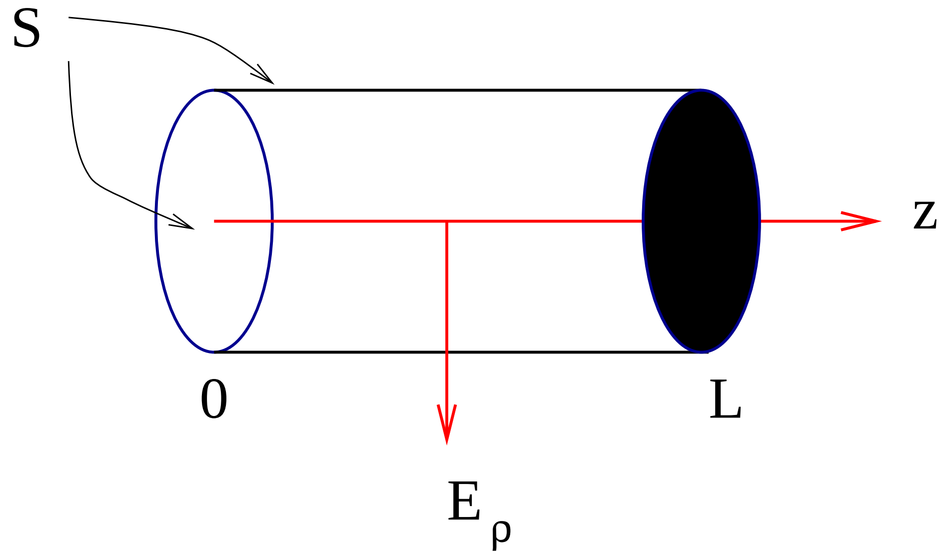
- **Outside** the shell, the electrostatic field is the same as that of the equivalent point charge Q at its centre:

$$\underline{E}(\underline{r}) = \frac{kQ}{r^2}\underline{e_r}$$

- **Inside** the shell, the field is **zero**.

Cylindrical Symmetry

Suppose we have a infinitely long, **cylindrically symmetric** distribution of charge, with the axis of symmetry along the z axis. Introduce cylindrical coordinates (ρ, θ, z). *Note:* we use θ rather than ϕ for the axial coordinate, to avoid confusion with the *potential* that we will be introducing next.
Consider an *element* of length L, and radius ρ, containing a charge $Q(\rho, L)$:



The field will depend solely on ρ, and therefore must be in the $\underline{e_\rho}$ direction, $\underline{E}(\underline{r}) = E(\rho)\underline{e_\rho}$. Applying Gauss' Law to the cylinder we have

$$\int_S \underline{E} \cdot \underline{dS} = 4\pi k Q(\rho, L)$$

Now on the "end-caps", $z = 0$ and $z = L$, $\underline{E} \cdot \underline{dS} = 0$, and therefore

$$\int_S \underline{E} \cdot \underline{dS} = \int_S E(\rho)\underline{e_\rho} \cdot \underline{dS} = E(\rho) \int_S dS = E(\rho) 2\pi\rho L.$$

Thus

$$E(\rho) = \frac{2kQ(\rho, L)}{\rho L} = \frac{2Q(\rho, L)}{4\pi\epsilon_0 \rho L} \qquad \text{in SI units}$$

Example: Infinitely long, thin rod carrying charge λ per unit length. Thus, $Q(\rho, L) = \lambda L$ and we have

$$E(\rho) = \frac{\lambda}{2\pi\epsilon_0 \rho}.$$

We expect the treatment of the rod as *infinitely long* to be a good approximation for a rod of finite length providing

$$w < \rho << l$$

where w and l and the *width* and the *length* of the rod respectively.

1.4 Maxwell's First Equation (ME1)

Our starting point is Gauss' Law:

$$\int_S \underline{E} \cdot \underline{dS} = 4\pi k \int_V \rho(\underline{r}')\, dV'$$

where $\rho(\underline{r}')$ is the charge density. By the *divergence theorem*, we have

$$\int_S \underline{E} \cdot \underline{dS} = \int_V \underline{\nabla} \cdot \underline{E}\, dV',$$

and thus

$$\int_V \{\underline{\nabla} \cdot \underline{E} - 4\pi k \rho\}\, dV' = 0.$$

This applies for *any* volume V, and therefore the integrand itself must vanish:

$$\underline{\nabla} \cdot \underline{E} = 4\pi k \rho = \frac{\rho}{\epsilon_0}. \tag{1.1}$$

This is **Maxwell's First Equation** (ME1). ME1 is essentially an expression of Gauss' law in differential form.

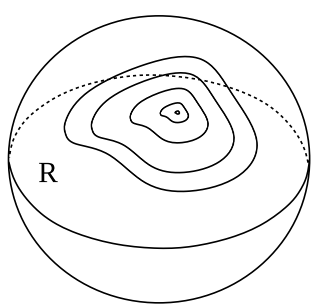
1.5 The Scalar Potential

ME1 has provided us with a differential equation to describe the electric field, $\underline{E}(\underline{r})$, but it would be easier were we able to work with a *scalar* quantity. The **scalar potential** provides a means of so doing.

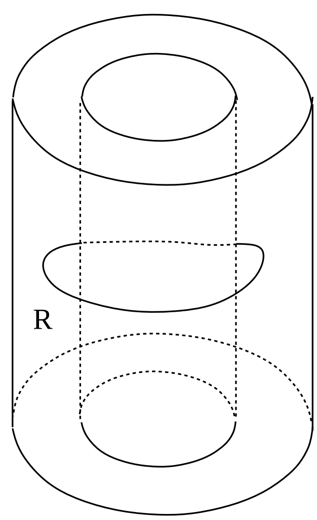
Scalar Potential

- Given a vector field $\underline{A}(\underline{r})$, under what conditions can we write $\underline{A}$ as the **gradient** of a **scalar** field ϕ, viz. $\underline{A}(\underline{r}) = -\underline{\nabla}\,\phi(\underline{r})$, where the minus sign is conventional?
- What can we say about the *uniqueness* of $\phi(\underline{r})$.

Definition: A **simply connected** region R is a region where every **closed** curve in R can be shrunk continuously to a point whilst remaining entirely in R.
Examples:



The inside of a sphere is **simply connected**



The region between two cylinders is **not** simply connected: it's **doubly connected**

1.5.1 Theorems on Scalar Potentials

Let $\underline{A}(\underline{r})$ be a continuously differentiable vector field defined in a simply connected region R. Then the following three statements are equivalent, i.e. **any one implies the other two**:-

1. $\underline{\nabla} \times \underline{A}(\underline{r}) \;=\; 0$ for all points $\underline{r} \in R$

2. (a) $\oint_C \underline{A}(\underline{r}') \cdot \underline{dr}' \;=\; 0,$ where C is any **closed** curve in R

   (b) $\phi(\underline{r}) \equiv -\int_{\underline{r}_0}^{\underline{r}} \underline{A}(\underline{r}') \cdot \underline{dr}'$ does not depend on the path between $\underline{r}_0$ and $\underline{r}$.

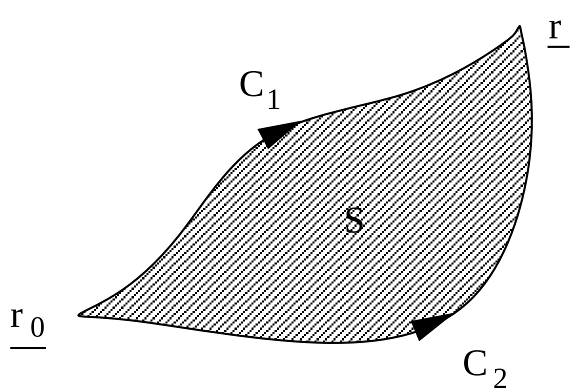
3. $\underline{A}(\underline{r})$ can be written as the **gradient** of a **scalar potential** $\phi(\underline{r})$

$$\underline{A}(\underline{r}) \;=\; -\underline{\nabla}\,\phi(\underline{r}) \quad \text{with} \quad \phi(\underline{r}) \;=\; -\int_{\underline{r}_0}^{\underline{r}} \underline{A}(\underline{r}') \cdot \underline{dr}'$$

   where $\underline{r}_0$ is some **arbitrary fixed point** in R.

Proof that (1) implies (2):

Let $\underline{\nabla} \times \underline{A}(\underline{r}) \;=\; 0$ in R, and consider any two curves, C_1 and C_2 from the point $\underline{r}_0$ to the point $\underline{r}$ in R. Introduce the *closed* curve $C = C_1 - C_2$, and let S be a surface spanning C.



Apply Stokes' theorem:

$$\oint_C \underline{A}(\underline{r}') \cdot \underline{dr}' = \int_S \underline{\nabla} \times \underline{A} \cdot \underline{dS} = 0$$

since $\underline{\nabla} \times \underline{A} = 0$ everywhere. Note that we use $\underline{r}'$ as integration variable to distinguish it from the **end-points** of C_1 and C_2, $\underline{r}_0$ and $\underline{r}$.

Thus we have:

$$\underline{\nabla} \times \underline{A} = 0 \Rightarrow \oint_C \underline{A}(\underline{r}') \cdot \underline{dr}' \; = \; 0 \tag{1.2}$$

for any curve C in R, and the first part of the proof is done.
For the second part of the proof, we observe

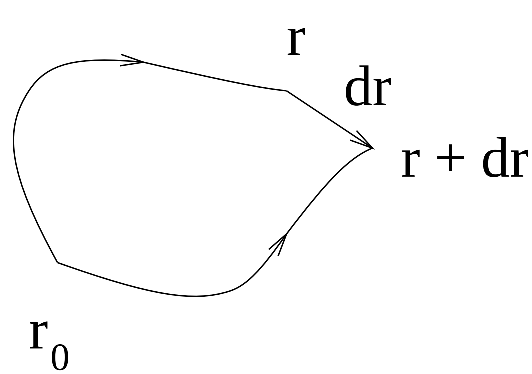
$$\int_{C_1} \underline{A}(\underline{r}') \cdot \underline{dr}' - \int_{C_2} \underline{A}(\underline{r}') \cdot \underline{dr}' = \oint_C \underline{A}(\underline{r}') \cdot \underline{dr}' = 0.$$

Thus the **scalar potential** $\phi(\underline{r})$ of the vector field $\underline{A}(\underline{r})$ defined by

$$\phi(\underline{r}) = -\int_{\underline{r}_0}^{\underline{r}} \underline{A}(\underline{r}') \cdot \underline{dr}'$$

is **independent of the path of integration** joining $\underline{r}_0$ and $\underline{r}$.
Proof that (2) implies (3)
Consider two neighbouring points $\underline{r}$ and $\underline{r} + \underline{dr}$. Define the scalar potential as before:

$$\phi(\underline{r}) \; = \; -\int_{\underline{r}_0}^{\underline{r}} \underline{A}(\underline{r}') \cdot \underline{dr}'$$



Now define the quantity $\delta\phi(\underline{r})$:

$$\begin{aligned}
\delta\phi(\underline{r}) &= \phi(\underline{r}+\underline{dr}) - \phi(\underline{r}) \\
&= \left\{ -\int_{\underline{r}_0}^{\underline{r}+\underline{dr}} \underline{A}(\underline{r}')\cdot\underline{dr}' + \int_{\underline{r}_0}^{\underline{r}} \underline{A}(\underline{r}')\cdot\underline{dr}' \right\} \quad \text{(by definition)} \\
&= -\left\{ \int_{\underline{r}_0}^{\underline{r}+\underline{dr}} \underline{A}(\underline{r}')\cdot\underline{dr}' + \int_{\underline{r}}^{\underline{r}_0} \underline{A}(\underline{r}')\cdot\underline{dr}' \right\} \quad \text{(swapped limits on 2nd } \textstyle\int) \\
&= -\int_{\underline{r}}^{\underline{r}+\underline{dr}} \underline{A}(\underline{r}')\cdot\underline{dr}' \quad \text{(Integral around closed curve vanishes)} \\
&= -\Big[\underline{A}(\underline{r}')\cdot\underline{r}'\Big]_{\underline{r}}^{\underline{r}+\underline{dr}} \quad \text{(for \textit{infinitesimal} } \underline{dr}) \\
&= \underline{A}(\underline{r})\cdot\{-(\underline{r}+\underline{dr}) + \underline{r}\}
\end{aligned}$$

So $$\delta\phi(\underline{r}) = -\underline{A}(\underline{r})\cdot\underline{dr} \tag{1.3}$$

To perform the integral, we used path independence and integrated along the infinitesimal straight line between $\underline{r}$ and $\underline{r}+\underline{dr}$ along which $\underline{A}(\underline{r}')$ is **constant** up to effects of $O(\underline{dr})$.

But, by Taylor's theorem, we also have

$$\delta\phi(\underline{r}) = \frac{\partial\phi(\underline{r})}{\partial x_i}\, dx_i = \underline{\nabla}\,\phi(\underline{r})\cdot\underline{dr} \tag{1.4}$$

Comparing equations (1.3) and (1.4), we obtain

$$\underline{A}(\underline{r}) = -\underline{\nabla}\,\phi(\underline{r})$$

Thus we have shown that **path independence** implies the existence of a **scalar potential** ϕ for the vector field $\underline{A}$.

Proof that (3) implies (1)

$$\underline{A} = \underline{\nabla}\,\phi \quad \Rightarrow \quad \underline{\nabla}\times\underline{A} = \underline{\nabla}\times(\underline{\nabla}\,\phi) \equiv 0$$

because curl (grad ϕ) is identically zero (i.e. it's zero for *any* scalar field ϕ).

1.5.2 Terminology

Such a vector field is called

- **Irrotational:** $\underline{\nabla} \times \underline{A}(\underline{r}) \; = \; 0 \quad \Leftrightarrow \quad \oint_C \underline{A}(\underline{r}') \cdot \underline{dr}' \; = \; 0$
  If you look in older textbooks, you will sometimes see *rot* rather than *curl.*

- **Conservative:** e.g. if $\underline{A}$ = force, then ϕ is potential energy and total energy is conserved (see later).

- The field $\phi(\underline{r})$ is the **scalar potential** for the vector field $\underline{A}(\underline{r})$.

1.5.3 Uniqueness

$\phi(\underline{r})$ is **uniquely determined** up to a **constant**.

Proof:

Let ϕ and ψ be scalar potentials obtained by **different** choices of $\underline{r}_0$. Then

$$\underline{\nabla}\,\phi \; - \; \underline{\nabla}\,\psi \; = \; \underline{A} \; - \; \underline{A} \; = \; 0$$

Therefore

$$\underline{\nabla}\,(\psi \; - \; \phi) \; = \; 0$$

Integration of this equation wrt any of x, y, or z gives

$$\psi \; - \; \phi \; = \; \text{constant}$$

Therefore

$$\psi \; = \; \phi \; + \; \text{constant}$$

The **absolute value** of a scalar potential has no meaning, only **potential differences** are significant.

1.5.4 Existence of Scalar Potential for Electrostatic Field

After the digression on subject of the scalar potentials, it is time to show that the electrostatic field is, indeed, irrotational.
The central result of this chapter was the expression for the electrostatic field due to a continuous charge distribution

$$\underline{E}(\underline{r}) = k \int_V \frac{\rho(\underline{r}')(\underline{r} - \underline{r}')dV'}{| \underline{r} - \underline{r}' |^3}.$$

Thus we have

$$\begin{aligned}
\underline{\nabla} \times \underline{E}(\underline{r}) &= \int \underline{\nabla} \times \left\{ \frac{\rho(\underline{r}')(\underline{r} - \underline{r}')}{| \underline{r} - \underline{r}' |^3} \right\} dV' \\
&= \int_V \rho(\underline{r}') \left\{ \underline{\nabla} \left(\frac{1}{| \underline{r} - \underline{r}' |^3} \right) \times (\underline{r} - \underline{r}') + \frac{1}{| \underline{r} - \underline{r}' |^3} \underline{\nabla} \times (\underline{r} - \underline{r}') \right\} dV' \\
&= \int \rho(\underline{r}') \left\{ \frac{-3(\underline{r} - \underline{r}')}{| \underline{r} - \underline{r}' |^5} \times (\underline{r} - \underline{r}') + 0 \right\} dV' \\
&= 0
\end{aligned}$$

where the derivatives act only on the *unprimed* indices.

> **The electrostatic field $\underline{E}(\underline{r})$ can be written in terms of a scalar potential $\underline{E}(\underline{r}) = -\underline{\nabla}\phi(\underline{r})$**

1.5.5 Methods for finding Scalar Potentials

We have shown that the scalar potential $\phi(\underline{r})$ for an irrotational vector field $\underline{A}(\underline{r})$ can be constructed via

$$\phi(\underline{r}) = - \int_{\underline{r}_0}^{\underline{r}} \underline{A}(\underline{r}') \cdot \underline{dr}'$$

for some suitably chosen $\underline{r}_0$ and any path which joins $\underline{r}_0$ and $\underline{r}$. Sensible choices for $\underline{r}_0$ are often $\underline{r}_0 = 0$ or $\underline{r}_0 = \infty$.

We have also shown that the line integral is *independent* of the path of integration between the endpoints. Therefore, a convenient way of evaluating such integrals is to integrate along a **straight line** between the points $\underline{r}_0$ and $\underline{r}$. Choosing $\underline{r}_0 = 0$, we can write this integral in parametric form as follows:

$$\begin{aligned} \underline{r}' &= \lambda\, \underline{r} \quad \text{where} \quad \{0 \leq \lambda \leq 1\} \\ \text{so} \quad \underline{dr}' &= d\lambda\, \underline{r} \qquad \text{and therefore} \\ \phi(\underline{r}) &= -\int_{\lambda=0}^{\lambda=1} \underline{A}(\lambda\, \underline{r}) \cdot (d\lambda\, \underline{r}) \end{aligned}$$

Example:

Let $\underline{A}(\underline{r}) \;=\; (\underline{a} \cdot \underline{r})\, \underline{a}$ where $\underline{a}$ is a constant vector.

It is easy to show that $\underline{\nabla} \times ((\underline{a} \cdot \underline{r})\, \underline{a}) = 0$. Thus

$$\begin{aligned} \phi(\underline{r}) &= -\int_0^{\underline{r}} \underline{A}(\underline{r}') \cdot \underline{dr}' \\ &= -\int_0^{\underline{r}} \Big((\underline{a} \cdot \underline{r}')\, \underline{a}\Big) \cdot \underline{dr}' \\ &= -\int_0^{1} \Big((\underline{a} \cdot \lambda\, \underline{r})\, \underline{a}\Big) \cdot (d\lambda\, \underline{r}) \\ &= -(\underline{a} \cdot \underline{r})^2 \int_0^1 \lambda\, d\lambda \\ &= -\frac{1}{2}(\underline{a} \cdot \underline{r})^2 \end{aligned}$$

Of course, this is all rather artifical. What we really want to do is to obtain ϕ and $\underline{A}$ *from first principles.*

1.5.6 Singular Fields

We have seen that, for the case of a point-charge at the origin, the electric field is *singular* at $\underline{r} = 0$. In such cases, it is not possible to obtain the corresponding scalar potential at $\underline{r}$ by integration along a path from the origin. All is not lost - remember that the starting point for our path is arbitrary, and often it is convenient to take it at infinity.

Example: Electric field due to point charge at $\underline{r} = \underline{0}$: $\underline{E}(\underline{r}) = kq\underline{r}/r^3$, so that $\underline{E}(\underline{r} = \underline{0})$ is singular, and hence undefined. As in the proof of Gauss' law, our region R must exclude an infinitessimal sphere centred at $\underline{r} = 0$.

Here we choose a path from $r_0 = \infty$, yielding

$$\begin{aligned}\phi(\underline{r}) &= -\int_\infty^{\underline{r}} \underline{E}(\underline{r}') \cdot \underline{dr'} = -\int_\infty^1 \underline{E}(\lambda \underline{r}) \cdot d\lambda\, \underline{r} \\ &= -kq \int_\infty^1 \frac{d\lambda}{\lambda^2} \frac{r^2}{r^3} \\ &= kq\frac{1}{r}\end{aligned}$$

Thus we have the famous $1/r$ potential due to a point charge.

Because of the *linearity* of the gradient operation, we can impose the linear superposition principle on the potential, and hence obtain an expression for the potential due to an extended charge distribution:

$$\phi(\underline{r}) = k \int_V \frac{\rho(\underline{r}')dV'}{|\ \underline{r} - \underline{r}'\ |} \tag{1.5}$$

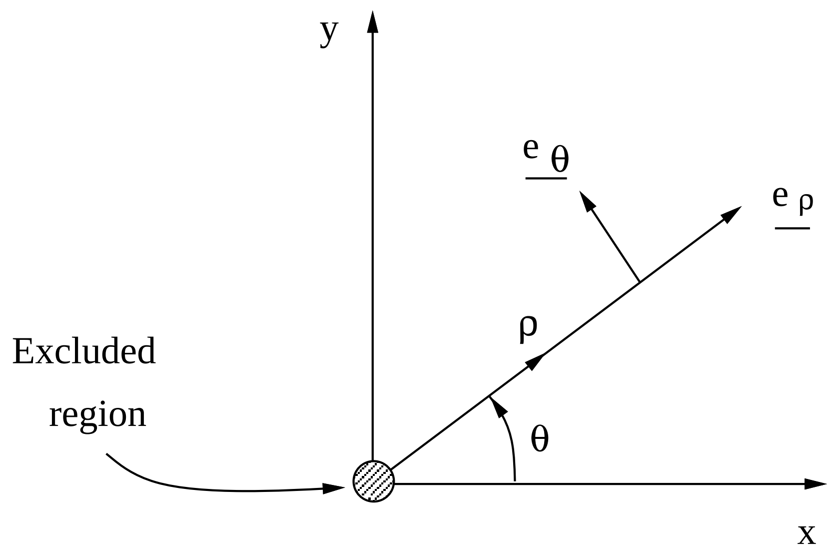
1.5.7 Multiply-connected Regions

In this case, $\underline{\nabla} \times \underline{A} = 0$ does not imply the existence of a scalar potential function.

Example: Work using **cylindrical coordinates** (ρ, ϕ, z). A vector field $\underline{A}$, with

$$A_\rho = A_z = 0, A_\phi = \frac{a}{\rho}$$

where a is a constant, is defined outside an infinitessimal cylinder about the z-axis, where A_ϕ is **singular**. This region is **doubly connected** (c.f. example above where we exclude an infinitessimal sphere).



Then we have (**Exercise!**):

- $\underline{\nabla} \times \underline{A} = 0$
- $\oint_C \underline{A} \cdot \underline{dr} \neq 0$ where C is a circular path enclosing the z-axis

In this case, the "potential" would depend on the choice of path, and in particular the *winding number* - the number of times that a path wraps around the z-axis.
Examples: Vortices in superconductors, Cosmic strings...

1.5.8 Conservative Forces and Physical Interpretation of Potenital

To see how the name *conservative field* arises, consider a vector field $\underline{F}(\underline{r})$ corresponding to the only force acting on some test particle of mass m. The **work done** by the force in going around a closed curve C is

$$W = \oint_C \underline{F}(\underline{r}) \cdot \underline{dr}$$

For a *conservative* force, $\underline{\nabla} \times \underline{F} = 0$, the earlier theorems tell us:

- The total work done by the force in moving the particle around a closed curve is zero.

- We can write the force in terms of a scalar potential

$$\underline{F}(\underline{r}) = -\underline{\nabla}U(\underline{r}).$$

  where the minus sign is conventional (see later).

We will now show that for a conservative force, the **total energy** is **constant** in time.

Proof

The particle moves under the influence of Newton's Second Law:

$$m\underline{\ddot{r}} = \underline{F}(\underline{r}).$$

Consider a small displacement $\underline{dr}$ taking time dt along the path followed by the particle. Then we have

$$m\underline{\ddot{r}} \cdot \underline{dr} = \underline{F}(\underline{r}) \cdot \underline{dr} = -\underline{\nabla}U(\underline{r}) \cdot \underline{dr}.$$

Integrating this expression along the path from $\underline{r_A}$ at time $t = t_A$ to $\underline{r_B}$ at time $t = t_B$ yields

$$m\int_{\underline{r_A}}^{\underline{r_B}} \underline{\ddot{r}} \cdot \underline{dr} = -\int_{\underline{r_A}}^{\underline{r_B}} \underline{\nabla}U(\underline{r}) \cdot \underline{dr}. \tag{1.6}$$

We can simplify the left-hand side of equation 1.6 to obtain

$$m\int_{\underline{r_A}}^{\underline{r_B}} \underline{\ddot{r}} \cdot \underline{dr} = m\int_{t_A}^{t_B} \underline{\ddot{r}} \cdot \underline{\dot{r}}dt = m\int_{t_A}^{t_B} \frac{1}{2}\frac{d}{dt}\underline{\dot{r}}^2 dt = \frac{1}{2}m[v_B^2 - v_A^2],$$

where v_A and v_B are the magnitudes of the velocities at the points labelled by A and B respectively.

To integrate the right-hand side of equation 1.6, we appeal to Taylor's theorem to note that

$$\underline{\nabla}U(\underline{r}) \cdot \underline{dr} = \frac{\partial U}{\partial x_i}dx_i$$

is the change is U when we move from $\underline{r}$ to $\underline{r} + \underline{dr}$. Thus we have

$$-\int_{\underline{r_A}}^{\underline{r_B}} \underline{\nabla} U(\underline{r}) \cdot \underline{dr} = -\int_{\underline{r_A}}^{\underline{r_B}} dU = U_A - U_B$$

where U_A and U_B are the values of the potential U at $\underline{r_A}$ and $\underline{r_B}$, respectively. Thus we have that

$$\frac{1}{2}mv_A^2 + U_A = \frac{1}{2}mv_B^2 + U_B$$

- The first term on both sides we recognise as the **kinetic energy**
- The second term we identify as the **potential energy**

The **Total Energy**

$$E = \frac{1}{2}mv^2 + U$$

is **conserved**, i.e. *constant in time.*

We have seen that the existence of a scalar potential is associated with the *irrotational* or *conservative* nature of a vector field. Where the vector field corresponds to a *force*, we have a neat physical motivation for the name: a force is conservative if the work done in going around a closed path is zero, and if a particle moves solely under the influence of that force, then the energy is conserved.

Physical Interpretation of $\phi(\underline{r})$

In electrostatics, the force $\underline{F}$ acting on a charge q due to an electrostatic field $\underline{E}$ is $\underline{F}(\underline{r}) = q\underline{E}(\underline{r})$. Now $\underline{E}(\underline{r}) = -\underline{\nabla}\phi(\underline{r})$ so that

$$\underline{F}(\underline{r}) = -\underline{\nabla}(q\phi(\underline{r})).$$

We have seen that the (conservative) force acting on a particle is minus the gradient of its potential energy: $\underline{F}(\underline{r}) = -\underline{\nabla}U(\underline{r})$.

> The **potential energy** $U(\underline{r})$ of a charge q situated at $\underline{r}$ in an electrostatic potential $\phi(\underline{r})$ is
>
> $$U(\underline{r}) = q\phi(\underline{r}). \qquad (1.7)$$

1.5.9 Potential Energy of Charge Distribution

For the case where ϕ vanishes at infinity, the potential $U(\underline{r})$ is the work done, W, in bringing the charge q from infinity to the point $\underline{r}$. We will now consider the work done in assembling a set of point charges q_i at $\underline{r}_i$, $i = 1, \ldots, N$.

We do this by bringing each charge i in turn, one at a time, to position $\underline{r}_i$, and then fixing it in position. The work done in bringing charge i is

$$W_i = \frac{q_i}{4\pi\epsilon_0} \sum_{j=1}^{i-1} \frac{q_j}{|\underline{r}_i - \underline{r}_j|}$$

and thus the total work done in assembling the charges is

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=2}^{N} \sum_{j=1}^{i-1} \frac{q_i q_j}{|\underline{r}_i - \underline{r}_j|} = U,$$

where U is the potential energy of the system. We can write this in a more symmetric form as

$$U = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{q_i q_j}{|\underline{r}_i - \underline{r}_j|}$$

where we do not include the **self-energy** term, $i = j$.

We can generalise this to a *continuous* charge distribution in the usual way, viz

$$U = \frac{1}{8\pi\epsilon_0} \int d^3r\, d^3r' \frac{\rho(\underline{r})\rho(\underline{r}')}{|\underline{r} - \underline{r}'|},$$

and we now use eqn. 1.5 to write

$$U = \frac{1}{2} \int \rho(\underline{r})\phi(\underline{r}) dV, \qquad (1.8)$$

analogous to eqn. 1.7.

We can also interpret the potential energy in terms of the *electric field*, by using **ME1**

$$\begin{aligned} U &= \frac{\epsilon_0}{2}\int dV\, \underline{\nabla}\cdot\underline{E}(\underline{r})\phi(\underline{r}) \\ &= -\frac{\epsilon_0}{2}\int dV\, \underline{E}(\underline{r})\cdot\underline{\nabla}\phi(\underline{r}) \quad (\textit{Integration by parts}) \\ &= \frac{\epsilon_0}{2}\int dV\, |\underline{E}|^2. \end{aligned} \tag{1.9}$$

We now identify the integrand as the **energy density**

$$u(\underline{r}) = \frac{\epsilon_0}{2}|\underline{E}(\underline{r})|^2.$$

1.6 Laplace's and Poisson's Equation

We are now ready to derive a differential equation for the potential. Our starting point is **Maxwell's First Equation** (ME1), derived earlier:

$$\underline{\nabla} \cdot \underline{E} = 4\pi k \rho = \frac{\rho}{\epsilon_0}.$$

We now make use of the irrotational nature of $\underline{E}(\underline{r})$ to write $\underline{E} = -\underline{\nabla}\phi(\underline{r})$. Thus ME1 becomes

$$\boxed{\nabla^2 \phi(\underline{r}) = -4\pi k \rho(\underline{r}) = -\rho(\underline{r})/\epsilon_0 \quad \text{in SI units}}$$

where $\nabla^2 \phi(\underline{r}) \equiv \underline{\nabla} \cdot (\underline{\nabla}\phi(\underline{r})) \equiv \partial^2 \phi(\underline{r})/\partial x_i^2$.

- This equation is **Poisson's Equation**. $\rho(\underline{r})$ is the **source** for the electrostatic potential $\phi(\underline{r})$.
- If we have that the **source** $\rho(\underline{r}) \equiv 0$ everywhere, then this equation becomes

  $$\nabla^2 \phi = 0.$$

  This is **Laplace's Equation**.

These are two of the most important equations in physics. They, or close variants, occur in:

- Electromagnetism, as above
- Gravitation, with $k \to -G$, ρ the *mass density*, and ϕ the gravitational potential
- Fluid dynamics, for the *irrotational* flow of a fluid.

1.6.1 Uniqueness of Solutions of Laplace's and Poisson's Equation

Laplace's and Poisson's equations are *linear, second order, partial differential equations*; to determine a solution we have also to specify **boundary conditions**.
Example: One-dimensional problem

$$\frac{d^2\phi(x)}{dx^2} = \lambda$$

for $x \in [0, L]$, where λ is a constant. This has solution

$$\phi(x) = \frac{1}{2}\lambda x^2 + Ax + B$$

where A, B are constants. To determine these constants, we might specify the values of $\phi(x = 0)$ and $\phi(x = L)$, i.e. the values on the boundary.

Consider the solution of Poisson's Equation within a finite volume V, bounded by a *closed* surface S. Boundary conditions are classified as:

- **Dirichlet** boundary conditions, where we require

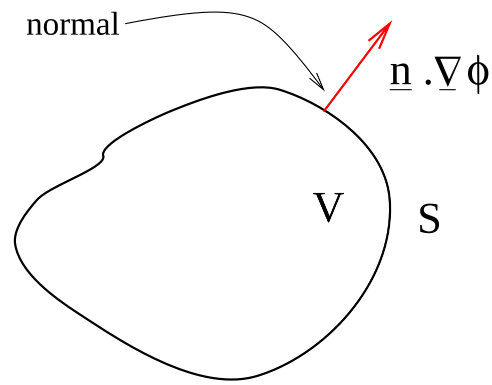
  $$\phi(\underline{r}) = f(\underline{r}) \qquad \text{on surface } S,$$

  i.e. we specify the *value* of $\phi(\underline{r})$ on the boundary. **Example:** Electrostatic potential inside a *conductor*, with ϕ specified on the boundaries.

- **Neumann** boundary conditions, where we require

  $$\underline{n} \cdot \underline{\nabla}\phi(\underline{r}) = \frac{\partial \phi}{\partial n} = g(\underline{r}) \qquad \text{on surface } S,$$

  where $\underline{n}$ is a unit vector normal to the surface S, i.e. we specify the *normal derivative* of ϕ on the boundary.



Example: Electrostatic potential inside S, with *charge* on S specified on the boundaries.

We will proceed to show that the solutions of Laplace's and Poisson's are **unique**, up to a constant (*Neumann*), if subject to either of the above boundary conditions. We begin with a couple of useful vector identities

Green's First Identity and Green's Theorem

We begin with a couple of identities that will be useful both in this proof and later.
Let ψ_1 and ψ_2 be two continuously differentiable, arbitrary scalar fields defined in a volume V bounded by a closed surface S. Introduce the vector field $\underline{A}(\underline{r}) = \psi_1 \underline{\nabla} \psi_2$. From the divergence theorem, we have

$$\int_V \underline{\nabla} \cdot \underline{A}\, dV = \int_S \underline{A} \cdot \underline{n}\, dS$$

where $\underline{n}$ is the unit *outward* normal to the surface S.
We now apply the vector identity

$$\underline{\nabla} \cdot \underline{A} = \psi_1 \nabla^2 \psi_2 + \underline{\nabla} \psi_1 \cdot \underline{\nabla} \psi_2,$$

to obtain

$$\int_V (\psi_1 \nabla^2 \psi_2 + \underline{\nabla} \psi_1 \cdot \underline{\nabla} \psi_2)\, dV = \int \psi_1 \underline{\nabla} \psi_2 \cdot \underline{n}\, dS. \tag{1.10}$$

This is known as **Green's first identity**.
If we write down eqn. 2.5 with ψ_1 and ψ_2 interchanged, and take the difference of the two equations, we obtain

$$\int_V (\psi_1 \nabla^2 \psi_2 - \psi_2 \nabla^2 \psi_1)\, dV = \int_S (\psi_1 \underline{\nabla} \psi_2 - \psi_2 \underline{\nabla} \psi_1) \cdot \underline{n}\, dS. \tag{1.11}$$

This identity is **Green's Theorem**.

1.6.2 Proof of Uniqueness of Solutions of Laplace's and Poisson's Equations

We now proceed to the formal proof. Let $\phi_1(\underline{r})$ and $\phi_2(\underline{r})$ be solutions of Poisson's equation $\nabla^2\phi_i = -\rho/\epsilon_0$ inside a volume V bounded by surface S, satisfying **either**:

1. Dirichlet boundary conditions

$$\phi_i(\underline{r}) = f(\underline{r}) \qquad \text{for } \underline{r} \text{ on surface } S$$

2. Neumann boundary conditions

$$\underline{n}\cdot\underline{\nabla}\phi_i(\underline{r}) = g(\underline{r}) \qquad \text{for } \underline{r} \text{ on surface } S$$

where $f(\underline{r})$ and $g(\underline{r})$ are continuous functions defined on the surface S.
Consider the function

$$\psi(\underline{r}) = \phi_1(\underline{r}) - \phi_2(\underline{r}).$$

Then ψ satisfies *Laplace's* equation:

$$\nabla^2\psi(\underline{r}) = 0 \qquad \text{in } V$$

with **either**

1. $\psi(\underline{r}) = 0$ for $\underline{r}$ on surface S - **Dirichlet**.
2. $\underline{n}\cdot\underline{\nabla}\psi(\underline{r}) = 0$ for $\underline{r}$ on surface S - **Neumann**

We now apply *Green's first identity* for the case $\psi_1 = \psi_2 = \psi$, and obtain

$$\begin{aligned}
\int_V |\underline{\nabla}\psi|^2\, dV &= \int_V (\psi\nabla^2\psi + |\underline{\nabla}\psi|^2)\, dV \qquad (\text{since } \nabla^2\psi = 0 \text{ in } V)\\
&= \int_S \psi\underline{\nabla}\psi\cdot\underline{n}\, dS \qquad (\text{from eqn. 2.5})\\
&= 0
\end{aligned} \tag{1.12}$$

since either $\psi(\underline{r}) = 0$ or $\underline{\nabla}\psi \cdot \underline{n} = 0$ on surface S. Now $|\underline{\nabla}\psi(\underline{r})|^2$ is **positive indefinite**, i.e.

$$|\underline{\nabla}\psi(\underline{r})|^2 \geq 0$$

for all $\underline{r} \in V$. Therefore, using equation (1.12), we have that $\underline{\nabla}\psi(\underline{r}) = 0$ everywhere in V, and thus

$$\psi(\underline{r}) = \text{constant}$$

for all $\underline{r} \in V$.
Thus we have

- **Dirichlet Problem**: $\psi(\underline{r})$ is continuous at surface S, and $\psi(\underline{r}) = 0$ on the surface. Therefore $\psi(\underline{r}) = 0$ everywhere, and solution is **unique**.
- **Neumann Problem**: $\underline{\nabla}\psi(\underline{r}) \cdot \underline{n} = 0$ on the surface S, and the constant undetermined. Solution is **unique up to an additive constant**.

Some observations on the proof:

- We can specify **either** Dirichlet or Neumann boundary conditions at each point on the boundary, but not **both**. To specify both is inconsistest, since the solution is then overdetermined.
- However, we can specify either Dirichlet **or** Neumann boundary conditions on **different** parts of the surface.
- The uniqueness property means we can use any method we wish to obtain the solution - if it satisfies the correct boundary conditions, and is a solution of the equation, then it is the correct solution. A good example: *Method of Images*, to be covered in the next chapter.

1.6.3 Uniqueness Theorem in an Infinite Region

We need a slight refinement of the proof if the region is infinite, ie if S contains a "surface at infinity".

The two solutions are the same provided they agree to $O(1)$ at infinity.

Proof:

We merely need to show that this is a sufficient condition to ensure that the surface integral vanishes at infinity.

Consider a **sphere**, radius r, area $S = 4\pi r^2$. Suppose

$$\psi = \phi_1 - \phi_2 = O(1/r) \quad \text{as} \quad r \to \infty \qquad \text{so that} \qquad \underline{\nabla}\,\psi = O(1/r^2)$$

then $$\int_S \psi\,\underline{\nabla}\,\psi \cdot \underline{dS} = O(1/r)$$

which vanishes as $r \to \infty$. The remainder of the proof is unchanged.

- If the potential is due to a **localised** charge distribution, then, by the multipole theorem, it falls off as least as fast as $(1/r)$ as $r \to \infty$. Hence, the difference $\psi = \phi_1 - \phi_2$ must also fall off as least as fast as $(1/r)$, and the uniqueness theorem applies.

- Sometimes a **uniform field** is specified at infinity. For example, if the uniform field $\underline{E}$ is in the z direction, then

  $$\phi(\underline{r}) = K - E\,z$$

  where K is a constant. In this case, the uniqueness theorem holds because the 'two' solutions must satisfy the boundary condition

  $$\phi(\underline{r}) + E\,z \to K + O(1/r)$$

  as $r \to \infty$.

The next couple of chapters of this course will be concerned with solving such *boundary-value* problems. We will conclude this chapter by discussing the boundary conditions to impose on our solutions, and in particular the boundary conditions at a conductor.

1.7 Boundary Conditions at a Conductor

- In a conductor, electrons are able to move freely so as to set up a charge distribution.

- In the presence of an external electrostatic field, a charge distribution is generated under the influence of this field, and itself give rise to an electrostatic field.

- Once equilibrium is attained (about 10^{-18} secs. for a good conductor), no current flows, and thus *the electric field $\underline{E}$ is zero throughout the body of a conductor.*

- If the electric field vanishes in a conductor, the potential must be constant. This provides the defining property of a conductor, namely that the boundary of a conductor is an equipotential surface.

On the boundary of a conductor, $\phi(\underline{r}) = \text{const}$.

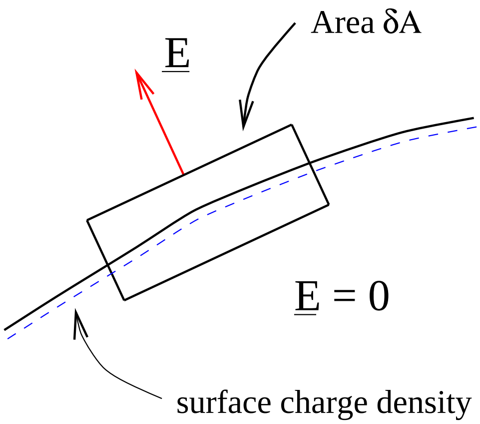
- Conventionally, we take $\phi = 0$ for an *earthed* conductor.

- The electric field at the surface of a conductor is **normal** to the surface; a tangential field would give rise to a charge flow along the surface.

1.7.1 Surface Charge Density at a Conductor

Within a conductor, the electrostatic field $\underline{E}$ must be zero. However, the field is zero because of an induced charge density sufficient to annul the external field. Now ME1 tells us that $\underline{\nabla} \cdot \underline{E} = \rho/\epsilon_0$, where ρ is the charge density. So if $\underline{E}$ is zero within the conductor, the charge density must be zero. So where does the induced charge density reside?
The charge density is confined solely to the surface of the conductor

We can compute this *surface charge density* using Gauss' Law.



Consider applying Gauss' Law to the infinitesimal "pill-box" of height δh and area δA, as shown. Within the conductor, $\underline{E} = 0$, and at the surface of the conductor $\underline{E}$ is *normal* to the surface.

Therefore we have

$$\underline{E}.\underline{n}\,\delta A = \delta A\,\sigma/\epsilon_0$$

where σ is the surface charge density, and $\underline{n}$ is the outward normal to the surface of the conductor.

Thus we have that the *surface charge density* is proportional to the **discontinuity** in the normal electrostatic field at the conductor.

$$\boxed{\underline{E}\cdot\underline{n} = \sigma/\epsilon_0}$$

Note: the **surface charge density** discussed here is different to a **sheet** of charge of density σ per unit area discussed earlier in the course. The latter may best be viewed as a charge distribution in an insulator, i.e. a fixed charge distribution. Unfortunately, the two terms are often confused in the literature, and indeed probably in these lectures!

1.7.2 Capacitance and Potential Energy of Conductors

Consider now a set of N isolated conductors, with charge q_i, $i = 1 \ldots N$, and with no external electric field. Then each conductor is an equipotential ϕ_i, and the charges reside on the surface of the conductor.

Thus the potential energy of this system is

$$U = \frac{1}{2}\int dV\,\rho(\underline{r})\phi(\underline{r}) = \frac{1}{2}q_i\phi_i.$$

The potentials ϕ_i and the charges q_i are not independent. In particular, for a given set of charges q_i the potentials are determined by the solutions of the field equations. Because of the linearity of the field equations, the relationship between the ϕ's and the q's must be linear, i.e.

$$\phi_i = \sum_{j=1}^{N} P_{ij}q_j,$$

which in matrix form may be written

$$\vec{\phi} = P\vec{q}.$$

We can invert this equation to obtain

$$q_i = \sum_{j=1}^{N} C_{ij}\phi_j \tag{1.13}$$

where, formally, $C = P^{-1}$.

The diagonal elements of this matrix C_{ii} are the **capacitances**, whilst the off-diagonal elements $C_{ij}, i \neq j$ are the **coefficients of induction**. We can use eqn. 1.13 to write the *potential energy* of a system of conductors in terms either of the potentials or charges alone:

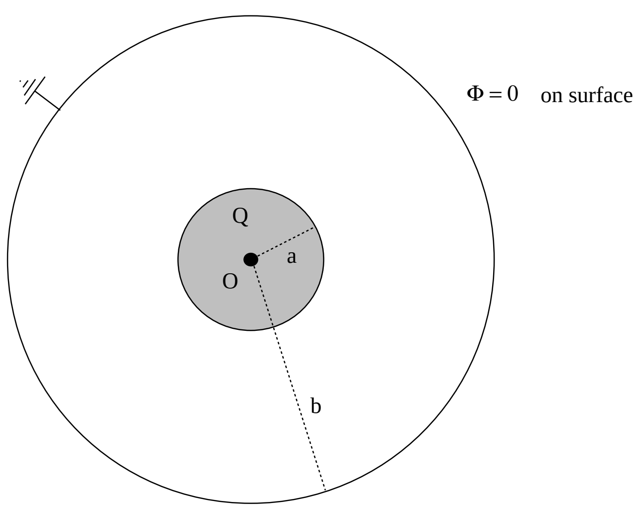
$$U = \frac{1}{2}\sum_{ij}\phi_i C_{ij}\phi_j \equiv \frac{1}{2}\sum_{ij} q_i C^{-1}_{ij} q_j$$

Chapter 2

Boundary-Value Problems in Electrostatics

In this chapter we will examine solutions to Poisson's and Laplace's equations in electrostatics. Before we proceed to a formal solution of Poisson's equation, we will look at a few simple solutions. In the next section we will exploit the *uniqueness theorem* in a particularly neat way through the *Method of Images*, but first, back to Gauss' Law for a simple example...

Example: Charged sphere inside grounded, conducting shell.
A sphere of radius a, carrying a charge Q, is placed inside an grounded, conducting sphere of radius b ($b > a$). Find the potential in the region $a \leq r \leq b$.



Thus we have to solve Poisson's equation, subject to the boundary conditions $\phi(\underline{r}) = 0$ for $r = b$. Apply Gauss' Law to the region $a < r < b$:

$$\underline{E}(\underline{r}) = \frac{Q}{4\pi\epsilon_0 r^2}\underline{e_r}; \qquad a \le r \le b$$

for which the potential is

$$\phi = \frac{Q}{4\pi\epsilon_0 r} + \phi_0; \qquad a \le r \le b$$

where ϕ_0 is a constant.

The boundary conditions tell us that ϕ vanishes at $r = b$. Thus we have

$$\phi = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{b}\right); \qquad a \le r \le b.$$

Let us check that our solution for $\phi(\underline{r})$ satisfies Poission's equation for $a \le r \le b$. We are implicitly working in spherical polars $(r,\,\theta,\,\psi)$, therefore (from your favourite vector-calculus course, or back of *Jackson*):

$$\begin{aligned}
\nabla^2\phi(r,\theta,\psi) &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\left\{\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{\partial^2\phi}{\partial\psi^2}\right\} \\
&= \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\left(\frac{-1}{r^2}\right)\right\} \\
&= \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}\frac{\partial}{\partial r}\Big(-1\Big) = 0
\end{aligned}$$

Hence $\phi(\underline{r})$ satisfies $\nabla^2\phi(\underline{r}) = 0$ in the charge free region $a \le r \le b$, and satisfies the boundary condition $\phi(b) = 0$ on the surface. Therefore, *it is the unique solution of Poisson's equation in this region.* Of course, due to spherical symmetry, $\phi(\underline{r})$ doesn't depend on θ or ψ, and therefore the calculation of the $\nabla^2\phi(\underline{r})$ is particularly simple.

Finally, let us find the surface charge density on the conductor. At the boundary of the conductor,

$$\underline{E} = \frac{Q}{4\pi\epsilon_0 b^2}\underline{e_r}$$

Thus the surface charge density is given by

$$\sigma = -\frac{Q}{4\pi b^2}$$

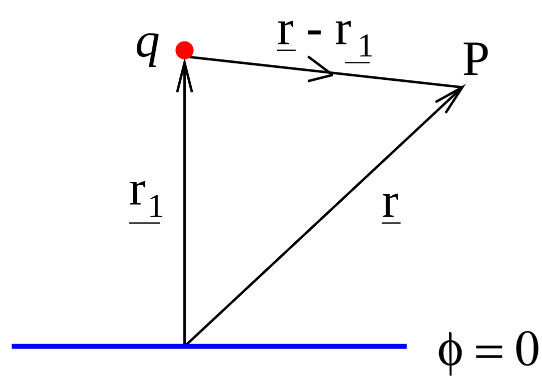
which is negative, as expected. Indeed the total induced charge on the conductor is equal and opposite to that of the charge distribution.

Once again, the method was particularly simple in this case because of *spherical symmetry*. Similar simplifications occur in the case of *cylindrical symmetry*.

2.1 Method of Images

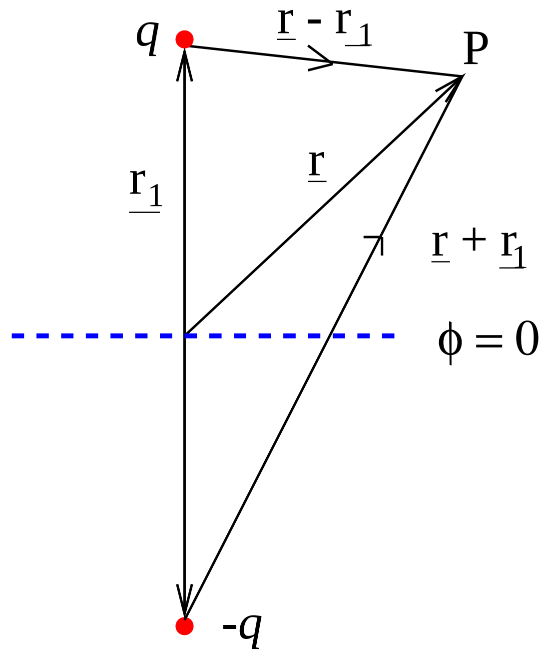
The uniqueness property of the solutions of Laplace's and Poisson's Equations leads to a neat method of obtaining their solution in particular geometric cases.

Consider a charge q placed at $\underline{r_1} = h\underline{k}$ above an infinite grounded conducting plane at $z = 0$, as shown on the right. Then on the conducting plane the potential must vanish.



Now consider a system with a charge q placed at $\underline{r}_1$, and a charge $-q$ placed at $-\underline{r}_1$ in the absence of the conducting plane, as shown on the right. The potential $\phi(\underline{r})$ is

$$\phi(\underline{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{|\underline{r} - \underline{r}_1|} + \frac{-q}{4\pi\epsilon_0}\frac{1}{|\underline{r} + \underline{r}_1|}.$$

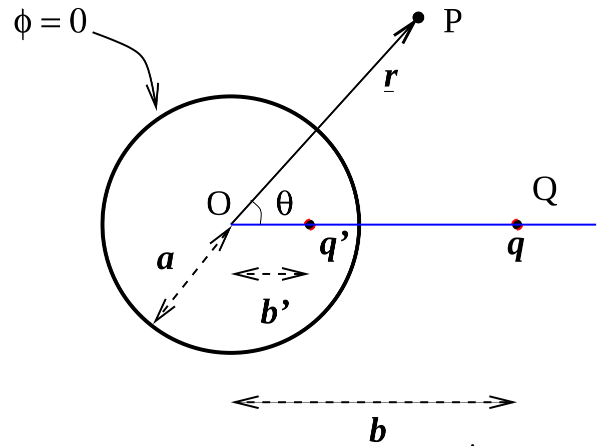


At $z = 0$, the potential vanishes because here points are equidistant from the positive and negative charges. Furthermore, in the *upper* half plane ϕ must satisfy Poisson's equation for a point charge at $\underline{r}_1$, since no further changes have been introduced in this region (the only charge we have introduced is in the *lower* half plane).

Thus, by our uniqueness theorem, the potential in the **upper half plane** is the same as that of a charge q placed above an grounded sheet at $z = 0$.

2.1.1 Point Charge near grounded Sphere

Consider a point charge q placed at a distance b from the centre of an grounded conducting sphere of radius $a < b$. We will now show that an equivalent problem is to place an *image charge* $q' = -qa/b$ as shown.



By symmetry, the image charge q' must lie along OQ, at a distance b', say, from the centre of the sphere. Thus the resultant potential of the image system is

$$\phi(\underline{r}) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{|\underline{r}-\underline{b}|} + \frac{q'}{|\underline{r}-\underline{b}'|}\right\}.$$

We need two equations to determine q' and b'; we will obtain these by imposing that ϕ vanish at the two points where OQ intersects the sphere

$$\begin{aligned} \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{b-a} + \frac{q'}{a-b'}\right\} &= 0 \\ \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{a+b} + \frac{q'}{a+b'}\right\} &= 0. \end{aligned}$$

Eliminating q', we obtain

$$\frac{a+b'}{a-b'} = \frac{a+b}{b-a}$$

and hence

$$b' = a^2/b.$$

We can substitute this into either equation to obtain

$$q' = -qa/b.$$

Finally, let us verify that ϕ does indeed vanish for *all* points on the surface of the sphere. On the surface,

$$|\underline{r}-\underline{b}'|^2 = a^2 - 2a\frac{a^2}{b}\cos\theta + \frac{a^4}{b^2}$$

$$= \frac{a^2}{b^2}\left\{a^2 - 2ab\cos\theta + b^2\right\}$$
$$= \frac{a^2}{b^2}|\underline{r} - \underline{b}|$$

and hence

$$\phi(\underline{r})|_{r=a} = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{|\underline{r}-\underline{b}|} - \frac{qa}{b}\frac{1}{a/b|\underline{r}-\underline{b}|}\right\} = 0.$$

Thus we have

1. The image system satisfies the original Poisson's equation for $r \geq a$ since the only additional charge we have introduced is in the region $r < a$.
2. The potential for the image system satisfies the condition $\phi = 0$ at $r = a$.

Thus, by the uniqueness theorem, the required potential is

$$\phi(\underline{r}) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{|\underline{r}-\underline{b}|} - \frac{qa}{b}\frac{1}{|\underline{r}-\underline{b}'|}\right\} \tag{2.1}$$

with $b' = a^2/b$.

Induced charge density

In Chapter 1, we showed that the induced charge density on the surface of a conductor is

$$\sigma = \epsilon_0\underline{E}\cdot\underline{n} = -\epsilon_0\underline{n}\cdot\underline{\nabla}\phi$$

where $\underline{n}$ is the outward normal to the surface.
From eqn. 2.1, we have

$$\underline{\nabla}\phi = \frac{1}{4\pi\epsilon_0}\left\{\frac{-q}{|\underline{r}-\underline{b}|^3}(\underline{r}-\underline{b}) + \frac{qa}{b}\frac{1}{|\underline{r}-a^2/b^2\underline{b}|^3}(\underline{r}-a^2/b^2\underline{b})\right\}$$

At the surface, $\underline{r} = \underline{a} = a\underline{e}_r$ and $\underline{n} = \underline{e}_r$, yielding

$$\sigma = -\frac{1}{4\pi}\left\{\frac{-q}{|\underline{a}-\underline{b}|^3}(a - \underline{b}\cdot\underline{e}_r) + \frac{qa}{b}\frac{1}{|\underline{a}-a^2/b^2\underline{b}|^3}(a - a^2/b^2\underline{b}\cdot\underline{e}_r)\right\}.$$

Using

$$|\underline{a} - a^2/b^2\underline{b}|_{r=a} = a/b|\underline{a} - \underline{b}|_{r=a}$$

we find

$$\sigma = -\frac{q}{4\pi}\frac{a}{|\underline{a} - \underline{b}|^3}\left\{b^2/a^2 - 1\right\}a.$$

Note that the surface charge density is not uniform, but that

$$\int_S \sigma\, dS = q'$$

as expected.

2.1.2 Point charge near insulated conducting sphere at potential V

This is a simple modification of the method above. We introduce an additional image charge $\hat{q} = Va4\pi\epsilon_0$ at the centre of the sphere yielding $\phi = V$ at $r = a$. Because we have introduced no additional charges in the region $r \geq a$, we apply the uniqueness theorem to say that the resultant potential is

$$\phi(\underline{r}) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{|\underline{r} - \underline{b}|} - \frac{qa}{b}\frac{1}{\underline{r} - a^2/b^2\underline{b}|}\right\} + \frac{Va}{r}.$$

2.1.3 Point charge near insulated, conducting sphere with total charge Q

This problem is a slightly more complicated. Our starting point is the point charge near the grounded conducting sphere, together with the superposition principle.

1. Start with an grounded conducting sphere. We have shown that a total surface charge q' is induced, distributed to balance the electrostatic forces due to q.

2. Disconnect the sphere from earth, and add a charge $Q - q'$ to the sphere. This charge will be uniformly distributed, since the charge q' is already distributed to balance the forces due to q.

Appealing to the uniqueness theorem, and noting that, once again, no charges have been introduced in $r \geq a$, we have

$$\phi(\underline{r}) = \frac{Q - q'}{4\pi\epsilon_0 r} + \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{|\underline{r} - \underline{b}|} + \frac{q'}{\underline{r} - a^2/b^2\underline{b}|}\right\}.$$

We will now proceed to calculate the **Force** on the charge q; this is just given by Coulomb's law for the forces between q and the two image charges:

$$\begin{aligned} \underline{F} &= \frac{1}{4\pi\epsilon_0} q \left\{\frac{Q - q'}{b^3}\underline{b} + \frac{q'}{|\underline{b} - \underline{b}'|^3}(\underline{b} - \underline{b}')\right\} \\ &= \frac{1}{4\pi\epsilon_0}\frac{q\,\underline{b}}{b^3}\left\{Q - \frac{qa^3(2b^2 - a^2)}{b(b^2 - a^2)^2}\right\} \end{aligned}$$

Note that the force is **always attractive** at sufficiently small distances irrespective of Q due to the induced surface charge density on the conductor.

2.2 Formal solution of Poisson's Equation: Preliminaries

We will now proceed to a formal solution using *Green functions.* First, however, a mathematical digression...

2.2.1 Dirac δ-Function

The *Dirac δ-function* is defined as follows:

1.

$$\delta(x-a) = 0 \qquad \text{if} \qquad x \neq a.$$

2.

$$\int_R dx\, \delta(x-a) = \begin{cases} 1 & \text{if } a \in R \\ 0 & \text{otherwise} \end{cases}$$

The delta function is not strictly a function but rather a *distribution*; it is defined purely through its effect under an integral. It immediately follows from the definition that

$$\int dx\, f(x)\delta(x-a) = f(a) \tag{2.2}$$

if a lies within the region of integration.

The δ-function $\delta(x-a)$ may be thought of as the limit of a *Gaussian* centred at a in which the width tends to zero whilst the area under the Gaussian remains unity.

$$\begin{aligned} \delta(x-a) &= \lim_{\epsilon\to 0} \delta_\epsilon(x-a) \\ \delta_\epsilon(x-a) &= \frac{1}{\sqrt{\pi\epsilon}} e^{-\frac{(x-a)^2}{\epsilon}} \end{aligned} \tag{2.3}$$

It is easy to see that $\lim_{\epsilon\to 0} \delta_\epsilon(x) = 0$ if $x \neq a$ and $\int_{-\infty}^{\infty} \delta_\epsilon(x-a) = 1$. Let us check the property (2.2)

$$\lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \delta_\epsilon(x-a)\; f(x)$$

$$\begin{aligned} &= \lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi\epsilon}}e^{-\frac{(x-a)^2}{\epsilon}}[f(a)+(x-a)f'(a)+\frac{1}{2}(x-a)^2f"(a)+...] \\ &= \lim_{\epsilon\to 0}[f(a)+\epsilon^2 f"(a)+O(\epsilon^4)] \;=\; f(a) \end{aligned}$$

There are some simple relations that follow from the Eq. (2.2)

1. The δ-function is a derivative of a *step* function $\theta(x)$:

$$\theta(x) \;=\; \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{2.4}$$

   Indeed, if $f(x)$ vanishes at infinity

$$\begin{aligned} &\int_{-\infty}^{\infty} f(x)\;\; \theta'(x) = -\int_{-\infty}^{\infty} f'(x)\;\theta(x) = -\int_{0}^{\infty} f'(x) = f(0) \\ &\int_{-\infty}^{\infty} f(x)\;\delta(x) = f(0) \end{aligned}$$

   $\Rightarrow \delta(x) = \theta'(x)$ (and $\delta(x-a) = \theta'(x-a)$).

2.

$$\begin{aligned} \int dx\, f(x)\delta'(x-a) &= -\int dx\, f'(x)\delta(x-a) \quad \text{integ. by parts} \\ &= -f'(a) \end{aligned}$$

3.

$$\begin{aligned} \int dx\, f(x)\delta(g(x)) &= \sum_i \int dy \left|\frac{1}{g'(x_i(y))}\right| f(x_i(y))\delta(y) \\ &= \sum_i \frac{f(x_i)}{|g'(x_i)|} \end{aligned}$$

   where y is defined in a small region of each x_i.

4. The definition extends naturally to three (or higher) dimensions:

$$\delta(\underline{x}-\underline{X}) = \delta(x_1-X_1)\delta(x_2-X_2)\delta(x_3-X_3)$$

   so that

$$\int_V d^3x\,\delta(\underline{x}-\underline{X}) = \begin{cases} 1 & \text{if } \underline{X}\in V \\ 0 & \text{otherwise} \end{cases}$$

Note that it is this last property that defines the multi-dimensional δ-function, with this simple representation in a Cartesian basis; you have to be a little careful when working in curvilinear coordinates.

As a simple illustration of the power of the δ-function, let us return to the expression, eqn. (1.5), for the potential due to a continuous charge distribution

$$\phi(\underline{x}) = \frac{1}{4\pi\epsilon_0}\int_V d^3\underline{x}'\,\frac{\rho(\underline{x}')}{|\underline{x}-\underline{x}'|}.$$

We now introduce the δ-function to enable us to write a set of N discrete charges q_i at $\underline{x}_i$ as a charge distribution

$$\rho(\underline{x}') = \sum_i q_i \delta^{(3)}(\underline{x}' - \underline{x}_i)$$

so that

$$\begin{aligned}\phi(\underline{x}) &= \frac{1}{4\pi\epsilon_0}\int_V d^3x' \frac{\sum_i q_i\delta^{(3)}(\underline{x}'-\underline{x}_i)}{|\underline{x}-\underline{x}'|}\\ &= \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{|\underline{x}-\underline{x}_i|}\end{aligned}$$

which is our familiar expression for the potential due to a set of point charges.

Poisson's Equation for a Point Charge

It is easy to see that

$$\nabla^2(1/r) = 0 \qquad r \neq 0.$$

Furthermore, from our proof of Gauss' law, we can see that

$$\int dV\,\nabla^2(1/r) = -4\pi.$$

Thus we can write formally

$$\boxed{\nabla^2\left(\frac{1}{|\underline{x}-\underline{x}'|}\right) = -4\pi\delta^{(3)}(\underline{x}-\underline{x}')}$$

2.3 Formal Solution of Boundary-Value Problem using Green Functions

Our starting point is Green's theorem, eqn. (1.11):

$$\int_V d^3x'(\psi_1(\underline{x}')\nabla'^2\psi_2(\underline{x}')-\psi_2(\underline{x}')\nabla'^2\psi_1(\underline{x}')) = \int_S(\psi_1(\underline{x}')\underline{\nabla}'\psi_2(\underline{x}')-\psi_2(\underline{x}')\underline{\nabla}'\psi_1(\underline{x}'))\cdot\underline{n}\,dS.$$

where the "primed" denotes differentiation with respect to the primed indices. Let us apply this for the case $\psi_1(\underline{x}') = \frac{1}{|\underline{x}-\underline{x}'|}$ and $\psi_2(\underline{x}') = \phi(\underline{x}')$ where

$$\nabla'^2\phi(\underline{x}') = -\rho(\underline{x}')/\epsilon_0.$$

and

$$\nabla'^2\psi_1(\underline{x}') = -4\pi\delta^{(3)}(\underline{x}-\underline{x}')$$

yielding

$$\begin{aligned}\int d^3x' \left\{\frac{1}{|\underline{x}-\underline{x}'|}\left(\frac{-\rho(\underline{x}')}{\epsilon_0}\right) + \phi(\underline{x}')4\pi\delta^{(3)}(\underline{x}-\underline{x}')\right\} = \\ \int dS'\,\underline{n}\cdot\left\{\frac{1}{|\underline{x}-\underline{x}'|}\underline{\nabla}'\phi(\underline{x}') - \phi(\underline{x}')\underline{\nabla}'\left(\frac{1}{|\underline{x}-\underline{x}'|}\right)\right\}.\end{aligned}$$

Applying our rule for integrating over δ-functions, we obtain

$$\begin{aligned}\phi(\underline{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x'\frac{\rho(\underline{x}')}{|\underline{x}-\underline{x}'|} + \\ \frac{1}{4\pi}\int dS'\left\{\frac{1}{|\underline{x}-\underline{x}'|}\frac{\partial\phi(\underline{x}')}{\partial n'} - \phi(\underline{x}')\frac{\partial}{\partial n'}\left(\frac{1}{|\underline{x}-\underline{x}'|}\right)\right\}.\end{aligned} \quad (2.5)$$

The function $1/|\underline{x}-\underline{x}'|$ is said to be a **Green function** for the problem. The Green function is not unique, and is just a function satisfying

$$\nabla'^2G(\underline{x},\underline{x}') = -4\pi\delta^{(3)}(\underline{x}-\underline{x}').$$

In general, it has the form

$$G(\underline{x},\underline{x}') = \frac{1}{|\underline{x}-\underline{x}'|} + F(\underline{x},\underline{x}'),$$

where $F(\underline{x}, \underline{x}')$ is a solution of Laplace's equation

$$\nabla'^2 F(\underline{x}, \underline{x}') = 0.$$

Thus our expression for the potential can be generalised to

$$\boxed{\begin{aligned}\phi(\underline{x}) = \frac{1}{4\pi\epsilon_0}\int_V d^3x'\, G(\underline{x}, \underline{x}')\rho(\underline{x}') + \\ \frac{1}{4\pi}\int_{S=\partial V} dS' \left\{ G(\underline{x}, \underline{x}')\frac{\partial\phi(\underline{x}')}{\partial n'} - \phi(\underline{x}')\frac{\partial G(\underline{x}, \underline{x}')}{\partial n'} \right\} \qquad (2.6)\end{aligned}}$$

The utility of this generalisation is the following. In eqn. 2.5, the surface integral involved both $\phi(\underline{x}')$, and $\partial\phi(\underline{x}')/\partial n'$; in general we cannot specify *both* simultaneously at a point on the surface, since the problem is then overdetermined. Thus in eqn. 2.5 we have an implicit equation for $\phi(\underline{x})$, with the unknown also appearing under the integral on the right-hand side. In eqn. 11.79, we can choose $G(\underline{x}, \underline{x}')$ so that the surface integral depends only on the proscribed boundary values of ϕ (Dirichlet) or $\partial\phi/\partial n'$ (Neumann).

2.3.1 Boundary Conditions on Green Functions

We will now consider the boundary conditions we have to impose on the Green Functions to accomplish the above aim.

Dirchlet Problem

Here the value of $\phi(\underline{x}')$ is specified on the surface, and therefore it is natural to impose that the Green function $G_D(\underline{x}, \underline{x}')$ satisfy

$$G_D(\underline{x}, \underline{x}') = 0 \quad \text{for } \underline{x}' \text{ on } S,$$

and thus

$$\phi(\underline{x}) = \frac{1}{4\pi\epsilon_0}\int_V d^3x'\, G_D(\underline{x}, \underline{x}')\rho(\underline{x}') - \frac{1}{4\pi}\int_S dS'\, \phi(\underline{x}')\frac{\partial G_D(\underline{x}, \underline{x}')}{\partial n'}. \qquad (2.7)$$

Thus the surface integral only involves $\phi(\underline{x}')$, and not the unknown $\partial\phi(\underline{x}')/\partial n'$.

Neumann Problem

Here it is tempting to construct the Green function $G_N(\underline{x}, \underline{x}')$ such that

$$\frac{\partial G_N(\underline{x}, \underline{x}')}{\partial n'} = 0 \quad \text{for } \underline{x}' \text{ on } S.$$

However, recall that the Green function satisfies

$$\int_S dS' \, \frac{\partial G_N(\underline{x}, \underline{x}')}{\partial n'} = \int d^3x' \, \nabla'^2 G_N(\underline{x}, \underline{x}') = -4\pi,$$

and thus $\partial G_N(\underline{x}, \underline{x}')/\partial n'$ cannot vanish everywhere. The simplest solution is to impose

$$\frac{\partial G_N(\underline{x}, \underline{x}')}{\partial n'} = -\frac{4\pi}{S} \, \forall \underline{x}' \in S$$

where S is the total area of the surface. Thus the solution is

$$\phi(\underline{x}) = \frac{1}{4\pi\epsilon_0} \int_V d^3x' \, G_N(\underline{x}, \underline{x}') \rho(\underline{x}') + \\ \frac{1}{4\pi} \int_S dS' \, G_N(\underline{x}, \underline{x}') \frac{\partial \phi(\underline{x}')}{\partial n'} + \frac{1}{S} \int_S dS' \, \phi(\underline{x}') \tag{2.8}$$

where the final term is just the average value of $\phi(\underline{x}')$ on the surface S. The inclusion of this term is perhaps not surprising; recall that the solution to the Neumann problem is unique only up to an additive constant.

2.3.2 Reciprocity relation for $G_D(\underline{x}, \underline{y})$

For the Dirichlet problem, we have $G_D(\underline{x}, \underline{y}) = G_D(\underline{y}, \underline{x})$.

Proof

Apply Green's theorem for the case $\psi_1(\underline{x}') = G_D(\underline{x}, \underline{x}')$, and $\psi_2(\underline{x}') = G_D(\underline{y}, \underline{x}')$:

$$\int_V d^3x' \, (G_D(\underline{x}, \underline{x}')\nabla'^2 G_D(\underline{y}, \underline{x}') - G_D(\underline{y}, \underline{x}')\nabla'^2 G_D(\underline{x}, \underline{x}')) = \\ \int_S dS' \, \underline{n} \cdot (G_D(\underline{x}, \underline{x}')\underline{\nabla}' G_D(\underline{y}, \underline{x}') - G_D(\underline{y}, \underline{x}')\underline{\nabla}' G_D(\underline{x}, \underline{x}')).$$

But for the Dirichlet problem $G_D(\underline{x}, \underline{x}')$ vanishes for all $\underline{x}' \in S$, and hence the right-hand side of the above is zero. Thus we have

$$\int d^3x' \left\{G_D(\underline{x}, \underline{x}')\{-4\pi\delta^{(3)}(\underline{y} - \underline{x}')\} - G_D(\underline{y}, \underline{x}')\{-4\pi\delta^{(3)}(\underline{x} - \underline{x}')\}\right\} = 0$$

and hence

$$\boxed{G_D(\underline{x}, \underline{y}) = G_D(\underline{y}, \underline{x})}$$

2.4 Methods of Finding Green Functions

The secret, then, to the solution of boundary value problems is determining the correct Green function, or equivalently obtaining the function $F(\underline{x}, \underline{x}')$. They are several techniques

1. Make a guess at the form of $F(\underline{x}, \underline{x}')$. Here we recall that F is just the solution of the homogeneous Laplace's equation $\nabla'^2 F(\underline{x}, \underline{x}') = 0$ inside V, and therefore is just the solution of the potential for a system of charges *external* to V. In particular, for the Dirichlet problem, since $G_D(\underline{x}, \underline{x}')$ vanishes at $\underline{x}' \in S$, we have that $F(\underline{x}, \underline{x}')$ is just that system of charges external to V that, when combined with a point charge at $\underline{x}$, assures that the potential vanishes on the surface. And finding that system of charges is precisely what we were doing in the *Method of Images*...

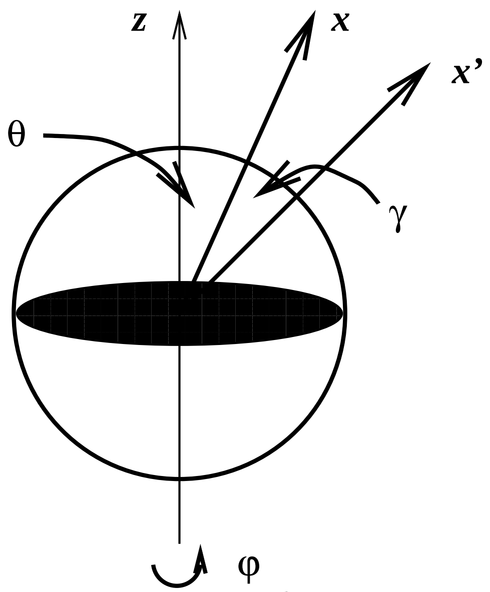
2. Expand the Green function as a series of orthonormal *eigenfunctions* of the Laplacian operator. We will be exploring this method later in the chapter.

2.4.1 Dirichlet Green Function for the Sphere

We saw at the beginning of this chapter how to use the method of images to construct the potential $\phi(\underline{x}')$ for a point charge at $\underline{x}$ outside an grounded conducting sphere of radius a. In particular, for a charge $q = 4\pi\epsilon_0$, the potential satisfies

$$\nabla'^2 \phi(\underline{x}') = -4\pi\delta^{(3)}(\underline{x} - \underline{x}')$$

with $\phi(\underline{x}') = 0$ for $\underline{x}'$ on S. Thus now see that $\phi(\underline{x}')$ is precisely the Green function $G_D(\underline{x}, \underline{x}')$ that we need. Note that you have to be careful to distinguish the variable we are integrating over, $\underline{x}'$, and the variable at which we are evaluating the potential, $\underline{x}$. Perhaps counter-intuitively, it is at the point $\underline{x}$ that we place our point charge.



From eqn. 2.1, we have that the Green function is

$$G(\underline{x}, \underline{x}') = \frac{1}{|\underline{x}' - \underline{x}|} - \frac{a}{x|\underline{x}' - a^2/x^2\underline{x}|},$$

and it is easy to check that, indeed, $G(\underline{x}, \underline{x}') = G(\underline{x}', \underline{x})$.

We can rewrite this as

$$G(\underline{x}, \underline{x}') = \left\{ \frac{1}{(x^2 + x'^2 - 2xx'\cos\gamma)^{1/2}} - \frac{1}{x^2x'^2/a^2 + a^2 - 2xx'\cos\gamma)^{1/2}} \right\}$$

where γ is the angle between $\underline{x}$ and $\underline{x}'$.
The general solution for the potential is then

$$\phi(\underline{x}) = \frac{1}{4\pi\epsilon_0}\int_V d^3x'\, G(\underline{x}, \underline{x}')\rho(\underline{x}') - \frac{1}{4\pi}\int_S dS'\, \phi(\underline{x}')\frac{\partial G(\underline{x}, \underline{x}')}{\partial n'}. \qquad (2.9)$$

Thus we need the normal gradient of the Green function to the surface, *which points inward*,

$$\begin{aligned} \left|\frac{\partial G}{\partial n'}\right|_{\text{surface}} &= -\left|\frac{\partial G}{\partial x'}\right|_{x'=a} \\ &= -\frac{1}{2}\left\{ \frac{-2a - 2x\cos\gamma}{(x^2 + a^2 - 2ax\cos\gamma)^{3/2}} + \frac{2x^2a/a^2 - 2x\cos\gamma}{(x^2a^2/a^4 + a^2 - 2ax\cos\gamma)^{3/2}} \right\} \\ &= -\frac{x^2 - a^2}{a(x^2 + a^2 - 2ax\cos\gamma)} \end{aligned}$$

Thus we have all the ingredients to solve the Dirichlet problem outside a sphere of radius a.

2.4.2 Solution of Laplace's equation outside a sphere comprising two hemispheres at equal and opposite potentials V

Because the source is zero, we only need the surface term from eqn. 2.9

$$\phi(\underline{x}) = -\frac{1}{4\pi}\int_S dS'\,\phi(\underline{x}')\frac{\partial G(\underline{x},\underline{x}')}{\partial n'}.$$

Now $dS' = a^2 d\psi d(\cos\theta')$, yielding

$$\begin{aligned}\phi(\underline{x}) &= -\frac{1}{4\pi}a^2\int_0^{2\pi} d\psi'\left\{V\int_0^1 d(\cos\theta')\frac{\partial G}{\partial n'} + (-V)\int_{-1}^0 d(\cos\theta')\frac{\partial G}{\partial n'}\right\} \\ &= \frac{V}{4\pi}\int_0^{2\pi} d\psi'\left\{\int_0^1 d(\cos\theta')\frac{a(x^2-a^2)}{(a^2+x^2-2ax\cos\gamma)^{3/2}} - \right. \\ &\quad \left.\int_{-1}^0 d(\cos\theta')\frac{a(x^2-a^2)}{(a^2+x^2-2ax\cos\gamma)^{3/2}}\right\}.\end{aligned}$$

We can express $\cos\gamma$ in terms of the spherical polar coordinates of $\underline{x}$ and $\underline{x}'$ by noting that

$$\begin{aligned}\cos\gamma = \underline{n}\cdot\underline{n}' &= (\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta)\cdot(\sin\theta'\cos\psi', \sin\theta'\sin\psi', \cos\theta') \\ &= \sin\theta\sin\theta'\cos(\psi-\psi') + \cos\theta\cos\theta',\end{aligned}$$

where $\underline{n}$ and $\underline{n}'$ are unit vectors in the directions of $\underline{x}$ and $\underline{x}'$ respectively. Finally, we can combine the two integrals through by making the substitution $\theta' \to \pi - \theta'$ and $\psi' \to \psi' + \pi$ in the second integral, giving

$$\boxed{\begin{aligned}\phi(\underline{x}) &= \frac{V}{4\pi}a(x^2-a^2)\int_0^{2\pi} d\psi'\int_0^1 d(\cos\theta')\left\{\frac{1}{(a^2+x^2-2ax\cos\gamma)^{3/2}}\right. \\ &\quad \left. - \frac{1}{(a^2+x^2+2ax\cos\gamma)^{3/2}}\right\}\end{aligned}}$$

In general, we cannot obtain the solution in closed form; γ is just too complicated a function of θ' and ψ'. However, we can study the solution in specific cases.

Solution above North Pole

Here $\theta = 0$, so that $\cos\gamma = \cos\theta'$, and $|\underline{x}| = z$. Thus

$$\phi(z) = \frac{V}{4\pi} a(z^2 - a^2) 2\pi \int_0^1 du \left\{ \frac{1}{(a^2 + z^2 - 2azu)^{3/2}} - \frac{1}{(a^2 + z^2 + 2azu)^{3/2}} \right\}.$$

The integration can be performed easily, by making the substitution $y = a^2 + z^2 - 2azu$ and $y = a^2 + z^2 + 2azu$ for the first and second terms respectively, yielding

$$\phi(z)|_{\theta=0} = V \left\{ 1 - \frac{(z^2 - a^2)}{z\sqrt{z^2 + a^2}} \right\}.$$

Note that for $z \gg a$, we have

$$\phi(z) \sim \frac{3Va^2}{2z^2},$$

and the boundary conditions are trivially satisfied at $z = a$.

Solution at Large Distances

We can also obtain the solution for $x >> a$, by means of a Taylor expansion. We begin by writing

$$a^2 + x^2 \pm 2ax\cos\gamma = (a^2 + x^2)(1 \pm 2\alpha\cos\gamma)$$

where

$$\alpha = \frac{ax}{a^2 + x^2},$$

yielding

$$\phi(\underline{x}) = \frac{V}{4\pi} \frac{a(x^2 - a^2)}{(a^2 + x^2)^{3/2}} \int_0^{2\pi} d\psi' \int_0^1 d(\cos\theta') \left\{ \frac{1}{(1 - 2\alpha\cos\gamma)^{3/2}} - \frac{1}{(1 + 2\alpha\cos\gamma)^{3/2}} \right\}.$$

We now expand the integrand as a power series in α, yielding

$$\{\} = 6\alpha\cos\gamma + 35\alpha^3\cos^3\gamma + \mathcal{O}(\alpha^5).$$

The integrals for the first two terms in the expansion are perfectly tractable. Recalling that $\cos\gamma = \sin\theta\sin\theta'\cos(\psi - \psi') + \cos\theta\cos\theta'$, we find

1.

$$\int_0^{2\pi} d\psi' \int_0^1 d(\cos\theta')\cos\gamma = \int_0^{2\pi} d\psi' \int_0^1 d(\cos\theta')\cos\theta\cos\theta' = \pi\cos\theta$$

2.

$$\int_0^{2\pi} d\psi' \int_0^1 d(\cos\theta')\cos^3\gamma = \pi/4\cos\theta(3-\cos^2\theta)$$

and thus

$$\phi(\underline{x}) = \frac{3Va^2x(x^2-a^2)}{2(a^2+x^2)^{5/2}}\cos\theta\left\{1+\frac{35}{24}\frac{a^2x^2}{(a^2+x^2)^2}(3-\cos^2\theta)+\mathcal{O}(a^4/x^4)\right\}.$$

Note that we can express this power series as a series in a^2/x^2, rather than α, yielding

$$\phi(x,\theta,\psi) = \frac{3Va^2}{x^2}\left\{\cos\theta - \frac{7a^2}{12x^2}\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right)+\mathcal{O}(a^4/x^4)\right\}.$$

and we can verify that this gives the correct expression for $\theta = 0$.
As we go to higher order terms in the expansion, the angular integrals become increasingly intractable, and this approach fails. However, the eagle-eyed amongst you may recognise the angular terms as the Legendre polynomials $P_1(\cos\theta)$ and $P_3(\cos\theta)$, and this brings us to the next section.

2.5 Orthogonal Functions

The expansion of the solution of a linear differential equation in terms of **orthogonal functions** is one of the most powerful techniques in mathematical physics. Consider a set of functions $\mathcal{U}_n(\xi)$, $n = 0, 1, \ldots$, defined on $a \le \xi \le b$.

1. The set $\{\mathcal{U}_n(\xi)\}$ is **orthonormal** iff (*if and only if*)

$$\int_a^b d\xi\, \mathcal{U}_n(\xi)\mathcal{U}_m^*(\xi) = \delta_{mn} \tag{2.10}$$

2. The set is set to be **complete** iff

$$\sum_{n=0}^{\infty} \mathcal{U}_n(\xi)\mathcal{U}_n^*(\xi') = \delta(\xi - \xi'). \tag{2.11}$$

The completeness relation is important because it implies that **any** square-integrable function $f(\xi)$ defined over the interval $a \leq \xi \leq b$ can be expressed as a series in the orthogonal functions $\mathcal{U}(\xi)$. This is easy to see:

$$\begin{aligned} f(\xi) &= \int d\xi' \, f(\xi')\delta(\xi - \xi') \quad \text{(defn. of } \delta\text{-func.)} \\ &= \int d\xi' \, f(\xi') \sum_{n=0}^{\infty} \mathcal{U}_n(\xi)\mathcal{U}_n^*(\xi') \quad \text{(completeness)} \\ &= \sum_{n=0}^{\infty} \mathcal{U}_n(\xi) \int d\xi' \, f(\xi')\mathcal{U}_n^*(\xi'). \end{aligned}$$

Thus we may write

$$f(\xi) = \sum_{n=0}^{\infty} \mathcal{U}_n(\xi) a_n$$

where

$$a_n = \int d\xi' f(\xi')\mathcal{U}_n^*(\xi').$$

2.5.1 Fourier Series

One of the best-known cases where we expand in terms of orthogonal functions is the *Fourier expansion*. Consider the expansion applied to the interval $-a/2 \leq x \leq a/2$. The set of ortonormal functions is provided by the *sines* and *cosines*:

$$\begin{aligned} C_m(x) &= \sqrt{2/a} \cos\left(\frac{2\pi m x}{a}\right), \quad m = 1, 2, \ldots \\ S_m(x) &= \sqrt{2/a} \sin\left(\frac{2\pi m x}{a}\right), \quad m = 1, 2, \ldots \\ C_0(x) &= \sqrt{1/a}. \end{aligned}$$

It is easy to show that the set $C_m(x), S_m(x)$ forms an orthonormal set of functions, viz.

$$\begin{aligned} \int dx \, S_m(x)S_n(x) = \int dx \, C_m(x)C_n(x) &= \delta_{mn}, \\ \int dx S_m(x)C_n(x) &= 0. \end{aligned}$$

Later we will prove *completeness*,

$$\frac{1}{a}+\frac{2}{a}\sum_1^\infty \cos\left(\frac{2\pi mx}{a}\right)\cos\left(\frac{2\pi mx'}{a}\right)+\frac{2}{a}\sum_1^\infty \sin\left(\frac{2\pi mx}{a}\right)\sin\left(\frac{2\pi mx'}{a}\right) \;=\; \delta(x-x') \tag{2.12}$$

and thus we can write **any** function $f(x)$ on the interval $-a/2 \le x \le a/2$ as

$$f(x)=\frac{A_0}{2}+\sum_{m=1}^{\infty}\left\{A_m\cos\left(\frac{2\pi mx'}{a}\right)+B_m\sin\left(\frac{2\pi mx'}{a}\right)\right\},$$

where

$$\begin{aligned} A_m &= \frac{2}{a}\int_{-a/2}^{a/2} dx\, f(x)\cos\left(\frac{2\pi mx}{a}\right) \qquad m=0,1,2,\ldots \\ B_m &= \frac{2}{a}\int_{-a/2}^{a/2} dx\, f(x)\sin\left(\frac{2\pi mx}{a}\right) \qquad m=1,2,\ldots \end{aligned}$$

We can combine the sine and cosine terms by noting

$$\begin{aligned} \cos x &= \frac{1}{2}\left[e^{ix}+e^{-ix}\right] \\ \sin x &= \frac{1}{2i}\left[e^{ix}-e^{-ix}\right], \end{aligned}$$

and introducing a new set of functions

$$\mathcal{U}_m(x)=\frac{1}{\sqrt{a}}e^{i2\pi mx/a} \qquad m=0,\pm1,\pm2,\ldots,$$

We get an expansion

$$f(x)=\sum_{m=-\infty}^{\infty} A_m\mathcal{U}_m(x),$$

where

$$A_m=\frac{1}{\sqrt{a}}\int_{-a/2}^{a/2} dx' f(x')e^{-2\pi imx'/a}.$$

Proof of completeness

$$\sum_{-\infty}^{\infty} e^{in(x-x')}=2\pi\delta(x-x')$$

for $x, x' \in [-\pi,\pi]$:

For simplicity, take x instead of $x - x'$. We have

$$\sum_{-\infty}^{\infty} e^{inx} = \sum_{0}^{\infty} e^{inx} + \sum_{1}^{\infty} e^{-inx} = \frac{1}{1 - e^{ix}} + \frac{e^{-ix}}{1 - e^{-ix}} = 0$$

if $x \neq 0$. To check for the δ-function contribution, calculate

$$\int_{-\pi}^{\pi} \sum_{-\infty}^{\infty} e^{inx} = \sum_{-\infty}^{\infty} \int_{-\pi}^{\pi} e^{inx} = 2\pi$$

$\Rightarrow \sum_{-\infty}^{\infty} e^{inx} = 2\pi\delta(x)$, Q.E.D.
For the interval $[-a/2, a/2]$ we get:

$$\sum_{-\infty}^{\infty} e^{in\frac{2\pi}{a}(x-x')} = a\delta(x - x') \tag{2.13}$$

Taking the real part of both sides of this equation we reproduce Eq. (2.12).

An orthonormal set $\sin \frac{\pi}{a} mx$

If we have to expand a function $f(x)[0, a] \to R$ which vanishes at the ends of the interval $[0, a]$ we can use an orthonormal set of $\sin' s$ only: $\mathcal{U}_n(x) = \sqrt{\frac{2}{a}} \sin \frac{\pi}{a} nx$. It is easy to check that

$$\frac{2}{a} \int_0^a dx \sin \frac{\pi}{a} mx \sin \frac{\pi}{a} nx = \delta_{mn} \tag{2.14}$$

and

$$\frac{2}{a} \sum_{1}^{\infty} \sin \frac{\pi}{a} nx \ \sin \frac{\pi}{a} nx' = \delta(x - x') \tag{2.15}$$

(Strictly speaking, in the r.h.s of the eq. (2.15) we get $\delta(x - x') - \delta(x + x')$ but the last term does not contribute for $x, x' \in [0, a]$).
Thus, we get an expansion

$$\begin{aligned} f(x) &= \sqrt{2/a} \sum_{1}^{\infty} f_n \ \sin \frac{\pi}{a} nx \\ f_n &= \sqrt{2/a} \int_0^a dx \ f(x) \sin \frac{\pi}{a} nx \end{aligned} \tag{2.16}$$

2.5.2 Fourier transformation

Suppose we now let $a \to \infty$, so that the discrete sum over m becomes an integral over a continuous variable k where

$$\frac{2\pi m}{a} \to k.$$

Then we have

$$\sum_m \to \frac{a}{2\pi} \int dk$$

and the discrete coefficients become a continuous function

$$A_m \to \sqrt{\frac{2\pi}{a}} A(k).$$

Thus we may express the *Fourier Transforms* as

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int dk\, A(k) e^{ikx} \\ A(k) &= \frac{1}{\sqrt{2\pi}} \int dx\, f(x) e^{-ikx}. \end{aligned}$$

Note that the assignment of the coefficients outside the integrals depends on the convention adopted; in all cases the product is $1/2\pi$.
The orthogonality and completeness relations assume the continuous, and symmetric, forms

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, e^{i(k-k')x} &= \delta(k-k') \\ \frac{1}{2\pi} \int dk\, e^{ik(x-x')} &= \delta(x-x') \end{aligned}$$

2.5.3 Sturm-Liouville Equation

How does one obtain a complete set of orthonormal functions? We will now show that, for a certain class of differential equations, the solutions are orthogonal, for specific boundary conditions.

The **Sturm-Liouville Equation** is the differential equation

$$p(x)\frac{d^2\psi_\lambda}{dx^2} + \frac{dp(x)}{dx}\frac{d\psi_\lambda}{dx} + q(x)\psi_\lambda(x) = -\lambda r(x)\psi_\lambda(x)$$

which we may write in the more compact form

$$\frac{d}{dx}\left[p(x)\frac{d\psi_\lambda}{dx}\right] + q(x)\psi_\lambda = -\lambda r(x)\psi_\lambda.$$

Here the parameter λ identifies the solution, and plays the rôle of an **eigenvalue**, with ψ_λ the corresponding **eigenvector**. In the next couple of lectures we will encounter several equations of this form - the **Legendre** and **Bessel** equations, and of course you are familiar with the time-independent **Schrödinger** equation.

2.5.4 Theorem

For the Sturm-Liouville equation, with p, q, r real functions of x, the integral

$$(\lambda^* - \lambda')\int_a^b dx r(x)\psi_\lambda^*(x)\psi_{\lambda'}(x)$$

is zero provided the following boundary condition is satisfied:

$$\left[p(x)\left(\psi_\lambda^*\frac{d\psi_{\lambda'}}{dx} - \psi_{\lambda'}\frac{d\psi_\lambda^*}{dx}\right)\right]_a^b = 0.$$

Proof

ψ_λ and $\psi_{\lambda'}$ satisfy

$$\frac{d}{dx}\left[p(x)\frac{d\psi_\lambda}{dx}\right] + q(x)\psi_\lambda = -\lambda r(x)\psi_\lambda \tag{2.17}$$

$$\frac{d}{dx}\left[p(x)\frac{d\psi_{\lambda'}}{dx}\right] + q(x)\psi_{\lambda'} = -\lambda' r(x)\psi_{\lambda'}, \tag{2.18}$$

respectively. Multiplying eqn. 2.17 by $\psi_{\lambda'}^*$ and eqn. 2.18 by ψ_λ^* and integrating, we obtain

$$\int_a^b \psi_{\lambda'}^*\frac{d}{dx}\left[p(x)\frac{d\psi_\lambda}{dx}\right] + \int_a^b dx\,\psi_{\lambda'}^* q\psi_\lambda = -\lambda\int_a^b dx\,\psi_{\lambda'}^* r\psi_\lambda$$

$$\int_a^b \psi_\lambda^*\frac{d}{dx}\left[p(x)\frac{d\psi_{\lambda'}}{dx}\right] + \int_a^b dx\,\psi_\lambda^* q\psi_{\lambda'} = -\lambda'\int_a^b dx\,\psi_\lambda^* r\psi_{\lambda'}.$$

Integrating by parts yields

$$-\int_a^b dx\,\frac{d\psi_{\lambda'}^*}{dx}p\frac{d\psi_\lambda}{dx}+\int_a^b \psi_{\lambda'}^* q\psi_\lambda \;=\; -\left[p\psi_{\lambda'}^*\frac{d\psi_\lambda}{dx}\right]_a^b-\lambda\int dx\,\psi_{\lambda'}^* r\psi_\lambda \quad (2.19)$$

$$-\int_a^b dx\,\frac{d\psi_{\lambda}^*}{dx}p\frac{d\psi_{\lambda'}}{dx}+\int_a^b \psi_{\lambda}^* q\psi_{\lambda'} \;=\; -\left[p\psi_{\lambda}^*\frac{d\psi_{\lambda'}}{dx}\right]_a^b-\lambda'\int dx\,\psi_{\lambda}^* r\psi_{\lambda'} \quad (2.20)$$

Observing that, since q, p, r are real, the l.h.s. of eqn. 2.19 is the complex conjugate of the l.h.s. of eqn. 2.20 we can take the difference to obtain

$$(\lambda^* - \lambda')\int dx\, r(x)\psi_\lambda^*\psi_{\lambda'} = 0,$$

providing

$$\left[p(x)\left(\psi_\lambda^*\frac{d\psi_{\lambda'}}{dx}-\psi_{\lambda'}\frac{d\psi_\lambda^*}{dx}\right)\right]_a^b = 0.$$

Corollaries

1. If $r(x)$ does not change sign in (a, b)

$$\int_a^b r(x)|\psi_\lambda|^2 \neq 0$$

   and hence $\lambda^* = \lambda$.

2. For $\lambda' \neq \lambda$,

$$\int_a^b dx r(x)\psi_\lambda^*\psi_{\lambda'} = 0,$$

   i.e. the functions ψ_λ are **orthogonal**.

2.6 Separation of Variables in Cartesian Coordinates

We will now see how the Sturm-Liouville equation arises in the solution of Laplace's equation, and how we can then use the Sturm-Liouville theorem to provide an orthonormal set of functions. The method we will use will be the **separation of variables**. It is best shown by illustration.

Consider the solution of Laplace's equation in a box $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$, with the values of the potential prescribed on the boundary. In particular, let us consider the case where ϕ vanishes on the boundary, except on the plane $z = c$ where $\phi(x, y, z = c) = V(x, y)$.

In Cartesian coordinates, the natural coordinate system for the problem, Laplace's equation assumes the form

$$\frac{\partial^2}{\partial x^2}\phi(x,y,z) + \frac{\partial^2}{\partial y^2}\phi(x,y,z) + \frac{\partial^2}{\partial z^2}\phi(x,y,z) = 0.$$

We will seek solutions to this equation that are **factorisable**, i.e.

$$\phi(x,y,z) = X(x)Y(y)Z(z),$$

and build up our final solution from such factorisable solutions. Substituting this form into Laplace's equation, we obtain

$$\frac{d^2X(x)}{dx^2}Y(y)Z(z) + X(x)\frac{d^2Y(y)}{dy^2}X(z) + X(x)Y(y)\frac{d^2Z(z)}{dz^2} = 0,$$

which we may write as

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0.$$

We have separated the equation into three terms, each dependent on a different variable. Since the equation holds for all x, y, z, we can say that each term must separately be constant. Thus

$$\frac{1}{X}X'' = C_1 \qquad (2.21)$$

$$\frac{1}{Y}Y'' = C_2 \tag{2.22}$$
$$\frac{1}{Z}Z'' = C_3 \tag{2.23}$$

where $C_1 + C_2 + C_3 = 0$.

Let us consider eqn. 2.21

$$\frac{d^2X(x)}{dx^2} - C_1X = 0,$$

and choose a trial solution

$$X(x) = e^{\alpha x}.$$

Then we have that $\alpha^2 = C_1$.

1. If $C_1 > 0$, α is **real**, and the trial solution is **exponential**.
2. If $C_1 < 0$, α is **imaginary**, and the trial solution is **oscillatory**.

The boundary conditions require that X vanish at $x = 0, a$, and this is only possible for the oscillating solutions. Thus if we choose $C_1 = -\alpha^2$, where α real, the general solution will be of the form

$$X(x) = A\cos\alpha x + B\sin\alpha x.$$

Since X must vanish at $x = 0$,

$$X(x) = \sin\alpha x.$$

Furthermore, X also vanishes at $x = a$, and thus

$$\alpha = \alpha_n = \frac{n\pi}{a}, \quad n = 1, 2, \ldots.$$

Thus we have a set of solutions

$$X_n(x) = \sin\alpha_n x.$$

Eqn. 2.21 is a Sturm-Liouville equation, with $p(x) = 1$, $q(x) = 0$, $r(x) = 1$ and $\lambda = \alpha^2$. It satisfies the conditions required for the Sturm-Liouville theorem, and hence we immediately know that the functions $X_n(x)$ are **orthogonal**. We can treat $Y(y)$ similarly, and obtain

$$Y_m(y) = \sin\beta_m y; \quad \beta_m = \frac{m\pi}{b}, m = 1, 2, \ldots$$

Finally, we obtain Z from

$$\frac{Z''}{Z} = \alpha_n^2 + \beta_m^2 = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2} > 0.$$

In this case, the solution is a real exponential, and imposing the boundary condition $Z(0) = 0$ we have

$$Z(z) = \sinh\gamma_{nm} z$$

where

$$\gamma_{nm} = \pi\sqrt{n^2/a^2 + m^2/b^2}.$$

Thus the general solution, using the completeness property, is

$$\phi(x, y, z) = \sum_{m,n=1}^{\infty} A_{nm} \sin\alpha_n x \sin\beta_m y \sinh\gamma_{nm} z.$$

We obtain the coefficients A_{mn} by imposing the boundary conditions on the plane $z = c$:

$$V(x, y) = \sum_{m,n=1}^{\infty} A_{nm} \sin\alpha_n x \sin\beta_m y \sinh\gamma_{nm} c.$$

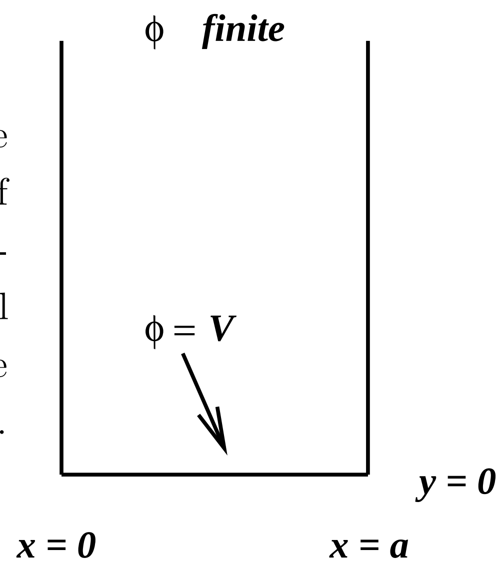
Using the orthonormal property of the basis functions, we have

$$\begin{aligned}
&\int_0^a dx \sin\frac{n\pi x}{a} \int_0^b dy \sin\frac{m\pi y}{b} V(x, y) \\
&= \sum_{m',n'} A_{n'm'} \int_0^a dx \sin\frac{n\pi x}{a} \sin\frac{n'\pi x}{a} \int_0^b dy \sin\frac{m\pi y}{b} \sin\frac{m'\pi y}{b} \sinh\gamma_{n'm'} c \\
&= \sum_{n',m'} A_{n'm'} \frac{a}{2}\delta_{n'n} \frac{b}{2}\delta_{m'm} \sinh\gamma_{n'm'} c \\
&= \frac{ab}{4} A_{nm} \sinh\gamma_{nm} c
\end{aligned}$$

Thus we have

$$A_{nm} = \frac{4}{ab \sinh \gamma_{nm} c} \int_0^a dx \int_0^b dy V(x,y) \sin \alpha_n x \sin \beta_m y$$

2.6.1 Two-dimensional Square Well



This is the two-dimensional version of the above problem. We have a square well, of width a, with the potential at the bottom constrained to be $\phi(x,0) = V$, and zero potential on the sides, with ϕ vanishing as $y \to \infty$. We wish to calculate the potential inside the well.

Laplace's equation becomes

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

subject to the boundary conditions

$$\begin{aligned} \phi(0,y) = \phi(a,y) &= 0 \\ \phi(x,0) &= V \\ \phi(x,y) &\to 0 \quad \text{as } y \to \infty \end{aligned}$$

As before, we look for separable solutions $\phi(x,y) = X(x)Y(y)$, yielding

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0,$$

so that each of the above terms must separately be constant.
Since $X(0) = X(a) = 0$, the solution for X must be oscillatory,

$$X'' + \alpha^2 X = 0$$

giving $X(x) = \sin \alpha x$. The boundary condition at $x = a$ then yields

$$X_n(x) = \sin \alpha_n x; \text{where } \alpha_n = \tfrac{n\pi}{a}, n = 1, 2, \ldots .$$

The corresponding function $Y_n(y)$ must satisfy

$$Y_n'' - \alpha_n^2 Y_n = 0$$

with *exponential* solutions $Y_n(y) = \exp \pm\alpha_n y$. The boundary condition $\phi \to 0$ as $y \to \infty$ requires that we take the exponentially falling solution, and thus

$$Y_n(y) = e^{-\alpha_n y}.$$

Thus the factorisable solutions are of the form

$$\phi_n(x, y) = e^{-\alpha_n y} \sin \alpha_n x$$

so that the general solution is

$$\phi(x, y) = \sum_n A_n e^{-\alpha_n y} \sin \alpha_n x; \quad \alpha_n = \frac{n\pi}{a}.$$

We determine the coefficients A_n by imposing the boundary condition at $y = 0$:

$$V = \sum_n A_n \sin \alpha_n x,$$

and using the orthogonality of the sin functions, we obtain

$$\begin{aligned} \int_0^a V \sin \frac{n\pi x}{a} &= \sum_n A_n \int_0^a dx \, \sin \frac{n\pi x}{a} \sin \frac{n'\pi x}{a} \\ &= \frac{a}{2} A_n. \end{aligned}$$

The integral is straightforward:

$$\begin{aligned} A_n &= \frac{2V}{a} \int_0^a dx \, \sin \frac{n\pi x}{a} \\ &= -\frac{2V}{a} \frac{a}{n\pi} \left[\cos \frac{n\pi x}{a} \right]_0^a \\ &= \frac{2V}{n\pi} [1 - (-1)^n], \end{aligned}$$

and thus

$$A_n = \begin{cases} 4V/n\pi & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

with

$$\boxed{\phi(x,y) = \frac{4V}{\pi} \sum_{n \text{ odd}} \frac{1}{n} e^{-n\pi y/a} \sin \frac{n\pi x}{a}.}$$

For $y/a \ll 1$, we can expand this as a series, and we converge to an accurate solution within a few terms - remember that exponential! But in this case, we can actually sum the series. We begin by recalling that

$$e^{ix} = \cos x + i \sin x.$$

yielding

$$\sin \frac{n\pi x}{a} = \Im e^{in\pi x/a}.$$

Thus we may write the general solution as

$$\begin{aligned} \phi(x,y) &= \frac{4V}{\pi} \sum_{n \text{ odd}} \frac{1}{n} e^{-n\pi y/a} \Im e^{in\pi x/a} \\ &= \frac{4V}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \Im e^{(n\pi i/a)(x+iy)}. \end{aligned}$$

We now introduce the variable

$$Z = e^{(i\pi/a)(x+iy)},$$

so that the solution becomes

$$\phi = \frac{4V}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \Im Z^n.$$

To sum this series, we recall that

$$\begin{aligned} \ln(1-Z) &= -Z - \frac{Z^2}{2} - \frac{Z^3}{3} + \ldots, \\ \ln(1+Z) &= Z - \frac{Z^2}{2} + \frac{Z^3}{3} + \ldots, \end{aligned}$$

and thus

$$\begin{aligned}\sum_{n \text{ odd}} \frac{1}{n} Z^n &= -\frac{1}{2}\{\ln(1-Z) - \ln(1+Z)\} \\ &= \frac{1}{2} \ln \frac{1+Z}{1-Z}.\end{aligned}$$

Hence we may write the general solution as

$$\phi(x,y) = \frac{2V}{\pi} \Im \ln \frac{1+Z}{1-Z}.$$

We will conclude by writing this solution explicitly in terms of x and y. We begin by noting that $Z = |Z| \exp i\theta$ where θ is the phase of Z, i.e. $\tan\theta = \Im Z / \Re Z$. Thus

$$\ln Z = \ln |Z| + i\theta \Longrightarrow \Im \ln Z = \theta.$$

Now,

$$\frac{1+Z}{1-Z} = \frac{(1+Z)(1-Z^*)}{|1-Z|^2} = \frac{1-|Z|^2 + 2i\Im Z}{|1-Z|^2}$$

and thus

$$\Im \ln \frac{1+Z}{1-Z} = \tan^{-1}\left(\frac{2\Im Z}{1-|Z|^2}\right).$$

But we have

$$\begin{aligned}\Im Z &= e^{-\pi y/a} \sin \frac{\pi x}{a} \\ |Z|^2 &= e^{-2\pi y/a}\end{aligned}$$

and thus

$$\phi(x,y) = \frac{2V}{\pi} \tan^{-1}\left[\frac{2e^{-\pi y/a} \sin \frac{\pi x}{a}}{1 - e^{-2\pi y/a}}\right]$$

which, after some simplification, becomes

$$\boxed{\phi(x,y) = \frac{2V}{\pi} \tan^{-1}\left(\frac{\sin \pi x/a}{\sinh \pi y/a}\right).}$$

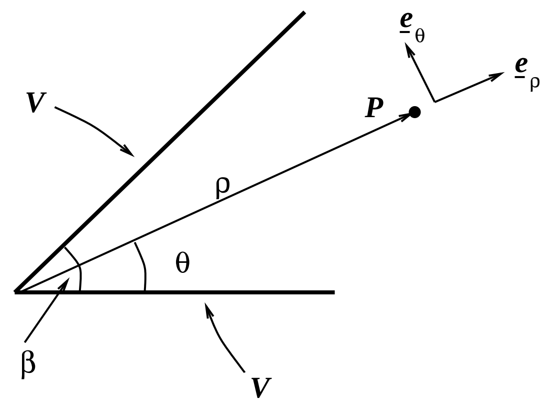
In practice, such problems can be done in a much simpler way, by observing that the real and imaginary components, u and v respectively, of an **analytic** complex function $f(z = x + iy)$ satisfy the two-dimensional Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0; \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

This is a direct consequence of the Cauchy-Riemann eqnations.

2.6.2 Field and Charge Distribution in Two-dimensional Corners

Consider two conducting planes meeting at an angle β, with potential V on the planes. The most appropriate coordinate system for the problem is that of cylindrical polars (s, θ, z), with the z axis along the line of intersection of the planes. Note that if we consider the problem sufficiently close to the intersection, the shape of the surface at larger distances will be unimportant.



Then Laplace's equation assumes the form

$$\nabla^2 \phi(s, \theta) = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial \phi}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

where we have suppressed the z variable. As before we look for factorizing solutions of the form

$$\phi(s, \theta) = R(s)T(\theta).$$

Then we have

$$\frac{s}{R}\frac{\partial}{\partial s}\left(s\frac{\partial R}{\partial s}\right) + \frac{1}{T}\frac{\partial^2 T}{\partial \theta^2} = 0.$$

Each term depends on a different variable, and this must hold for all s and z. Thus each term is separably constant. For the function $T(\theta)$, let us take

$$\frac{1}{T}\frac{\partial^2 T}{\partial\theta^2} = -\nu^2.$$

Since T must attain the same value at $\theta = 0$ and $\theta = \beta$, the solution must be oscillatory rather than exponential, and hence ν^2 must be **positive**. Thus the solution is

$$T_\nu(\theta) = \begin{cases} A_\nu \cos\nu\theta + B_\nu \sin\nu\theta; & \nu \neq 0 \\ A_0 + B_0\theta; & \nu = 0 \end{cases}$$

For the radial function, we have

$$s\frac{\partial}{\partial s}\left(s\frac{\partial R}{\partial s}\right) - \nu^2 R = 0.$$

For $\nu \neq 0$, let us take as trial solution $R \sim s^\alpha$,

$$(\alpha^2 - \nu^2)s^\alpha = 0,$$

yielding $\alpha = \pm\nu$. We need to consider the case $\nu = 0$ separately. Here we have

$$\frac{\partial}{\partial s}\left(s\frac{\partial R}{\partial s}\right) = 0$$

with solution

$$R_0(s) = a_0 + b_0 \ln s.$$

Thus the general form of R_ν is

$$R_\nu(s) = \begin{cases} a_\nu s^\nu + b_\nu s^{-\nu}; & \nu > 0 \\ a_0 + b_0 \ln s; & \nu = 0 \end{cases}$$

so the general solution for the potential has the form

$$\phi(s,\theta) = (a_0 + b_0 \ln s)(A_0 + B_0\theta) + \sum_{\nu>0}(a_\nu s^\nu + b_\nu s^{-\nu})(A_\nu \cos\nu\theta + B_\nu \sin\nu\theta) \quad (2.24)$$

The solution must be valid as $s \to 0$ (note that we are not interested in the solution for s large), and therefore the terms proportional to $\ln s$ and $s^{-\nu}$ cannot contribute. Thus our solution is of the form

$$\phi(s,\theta) = \begin{cases} A_0 + B_0\theta; & \nu = 0 \\ s^{\nu}(A_\nu \cos\nu\theta + B_\nu \sin\nu\theta); & \nu > 0 \end{cases}$$

We will now use the boundary conditions on the planes to further constrain the solution. At $\theta = 0, \beta$, we have $\phi = V$, independent of s, and therefore we have

$$\begin{aligned} A_\nu &= 0 \\ \sin\nu\beta &= 0; \quad \nu = \frac{n\pi}{\beta}, \; n = 1, 2, \ldots, \end{aligned}$$

yielding

$$\phi(s,\theta) = A_0 + B_0\theta + \sum_n B_n s^{n\pi/\beta} \sin\frac{n\pi\theta}{\beta}.$$

Finally, we impose that the potential be V on the two planes

$$\begin{aligned} \theta = 0, \phi = V &\implies A_0 = V \\ \theta = \beta, \phi = V &\implies B_0 = 0, \end{aligned}$$

and thus our final result is

$$\boxed{\phi(s,\theta) = V + \sum_n B_n s^{n\pi/\beta} \sin\frac{n\pi\theta}{\beta}.}$$

As we get closer into the corner, the first term will dominate,

$$\phi(s,\theta) \sim V + B_1 s^{\pi/\beta} \sin\frac{\pi\theta}{\beta}.$$

Taking the gradient, we obtain

$$\underline{E} = -\underline{\nabla}\phi = -\frac{\pi B_1}{\beta} s^{\pi/\beta - 1} \sin\frac{\pi\theta}{\beta}\underline{e_s} - \frac{\pi B_1}{\beta} s^{\pi/\beta - 1} \cos\frac{\pi\theta}{\beta}\underline{e_\theta}$$

and the induced surface charge density is

$$\sigma = \epsilon_0 \left[\underline{E} \cdot \underline{n}\right] = -\frac{\pi B_1 \epsilon_0}{\beta} s^{\pi/\beta - 1}$$

1. For $\beta < \pi$, we have that $\underline{E}$ and σ vanish as $s \to 0$.
2. For $\beta > \pi$, $\underline{E}$ and σ become **singular** as $s \to 0$.

Thus we see behaviour familiar from our knowledge of "action at points" - the fields and surface charge densities become singular near sharp edges.

Chapter 3

Boundary-value Problems in Curvilinear Coordinates

In the previous chapter, we saw how we could look for factorizable solutions to Laplace's Equation in Cartesian coordinates, and then construct the solution for more general boundary values using the completeness property of the such factorisable solutions. In this chapter we will employ analogous methods in *spherical polar* and *cylindrical* coordinate systems. In practice, the coordinate system that is appropriate depends on the *symmetry* or *geometry* of the problem.

3.1 Laplace's Equation in Spherical Polar Coordinates

We will denote our coordinates by (r, θ, φ), in terms of which Laplace's equation assumes the form

$$\nabla^2\phi(s,\theta,\varphi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}.$$

We will now seek *factorisable* solutions of the form

$$\phi(r,\theta,\varphi) = \frac{U(r)}{r}P(\theta)Q(\varphi),$$

where the factor of $1/r$ is conventional. Substituting this into Laplace's equation, we have

$$P(\theta)Q(\varphi)\frac{1}{r^2}\frac{d}{dr}\left[r^2\left(-\frac{1}{r^2}U(r)+\frac{1}{r}\frac{dU(r)}{dr}\right)\right]$$
$$+\quad\frac{U(r)Q(\varphi)}{r}\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP(\theta)}{d\theta}\right)+\frac{U(r)P(\theta)}{r}\frac{1}{r^2\sin^2\theta}\frac{d^2Q(\varphi)}{d\varphi^2}=0,$$

yielding

$$\frac{PQ}{r}\frac{d^2U}{dr^2}+\frac{UQ}{r^3\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\frac{UP}{r^3\sin^2\theta}\frac{d^2Q}{d\varphi^2}=0,$$

which we may write as

$$\frac{1}{Q}\frac{d^2Q}{d\varphi^2}+r^2\sin^2\theta\left[\frac{1}{U}\frac{d^2U}{dr^2}+\frac{1}{r^2\sin\theta}\frac{1}{P}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)\right]=0. \qquad (3.1)$$

The first term is a function of φ alone, and the remaining term is a function of (r,θ) alone. Thus they must be separately constant, and we may write

$$\frac{1}{Q}\frac{d^2Q}{d\varphi^2}=-m^2, \qquad (3.2)$$

where m is a constant. Eqn. 3.2 has solution

$$Q=e^{\pm im\varphi}.$$

We now observe that the solution must be periodic, with period 2π, in the azimuthal variable φ. Thus m must be an **integer**, and, of course, real. Thus we may write eqn. 3.1 as

$$\frac{r^2}{U}\frac{d^2U}{dr^2}+\frac{1}{\sin\theta}\frac{1}{P}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)-\frac{m^2}{\sin^2\theta}=0.$$

We now observe that the first term is purely a function of r, whilst the remaining terms are purely a function of θ. Thus we may write

$$\frac{r^2}{U}\frac{d^2U}{dr^2}=l(l+1)$$

where l is a constant - we will see the reason for expressing the constant in this way later. To solve this equation, we will take a trial solution

$$U(r) = r^\alpha,$$

yielding

$$\alpha(\alpha - 1) = l(l + 1)$$

with solutions $\alpha = l + 1, -l$. Thus we have

$$\boxed{U(r) = Ar^{l+1} + Br^{-l}.}$$

The equation for the polar coordinate θ now assumes the form

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]P = 0.$$

It is convenient to introduce the variable $x = \cos\theta$, with $-1 \leq x \leq 1$ and

$$\frac{d}{d\theta} = -\sin\theta\frac{d}{dx}.$$

After a little algebra, we have

$$\boxed{\frac{d}{dx}\left[(1 - x^2)\frac{dP}{dx}\right] + \left[l(l+1) - \frac{m^2}{1 - x^2}\right]P = 0}$$

This is the **Generalised Legendre Equation**, and is, once again, an equation of *Sturm-Liouville* type, with $p(x) = 1 - x^2$, $q(x) = -m^2/(1 - x^2)$, $\lambda = l(l + 1)$, and $r(x) = 1$.

We will now seek solutions of this equation, first for the case $m = 0$, where the equation is known as the **Ordinary Legendre Equation**

$$\boxed{\frac{d}{dx}\left[(1 - x^2)\frac{dP}{dx}\right] + l(l+1)P = 0.}$$

We begin by noting that the solutions must be both **continuous** and **single-valued** in the region $-1 \leq x \leq 1$, corresponding to $0 \leq \theta \leq \pi$. We will obtain the solutions through **series substitution**, i.e. by trying a solution of the form

$$P = \sum_{n=0}^{\infty} c_n x^{\gamma+n},$$

from which

$$\begin{aligned}
\frac{dP}{dx} &= \sum_{n=0}^{\infty} c_n(\gamma+n)x^{\gamma+n-1} \\
(1-x^2)\frac{dP}{dx} &= \sum_{n=0}^{\infty} c_n(\gamma+n)x^{\gamma+n-1} - \sum_{n=0}^{\infty} c_n(\gamma+n)x^{\gamma+n+1}, \\
\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] &= \sum_{n=0}^{\infty} c_n(\gamma+n)(\gamma+n-1)x^{\gamma+n-2} - \sum_{n=0}^{\infty} c_n(\gamma+n)(\gamma+n+1)x^{\gamma+n}.
\end{aligned}$$

Thus Legendre's equation becomes

$$\sum_{n=0}^{\infty} c_n(\gamma+n)(\gamma+n-1)x^{\gamma+n-2} + \sum_{n=0}^{\infty} c_n\left[l(l+1) - (\gamma+n)(\gamma+n+1)\right]x^{\gamma+n} = 0.$$

As this equation must be valid $\forall x \in [-1,1]$, we can equate the coefficients of the powers of x to zero. The leading power of x is $x^{\gamma-2}$, and we use this equation, the **indicial equation**, to determine γ. Thus

- $x^{\gamma-2}$:

$$c_0\gamma(\gamma-1) = 0 \implies \gamma = 0 \,\text{or}\, \gamma = 1$$

- $x^{\gamma-1}$:

$$c_1(\gamma+1)(\gamma+1-1) = 0 \implies \begin{cases} c_1 \text{ undetermined} & : \gamma = 0 \\ c_1 = 0 & : \gamma = 1 \end{cases}$$

- $x^{\gamma+n}$, $n \geq 0$:

$$c_{n+2} = \frac{(\gamma+n)(\gamma+n+1) - l(l+1)}{(\gamma+n+2)(\gamma+n+1)}c_n.$$

Note that the recursion relation relates only even (odd) polynomials for $\gamma = 0$ ($\gamma = 1$).

We have already noted that the solution must be valid for $x \in [-1, 1]$, and in particular at the end points $x = \pm 1$. Thus the series must be finite at $x = \pm 1$. To explore the convergence properties, we note that

$$c_{n+2}/c_n \longrightarrow 1 \text{ as } n \longrightarrow \infty,$$

and thus the series resembles a geometrical expansion $\sum x^{2n}$. This diverges at $x = \pm 1$ unless the series terminates, i.e. unless $c_n = 0$ for some n. Thus our requirement for convergence is

$$(\gamma + n)(\gamma + n - 1) - l(l + 1) = 0 \text{ for some } n.$$

- $\gamma = 0$:

$$n(n + 1) = l(l + 1) \Longrightarrow n = l$$

- $\gamma = 1$:

$$(n + 1)(n + 2) = l(l + 1) \Longrightarrow n = l - 1.$$

Note that in both cases the highest power of x is x^l; the two cases are the same. We call the corresponding solutions $P_l(x)$ the **Legendre Polynomials**, and conventially we take $P_l(1) = 1$. The first few are

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}(3x^2 - 1) \\ P_3(x) &= \frac{1}{2}(5x^3 - 3x). \end{aligned}$$

3.1.1 Rodriques' Formula and Generating Function

We can write the Legendre polynomials in a more memorable form through **Rodrigues' Formula**:

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l.$$

An equally useful means of determining the Legendre polynomials is through the **generating function**

$$g(t,x) \equiv (1-2xt+t^2)^{-1/2} = \sum_{l=0}^{\infty} P_l(x)t^l,\, |t|<1. \tag{3.3}$$

3.1.2 Orthogonality and Normalisation of Legendre Polynomials

Applying our theorem concerning the orthogonality of the solutions of the Sturm-Liouville equation yields

$$[l(l-1) - l'(l'+1)] \int_{-1}^{1} dx\, P_l(x)P_{l'}(x) = 0 \Longrightarrow \int_{-1}^{1} dx\, P_l(x)P_{l'}(x) = 0,\, l \neq l',$$

i.e. the Legendre polynomials are *orthogonal*. *N.B.* it is easy to check that our solutions satisfy the required boundary conditions.

To determine their normalisation, we can use either Rodrigues' formula, or the generating function; we use the latter. From eqn. 3.3, we have

$$\begin{aligned} \int_{-1}^{1} dx\, g(t,x)^2 &= \int_{-1}^{1} dx\, \frac{1}{1-2xt+t^2} \\ &= \left\{ -\frac{1}{2t} \ln(1-2xt+t^2) \right\}_{-1}^{1} \\ &= -\frac{1}{2t} \ln \frac{(1-t)^2}{(1+t)^2} \\ &= \sum_{l=0}^{\infty} \frac{2t^{2l}}{2l+1}, \end{aligned}$$

where we have used the series expansion of $\ln(1+t)$. However, we also have

$$\begin{aligned} \int_{-1}^{1} dx\, g(t,x)^2 &= \sum_{l,l'=0}^{\infty} \int dx\, P_l(x)P_{l'}(x)t^{l+l'} \\ &= \sum_{l=0}^{\infty} t^{2l} \int_{-1}^{1} dx\, P_l(x)^2. \end{aligned}$$

Equating the coefficients in these two expansions yields

$$\boxed{\int_{-1}^{1} dx\, P_l(x)P_l(x) = \frac{2}{2l+1}}$$

3.1.3 Recurrence Relations

Rodrigues' formula provides a means to obtain various **recurrence relations** between the Legendre Polynomials, for example:

$$\begin{aligned}
(l+1)P_{l+1}(x) - (2l+1)xP_l(x) + lP_{l-1}(x) &= 0 \\
\frac{d}{dx}P_{l+1}(x) - x\frac{dP_l(x)}{dx} - (l+1)P_l(x) &= 0 \\
(x^2-1)\frac{dP_l(x)}{dx} - lxP_l(x) + lP_{l-1}(x) &= 0 \\
\frac{d}{dx}P_{l+1}(x) - \frac{dP_{l-1}(x)}{dx} - (2l+1)P_l(x) &= 0.
\end{aligned}$$

Such recurrence relations allow us to evaluate many of the integrals we will encounter in the problems.

3.1.4 Completeness

Since the Legendre Polynomials form a complete set, we may write any function $f(x)$, $x \in [-1, 1]$ as

$$f(x) = \sum_{l=0}^{\infty} A_l P_l(x).$$

We obtain the coefficients using the orthogonality relations

$$\begin{aligned}
\int dx\, f(x)P_l(x) &= \sum_{l'=0}^{\infty} A_l \int_{-1}^{1} dx\, P_l(x)P_{l'}(x) \\
&= A_l \frac{2}{2l+1}
\end{aligned}$$

whence

$$A_l = \frac{2l+1}{2}\int_{-1}^{1} dx\, f(x)P_l(x)$$

Example

Consider the step-function $f(x)$ defined by

$$f(x) = \begin{cases} 1 & 0 < x \leq 1 \\ -1 & -1 \leq x < 0 \end{cases}$$

Then we have that

$$\begin{aligned} A_l &= \frac{2l+1}{2}\int_{-1}^{1} dx f(x) P_l(x) \\ &= \frac{2l+1}{2}\left\{\int_0^1 dx\, P_l(x) - \int_{-1}^0 dx\, P_l(x)\right\} \\ &= \frac{2l+1}{2}\int_0^1 dx\,\{P_l(x) - P_l(-x)\}. \end{aligned}$$

Thus we see that A_l is non-zero only for l odd:

$$A_l = \begin{cases} (2l+1)\int_0^1 dx\, P_l(x) & : l \text{ odd} \\ 0 & : l \text{ even} \end{cases}$$

Now by the last of our recurrence relations

$$\begin{aligned} A_l &= \int_0^1 dx \left\{\frac{d}{dx}P_{l+1}(x) - \frac{d}{dx}P_{l-1}(x)\right\} \\ &= P_{l+1}(1) - P_{l+1}(0) - P_{l-1}(1) + P_{l-1}(0) \\ &= P_{l-1}(0) - P_{l+1}(0) \end{aligned}$$

where we have used the normalisation condition $P_l(1) = 1$. But we have (from Rodrigues's formula, with a little work)

$$P_l(0) = \begin{cases} \frac{(-1)^{l/2}(l-1)!!}{2^{l/2}(l/2)!} & : l \text{ even} \\ 0 & : l \text{ odd} \end{cases}$$

where $(l-1)!! = (l-1)(l-3)\ldots 3.1$. Thus

$$\begin{aligned} A_l &= -\frac{(-1)^{(l+1)/2} l!!}{2^{(l+1)/2}((l+1)/2)!} + \frac{(-1)^{(l-1)/2}(l-2)!!}{2^{(l-1)/2}((l-1)/2)!} \\ &= \frac{(-1)^{(l-1)/2}(l-2)!!}{2^{(l-1)/2}((l-1)/2)!}\left\{1 + \frac{l}{l+1}\right\} \end{aligned}$$

Thus

$$A_l = \begin{cases} \left(-\frac{1}{2}\right)^{(l-1)/2} \frac{(l-2)!!(2l+1)}{2(\frac{l+1}{2})!} & : l \text{ odd} \\ 0 & : l \text{ even} \end{cases}$$

and we have

$$f(x) = \frac{3}{2}P_1(x) - \frac{7}{8}P_3(x) + \frac{11}{16}P_5(x)\ldots$$

3.2 Boundary-Value Problems with Azimuthal Symmetry

We may now write our general solution for the boundary-value problem in spherical coordinates with azimuthal symmetry, i.e. no φ dependence, as

$$\boxed{\phi(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-l-1}\right) P_l(\cos\theta),}$$

where the coefficients A_l and B_l are determined from the boundary conditions.

Example:

Consider the case of a sphere, of radius a, with no charge inside but potential $V(\theta)$ specified on the surface.
Since there are no charges inside the sphere, the potential ϕ **inside** must be regular everywhere. Thus $B_l = 0\,\forall l$, and we may write the solution as

$$\phi(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta).$$

Imposing the boundary conditions at $r = a$ yields

$$V(\theta) = \sum_{l=0}^{\infty} A_l\, a^l P_l(\cos\theta),$$

so that, using the normalisation condition on the Legendre polynomials, we have

$$A_l = \frac{2l+1}{2a^l} \int_0^{\pi} d\theta\, \sin\theta\, V(\theta) P_l(\cos\theta).$$

Suppose now that we require the solution **outside** the sphere. Then the solution must be finite as $r \to \infty$, and thus

$$\phi(r,\theta) = \sum_{l=0}^{\infty} B_l r^{-l-1} P_l(\cos\theta)$$

with

$$V(\theta) = \sum_{l=0}^{\infty} B_l a^{-l-1} P_l(\cos\theta),$$

so that

$$B_l = \frac{2l+1}{2} a^{l+1} \int_0^\pi d\theta \, \sin\theta \, V(\theta) P_l(\cos\theta).$$

Let us now go back to the problem in Section 2.4.2:

$$V(\theta) = \begin{cases} V & : 0 \le \theta \le \pi/2 \\ -V & : \pi/2 \le \theta \le \pi \end{cases}$$

Then we have

$$\begin{aligned} B_l &= \frac{2l+1}{2} a^{l+1} V \left\{ \int_0^{\pi/2} P_l(\cos\theta) \sin\theta d\theta - \int_{-\pi/2}^{\pi} P_l(\cos\theta) \sin\theta d\theta \right\} \\ &= \frac{2l+1}{2} a^{l+1} V \left\{ \int_0^1 dx P_l(x) - \int_{-1}^0 dx P_l(x) \right\} \\ &= \frac{2l+1}{2} a^{l+1} V \left\{ \int_{-1}^1 dx f(x) P_l(x) \right\} \end{aligned}$$

where

$$f(x) = \begin{cases} 1 & 0 < x \le 1 \\ -1 & -1 \le x < 0 \end{cases}$$

This is just the expression we evaluated in Section 3.1.4, and thus we have:

$$B_l = \begin{cases} V a^{l+1} (-\frac{1}{2})^{\frac{l-1}{2}} \frac{(l-2)!!(2l+1)}{2(\frac{l+1}{2})!} & l \text{ odd} \\ 0 & l \text{ even} \end{cases}$$

so that

$$\phi(r,\theta) = V \left\{ \frac{3}{2} \frac{a^2}{r^2} P_1(\cos\theta) - \frac{7}{8} \frac{a^4}{r^4} P_3(\cos\theta) + \frac{11}{16} \frac{a^6}{r^6} P_5(\cos\theta) + \ldots \right\}. \qquad (3.4)$$

Recall that in Section 2.4.2 we obtained

$$\begin{aligned} \phi(r,\theta,\varphi) &= \frac{3Va^2}{r^2} \left\{ \cos\theta - \frac{7a^2}{12r^2} \left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta \right) + \mathcal{O}(\frac{a^4}{r^4}) \right\} \\ &= V \left\{ \frac{3}{2} \frac{a^2}{r^2} P_1(\cos\theta) - \frac{7}{8} \frac{a^4}{r^4} P_3(\cos\theta) + \ldots \right\}, \end{aligned}$$

which is precisely the first two terms in the expansion of eqn. 3.4.

The crucial observation in such problems is that the series expansion

$$\phi(r,\theta) = \sum_l \left(A_l\, r^l + B_l\, r^{-l-1}\right) P_l(\cos\theta) \tag{3.5}$$

is unique. Thus it is possible to determine the coefficients A_l and B_l from a knowledge of the solution in some limited domain. As an illustration, we recall that we obtained a closed solution to the above problem above the North pole, i.e. at $\theta = 0$:

$$\phi(z = r, \theta = 0) = V\left\{1 - \frac{r^2 - a^2}{r\sqrt{r^2 + a^2}}\right\}.$$

We can use the binomial expansion to express this as a series in a/r:

$$\phi(z = r, \theta = 0) = V\left\{\frac{a^2}{r^2} - (1 - a^2/r^2)\sum_{j=1}^{\infty}\frac{\Gamma(\frac{1}{2})}{\Gamma(j+1)\Gamma(\frac{1}{2} - j)}\left(\frac{a}{r}\right)^{2j}\right\}.$$

If we use the property

$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z}$$

and note that $\Gamma(1/2) = \sqrt{\pi}$, we obtain, after a little manipulation (*exercise*),

$$\phi(r, \theta = 0) = \frac{V}{\sqrt{\pi}}\sum_{j=1}^{\infty}(-1)^{j-1}\frac{(2j - \frac{1}{2})\Gamma(j - \frac{1}{2})}{j!}\left(\frac{a}{r}\right)^{2j}.$$

We now compare this series solution with eqn. 3.5, evaluated at $\theta = 0$, and observe that only terms with $l = 2j - 1$ enter, and that

$$B_{2j-1} = \frac{V}{\sqrt{\pi}}(-1)^{j-1}\frac{(2j - \frac{1}{2})\Gamma(j - \frac{1}{2})}{j!}a^{2j}.$$

Let us try the first couple of term

$$\begin{aligned} j = 1:&\quad B_1 = \tfrac{V}{\sqrt{\pi}}(-1)^0\tfrac{(3/2)\Gamma(1/2)}{1!}a^2 = 3Va^2/2 \\ j = 2:&\quad B_3 = \tfrac{V}{\sqrt{\pi}}(-1)^1\tfrac{(5/2)\Gamma(3/2)}{2!}a^4 = -\tfrac{7}{8}Va^4, \end{aligned}$$

and once again we reproduce the expression eqn. 3.4.

3.2.1 Expansion of $\frac{1}{|\underline{x}-\underline{x}'|}$

Let us conclude this section by looking at the expansion of this critical quantity that occurs in the construction of the Green's function. We begin by observing that the result can depend only on r, r' and γ, the angle between $\underline{x}$ and $\underline{x}'$. We may thus simplify the problem by choosing the azimuthal direction (z axis) along the $\underline{x}'$ axis. The problem then displays manifest azimuthal symmetry, and we may write

$$\frac{1}{|\underline{x}-\underline{x}'|} = \sum_{l=0}^{\infty}\left(A_l(r')r^l + B_l(r')r^{-l-1}\right)P_l(\cos\gamma) \tag{3.6}$$

We now consider the case where $\underline{x}$ lies parallel to $\underline{x}'$, when $\cos\gamma = 1$. Then the l.h.s. of eqn. 3.6 becomes

$$\frac{1}{|\underline{x}-\underline{x}'|} = \frac{1}{|r-r'|}.$$

There are two cases:

$$r > r' \ : \ \frac{1}{|r-r'|} = \frac{1}{r-r'} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l = \sum_{l=0}^{\infty}\frac{r'^l}{r^{l+1}}$$
$$r < r' \ : \ \frac{1}{|r-r'|} = \frac{1}{r'-r} = \frac{1}{r'}\sum_{l=0}^{\infty}\left(\frac{r}{r'}\right)^l = \sum_{l=0}^{\infty}\frac{r^l}{r'^{l+1}}$$

Let us introduce $r_> = \max(r, r')$ and $r_< = \min(r, r')$. Then we may write

$$\frac{1}{|r-r'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}$$

and, comparing with eqn. 3.6, we have

$$\boxed{\frac{1}{|\underline{x}-\underline{x}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma)}$$

3.3 Solution of the Generalised Legendre Equation

Let us now consider the case where we no longer assume azimuthal symmetry. Then we are concerned with solutions of the *Generalised Legendre Equation*,

$$\frac{d}{dx}\left[(1-x^2)\frac{dP(x)}{dx}\right]+\left[l(l+1)-\frac{m^2}{1-x^2}\right]P(x)=0. \tag{3.7}$$

We can obtain a series solution in an analogous way to that of the *ordinary Legendre equation.* For solutions to be finite at $x=\pm1$, corresponding to $\theta=0,\pi$, we require that l must be a positive integer or zero, **and** that m takes the values

$$m=-l,-l+1,\ldots,l-1,l.$$

Recall that we already know that m must be an integer by the requirement that the azimuthal function $Q(\varphi)$ be single-valued.
For the case where m is *positive*, we can write the solutions $P_l^m(x)$ as

$$P_l^m(x)=(-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_l(x)$$

or for both positive and negative m by adopting Rodrigues' formula:

$$P_l^m(x)=\frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l.$$

Note that eqn. 3.7 depends only on m^2. Thus we have that $P_l^{-m}(x)$ must be proportional to $P_l^m(x)$, and in fact

$$P_l^{-m}(x)=(-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x).$$

Eqn. 3.7 is an equation of Sturm-Liouville class, with eigenvalues $l(l+1)$. We can apply the orthogonality theorem *at fixed* m, and we have

$$\int_{-1}^{1}dx P_{l'}^m(x)P_l^m(x)=\frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{ll'}.$$

3.4 Spherical Harmonics

We began by looking at separable solutions in spherical polar coordinates, and writing

$$\phi(r,\theta,\varphi) = \frac{1}{r}U(r)P(\theta)Q(\varphi).$$

It is convenient to combine the angular functions into solutions on the unit sphere:

$$Y_{lm}(\theta,\varphi) = \sqrt{\frac{(l-m)!(2l+1)}{4\pi(l+m)!}}P_l^m(\cos\theta)e^{im\varphi}. \tag{3.8}$$

The spherical harmonics (3.8) satisfy the equation

$$-\nabla^2 Y_{lm}(\theta,\varphi) = l(l+1)r^2 Y_{lm}(\theta,\varphi) \tag{3.9}$$

or, in explicit form

$$\left[-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]Y_{lm}(\theta,\varphi) = l(l+1)r^2 Y_{lm}(\theta,\varphi) \tag{3.10}$$

(A person familiar with quantum mechanics may recognize the expression in square brackets in l.h.s. of this eqn as a square of operator of anglular momentum L^2).

Using our relation between $P_l^{-m}(\cos\theta)$ and $P_l^m(\cos\theta)$ we have

$$Y_{l,-m}(\theta,\varphi) = (-1)^m Y_{lm}(\theta,\varphi)^*$$

and the normalisation condition is

$$\int_0^{2\pi}\int_0^{\pi} d\varphi\, d\theta \sin\theta Y^*_{lm}(\theta,\varphi)Y_{l'm'}(\theta,\varphi) = \delta_{ll'}\delta_{mm'},$$

i.e.

$$\int d\Omega\, Y_{lm}(\theta,\varphi)Y^*_{l'm'}(\theta,\varphi) = \delta_{mm'}\delta_{ll'}.$$

For the case $m = 0$, the solution clearly reduces to the Legendre polynomial, up to some normalisation:

$$Y_{l0}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)$$

3.4.1 Completeness

Any arbitrary function $g(\theta, \varphi)$ defined on $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$ may be expressed in terms of Y_{lm}:

$$g(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} Y_{lm}(\theta, \varphi)$$

where

$$\begin{aligned} A_{lm} &= \int_0^{\pi} d\theta \, \sin\theta \int_0^{2\pi} g(\theta, \varphi) Y_{lm}^*(\theta, \varphi) \\ &= \int d\Omega \, Y_{lm}^*(\theta, \varphi) g(\theta, \varphi) \end{aligned}$$

3.4.2 General Solution

We can now write the general solution of the Laplace boundary value problem as

$$\phi(r, \theta, \varphi) = \sum_{l=}^{\infty} \sum_{m=-l}^{l} \left[A_{lm} r^l + B_{lm} r^{-l-1} \right] Y_{lm}(\theta, \varphi)$$

3.4.3 Addition Theorem for Spherical Harmonics

Consider two vectors $\underline{x}, \underline{x}'$, with coordinates (r, θ, φ) and (r', θ', φ') respectively. Let γ be the angle between $\underline{x}$ and $\underline{x}'$, so that

$$\cos\gamma = \frac{\underline{x}.\underline{x}'}{|\underline{x}||\underline{x}'|} = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi').$$

Then we have

$$\boxed{P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi)}$$

This is proved in Jackson, but is more easily proved using group theory. Note that we can rewrite this in the form

$$P_l(\cos\gamma) = P_l(\cos\theta) P_l(\cos\theta') + 2 \sum_{m=1}^{l} \frac{(l-m)!}{(l+m)!} P_l^m(\cos\theta) P_l^m(\cos\theta') \cos m(\varphi - \varphi')$$

Example

An important application is to the expansion of $\frac{1}{|\underline{x}-\underline{x}|}$, discussed in section 3.2.1:

$$\frac{1}{|\underline{x}-\underline{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma).$$

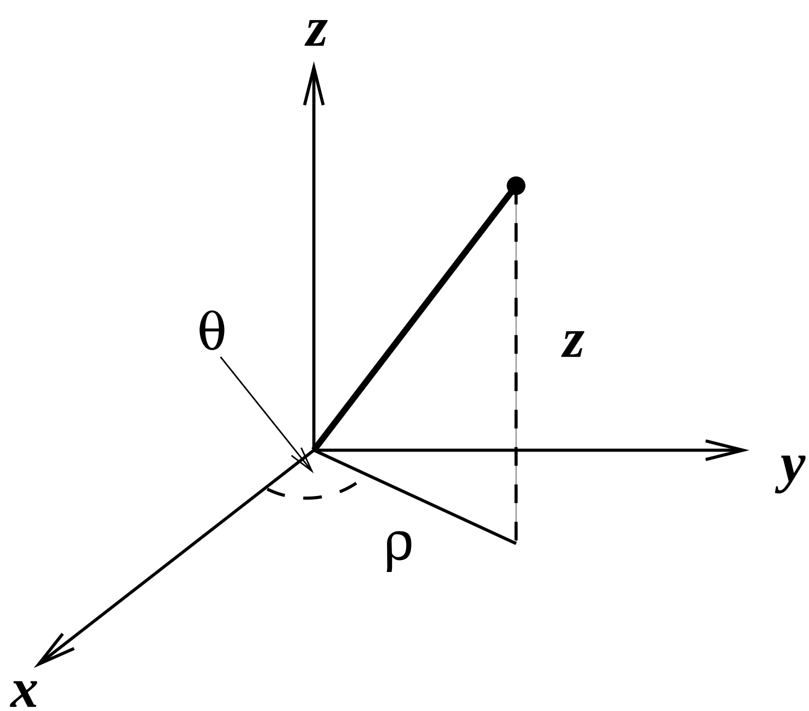
Using the addition theorem, we can rewrite this as

$$\frac{1}{|\underline{x}-\underline{x}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi).$$

Superficially, this looks like a much more complicated expression, since we have introduced an additional sum over m. But it is now a sum over terms that factorise into a function of (θ, φ) and a function of (θ', φ'), and thus much more useful.

3.5 Laplace's Equation in Cylindrical Polar Coordinates

We will denote the coordinates by (s, φ, z)



In terms of these coordinates, Laplace's equation assumes the form

$$\nabla^2\phi(s,\varphi,z) = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial\phi}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2} = 0.$$

As before, we look for separable solutions of the form

$$\phi(s,\varphi,z) = R(s)T(\varphi)Z(z),$$

so that Laplace's equation becomes

$$TZ\frac{1}{s}\frac{d}{ds}\left(s\frac{dR}{ds}\right) + RZ\frac{1}{s^2}\frac{d^2T}{d\varphi^2} + RT\frac{d^2Z}{dz^2} = 0,$$

which we may rewrite as

$$\frac{1}{Rs}\frac{d}{ds}\left(s\frac{dR}{ds}\right) + \frac{1}{s^2T}\frac{d^2T}{d\varphi^1} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0.$$

The third term is a function of z alone, whilst the others are a function of s and φ alone. Thus we may write

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = k^2$$

where k is a (not necessarily real) constant, with solution

$$Z(z) = e^{\pm kz}.$$

Thus we may now rewrite Laplace's equation as

$$\frac{s}{R}\frac{d}{ds}\left(s\frac{dR}{ds}\right) + \frac{1}{T}\frac{d^2T}{d\varphi^2} + k^2s^2 = 0,$$

and so for the angular term we have

$$\frac{1}{T}\frac{d^2T}{d\varphi^2} = -\nu^2$$

with solution

$$T(\varphi) = e^{\pm i\nu\varphi}.$$

For the solution to be single valued at $\varphi = 0$ and 2π, ν must be an **integer**.

Finally, the radial equation is

$$\frac{s}{R}\frac{d}{ds}\left(s\frac{dR}{ds}\right) - \nu^2 + k^2 s^2 = 0.$$

We can eliminate the constant k by the substitution $x = ks$, yielding

$$\frac{x}{R}\frac{d}{dx}\left(x\frac{dR}{dx}\right) - \nu^2 + x^2 = 0$$

which we write as

$$\boxed{\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right) R = 0}$$

This is the **Bessel Equation**.

As in the case of the Legendre equation, we find a solution by **series substitution**

$$R(x) = \sum_{n=0}^{\infty} c_n x^{s+n} : \quad c_0 \neq 0 \tag{3.11}$$

Aside: why do we have to introduce the power x^s, rather than just looking for a solution in terms of a Taylor expansion about $x = 0$? The reason is that there is a *regular singular point* at $x = 0$, i.e. the coefficents of R and its derivatives in the Bessel equation vanish at $x = 0$, and therefore the solution can have a singularity there. In the case of the Legendre equation, there are regular singular points at $x = \pm 1$.

From eqn. 3.11, we have

$$\begin{aligned}\frac{dR}{dx} &= \sum_{n=0}^{\infty} c_n(\gamma + n)x^{\gamma+n-1} \\ \frac{d^2R}{dx^2} &= \sum_{n=0}^{\infty} c_n(\gamma + n)(\gamma + n - 1)x^{\gamma+n-2},\end{aligned}$$

and substituting into the Bessel equation we have

$$\sum_{n=0}^{\infty} c_n(\gamma+n)(\gamma+n-1)x^{\gamma+n-2} + \sum_{n=0}^{\infty} c_n(\gamma+n)x^{\gamma+n-2} + \sum_{n=0}^{\infty} c_n x^{\gamma+n} - \nu^2 \sum_{n=0}^{\infty} c_n x^{\gamma+n-2} = 0.$$

The lowest power of x is $x^{\gamma-2}$, and equating the coefficients of this to zero gives the indicial equation which determines γ.

- $x^{\gamma-2}$:

$$c_0\gamma(\gamma-1)+c_0\gamma+\nu^2c_0=0 \Rightarrow \gamma=\pm\nu, \quad \text{since } c_0 \neq 0.$$

- $x^{\gamma-1}$:

$$\begin{aligned} 0 &= c_1(\gamma+1)\gamma+c_1(\gamma+1)-\nu^2c_1 \\ &= c_1(\gamma^2+2\gamma+1-\nu^2) \\ &= c_1(2\gamma+1) \quad \text{since } \gamma^2=\nu^2 \\ \Rightarrow c_1 &= 0 \quad \text{since } \nu \text{ is an integer.} \end{aligned}$$

- $x^{n+\gamma}, n \geq 0$:

$$\begin{aligned} c_{n+2}[(\gamma+n+2)(\gamma+n+1)+(\gamma+n+2)-\nu^2]+c_n &= 0 \\ \Rightarrow c_{n+2}[(\gamma+n+2)^2-\nu^2] &= -c_n \\ \Rightarrow c_{n+2} &= -\frac{1}{(n+2)(n+2+2\gamma)}c_n \end{aligned}$$

  where in the last line we have used $\gamma^2=\nu^2$.

As in the case of Legendre's equation, the recurrence relation relates either odd or even values of n. However, we have seen that $c_1=0$. Thus $c_n=0$ *for all odd* n. Therefore, let us make the substitution $n=2j$, and write the recurrence relation as

$$\begin{aligned} c_{2j+2} &= -\frac{1}{4(j+1)(j+1+\gamma)}c_{2j}, \quad j=0,1,2,\ldots \\ c_{2j+1} &= 0. \end{aligned}$$

We can now iterate this recursion relation to obtain

$$c_{2j}=(-1)^j\left(\frac{1}{2}\right)^{2j}\frac{\Gamma(\gamma+1)}{\Gamma(j+1)\Gamma(\gamma+j+1)}c_0.$$

Conventionally, we choose

$$c_0=\frac{1}{2^\gamma\Gamma(\gamma+1)},$$

so that the solutions may be written

$$\begin{aligned} J_\nu(x) &= \left(\frac{x}{2}\right)^{\nu} \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j+1)\Gamma(\nu+j+1)} \left(\frac{x}{2}\right)^{2j} \\ J_{-\nu}(x) &= \left(\frac{x}{2}\right)^{-\nu} \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j+1)\Gamma(j-\nu+1)} \left(\frac{x}{2}\right)^{2j}. \end{aligned}$$

These are the **Bessel Functions of the first kind of order** $\pm\nu$. Some observations:

- The series converge for all finite x
- If ν is *not* an integer, the solutions are *linearly independent.*
- If ν *is* an integer, they are *linearly dependent,* and in particular

$$J_{-m}(x) = (-1)^m J_m(x).$$

Proof: This is a consequence of the properties of the gamma function $\Gamma(z)$, which has singularities for $z = 0$ and for z a negative integer - recall the earlier relation

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

We have

$$J_{-m}(x) = \left(\frac{x}{2}\right)^{-m} \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j+1)\Gamma(j-m+1)} \left(\frac{x}{2}\right)^{2j}.$$

Now $\Gamma(j-m+1) \longrightarrow \infty$ as argument approaches 0 or a negative integer. Thus only those terms in the sum for which $j-m+1 \geq 1$ contribute, and we can write

$$\begin{aligned} J_{-m}(x) &= \left(\frac{x}{2}\right)^{-m} \sum_{j=m}^{\infty} \frac{(-1)^j}{\Gamma(j+1)\Gamma(j-m+1)} \left(\frac{x}{2}\right)^{2j} \\ &= \left(\frac{x}{2}\right)^{-m} \left(\frac{x}{2}\right)^{2m} \sum_{l=0}^{\infty} \frac{(-1)^{l+m}}{\Gamma(l+1)\Gamma(l+m+1)} \left(\frac{x}{2}\right)^{2l} \\ &= \left(\frac{x}{2}\right)^{m} (-1)^m \sum_{l=0}^{\infty} \frac{(-1)^l}{\Gamma(l+1)\Gamma(l+m+1)} \left(\frac{x}{2}\right)^{2l} \\ &= (-1)^m J_m(x) \end{aligned}$$

Because of the linear dependence of $J_{-m}(x)$ on $J_m(x)$, we introduce a second, linearly independent function

$$N_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}$$

known as the **Neumann Function** or the **Bessel Function of the second kind**. Conventionally, we choose as our linearly independent functions $J_\nu(x)$ and $N_\nu(x)$ even if ν is not an integer.

Bessel Function of the Third Kind

These are just another pair of linearly independent solutions of the Bessel equation:

$$\begin{aligned} H_\nu^{(1)}(x) &= J_\nu(x) + iN_\nu(x) \\ H_\nu^{(2)}(x) &= J_\nu(x) - iN_\nu(x) \end{aligned}$$

These are also known as **Hankel Functions**. Their utility is that they have a more straightforward integral representation than $J_\nu(x)$ and $N_\nu(x)$.

3.5.1 Recursion Relations

The sets of solutions of the Bessel equation are collectively known as **cylinder functions**, and satisfy recursion relations in the same manner as the Legendre polynomials, e.g.

$$\begin{aligned} \Omega_{\nu-1}(x) + \Omega_{\nu+1}(x) &= \frac{2\nu}{x}\Omega_\nu(x) \\ \Omega_{\nu-1}(x) - \Omega_{\nu+1}(x) &= 2\frac{d\Omega_\nu(x)}{dx} \end{aligned}$$

3.5.2 Limiting Behaviour of Solutions

In the limit $x \ll 1$, we have

$$J_\nu(x) \rightarrow \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu$$

$$N_\nu(x) \quad \rightarrow \quad \begin{cases} \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + \gamma_E + \ldots\right] & \nu = 0 \\ -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu & \nu \neq 0 \end{cases}$$

where ν is real and non-negative, and $\gamma_E = 0.5772\ldots$ is the *Euler-Mascheroni constant*. Note that, when constructing solutions of the boundary-value problem, only $J_\nu(x)$ is regular as $x \rightarrow 0$.
In the limit $x \gg 1, \nu$, we have

$$\begin{aligned} J_\nu(x) \quad &\rightarrow \quad \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \\ N_\nu(x) \quad &\rightarrow \quad \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right). \end{aligned} \tag{3.12}$$

The transition between these limiting forms occurs at $x \sim \nu$.

3.5.3 Roots of the Bessel functions

From the limiting forms 3.12, we see that each Bessel function has an infinite number of roots, which we denote $x_{\nu n}, n = 1, 2, 3, \ldots$ where

$$J_\nu(x_{\nu n}) = 0, \quad \text{for } x = 1, 2, 3, \ldots.$$

In particular, we have

$$\begin{aligned} \nu = 0 : x_{0n} \quad &= \quad 2.405, 5.520, 8.654, \ldots \\ \nu = 1 : x_{1n} \quad &= \quad 3.832, 7.016, 10.173, \ldots \\ \nu = 2 : x_{2n} \quad &= \quad 5.136, 8.417, 11.620, \ldots \end{aligned}$$

3.5.4 Ortogonality of the Bessel Functions

The roots of the Bessel function $J_\nu(x)$ are crucial when we consider its *orthogonality properties*, which take a rather unexpected form. We introduce the functions

$$\sqrt{s}J_\nu(x_{\nu n}s/a), n = 1, 2, 3, \ldots$$

and will now show that, for fixed $\nu \geq 0$, these functions, identified by n, form an orthogonal set on $0 \leq s \leq a$.

Proof

Substitute into the Bessel equation:

$$\frac{1}{s}\frac{d}{ds}\left[s\frac{d}{ds}J_\nu(x_{\nu n}s/a)\right] + \left(\frac{x_{\nu n}^2}{a^2} - \frac{\nu^2}{s^2}\right) J_\nu(x_{\nu n}s/a) = 0,$$

where we have made the change of variable $x \longrightarrow x_{\nu n}s/a$. We now rewrite this as

$$\frac{d}{ds}\left[s\frac{dJ_\nu}{ds}\right] - \frac{\nu^2}{s}J_\nu = -\frac{x_{\nu n}^2}{a^2}sJ_\nu.$$

This is the Sturm-Liouville equation, with

$$\begin{aligned} p(x) &= s, \\ q(x) &= -\nu^2/s, \\ r(x) &= s, \\ \lambda &= x_{\nu n}^2/a^2. \end{aligned}$$

Thus we have

$$(x_{\nu n}^2 - x_{\nu n'}^2)\int_0^a ds\, sJ_\nu(x_{\nu n'}s/a)J_\nu(x_{\nu n}s/a) = 0$$

providing

$$\left[s\left\{J_\nu x_{\nu n'}s/a)\frac{d}{ds}J_\nu(x_{\nu n'}s/a) - J_\nu(x_{\nu n}s/a)\frac{d}{ds}J_\nu(x_{\nu n'}s/a)\right\}\right]_0^a = 0. \qquad (3.13)$$

At the upper limit, $s = a$, this expression vanishes since $x_{\nu n}$ and $x_{\nu n'}$ are roots of the Bessel function, and at the lower limit, $s = 0$, the expression vanishes because of the factor of s. Thus we have

$$\int_0^a ds\, sJ_\nu\left(\frac{x_{\nu n}s}{a}\right)J_\nu\left(\frac{x_{\nu n'}s}{a}\right) = 0,\ n \neq n'$$

The integral can be evaluated for $n' = n$, with the result

$$\boxed{\int_0^a ds\, sJ_\nu\left(\frac{x_{\nu n}s}{a}\right)J_\nu\left(\frac{x_{\nu n'}s}{a}\right) = \frac{a^2}{2}\left[J_{\nu+1}(x_{\nu n})\right]^2\delta_{nn'}.}$$

3.5.5 Completeness

We now assume that the Bessel functions satisfy the completeness relation, and therefore we can expand *any* function on $0 \le s \le a$ as

$$f(s) = \sum_{n=1}^{\infty} A_{\nu n} J_\nu(x_{\nu n} s/a)$$

where

$$A_{\nu n} = \frac{2}{a^2 J_{\nu+1}^2(x_{\nu n})} \int_0^a ds\, s f(s) J_\nu\left(\frac{x_{\nu n} s}{a}\right).$$

This is a **Fourier-Bessel series**. This expansion is particularly useful for the case where $f(a) = 0$, e.g. the Dirichlet problem, since each term in the expansion satisfies the boundary conditions. An alternative set of basis functions is provided by

$$\sqrt{s} J_\nu\left(\frac{y_{\nu n} s}{a}\right),$$

where the $y_{\nu n}$ are the roots of $dJ_\nu/dx = 0$, because this set still satisfies the condition of eqn. 3.13. This choice is often more appropriate for the Neumann problem.

3.5.6 Modified Bessel Functions

Note that if we had chosed a separation constant such that the solution in the z-variable was

$$Z(z) = e^{\pm ikz},$$

then the equation for $R(s)$ would have been

$$\frac{d^2R}{ds^2} + \frac{1}{s}\frac{dR}{ds} - \left(k^2 + \frac{\nu^2}{s^2}\right) R = 0,$$

which, after our usual substitution $x = ks$, becomes

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{d^2R}{dx} - \left(1 + \frac{\nu^2}{x^2}\right) R = 0.$$

with solutions

$$\begin{aligned} I_\nu(x) &= i^{-\nu} J_\nu(ix), \\ K_\nu(x) &= \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix). \end{aligned}$$

These, like I_ν and N_ν, are real functions of a real variable x, with limiting forms:

$x \ll 1$

$$\begin{aligned} I_\nu(x) &\rightarrow \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu \\ K_\nu(x) &\rightarrow \begin{cases} -\left[\ln\left(\frac{x}{2}\right) + \gamma_E + \ldots\right] & \nu = 0 \\ \frac{\Gamma(\nu)}{2} \left(\frac{2}{x}\right)^\nu & \nu \neq 0 \end{cases} \end{aligned}$$

$x \gg 1, \nu$

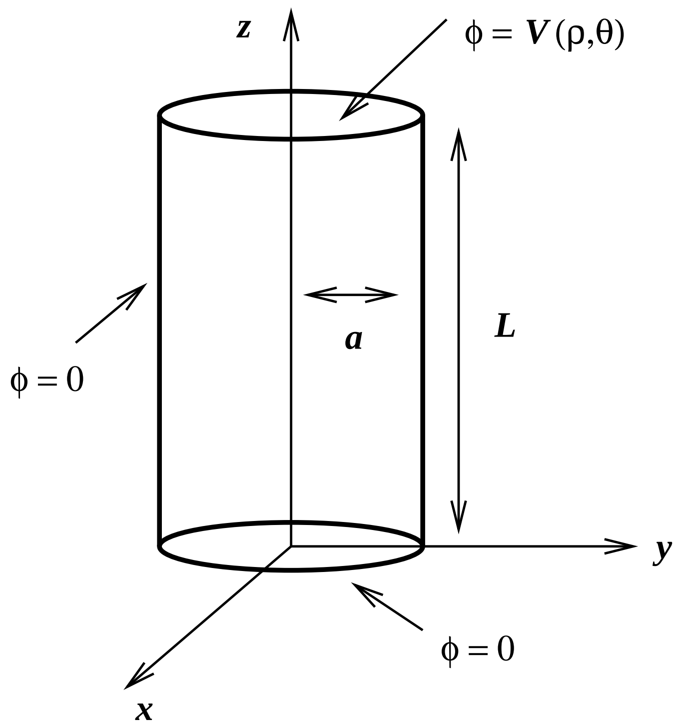
$$\begin{aligned} I_\nu(x) &\rightarrow \frac{1}{\sqrt{2\pi x}} e^x \left[1 + \mathcal{O}\left(\frac{1}{x}\right)\right] \\ K_\nu(x) &\rightarrow \sqrt{\frac{\pi}{2x}} e^{-x} \left[1 + \mathcal{O}\left(\frac{1}{x}\right)\right]. \end{aligned}$$

Note again that only $I_\nu(x)$ is regular as $x \rightarrow 0$.

3.6 Boundary-value Problems in Cylindrical Coordinates

Consider the solution of the boundary-value problem in a cylinder of radius a, and length L, subject to the boundary conditions

$$\begin{aligned}\phi(s,\varphi,0) &= 0\\ \phi(a,\varphi,z) &= 0;\ 0 \le z \le L\\ \phi(s,\varphi,L) &= V(s,\varphi)\end{aligned}$$



We look for separable solutions of the form

$$\phi(s,\varphi,z) = R(s)T(\varphi)Z(z).$$

The angular factor has the form

$$T_m(\varphi) = A\sin m\varphi + B\cos m\varphi$$

where m is an integer greater than or equal to zero. The z factor is of the form

$$Z(z) = \sinh kz$$

where k is the separation constant, and we have imposed the boundary condition $Z(0) = 0$. Finally, the radial component is of the form

$$R_m(s) = C_m J_m(ks) + D_m N_m(ks).$$

Since there are no charges in the region $s \leq a$, the solution must be regular there, and in particular must be finite at $s = 0$. Thus we have $D_m = 0$. Furthermore, R must vanish at $s = a$, and thus

$$J_m(ka) = 0$$

and hence the values of k are

$$k_{mn} = x_{mn}/a,\ n = 1, 2, 3, \ldots$$

where x_{mn} is the n-th root of $J_m(x) = 0$. Thus our general solution may be written

$$\begin{aligned}\phi(s,\varphi,z) &= \sum_{n=1}^{\infty} \frac{B_{0n}}{2} J_0(k_{0n}s)\sinh(k_{0n}z) \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m(k_{mn}s)\sinh(k_{mn}z)\left[A_{mn}\sin m\varphi + B_{mn}\cos m\varphi\right]. \quad (3.14)\end{aligned}$$

We now impose the boundary condition at $z = L$:

$$\begin{aligned}V(s,\varphi) &= \sum_{n=1}^{\infty} \frac{B_{0n}}{2} J_0(k_{0n}s)\sinh(k_{0n}L) \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m(k_{mn}s)\sinh(k_{mn}L)\left[A_{mn}\sin m\varphi + B_{mn}\cos m\varphi\right].\end{aligned}$$

This is a **Fourier series** in φ and a **Fourier-Bessel series** in s. We apply the orthogonality conditions, e.g., for A_{mn}:

$$\begin{aligned}&\int_0^a ds\, s \int_0^{2\pi} d\varphi\, V(s,\varphi) J_{m'}(k_{m'n'}s)\sin m'\varphi = \\ &\quad \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \sinh(k_{mn}L)\left\{\int_0^a ds\, s J_m(k_{mn}s) J_{m'}(k_{m'n'}s)\right\} \\ &\quad \times\left\{A_{mn}\int_0^{2\pi} d\varphi\, \sin m\varphi \sin m'\varphi + B_{mn}\int_0^{2\pi} d\varphi\, \cos m\varphi \sin m'\varphi\right\} \\ &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \sinh(k_{mn}L) A_{mn}\left\{\frac{a^2}{2}[J_{m+1}(x_{mn})]^2\delta_{nn'}\right\}\{\pi\delta_{mm'}\}\end{aligned}$$

and thus

$$
\begin{aligned}
A_{mn} &= \frac{2}{\pi a^2 \sinh(k_{mn}L)[J_{m+1}(x_{mn})]^2} \int_0^a ds\, s \int_0^{2\pi} d\varphi\, V(s,\varphi) J_m(k_{mn}s) \sin m\varphi \\
B_{mn} &= \frac{2}{\pi a^2 \sinh(k_{mn}L)[J_{m+1}(x_{mn})]^2} \int_0^a ds\, s \int_0^{2\pi} d\varphi\, V(s,\varphi) J_m(k_{mn}s) \cos m\varphi
\end{aligned}
$$

This form of the Fourier-Bessel seires is appropriate for problems confined to a finite region of s. Suppose, however, that we are interested in the solution for all $0 \le s \le \infty$.

Example

Determine $\phi(s, \varphi, z)$ in the upper-half plane $z \ge 0$, with $\phi(s, \varphi, 0) = V(s, \varphi)$, and ϕ finite as $z \to \infty$. Then the separable solutions are of the form

$$
e^{-kz}[A \sin m\varphi + B \cos m\varphi] J_m(kz)
$$

but there is now no restriction on the value of k other than it be positive (to ensure that ϕ is finite as $z \to \infty$). Thus the sum over *discrete* values of k becomes an integral over k, and our general solution is

$$
\begin{aligned}
\phi(s,\varphi,z) &= \int_0^\infty dk\, e^{-kz}\, \frac{B_0(k)}{2} J_0(ks) \\
&+ \sum_{m=1}^{\infty} \int_0^\infty dk\, e^{-kz} \left\{ A_m(k) \sin m\varphi + B_m(k) \cos m\varphi) \right\} J_m(ks).
\end{aligned}
$$

We still have a Fourier series in φ, but the Fourier-Bessel series has evolved to a **Bessel transform**.
Imposing the boundary conditions at $z = 0$, we have

$$
\begin{aligned}
V(s,\varphi) &= \int_0^\infty dk\, \frac{B_0(k)}{2} J_0(ks) \\
&+ \sum_{m=1}^{\infty} \int_0^\infty dk\, \left\{ A_m(k) \sin m\varphi + B_m(k) \cos m\varphi \right\} J_m(ks)
\end{aligned}
$$

and we can invert the Fourier series to obtain

$$\begin{aligned}\int_0^\infty dk\, A_m(k) J_m(ks) &= \frac{1}{\pi}\int_0^{2\pi} d\varphi\, V(s,\varphi)\sin m\varphi \\ \int_0^\infty dk\, B_m(k) J_m(ks) &= \frac{1}{\pi}\int_0^{2\pi} d\varphi\, V(s,\varphi)\cos m\varphi. \qquad (3.15)\end{aligned}$$

We can invert the **Hankel transforms** on the left-hand side using the completeness relation

$$\int_0^\infty dx\, x J_m(kx) J_m(k'x) = \frac{1}{k}\delta(k'-k).$$

Applying this to the first line of eqn. 3.15, we have

$$\begin{aligned}\frac{1}{\pi}\int_0^\infty ds\, s\int_0^{2\pi} d\varphi\, V(s,\varphi)\sin m\varphi J_m(k's) &= \int_0^\infty ds\, s\int_0^\infty dk A_m(k) J_m(ks) J_m(k's) \\ &= \int_0^\infty dk\, A_m(k)\frac{1}{k}\delta(k-k') \\ &= \frac{1}{k'}A_m(k'),\end{aligned}$$

and thus we have

$$\begin{aligned}A_m(k) &= \frac{k}{\pi}\int_0^\infty ds\, s\int_0^{2\pi} d\varphi\, V(s,\varphi)\sin m\varphi J_m(ks) \\ B_m(k) &= \frac{k}{\pi}\int_0^\infty ds\, s\int_0^{2\pi} d\varphi\, V(s,\varphi)\cos m\varphi J_m(ks)\end{aligned}$$

3.7 Expansion of Green Functions in terms of Orthogonal Functions

The solutions found by separation of variables constituted complete sets of orthogonal functions satisfying the appropriate boundary conditions. We have shown that any function, and in particular the Green function, satisfying the same boundary conditions can be expanded as a series of these orthogonal functions. We will illustrate the basic principle for our old friend, the Dirichlet Green function for the sphere in spherical polar coordinates, and then proceed to discuss the general construction using cylindrical coordinates.

3.7.1 Green function for the Sphere in Spherical Harmonics

Recall that the Green function for the region V satisfies

$$\nabla'^2 G(\underline{x}, \underline{x}') = -4\pi\delta(\underline{x} - \underline{x}') \text{ for } \underline{x}, \underline{x}' \in V,$$

and has the general form

$$G(\underline{x}, \underline{x}') = \frac{1}{|\underline{x} - \underline{x}'|} + F(\underline{x}, \underline{x}'),$$

where $F(\underline{x}, \underline{x}')$ is a solution of Laplace's equation in V, chosen to satisfy the boundary conditions on G, e.g. for the Dirichlet Green function

$$G(\underline{x}, \underline{x}') = 0 \text{ for } x \in \partial V.$$

We have already seen the expansion of $1/|\underline{x} - \underline{x}'|$ in terms of spherical harmonics, viz.

$$\frac{1}{|\underline{x} - \underline{x}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\theta', \varphi') Y_{lm}(\theta, \varphi).$$

Suppose we wish to construct the Dirichlet Green function for the *outside* of a sphere of radius a. We use the *method of images*, and obtain

$$G(\underline{x}, \underline{x}') = \frac{1}{|\underline{x}' - \underline{x}|} - \frac{a}{r|\underline{x}' - a^2/r^2 \underline{x}|}$$

$$= 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi) \left\{ \frac{r_<^l}{r_>^{l+1}} - \frac{a}{r} \frac{(a^2/r)^l}{r'^{l+1}}. \right\}$$

where we note that, for the image charge, $r_> = r', r_< = a^2/r$, since the image charge is *always inside* the sphere. Thus we have

$$G(\underline{x}, \underline{x}') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi) \left\{ \frac{r_<^l}{r_>^{l+1}} - \frac{1}{a} \left(\frac{a^2}{rr'} \right)^{l+1} \right\}$$

We have thus accomplished our goal of expressing the Green function as an expansion over orthogonal functions. There are some important observations we can make by looking at the radial part

$$\{ \frac{r_<^l}{r_>^{l+1}} - \frac{1}{a} \left(\frac{a^2}{rr'} \right)^{l+1} \} = \begin{cases} \frac{1}{r'^{l+1}} \left[r^l - \frac{a^{2l+1}}{r^{l+1}} \right] & r < r' \\ \frac{1}{r^{l+1}} \left[r'^l - \frac{a^{2l+1}}{r'^{l+1}} \right] & r > r' \end{cases} .$$

- The radial part manifestly vanishes at $r = a$ and $r' = a$.
- It is symmetric under $r \leftrightarrow r'$.
- The solution is a linear combination of the solutions of Laplace's equation, regarded as a function of r' for fixed r, but a different linear combination for $r' > r$ and $r' < r$. We will see how this property arises below.

3.8 General Solution of Green Function in Cylindrical Polars

The Green function satisfies

$$\nabla'^2 G(\underline{x}, \underline{x}') = -4\pi \delta(\underline{x} - \underline{x}').$$

To express the r.h.s. in terms of cylindrical coordinates, we recall that

$$\delta[g(x)] = \sum_i \left| \frac{1}{g'(x_i)} \right| \delta(x_i)$$

where x_i are the roots of $g(x) = 0$. Thus, in three dimensions, we have

$$\delta(\underline{x} - \underline{x}') = \left|\frac{\partial(x,y,z)}{\partial(s,\varphi,z)}\right|^{-1} \delta(s-s')\delta(\varphi-\varphi')\delta(z-z') = \frac{1}{s'}\delta(s-s')\delta(\varphi-\varphi')\delta(z-z').$$

Hence the Green function satisfies

$$\nabla'^2 G(\underline{x} - \underline{x}') = -\frac{4\pi}{s'}\delta(s-s')\delta(\varphi-\varphi')\delta(z-z') \tag{3.16}$$

where

$$\nabla'^2 = \frac{1}{s'}\frac{\partial}{\partial s'}\left(s'\frac{\partial}{\partial s'}\right) + \frac{1}{s'^2}\frac{\partial^2}{\partial \varphi'^2} + \frac{\partial^2}{\partial z'^2}.$$

Note that in the following we will treat the *unprimed* indices as fixed parameters. We will now specialise to the case where we wish to obtain the Green function in a volume V encompassing the full angular range $0 \le \varphi \le 2\pi$. Then any solution can be expressed as a Fourier series in φ',

$$G(\underline{x}, \underline{x}') = G(s,\varphi,z;s',\varphi',z') = \sum_{m'=-\infty}^{\infty} F_{m'}(s,\varphi,z;s',z')e^{-im'\varphi'}.$$

Substituting this into eqn. 3.16, we have

$$\sum_{m'=-\infty}^{\infty} e^{-im'\varphi'}\left\{\frac{1}{s'}\frac{\partial}{\partial s'}\left[s'\frac{\partial}{\partial s'}F_{m'}(s,\varphi,z;s',z')\right] - m'^2\frac{1}{s'^2}F_{m'}(s,\varphi,z;s',z')\right.$$
$$\left. + \frac{\partial^2}{\partial z'^2}F_{m'}(s,\varphi,z;s',z')\right\} = -\frac{4\pi}{s'}\delta(s-s')\delta(\varphi-\varphi')\delta(z-z').$$

We now use the orthogonality properties of the $\exp im\varphi$ to obtain

$$\sum_{m'=-\infty}^{\infty} \int_0^{2\pi} d\varphi'\, e^{i(m-m')\varphi'}\left\{\frac{1}{s'}\frac{\partial}{\partial s'}\left[s'\frac{\partial}{\partial s'}F_{m'}(s,\varphi,z;s',z')\right] - m'^2\frac{1}{s'^2}F_{m'}(s,\varphi,z;s',z')\right.$$
$$\left. + \frac{\partial^2}{\partial z'^2}F_{m'}(s,\varphi,z;s',z')\right\} = -\frac{4\pi}{s'}\delta(s-s')\delta(z-z')\int_0^{2\pi} d\varphi' e^{im\varphi'}\delta(\varphi-\varphi'),$$

yielding

$$\frac{1}{s'}\frac{\partial}{\partial s'}\left[s'\frac{\partial F_m}{\partial s'}\right] - \frac{m^2}{s'^2}F_m + \frac{\partial^2 F_m}{\partial z'^2} = -\frac{2}{s'}\delta(s-s')\delta(z-z')e^{im\varphi}.$$

Thus we have explicitly exhibited the φ dependence, and can write

$$F_m(s,\varphi,z;s',z') = f_m(s,z;s',z')e^{im\varphi},$$

where f_m obeys the P.D.E.

$$\frac{1}{s'}\frac{\partial}{\partial s'}\left[s'\frac{\partial f_m}{\partial m'}\right] - \frac{m^2}{s'^2}f_m + \frac{\partial^2 f_m}{\partial z'^2} = -\frac{2}{s'}\delta(s-s')\delta(z-z'). \tag{3.17}$$

Thus from our original P.D.E. in *three* variables we now have a two-variable P.D.E..

To proceed further, we must say something about the boundary conditions, or at the very least specify the volume V. We will assume that is covers $-\infty \le z \le \infty$, and then introduce the Fourier transform (F.T.) of f_m, defined by

$$\begin{aligned}\tilde{f}_m(s,z;s',k) &= \int_{-\infty}^{\infty} dz'\, e^{ikz'} f_m(s,z;s',z') \\ f_m(s,z;s',z') &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{-ikz'}\tilde{f}_m(s,z;s',k).\end{aligned}$$

We apply the F.T. operator to eqn. 3.17,

$$\int_{\infty}^{\infty} dz'\, e^{ikz'}\left\{\frac{1}{s'}\frac{\partial}{\partial s'}\left[s'\frac{\partial f_m}{\partial s'}\right] - \frac{m^2}{s'^2}f_m + \frac{\partial^2 f_m}{\partial z'^2}\right\} = -\frac{2}{s'}\delta(s-s')\int_{-\infty}^{\infty} dz' e^{ikz'}\delta(z-z'),$$

yielding

$$\frac{1}{s'}\frac{\partial}{\partial s'}\left[s'\frac{\partial \tilde{f}_m}{\partial s'}\right] - \frac{m^2}{s'^2}\tilde{f}_m - k^2\tilde{f}_m = -\frac{2}{s'}\delta(s-s')e^{ikz}$$

where we have used the well-known properties concerning the F.T. of a derivative.

We have now exhibited the z dependence of the function, and may write

$$\tilde{f}_m(s,z;s',k) = \frac{1}{2\pi}e^{ikz}g_m(s,s';k),$$

giving

$$\frac{1}{s'}\frac{\partial}{\partial s'}\left[s\frac{\partial g_m}{\partial s'}\right] - \left(\frac{m^2}{s'^2}+k^2\right)g_m = \frac{4\pi}{s'}\delta(s-s').$$

This is just a **one-dimensional** Green function equation, which we may write in a more familiar form by substituting

$$\begin{aligned} x &= ks \\ x' &= ks', \end{aligned}$$

yielding

$$\frac{\partial}{\partial x'}\left[x'\frac{\partial g_m(x,x')}{\partial x'}\right] - x'\left(1+\frac{m^2}{x'^2}\right)g_m(x,x') = -4\pi\delta(x-x').$$

This is just the modified Bessel equation, with inhomogeneous source. As we noted earlier, the modified Bessel equation (like the Legendre equation) is of Sturm-Liouville type:

$$\frac{d}{dx'}\left[p(x')\frac{dg(x,x')}{dx'}\right] + q(x')g(x,x') = -4\pi\delta(x-x')$$

with

$$\begin{aligned} p(x') &= x' \\ q(x') &= -x'\left(1+\frac{m^2}{x'^2}\right). \end{aligned}$$

Thus we have finally reduced the problem to the solution of the Green function for the Sturm-Liouville equation.

3.8.1 Green Function for the Sturm-Liouville Equation

We wish to determine the Green function to the equation

$$\frac{d}{dx'}\left[p(x')\frac{dg(x,x')}{dx'}\right] + q(x')g(x,x') = -4\pi\delta(x-x'),$$

defined on the interval $x' \in [a,b]$, with homogeneous boundary conditions at a and b. Note that we regard x as some arbitrary, fixed parameter.
The Green function must possess the following properties:

1. For $x' \neq x$, $g(x,x')$ satisfies the homogeneous equation, i.e. the Sturm-Liouville equation with no source on the r.h.s..

2. $g(x, x'0$ satisfies the homogeneous boundary condition at $x' = a$ and $x' = b$, e.g. $g(x,x') = 0$.

3. $g(x,x')$ must be continuous at $x' = x$. This is subtle; otherwise dg/dx' would contain a δ-function, and d^2g/dx'^2 would contain the *derivative* of a δ-function at $x' = x$, which is more singular than the r.h.s. of the equation.

To see what happens at $x' = x$, we integrate the equation from $x - \epsilon$ to $x + \epsilon$:

$$\int_{x-\epsilon}^{x+\epsilon} dx' \left\{ \frac{d}{dx'} \left[p(x') \frac{dg(x,x')}{dx'} \right] + q(x')g(x,x') \right\} = -4\pi \int_{x-\epsilon}^{x+\epsilon} dx' \, \delta(x - x'),$$

leading to

$$\left[p(x') \frac{dg(x,x')}{dx'} \right]_{x-\epsilon}^{x+\epsilon} + \int_{x-\epsilon}^{x+\epsilon} dx' \, q(x')g(x,x') = -4\pi.$$

Both $q(x')$ and $g(x,x')$ are continuous at $x' = x$, and therefore we have

$$\lim_{\epsilon \to 0} \int_{x-\epsilon}^{x+\epsilon} dx' \, q(x')g(x,x') = 0,$$

and we may write

$$\lim_{\epsilon \to 0} \left[p(x') \frac{dg(x,x')}{dx'} \right]_{x-\epsilon}^{x+\epsilon} = -4\pi.$$

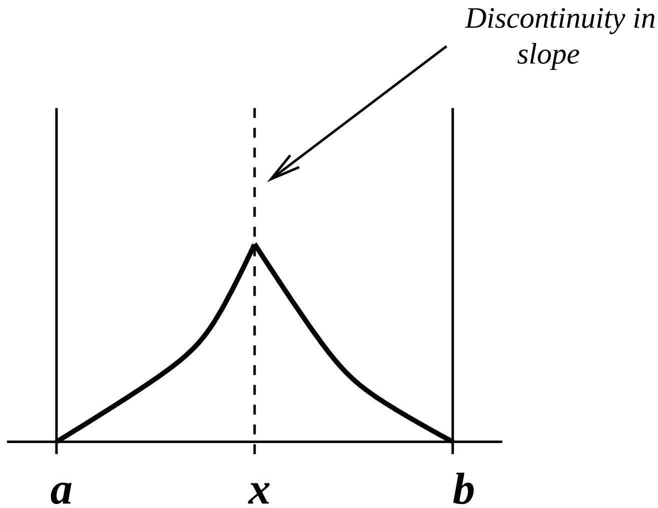
Thus function $p(x')$ is also continuous at $x' = x$, and thus

$$p(x) \times \lim_{\epsilon \to 0} \left\{ \frac{dg(x, x' = x + \epsilon)}{dx'} - \frac{dg(x, x' = x - \epsilon)}{dx'} \right\} = -4\pi$$

which we write as

$$\left[\frac{dg(x,x')}{dx'} \right]_{x'=x} = -\frac{4\pi}{p(x)},$$

i.e. there is a discontinuity in the slope of the Green function of magnitude $4\pi/p(x)$ at $x' = x$.



Thus we will write our Green function as

- $a \leq x' \leq x$:

$$g(x, x') = C_1(x) y_1(x'),$$

  where $y_1(x')$ is a solution of the homogeneous equation satisfying the appropriate boundary condition at $x' = a$.

- $x \leq x' \leq b$:

$$g(x, x') = C_2(x) y_2(x')$$

  where $y_2(x')$ is a solution of the homogeneous equation satisfying the appropriate boundary condition at $x' = b$.

We now impose the conditions on the Green function at $x' = x$

- $g(x, x')$ continuous at $x' = x$:

$$C_1(x) y_1(x) - C_2(x) y_2(x) = 0 \tag{3.18}$$

- Discontinuity in slope is $-4\pi/p(x)$:

$$C_2(x) y_2'(x) - C_1(x) y_1'(x) = -\frac{4\pi}{p(x)} \tag{3.19}$$

From eqn. 3.18, we have

$$C_2(x) = \frac{C_1(x)y_1(x)}{y_2(x)}.$$

Substituting into eqn. 3.19, we find

$$\begin{aligned} \frac{C_1(x)y_1(x)y_2'(x)}{y_2(x)} - c_1(x)y_1'(x) &= -\frac{4\pi}{p(x)} \\ \Rightarrow C_1(x) &= -\frac{4\pi}{p(x)}\frac{y_2(x)}{W[y_1(x), y_2(x)]}, \end{aligned}$$

where the W is the **Wronskian**,

$$W[y_1(x), y_2(x)] = y_1(x)y_2'(x) - y_2(x)y_1'(x).$$

Note that this method only works if y_1 and y_2 are *linearly independent*, since otherwise the Wronskian vanishes.

Thus our general form for the Green function is

$$g(x, x') = \begin{cases} -\frac{4\pi}{p(x)}\frac{y_2(x)y_1(x')}{W[y_1(x), y_2(x)]} & a \leq x' \leq x \\ -\frac{4\pi}{p(x)}\frac{y_2(x')y_1(x)}{W[y_1(x), y_2(x)]} & x \leq x' \leq b \end{cases}$$

So, as we have already observed for spherical polar coordinates, the Green function in the regions $x' < x$ and $x' > x$ comprises two different, linearly independent solutions of the homogeneous equation.

3.8.2 Green Function for Modified Bessel Equation

We will now return to the case of the modified Bessel equation with δ-function source

$$\frac{d}{dx'}\left[p(x')\frac{dg(x, x')}{dx'}\right] + q(x')g(x, x') = -4\pi\delta(x - x').$$

A pair of linearly independent solutions is provided by the modified Bessel functions $I_m(x)$ and $K_m(x)$. We will now consider the case where we require the solution over all space, i.e. $x \in [0, \infty]$. The solution must be *finite* at $x = 0$, and thus

$$y_1(x') = I_m(x').$$

If we further require that the solution be finite as $x' \to \infty$, then we have

$$y_2(x') = K_m(x'),$$

which we can see from the limiting behaviour quoted earlier. In this case, the Wronskian is (see Jackson)

$$W[I_m(x), K_m(x)] = -\frac{1}{x}$$

and thus our general solution for the Green function is

$$g_m(x, x') = \begin{cases} -\frac{4\pi}{x}\frac{K_m(x)I_m(x')}{-1/x} & 0 \le x' \le x \\ -\frac{4\pi}{x}\frac{K_m(x')I_m(x)}{-1/x} & x \le x' \le \infty \end{cases},$$

which we may express as

$$\boxed{g_m(x, x') = 4\pi I_m(x_<)K_m(x_>)}$$

where $x_< = \min(x, x')$ and $x_> = \max(x, x')$.

3.8.3 Reconstruction of the Full Green Function

We reconstruct the full Green function in four steps:

1.

$$\begin{aligned} \tilde{f}_m(s, z; s', k) &= g_m(s, s'; k)e^{ikz}/2\pi \\ &= 2I_m(ks_<)K_m(ks_>)e^{ikz} \end{aligned}$$

2.

$$f_m(s,z;s',z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-ikz'} \tilde{f}_m(s,z;s',z') = \frac{1}{\pi}\int_{-\infty}^{\infty} dk\, e^{ik(z-z')} I_m(ks_<)K_m(ks_>)$$

3.

$$F_m(s,\varphi,z;s',z') = \frac{1}{\pi}\int_{-\infty}^{\infty} dk\, e^{ik(z-z')} I_m(ks_<)K_m(ks_>)e^{im\varphi}$$

4.

$$G(\underline{x},\underline{x}') = \frac{1}{\pi}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi')}\int_{-\infty}^{\infty} dk\, e^{ik(z-z')} I_m(ks_<)K_m(ks_>).$$

Since we have evaluated the Green function with boundary conditions at infinity, this last expression is just the expansion of $|\underline{x}-\underline{x}'|^{-1}$ in cylindrical polar coordinates.

3.9 Expansion of Green Function in Spherical Polar Coordinates

This is performed in exactly the same way as for Cylindrical coordinates. In spherical polars, the Green function satisfiese

$$\nabla'^2 G(\underline{x}, \underline{x}') = -\frac{4\pi}{r'^2}\delta(r-r')\delta(\varphi-\varphi')\delta(\cos\theta-\cos\theta'),$$

where

$$\nabla'^2 = \frac{1}{r'^2}\frac{\partial}{\partial r'}\left(r'^2\frac{\partial}{\partial r'}\right) + \frac{1}{r'^2\sin\theta'}\frac{\partial}{\partial\theta'}\left(\sin\theta'\frac{\partial}{\partial\theta'}\right) + \frac{1}{r'^2\sin^2\theta'}\frac{\partial^2}{\partial\varphi'^2}.$$

We will consider the case where we require the Green function over the full angular range $0 \le \theta' \le \pi$, $0 \le \varphi \le 2\pi$. Thus we can expand the Green function, as a function of the *primed* variables with the unprimed variables fixed, in spherical harmonics:

$$G(\underline{x}, \underline{x}') = \sum_{l',m'} F_{l'm'}(r,\theta,\varphi;r')Y^*_{l'm'}(\theta',\varphi').$$

Substituting this into the inhomogeneous equation we have

$$\begin{aligned}
&\sum_{l',m'}\left\{\frac{1}{r'^2}\frac{\partial}{\partial r'}\left[r'^2\frac{\partial F_{l'm'}}{\partial r'}\right]Y^*_{l'm'}(\theta',\varphi')\right.\\
&+\ \left.\frac{F_{l'm'}}{r'^2}\left[\frac{1}{\sin\theta'}\frac{\partial}{\partial\theta'}\left(\sin\theta'\frac{\partial Y^*_{l'm'}}{\partial\theta'}\right)+\frac{1}{\sin^2\theta'}\frac{\partial^2 Y^*_{l'm'}}{\partial\varphi'^2}\right]\right\}\\
&=\ -\frac{4\pi}{r'^2}\delta(r-r')\delta(\varphi-\varphi')\delta(\cos\theta-\cos\theta').
\end{aligned}$$

Now the spherical harmonics are solutions of Laplace's equation on the unit sphere, and, from eqn. 3.1, satisfy

$$\frac{1}{\sin^2\theta'}\frac{\partial^2 Y^*_{l'm'}}{\partial\varphi'^2} + l(l+1)Y^*_{l'm'} + \frac{1}{\sin\theta'}\frac{\partial}{\partial\theta'}\left(\sin\theta'\frac{\partial Y^*_{l'm'}}{\partial\theta'}\right) = 0.$$

Thus our Green function equation becomes

$$\sum_{l'm'}\left\{\frac{1}{r'^2}\frac{\partial}{\partial r'}\left[r'^2\frac{\partial F_{l'm'}}{\partial r'}\right] - \frac{F_{l'm'}}{r'^2}l(l+1)\right\}Y^*_{l'm'}(\theta',\varphi') = -\frac{4\pi}{r'^2}\delta(r-r')\delta(\varphi-\varphi')\delta(\cos\theta-\cos\theta').$$

We now multiply by $Y^*_{lm}(\theta', \varphi')$, and use the orthogonality properties of the spherical harmonics:

$$\begin{aligned}
&\frac{1}{r'^2}\frac{\partial}{\partial r'}\left[r'^2\frac{\partial F_{lm}}{\partial r'}\right] - \frac{F_{lm}}{r'^2}l(l+1) \\
&= -\frac{4\pi}{r'^2}\int d\Omega'\, Y_{lm}(\theta', \varphi')\delta(r-r')\delta(\varphi-\varphi')\delta(\cos\theta - \cos\theta') \\
&= -\frac{4\pi}{r'^2}Y_{lm}(\theta, \varphi)\delta(r-r').
\end{aligned}$$

We may then write

$$F_{lm}(r, \theta, \varphi; r') = g_l(r, r')Y_{lm}(\theta, \varphi)$$

where $g_l(r, r')$ satisfies

$$\frac{d}{dr'}\left(r'^2\frac{d}{dr'}g_l(r, r')\right) - l(l+1)g_l(r, r') = -4\pi\delta(r-r').$$

This is just the radial part of Laplace's equation. To proceed further, we must specifiy boundary conditions.

3.9.1 Dirichlet Green Function between Spheres at $r = a$ and $r = b$

We require $g_l(r, r')$ subject to the boundary conditions $g_l(r, a) = g_l(r, b) = 0$.

1. $a \leq r' \leq r$: The solution $y_1(r')$ of the homogeneous equation must satisfy $y_1(a) = 0$. Now the general solution is of the form

   $$y_1(r') = A_1 r'^l + B_1 r'^{-l-1},$$

   and thus we have

   $$A_1 a^l + B_1 a^{-l-1} = 0 \;\Rightarrow\; B_1 = -A_1 a^{2l+1}$$

   yielding

   $$y_1(r') = A_1\left[r'^l - \frac{a^{2l+1}}{r'^{l+1}}\right].$$

2. $r \leq r' \leq b$: Then the solution $y_2(r')$ of the homogeneous equation must satisfy $y_2(b) = 0$, yielding

$$y_2(r') = B_2 \left[\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}}. \right]$$

We now construct the Wronskian

$$\begin{aligned} W[y_1(r), y_2(r)] &= y_1(r) y_2'(r) - y_2(r) y_1'(r) \\ &= -A_1 B_2 \frac{2l+1}{r^2} \left[1 - \frac{a^{2l+1}}{b^{2l+1}} \right]. \end{aligned}$$

Noting that $p(r) = r^2$, we observe that, once again, the Wronskian is independent of the evaluation point, and we have general solution

$$g_l(r, r') = \begin{cases} -4\pi \dfrac{A_1 \left(r'^l - \dfrac{a^{2l+1}}{r'^{l+1}} \right) B_2 \left(\dfrac{1}{r^{l+1}} - \dfrac{r^l}{b^{2l+1}} \right)}{-A_1 B_2 (2l+1) \left(1 - \dfrac{a^{2l+1}}{b^{2l+1}} \right)}; & a \leq r' \leq r \\ -4\pi \dfrac{A_1 \left(r^l - \dfrac{a^{2l+1}}{r^{l+1}} \right) B_2 \left(\dfrac{1}{r'^{l+1}} - \dfrac{r'^l}{b^{2l+1}} \right)}{-A_1 B_2 (2l+1) \left(1 - \dfrac{a^{2l+1}}{b^{2l+1}} \right)}; & a \leq r' \leq r \end{cases},$$

which we may write in the more compact form

$$g_l(r, r') = \frac{4\pi}{2l+1} \left(1 - \frac{a^{2l+1}}{b^{2l+1}} \right)^{-1} \left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}} \right) \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}} \right),$$

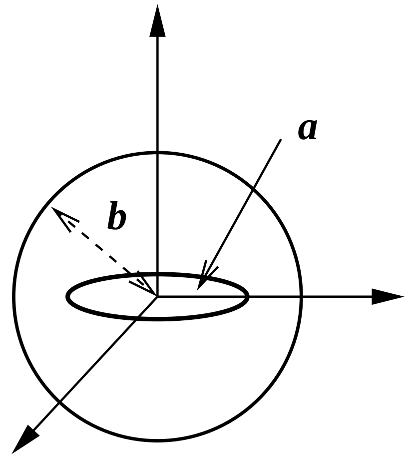
and hence

$$G(\underline{x}, \underline{x}') = \sum_{l,m} g_l(r, r') Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi). \tag{3.20}$$

Note that it is also possible to recover this result using the *method of images*, but in this case an infinite number of image charges are required.

Example:

Consider the potential inside an grounded, conducting sphere of radius b, due to a uniform ring of charge of radius $a < b$, and total charge Q, lying in the plane through the equator, and centred at the centre of the sphere.



We can obtain the Green function by taking the $a \to 0$ limit of eqn. 3.20:

$$G(\underline{x}, \underline{x}') = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \left(\frac{r_<^l}{r_>^{l+1}} - \frac{(r_< r_>)^l}{b^{2l+1}} \right) Y^*_{lm}(\theta', \varphi') Y_{lm}(\theta, \varphi).$$

The potential is then given by

$$\phi(\underline{x}) = \frac{1}{4\pi\epsilon_0} \int_V' d^3x' \, G(\underline{x}, \underline{x}') \rho(\underline{x}') - \frac{1}{4\pi} \int_{S=\partial V} dS' \, \phi(\underline{x}') \frac{\partial G(\underline{x}, \underline{x}')}{\partial n'}.$$

In our case the surface integral vanishes, because the potential vanishes there. The (linear) charge density is given by

$$\rho(\underline{x}') = \frac{Q}{2\pi a^2} \delta(r' - a) \delta(\cos\theta').$$

Exercise: verify that the total charge is indeed Q. Thus the potential is

$$\begin{aligned} \phi(\underline{x}) &= \frac{1}{4\pi\epsilon_0} \int d\varphi' d(\cos\theta') dr' \, r'^2 \frac{Q}{2\pi a^2} \delta(r' - a) \delta(\cos\theta') \\ &\times \; 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} Y^*_{lm}(\theta', \varphi') Y_{lm}(\theta, \varphi) \left\{ \frac{r_<^l}{r_>^{l+1}} - \frac{(r_< r_>)^l}{b^{2l+1}} \right\} \end{aligned}$$

In this case we have azimuthal symmetry, and the only non-vanishing integrals arise from the terms with $m = 0$, for which

$$Y_{l0}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta).$$

Thus we have

$$\begin{aligned}\phi(\underline{x}) &= \frac{1}{4\pi\epsilon_0}\int r' r'^2 \frac{Q}{a^2}\delta(r'-a)\sum_{l=0}^{\infty} P_l(0)P_l(\cos\theta)\left\{\frac{r_<^l}{r_>^{l+1}} - \frac{(r_< r_>)^l}{b^{2l+1}}\right\} \\ &= \frac{Q}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{2^n n!}\left\{\frac{r_<^{2n}}{r_>^{2n+1}} - \frac{r_>^{2n}}{b^{4n+1}}\right\} P_{2n}(\cos\theta),\end{aligned}$$

where we have used

$$\begin{aligned}P_{2n+1}(0) &= 0 \\ P_{2n}(0) &= \frac{(-1)^n(2n+1)!!}{2^n n!},\end{aligned}$$

and $r_< = \min(r,a)$, $r_> = \max(r,a)$.

3.10 Expansion of Green Function in terms of Eigenfunctions

A closely related method to those discussed above is the expansion of the Green function in terms of the *eigenfunctions* of some related problem. Consider the solution of

$$\nabla^2\varphi(\underline{x}) + [f(\underline{x}) + \lambda]\varphi(\underline{x}) = 0,$$

in a volume V bounded by a surface S, subject to φ satisfying certain *homogeneous* boundary conditions for $x \in S$. In general, consistent solutions can be obtained only for certain (possibly continuous) values of λ, which we will denote λ_n, the **eigenvalues**. The corresponding solutions, the **eigenfunctions**, we will denote $\varphi_n(\underline{x})$. The eigenvalue equation is then

$$\nabla^2\varphi_n + [f(\underline{x}) + \lambda_n]\varphi_n = 0. \tag{3.21}$$

The eigenfunctions form a complete, orthogonal set of functions (the proof of orthogonality follows that for the Sturm-Liouville equation), and we will assume that they are normalised:

$$\int d^3x\, \varphi_m^*\varphi_n = \delta_{mn}.$$

Then *any* function satisfying the *same homogeneous boundary conditions* may be expanded as a series in the eigenfunctions. Consider in particular a Green function, satisfying

$$\nabla'^2 G(\underline{x},\underline{x}') + [f(\underline{x}') + \lambda]G(\underline{x},\underline{x}') = -4\pi\delta(\underline{x}-\underline{x}') \tag{3.22}$$

where λ is, in general, not an eigenvalue. The corresponding eigenfunction expansion is

$$G(\underline{x},\underline{x}') = \sum_n a_n(\underline{x})\varphi_n(\underline{x}'),$$

and, inserting in eqn. 11.11.1, we obtain

$$\sum_n a_n(\underline{x})\{\nabla'^2\varphi_n(\underline{x}') + f(\underline{x}')\varphi_n(\underline{x}') + \lambda\varphi_n(\underline{x}')\} = -4\pi\delta(\underline{x}-\underline{x}').$$

We now use that φ_n is an eigenfunction of eqn. 3.21 with eigenvalue λ_n, and obtain

$$\sum_n a_n(\underline{x})[\lambda - \lambda_n]\varphi_n(\underline{x}') = -4\pi\delta(\underline{x}-\underline{x}').$$

Using the orthonormal property of the eigenfunctions, we obtain

$$a_n(\underline{x}) = 4\pi\frac{\varphi_n^*(\underline{x})}{\lambda_n - \lambda}$$

and hence

$$\boxed{G(\underline{x},\underline{x}') = 4\pi\sum_n \frac{\varphi_n^*(\underline{x})\varphi_n(\underline{x}')}{\lambda_n - \lambda}}$$

This is often referred to as the **spectral representation** of the Green function.

Example: Green function in free space

Let us now specialise to Poisson's equation, i.e. we set $f(\underline{x}) = 0$ and $\lambda = 0$ in eqn. 11.11.1. We will begin by considering the solution in free space, for which the most closely related eigenvalue equation is the wave equation

$$(\nabla^2 + k^2)\varphi_{\underline{k}}(\underline{x}) = 0$$

where k^2 is the (continuous) eigenvalue, and the corresponding normalised eigenfunction is

$$\varphi_{\underline{k}}(\underline{x}) = \left(\frac{1}{2\pi}\right)^{3/2} e^{i\underline{k}\cdot\underline{x}},$$

with normalisation

$$\int d^3x\, \varphi^*_{\underline{k}'}(\underline{x})\varphi_{\underline{k}}(\underline{x}) = \delta(\underline{k} - \underline{k}').$$

Then the expression for the Green function is

$$G(\underline{x}, \underline{x}') = 4\pi \int d^3k \frac{e^{i\underline{k}\cdot(\underline{x}'-\underline{x})}}{k^2} \left(\frac{1}{2\pi}\right)^2$$

which we observe may be written

$$\frac{1}{|\underline{x} - \underline{x}'|} = \frac{1}{2\pi^2} \int d^3k \frac{e^{i\underline{k}\cdot(\underline{x}'-\underline{x})}}{k^2}.$$

Example: Dirichlet Green function inside a rectangular box

We define the surface of the box to be the planes $x = 0, a$, $y = 0, b$, and $z = 0, c$. The most closely related eigenvalue problem is

$$\nabla^2\varphi + k^2_{lmn}\varphi_{lmn} = 0,$$

where the eigenvalues and normalised eigenfunctions are

$$\begin{aligned} k^2_{lmn} &= \pi^2 \left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right) \\ \varphi_{lmn}(\underline{x}) &= \sqrt{\frac{8}{abc}} \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{b} \sin\frac{n\pi z}{c}. \end{aligned}$$

Thus we can immediately write down the Green function as

$$G(\underline{x}, \underline{x}') = \frac{32}{\pi abc} \sum_{l,m,n} \frac{\sin\frac{l\pi x}{a} \sin\frac{m\pi y}{b} \sin\frac{n\pi z}{c} \sin\frac{l\pi x'}{a} \sin\frac{m\pi y'}{b} \sin\frac{n\pi z'}{c}}{\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}}.$$

Chapter 4

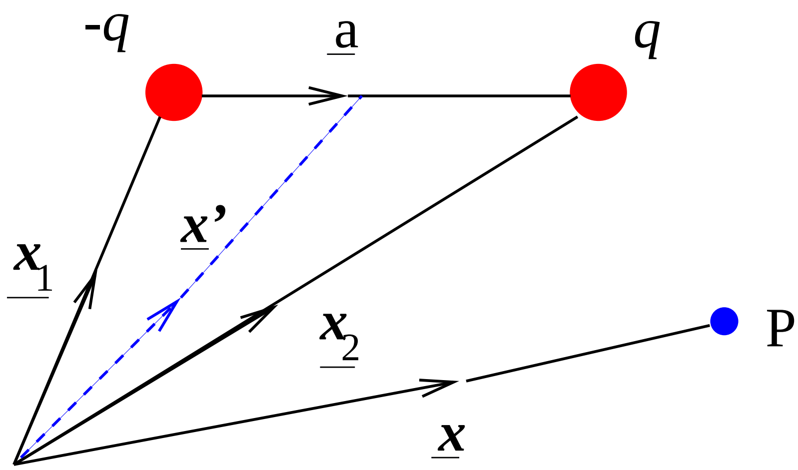
Multipoles and the Electrostatics of Macroscopic Media

The simplest source for an electrostatic field is a **point charge**; such a source is sometimes known as a **pole**. The arrangement of *two* point charges, of equal but opposite sign, is known as a **dipole**. The concept of a dipole plays a crucial rôle in electrostatics:

- Even in the case of a neutral atom or molecule, the positive and negative charges can become *separated*, e.g. by an applied external electric field. In that case, the atom or molecule gives rise to an electrostatic field that can be approximated by a **dipole**
- The concept of dipoles, and, more generally, **multipoles**, leads to an important method for obtaining the electrostatic field and potential far from a charge distribution, the **multipole expansion**.

4.1 Introduction and Revision: Electric Dipoles

Consider two charges $-q$ and q at $\underline{x_1}$ and $\underline{x_2}$ respectively, and let $\underline{a}$ be the position vector of q relative to $-q$.



Let $\underline{x}'$ be the mid point of the dipole, so that

$$\begin{aligned}\underline{x}_1 &= \underline{x}' - \underline{a}/2 \\ \underline{x}_2 &= \underline{x}' + \underline{a}/2\end{aligned}$$

Then the potential at the point P is

$$\phi(\underline{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mid \underline{x} - \underline{x}_2 \mid} + \frac{-q}{\mid \underline{x} - \underline{x}_1 \mid}\right) = \frac{1}{4\pi\epsilon_0} q \left(\frac{1}{\mid \underline{x} - \underline{x}' - \underline{a}/2 \mid} - \frac{1}{\mid \underline{x} - \underline{x}' + \underline{a}/2 \mid}\right)$$

We will now consider the case where the separation between the charges is much less than the distance of the point P from the charges, i.e. $|\underline{a}| \ll \mid \underline{x} - \underline{x}' \mid$. Then we have

$$\begin{aligned}|\underline{x} - \underline{x}' \pm \underline{a}/2|^{-1} &= \left\{|\underline{x} - \underline{x}'|^2 + \frac{|\underline{a}|^2}{4} \pm \underline{a}\cdot(\underline{x} - \underline{x}')\right\}^{-1/2} \\ &= |\underline{x} - \underline{x}'|^{-1}\left\{1 \pm \frac{\underline{a}\cdot(\underline{x} - \underline{x}')}{|\underline{x} - \underline{x}'|^2} + \frac{|\underline{a}|^2}{4|\underline{x} - \underline{x}'|^2}\right\}^{-1/2}.\end{aligned}$$

Expanding as a series in $|\underline{a}|^2/|\underline{x} - \underline{x}'|^2$ using the *binomial expansion*, we obtain

$$|\underline{x} - \underline{x}' \pm \underline{a}/2|^{-1} = |\underline{x} - \underline{x}'|^{-1}\left\{1 + \left(-\frac{1}{2}\right)\left[\pm\frac{\underline{a}\cdot(\underline{x} - \underline{x}')}{|\underline{x} - \underline{x}'|^2}\right] + O\left(\frac{|\underline{a}|^2}{|\underline{x} - \underline{x}'|^2}\right)\right\}$$

Thus

$$\phi(\underline{x}) = \frac{1}{4\pi\epsilon_0}\frac{q\underline{a}\cdot(\underline{x} - \underline{x}')}{|\underline{x} - \underline{x}'|^3}.$$

We now take the limit $|\underline{a}| \to 0$, $q \to \infty$, with $\underline{a}q = \underline{p}$ fixed and finite. This defines a **simple** or **ideal dipole** and we have

$$\boxed{\phi_D(\underline{x}) = \frac{1}{4\pi\epsilon_0}\frac{\underline{p}\cdot(\underline{x}-\underline{x}')}{|\underline{x}-\underline{x}'|^3}}$$

- $\underline{p}$ is the **vector moment** or **dipole moment** of the dipole.
- $\phi_D(\underline{p})$ is the potential at $\underline{x}$ due to a dipole of moment $\underline{p}$ at $\underline{r}'$.

We can obtain the **electrostatic field** due to a dipole by applying $\underline{E}(\underline{x}) = -\underline{\nabla}\phi_D(\underline{x})$, and obtain

$$\underline{E}(\underline{x}) = \frac{1}{4\pi\epsilon_0}\frac{3(\underline{p_1}\cdot\underline{x})\underline{x} - r^2\underline{p_1}}{r^5} \tag{4.1}$$

for a dipole at the origin.

4.1.1 Dipole in External Electrostatic Field

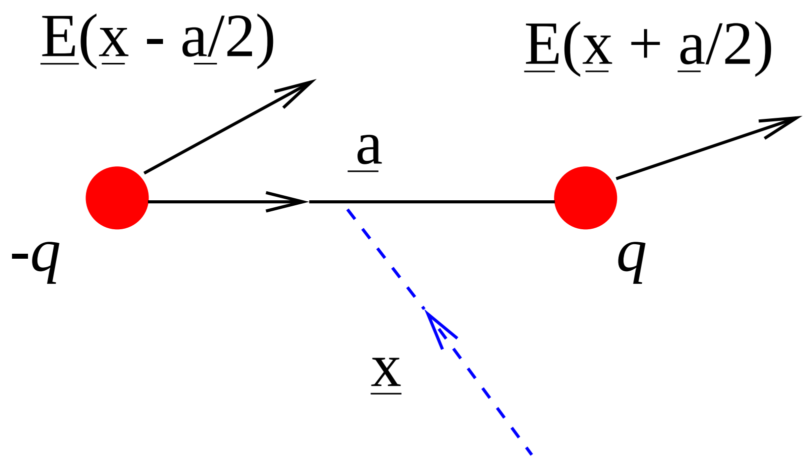
In this subsection, we will consider not the field due to a dipole, but rather the energy and forces on a dipole in an **external** field $\underline{E}(\underline{x}) = -\underline{\nabla}\phi(\underline{x})$.

Potential Energy of Dipole in External Electrostatic Field

Recall from Section 3.5 that for a charge q in an **electrostatic potential** $\phi(\underline{x})$, the **potential energy** is

$$U(\underline{x}) = q\phi(\underline{x})$$

Let us now apply this to the case of a **dipole** in an external field; once again, $\underline{a}$ is the separation of the charge q from $-q$.



The potential energy of the dipole is

$$U_D(\underline{x}) = (-q)\phi(\underline{x} - \underline{a}/2) + q\phi(\underline{x} + \underline{a}/2).$$

If the separation between the charges is small, we can expand about $\underline{x}$ to obtain

$$\begin{aligned} \phi(\underline{x} \pm \underline{a}/2) = & \\ & \phi(\underline{x}) \pm \frac{1}{2} a_i \frac{\partial}{\partial x_i} \phi(\underline{r}) + \frac{1}{2!} \frac{a_i a_j}{4} \frac{\partial^2}{\partial x_i \partial x_j} \phi(\underline{x}) + \ldots \\ = & \ \phi(\underline{x}) \pm \frac{1}{2} \underline{a} \cdot \underline{\nabla} \phi(\underline{x}) + O(a^2) \end{aligned}$$

Thus we have

$$\begin{aligned} U_D(\underline{x}) &= q\,[\phi(\underline{x}) + \frac{1}{2}\underline{a} \cdot \underline{\nabla}\phi(\underline{x}) - \phi(\underline{x}) + \frac{1}{2}\underline{a} \cdot \underline{\nabla}\phi(\underline{x}) + O(a^3)] \\ &= q\,\underline{a} \cdot \underline{\nabla}\phi(\underline{x})[1 + O(a^2)] \end{aligned}$$

Now take the **point dipole limit**, $a \to 0$, $q \to \infty$, $\underline{a}q = \underline{p}$ fixed. Then

$$U_D(\underline{x}) = \underline{p} \cdot \underline{\nabla}\phi(\underline{x})$$

Aside: why did I take $\underline{x}$ to be at the *mid point* of the dipole? Because for a simple dipole, all the corrections to the formula above involving *even* derivatives of $\phi(\underline{x})$ vanish. It just makes the expansion *neater*, but of course I could have performed the expansion about any point between the charges.

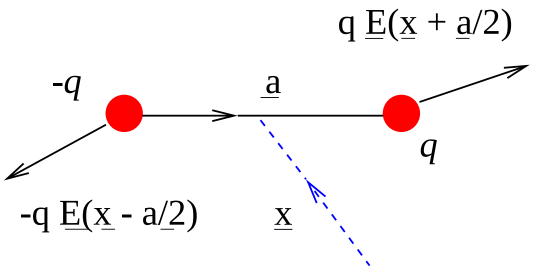
Recalling that $\underline{E}(\underline{x}) = \underline{\nabla}\phi(\underline{x})$, we have

$$\boxed{\underline{U_D}(\underline{x}) = -\underline{p} \cdot \underline{E}(\underline{x})}$$

Note that the potential energy of a dipole is a **minimum** when $\underline{E}$ and $\underline{p}$ are **parallel**

Force on Dipole in External Electrostatic Field

We will now consider the force on an electric dipole.



The force on the dipole is

$$\underline{F_D}(\underline{x}) = -q\underline{E}(\underline{x} - \underline{a}/2) + q\underline{E}(\underline{x} + \underline{a}/2)$$

Once again, we can expand about $\underline{r}$:

$$\underline{E}(\underline{x} \pm \underline{a}/2) = \underline{E}(\underline{x}) \pm \frac{1}{2}(\underline{a} \cdot \underline{\nabla})\underline{E} + O(a^2).$$

We thus obtain

$$\underline{F_D}(\underline{x}) = q(\underline{a} \cdot \underline{\nabla})\underline{E}(\underline{x}) = (\underline{p} \cdot \underline{\nabla})\underline{E}(\underline{x}) \tag{4.2}$$

Now since $\underline{E}(\underline{x})$ is an electrostatic field, it is *irrotational*:

$$\underline{\nabla} \times \underline{E}(\underline{x}) = 0.$$

Let $\underline{c}$ be a constant vector, and let $\underline{A}(\underline{x})$ be an arbitrary vector field. Then we have the identity

$$\underline{\nabla}(\underline{c} \cdot \underline{A}(\underline{x})) = \underline{c} \times (\underline{\nabla} \times \underline{A}(\underline{x})) + (\underline{c} \cdot \underline{\nabla})\underline{A}(\underline{x})$$

which we can apply to equation (4.2) to obtain

$$\underline{F_D}(\underline{x}) = \underline{\nabla}(\underline{p} \cdot \underline{E}(\underline{x})) = -\underline{\nabla} U_D(\underline{x})$$

using $\underline{\nabla} \times \underline{E}(\underline{x}) = 0$. Thus the force on a dipole is just minus the gradient of the potential energy, and furthermore for a *uniform* external field, independent of $\underline{x}$, the force is zero.

Torque on a Dipole in an External Field

We will now evaluate the **torque**, or moment of the force, τ on a simple dipole about its centre. This is just the moment of the forces acting on the two charges about the centre of the dipole:

$$\begin{aligned} \underline{\tau} &= \left(\frac{1}{2}\,\underline{a}\right) \times (+q)\,\underline{E}(\underline{x}+\underline{a}/2) \;+\; \left(-\frac{1}{2}\,\underline{a}\right) \times (-q\,)\,\underline{E}(\underline{x}-\underline{a}/2) \\ &= \left(\frac{1}{2}\,\underline{a}\right) \times q\left(\underline{E}(\underline{x}) \;+\; \frac{1}{2}(\underline{a}\cdot\underline{\nabla})\,\underline{E}(\underline{x}) \;+\; \underline{E}(\underline{x}) \;-\; \frac{1}{2}(\underline{a}\cdot\underline{\nabla})\,\underline{E}(\underline{x}) + O(a^2)\right) \\ \text{ie}\quad \underline{\tau} &= \underline{p}\times\underline{E}(\underline{x}) \quad \text{in the } \textbf{point dipole limit} \end{aligned}$$

- Note that the torque about some point other than the **centre** of the dipole will be different.
- $\tau = \underline{p} \times \underline{E}(\underline{x})$ is true for dipoles other than point dipoles if $\underline{E}(\underline{x})$ is constant over the dipole.

4.1.2 Force between Two Dipoles

Many materials are dipolar; the positive and negative materials are separated. Here we will consider the force between a dipole $\underline{p_1}$ at $\underline{x_1}$ and $\underline{p_2}$ at $\underline{x_2}$. The force $\underline{F_{21}}$ on the dipole at $\underline{x_2}$ due to the electrostatic field $\underline{E_1}$ produced by the dipole $\underline{p_1}$ is

$$\underline{F}_{21}(\underline{x}_2) \;=\; (\underline{p}_2\cdot\underline{\nabla}_2)\,\underline{E}_1(\underline{x}_2) \;=\; (\underline{p}_2\cdot\underline{\nabla}_2)\,C\left\{\frac{3\left(\underline{p}_1\cdot(\underline{x}_2-\underline{x}_1)\right)\,(\underline{x}_2-\underline{x}_1) \;-\; \underline{p}_1\,|\underline{x}_2-\underline{x}_1|^2}{|\underline{x}_2-\underline{x}_1|^5}\right\}$$

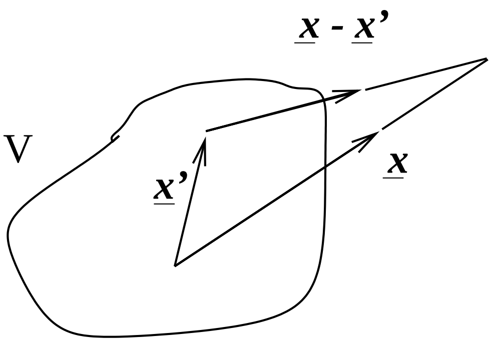
where $\underline{\nabla}_2$ means that we take derivatives with respect to $\underline{x}_2$ (the position vector of dipole $\underline{p}_2$), and we have used eqn. (4.1). As discussed above, we can express this as

$$\underline{F_{21}}(\underline{x_2}) = \underline{\nabla}_2(\underline{p_2}\cdot\underline{E_1}(\underline{x_2})).$$

Note that the force $\underline{F_{12}}(\underline{x_1})$ is *equal and opposite* to $\underline{F_{21}}(\underline{x_2})$.

4.2 Multipole Expansion

In this section, we will see why the concept of dipoles, and more generally multipoles, is so important in electrostatics. Consider the case of a charge distribution, localised to some volume V. For convenience we will take the origin for our vectors inside V.



We have that the potential due to the charge distribution within V at a point P outside the volume is:

$$\phi(\underline{x}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\underline{x}')\,dV'}{|\underline{x}-\underline{x}'|}$$

For r much larger than the extent of V, i.e. $r \gg r'$ for all $\underline{x}'$ such that $\rho(\underline{x}') \neq 0$, we can expand the denominator

$$\begin{aligned} |\underline{x}-\underline{x}'|^{-1} &= \{|r|^2 - 2\underline{x}\cdot\underline{x}' + r'^2\}^{-1/2} \\ &= r^{-1}\left\{1 - 2\frac{\underline{x}\cdot\underline{x}'}{r^2} + \frac{r'^2}{r^2}\right\}^{-1/2} \\ &= \frac{1}{r}\left\{1 + \frac{\underline{x}\cdot\underline{x}'}{r^2} + O(r'^2/r^2)\right\} \end{aligned}$$

Thus we have

$$\frac{1}{|\underline{x}-\underline{x}'|} = \frac{1}{r} + \frac{\underline{x}\cdot\underline{x}'}{r^3} + O(r'^2/r^3).$$

Hence we can write

$$\boxed{\phi(\underline{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} + \frac{\underline{P}\cdot\underline{x}}{r^3} + \frac{1}{2}\sum_{i,j=1}^{3} Q_{ij}\frac{x_i x_j}{r^5} + O(1/r^5)\right)}$$

where

$Q \;=\; \int_V \rho(\underline{x}')dV'$ is the **total charge** within V

$$\underline{P} = \int_V \rho(\underline{x}')\underline{x}' dV' \quad \text{is the } \textbf{dipole moment} \text{ of the charge about the origin}$$
$$Q_{ij} = \int_V \rho(\underline{x}')(3x'_i x'_j - r'^2 \delta_{ij}) dV' \quad \text{is the } \textbf{quadrupole moment} \text{ of the charge.}$$

- We have defined the moments with respect to a particular point, e.g. the **dipole moment** is the integral of the **displacement** $\underline{x}'$ times the charge density $\rho(\underline{x}')$. In general, the moments depend on the choice of "origin". *What about the total dipole moment when the total charge is zero?*

- At large distances from the charge distribution, only the first few moments (Q, $\underline{P}$, **quadrupole moment**, ...) are important.

- For a **neutral** charge distribution, the leading behaviour is given by the dipole moment.

Example:

The region inside the sphere: $r < a$, contains a charge density

$$\rho(x, y, z) = f\, z\, (a^2 - r^2)$$

where f is a constant. Show that at large distances from the origin the potential due to the charge distribution is given approximately by

$$\phi(\underline{x}) = \frac{2f\, a^7}{105\epsilon_0} \frac{z}{r^3}$$

Use the multipole expansion in SI units:

$$\phi(\underline{x}) = \frac{1}{4\pi\epsilon_0} \left(\frac{Q}{r} + \frac{\underline{P} \cdot \underline{x}}{r^3} + O\left(\frac{1}{r^3}\right) \right)$$

In **spherical polars** (r, θ, φ),

$$x = r \sin\theta \cos\varphi \quad ; \quad y = r \sin\theta \sin\varphi \quad ; \quad z = r\cos\theta$$

The **total charge** Q is

$$Q = \int_V \rho(\underline{x})\, dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \left(f r \cos\theta\, (a^2 - r^2) \right) r^2 \sin\theta\, dr\, d\theta\, d\varphi = 0.$$

The integral vanishes because

$$\int_0^{\pi} \cos\theta \sin\theta \, d\theta \;=\; \int_0^{\pi} \frac{1}{2}\, \sin(2\theta)\, d\theta \;=\; 0.$$

The **total dipole moment** $\underline{P}$ about the origin is

$$\begin{aligned} \underline{P} &= \int_V \underline{x}\, \rho(\underline{x})\, dV \;=\; \int_V r\, \underline{e}_r\, \rho(\underline{x})\, dV \\ &= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} r\,(\sin\theta\, \cos\varphi\, \underline{i} \,+\, \sin\theta\, \sin\varphi\, \underline{j} \,+\, \cos\theta\, \underline{k}) \\ &\qquad\qquad \Big(f r\cos\theta\, (a^2 \,-\, r^2)\Big)\, r^2 \sin\theta\, dr\, d\theta\, d\varphi. \end{aligned}$$

The x and y components of the φ integral vanish. The z component factorises:

$$P_z \;=\; f \int_0^{2\pi} d\varphi \quad \int_0^{\pi} \sin\theta \cos^2\theta\, d\theta \quad \int_0^{a} r^4\, (a^2 \,-\, r^2)\, dr \;=\; f \quad 2\pi \quad \frac{2}{3} \quad \frac{2a^7}{35}.$$

Putting it all together, we obtain

$$\phi(\underline{x}) \;=\; \frac{1}{4\pi\epsilon_0}\, \frac{8\pi a^7 f}{105}\, \frac{\underline{k}\cdot\underline{x}}{r^3} \;=\; \frac{2f\, a^7}{105\epsilon_0}\, \frac{z}{r^3}.$$

4.2.1 Multipole Expansion using Spherical Harmonics

To proceed further, we go back to our expansion of a pole in *spherical harmonics*

$$\frac{1}{|\underline{x} - \underline{x}'|} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi).$$

We assume that the charge is confined to a sphere of radius a, and take the centre of the sphere to be the origin for our vectors. Then for the case $r > a$, we have

$$\begin{aligned} r_< &= r' \\ r_> &= r, \end{aligned}$$

and we have

$$\phi(\underline{x}) = \frac{1}{\epsilon_0} \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1} \frac{Y_{lm}(\theta,\varphi)}{r^{l+1}} \int d\Omega' \int dr'\, r'^2 Y_{lm}^*(\theta', \varphi') r'^l \rho(\underline{x}').$$

We now write

$$q_{lm} = \int d\Omega' dr' r'^2 Y^*_{lm}(\theta', \varphi') r'^l \rho(\underline{x}')$$

so that the expansion may be written

$$\boxed{\phi(\underline{x}) = \frac{1}{\epsilon_0} \sum_{l,m} \frac{1}{2l+1} q_{lm} \frac{Y_{lm}(\theta, \varphi)}{r^{l+1}}.}$$

This is the multipole expansion using spherical harmonics. To make the connection with our previous expansion, it is useful to consider the few few terms in Cartesian coordinates

$$\begin{aligned}
q_{00} &= \frac{1}{\sqrt{4\pi}} \int d^3x' \rho(\underline{x}') = \frac{1}{4\pi} Q \\
q_{11} &= -\sqrt{\frac{3}{8\pi}} \int d^3x'\, \rho(\underline{x}')(x' - iy') = -\sqrt{\frac{3}{8\pi}} (P_x - iP_y) \\
q_{10} &= \sqrt{\frac{3}{4\pi}} \int d^3x'\, \rho(\underline{x}') z' = \sqrt{\frac{3}{4\pi}} P_z \\
q_{22} &= \frac{1}{4}\sqrt{\frac{15}{2\pi}} \int d^3x' \rho(\underline{x}')(x' - iy')^2 = \frac{1}{12}\sqrt{\frac{15}{2\pi}} (Q_{11} - 2iQ_{12} - Q_{22}) \\
q_{21} &= -\sqrt{\frac{15}{8\pi}} \int d^3x' \rho(\underline{x}') z'(x' - iy') = -\frac{1}{3}\sqrt{\frac{15}{8\pi}} (Q_{13} - iQ_{23}) \\
q_{20} &= \frac{1}{2}\sqrt{\frac{5}{4\pi}} \int d^3x'\, \rho(\underline{x}')(3z'^2 - r'^2) = \frac{1}{2}\sqrt{\frac{5}{4\pi}} Q_{33}.
\end{aligned}$$

Note that the components for negative m can be trivially obtained using

$$q_{l,-m} = (-1)^m q^*_{lm}.$$

In general, for the l-th multipole moment, there are $(l+1)(l+2)/2$ components in Cartesian coordinates, while only $2l+1$ components using spherical harmonics. There is no inconsistency here - the Cartesian tensors are **reducible** under rotations (i.e. mix with tensors having few indices under rotations) whilst the tensor moments expressed in spherical harmonics are **irreducible** (i.e. the q_{lm} for fixed

l mix only amongst themselves under rotations); that is why we *remove the trace* in the quadrupole moment Q_{ij}, to give us 5 irreducible components.
We can express the electric field components trivially in spherical harmonics. In particular, the contribution of definite l, m is

$$\begin{aligned} E_r &= \frac{1}{\epsilon_0}\frac{l+1}{2l+1}Y_{lm}(\theta,\varphi)q_{lm}\frac{1}{r^{l+2}} \\ E_\theta &= -\frac{1}{\epsilon_0}\frac{1}{2l+1}q_{lm}\frac{1}{r^{l+2}}\frac{\partial}{\partial\theta}Y_{lm}(\theta,\varphi) \\ E_\varphi &= \frac{1}{\epsilon_0}\frac{1}{2l+1}q_{lm}\frac{1}{r^{l+2}}\frac{im}{\sin\theta}Y_{lm}(\theta,\varphi). \end{aligned}$$

If we now consider the case of a ideal dipole $\underline{p}$ along the z-axis, then

$$\begin{aligned} q_{10} &= \sqrt{\frac{3}{4\pi}}p \\ q_{11} &= q_{1,-1} = 0 \end{aligned}$$

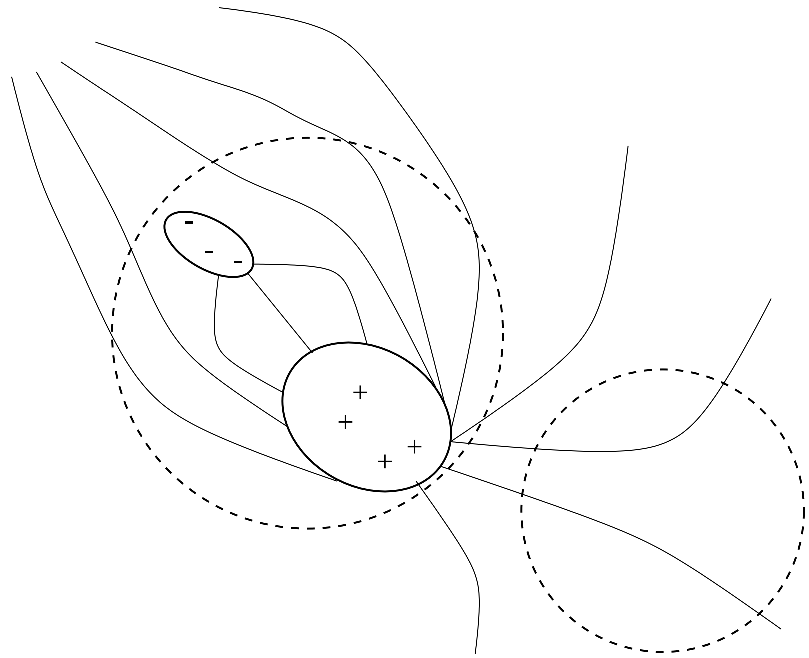
and we have

$$\begin{aligned} E_r &= \frac{1}{\epsilon_0}\frac{2}{3}\sqrt{\frac{3}{4\pi}}\sqrt{\frac{3}{4\pi}}\frac{p\cos\theta}{r^3} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3} \\ E_\theta &= \frac{1}{\epsilon_0}\frac{1}{3}\sqrt{\frac{3}{4\pi}}p\frac{1}{r^3}\sqrt{\frac{3}{4\pi}}\sin\theta = \frac{p\sin\theta}{4\pi\epsilon_0 r^3} \\ E_\varphi &= 0, \end{aligned}$$

which reduces to the expression we derived earlier for an ideal dipole, eqn. 4.1.

4.2.2 Point Dipole vs. Dipole Moment

There is a danger in using the expression for the electrostatic field due to an ideal, or point dipole. To see this, consider the electrostatic field $\underline{E}(\underline{x})$ due to a localised charge distribution $\rho(\underline{x})$. In partciular, consider the integral of $\underline{E}$ over some sphere of radius R, the centre of which we will take as the origin of our vectors.



We have

$$\int_{r<R} d^3x\, \underline{E} = -\int_{r<R} d^3x\, \underline{\nabla}\phi = -R^2 \int d\Omega \phi(\underline{x})\underline{n}$$

where $\underline{n}$ is a unit normal outward from the surface of the sphere, and we have used the generalisation of the divergence theorem.
Using Coulomb's law for an extended charge distribution, we may write

$$\int d^3x \underline{E} = \frac{R^2}{4\pi\epsilon_0} \int d^3x'\, \rho(\underline{x}') \int d\Omega \frac{\underline{n}}{|\underline{x} - \underline{x}'|}.$$

Now we can evaluate the x integration by writing the vector $\underline{n} = \sin\theta\cos\varphi\underline{i} + \sin\theta\sin\varphi\underline{j} + \cos\theta\underline{k}$, and then expressing these terms in spherical harmonics as

$$\begin{aligned} \sin\theta\cos\varphi &= -\sqrt{\frac{8\pi}{3}}\left(\frac{Y_{11}(\theta,\varphi) + Y_{1,-1}(\theta,\varphi)}{2}\right) \\ \sin\theta\sin\varphi &= -\sqrt{\frac{8\pi}{3}}\left(\frac{Y_{11}(\theta,\varphi) - Y_{1,-1}(\theta,\varphi)}{2i}\right) \\ \cos\theta &= \sqrt{\frac{4\pi}{3}} Y_{10}(\theta,\varphi) \end{aligned}$$

Thus only the $l = 1$ terms contribute, and using the orthogonality property of spherical harmonics we have

$$\int d\Omega \, \frac{\underline{n}}{|\underline{x} - \underline{x}'|} = \frac{4\pi}{3} \frac{r_<}{r_>^2} \underline{n}'$$

where $\underline{n}'$ is a unit vector in the direction of $\underline{x}'$. Hence we have

$$\begin{aligned} \int d^3x \, \underline{E} &= -\frac{R^2}{4\pi\epsilon_0} \int d^3x' \, \frac{r_<}{r_>^2} \frac{4\pi}{3} \underline{n}' \rho(\underline{x}') \\ &= -\frac{R^2}{3\epsilon_0} \int d^3x' \frac{r_<}{r_>^2} \underline{n}' \rho(\underline{x}') \end{aligned} \tag{4.3}$$

where $r_< = \min(r', R)$.

We now consider two cases

1. *Sphere completely encloses the charge density.* Then we have $r_< = r'$, and $r_> = R$, and we have, from eqn. 4.3,

$$\int d^3x \underline{E} = -\frac{\underline{P}}{3\epsilon_0}, \tag{4.4}$$

   where $\underline{P}$ is the electric dipole moment. Note that this expression is independent of the size of the sphere, provided it completely encloses the dipole.

2. *Charge density completely outside the sphere.* Then we have $r_< = R$, $r_> = r'$, and we have

$$\begin{aligned} \int d^3x \underline{E} &= -\frac{R^3}{3\epsilon_0} \int d^3x' \, \frac{\underline{n}'}{r'^2} \rho(\underline{x}') \\ &= \frac{4\pi}{3} R^3 \underline{E}(\underline{0}). \end{aligned}$$

   Thus the average value of the electric field over a spherical volume containing no charge is just the value of the field at the centre of the sphere.

Let us now consider the corresponding expression for the integrated $\underline{E}$ in the case of an *ideal dipole*, eqn. 4.1:

$$\begin{aligned} \int_{r<R} d^3x \underline{E}(\underline{x}) &= \int_{r<R} d^3x \frac{1}{4\pi\epsilon_0} \frac{3(\underline{p} \cdot \underline{n})\underline{n} - \underline{p}}{r^3} \\ &= 0 \quad \textit{Exercise}\text{: let } \underline{p} = p\underline{k}\text{, and work in spherical polars.} \end{aligned}$$

For this to be consistent with eqn. 4.4, our expression for the electrostatic field due to a dipole at $\underline{x_0}$ must be modified

$$\underline{E}(\underline{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{3\underline{n}(\underline{p}\cdot\underline{n}) - \underline{p}}{|\underline{x} - \underline{x_0}|^3} - \frac{4\pi}{3}\underline{p}\delta(\underline{x} - \underline{x_0})\right].$$

This expression only changes the electric field *at the position of the dipole*, and we can then, with some care, use the expression as if we were using ideal, or point, dipoles. The δ-function contains information about the finite distribution of the charge lost in the multipole expansion.

4.3 Energy of Charge Distribution in External Electrostatic Field

The energy is given by

$$W = \int d^3x\, \rho(\underline{x})\phi(\underline{x}).$$

We now suppose that ϕ is slowly varying, so that

$$\begin{aligned}\phi(\underline{x}) &= \phi(0) + \underline{x}\cdot\underline{\nabla}\phi + \frac{1}{2}\sum_{i,j} x_i x_j \frac{\partial^2\phi}{\partial x_i \partial x_j} + \ldots \\ &= \phi(\underline{0}) - \underline{x}\cdot\underline{E}(\underline{0}) - \frac{1}{2}\sum_{i,j} x_i x_j \frac{\partial E_i}{\partial x_j}.\end{aligned}$$

Now in the case of an *external* electrostatic field, we have $\underline{\nabla}\cdot\underline{E} = 0$, and thus we may write

$$\begin{aligned}\phi(\underline{x}) &= \phi - \underline{x}\cdot\underline{E} - \frac{1}{2}\sum_{i,j} x_i x_j \left\{\frac{\partial E_i}{\partial x_j} - \frac{1}{3}\delta_{ij}\underline{\nabla}\cdot\underline{E}\right\} \\ &= \phi - \underline{x}\cdot\underline{E} - \frac{1}{6}\sum_{i,j}[3x_i x_j - \delta_{ij} r^2]\frac{\partial E_i}{\partial x_j}\end{aligned}$$

where the derivatives are evaluated at $\underline{0}$. Thus we have

$$\begin{aligned}W &= \int d^3x\, \rho(\underline{x})\{\phi(0) - \underline{x}\cdot\underline{E} - \frac{1}{6}\sum_{i,j}[3x_i x_j - \delta_{ij} r^2]\frac{\partial E_i}{\partial x_j}\} \\ &= \phi(0)Q - \underline{E}(\underline{0})\cdot\underline{P} - \frac{1}{6}\sum_{i,j} Q_{ij}\frac{\partial E_i}{\partial x_j}(\underline{0})\end{aligned}$$

4.4 Electrostatics with Ponderable Media

So far we have only considered the case of electrostatics in free space. We will now consider the case of macroscopic materials in the presence of electric fields. Such materials are classified according to whether or not electrons, or charges, can flow over long distances. In the case of conductors, charges can move freely about the material, and, as we have already seen, generate an induced field that exactly cancels any applied external field.

In this chapter we consider the case of **dielectics**. Here the electrons are bound to atoms, and have only limited freedom to move. The material might have an inherent dipole moment, or a dipole moment might be generated by the presence of an external electric field. The crucial property of a dielectric is that

$$\underline{\nabla} \times \underline{E} = 0.$$

Thus

- We have a *conservative* electric force
- We can express the field as the gradient of a potential

In the following, we will assume the applied field induces a dipole moment, but no higher moments. Now consider the potential at $\underline{x}$ due to the charge, and dipole moment, of a volume ΔV at $\underline{x}'$:

$$\Delta\phi(\underline{x}, \underline{x}') = \frac{1}{4\pi\epsilon_0}\left[\frac{\rho(\underline{x}')\Delta V}{|\underline{x} - \underline{x}'|} + \frac{\underline{P}(\underline{x}') \cdot (\underline{x} - \underline{x}')}{|\underline{x} - \underline{x}'|^3}\Delta V\right],$$

where $\underline{x}$ is outside the volume ΔV. The dipole moment per unit volume is called *polarization*. We now pass to an integral in the usual way, and obtain

$$\begin{aligned} \phi(\underline{x}) &= \frac{1}{4\pi\epsilon_0}\int_V d^3x'\left[\frac{\rho(\underline{x}')}{|\underline{x} - \underline{x}'|} + \frac{\underline{P}(\underline{x}') \cdot (\underline{x} - \underline{x}')}{|\underline{x} - \underline{x}'|^3}\right] \\ &= \frac{1}{4\pi\epsilon_0}\int d^3x'\left[\frac{\rho(\underline{x}')}{|\underline{x} - \underline{x}'|} + \underline{P}(\underline{x}') \cdot \underline{\nabla}'\left(\frac{1}{|\underline{x} - \underline{x}'|}\right)\right] \quad \textit{(integ. by parts)} \\ &= \frac{1}{4\pi\epsilon_0}\int_V d^3x'\frac{1}{|\underline{x} - \underline{x}'|}\left[\rho(\underline{x}') - \underline{\nabla}' \cdot \underline{P}(\underline{x}')\right] + \frac{1}{4\pi\epsilon_0}\int_{S=\partial V} dS'\frac{\underline{P}(\underline{x}') \cdot \underline{n}}{|\underline{x} - \underline{x}'|} \end{aligned}$$

This expression can be rewritten as follows

$$\phi(\underline{x}) = \frac{1}{4\pi\epsilon_0}\int_V d^3x'\frac{\rho_f(\underline{x}') + \rho_b(\underline{x}')}{|\underline{x} - \underline{x}'|} + \frac{1}{4\pi\epsilon_0}\int_{S=\partial V} dS'\frac{\sigma_b(\underline{x}')}{|\underline{x} - \underline{x}'|}$$

where $\sigma_b \equiv \underline{P} \cdot \underline{n}$ is the surface density of the bound charge, $\rho_b \equiv -\underline{\nabla} \cdot \underline{P}$ is the volume density of the bound charge, and the "old" charge density ρ is called the free charge density ρ_f to distinguish from the density of the bound charge.

Thus Maxwell's equation becomes

$$\underline{\nabla} \cdot \underline{E} = \frac{1}{\epsilon_0}[\rho_f - \underline{\nabla} \cdot \underline{P}].$$

which we can write as

$$\underline{\nabla} \cdot \underline{D} = \rho_f$$

where

$$\underline{D} \equiv \epsilon_0 \underline{E} + \underline{P}$$

is the **electric displacement**. Note that $-\underline{\nabla} \cdot \underline{P}$ is the **polarisation charge density**.

We now suppose that the media is *isotropic*, i.e. no preferred direction. Then the induced dipole moment must be aligned with $\underline{E}$, and we set

$$\underline{P} = \epsilon_o \chi_e \underline{E}$$

where χ_e is the **electric susceptibility**. Thus we have

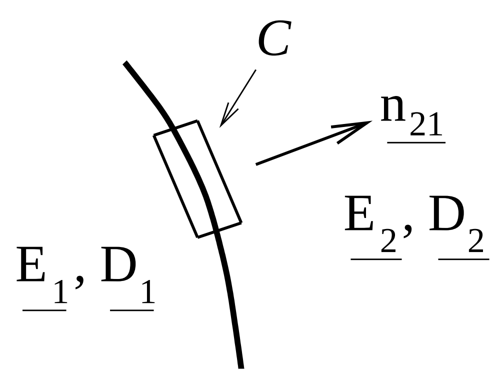
$$\underline{D} = \epsilon_0 \underline{E} + \epsilon_0 \chi_e \underline{E} = \epsilon \underline{E}$$

where $\epsilon = \epsilon_0(1 + \chi_e)$. Note that ϵ/ϵ_0 is the **dielectric constant**. Finally, if the material is **uniform**, then χ_e does not depend on position, and we have

$$\underline{D} = \epsilon \underline{E}, \quad \text{with} \quad \underline{\nabla} \cdot \underline{E} = \rho_f/\epsilon.$$

4.4.1 Boundary Conditions at Boundary between Materials

We will now consider the boundary conditions at the boundary between two materials, of permittivities ϵ_1 and ϵ_2, and with electric fields $\underline{E}_1, \underline{D}_1$ and $\underline{E}_2, \underline{D}_2$ respectively.



Tangential condition

We have that $\underline{\nabla} \times \underline{E} = 0$, and thus, applying Stoke's theorem to the closed curve C shown above, we have

$$\int_C \underline{E} \cdot \underline{dl} = 0,$$

yielding

$$\underline{E}_1^{||} = \underline{E}_2^{||}$$

which we can express as

$$\boxed{(\underline{E}_2 - \underline{E}_1) \times \underline{n}_{21} = \underline{0}}$$

where $\underline{n}_{21}$ is the normal from 1 to 2.

Normal condition

Applying Gauss' law to the usual elementary pill-box we have

$$\underline{\nabla} \cdot \underline{D} = \rho \quad \Rightarrow \int \underline{D} \cdot \underline{dS} = \int \rho_f \, dV$$

from which we find

$$\boxed{(\underline{D}_2 - \underline{D}_1) \cdot \underline{n}_{21} = \sigma_f}$$

where σ is the macroscopic *free* surface charge density at the interface.
To summarise, at the interface between two dielectrics:

- The **tangential component** of $\underline{E}$ is **continuous**.
- The **normal component** of $\underline{D}$ has a discontinuity given by

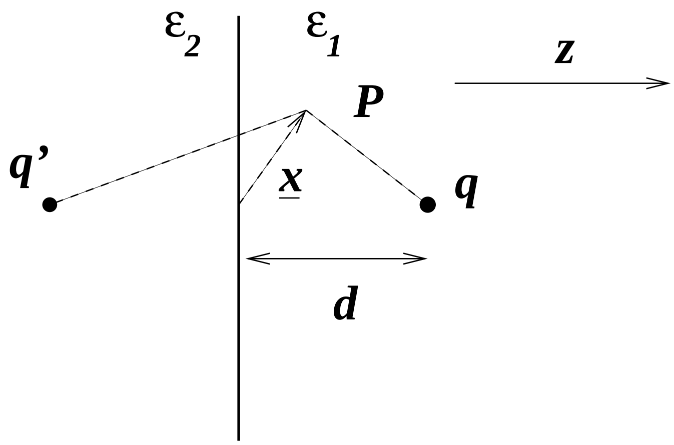
$$(\underline{D}_2 - \underline{D}_1) \cdot \underline{n}_{21} = \sigma_f$$

4.5 Boundary-value Problems with Dielectrics

The method we adopt here essentially follows that of the solution of boundary-value problems *in vacua*, with the boundaries given by conducting surfaces. The method is best illustrated by examples.

Example:

A point charge q in a material of permittivity ϵ_1 a distance d from the interface with a charge-free region of permittivity ϵ_2.



The boundary conditions at the interface $z = 0$ are

$$\begin{aligned} \epsilon_1 E_z(0+) &= \epsilon_2 E_z(0-) \quad (\textit{normal on } \underline{D}) \\ E_x(0+) &= E_x(0-) \quad (\textit{tangential}) \\ E_y(0+) &= E_y(0-) \quad (\textit{tangential}). \end{aligned}$$

In order to determine the potential in the region $z > 0$, let us try an image charge q' at $z = -d$. Then the potential at $\underline{x}$ is

$$\phi(\underline{x})|_{z>0} = \frac{1}{4\pi\epsilon_1} \left[\frac{q}{|\underline{x} - d\underline{e}_z|} + \frac{q'}{|\underline{x} + d\underline{e}_z|} \right].$$

We know that the potential in the region $z < 0$ must satisfy Laplace's equation in that region, and therefore, in particular, there cannot be any poles in the region

$z < 0$. Therefore, let us try the potential due to a charge q'' at the postion of our original charge q:

$$\phi(\underline{x})|_{z<0} = \frac{1}{4\pi\epsilon_2}\frac{q''}{|\underline{x} - d\underline{e}_z|}.$$

We now introduce cylindrical polar coordinates, so that

$$\phi(\rho,\theta,z) = \begin{cases} \dfrac{1}{4\pi\epsilon_2}\dfrac{q''}{\{\rho^2+(z-d)^2\}^{1/2}} & z<0 \\ \dfrac{1}{4\pi\epsilon_1}\left\{\dfrac{q}{\{\rho^2+(z-d)^2\}^{1/2}} + \dfrac{q'}{\{\rho^2+(z+d)^2\}^{1/2}}\right\} & z>0 \end{cases}$$

We have two unknowns, q' and q'', which we determine by imposing the boundary conditions at $z = 0$. We begin with the *tangential* condition. We have that $E_\rho = -\partial\phi/\partial\rho$, and thus

$$E_\rho = \begin{cases} \dfrac{1}{4\pi\epsilon_2}\dfrac{q''\rho}{(\rho^2+d^2)^{3/2}} & z=0- \\ \dfrac{1}{4\pi\epsilon_1}\left\{\dfrac{q\rho}{(\rho^2+d^2)^{3/2}} + \dfrac{q'\rho}{(\rho^2+d^2)^{3/2}}\right\} & z=0+ \end{cases}$$

Thus the tangential boundary condition is

$$\frac{1}{4\pi\epsilon_1}[q+q'] = \frac{1}{4\pi\epsilon_2}q'' \quad \Rightarrow \quad q''\epsilon_1 = (q+q')\epsilon_2. \tag{4.5}$$

To impose the normal boundary condition, we note that

$$E_z = \begin{cases} \dfrac{1}{4\pi\epsilon_2}\dfrac{-dq''}{(\rho^2+d^2)^{3/2}} & z=0- \\ \dfrac{1}{4\pi\epsilon_1}\dfrac{d}{(\rho^2+d^2)^{3/2}}(q'-q) & z=0+ \end{cases},$$

from which we find

$$q'' + q' = q. \tag{4.6}$$

We can solve for q' and q'' from eqns. 4.5 and 4.6, yielding

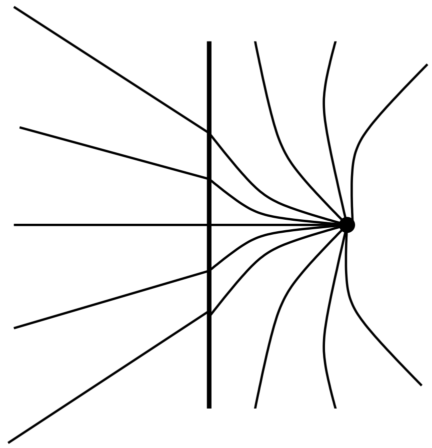
$$\begin{aligned} q' &= -\left(\frac{\epsilon_2-\epsilon_1}{\epsilon_1+\epsilon_2}\right)q \\ q'' &= \frac{2\epsilon_2}{\epsilon_1+\epsilon_2}q. \end{aligned}$$

Thus we have a solution that satisfies the Laplace's equation in $z < 0$, and Poisson's equation in $z > 0$, and the correct boundary conditions at $z = 0$. Thus, by our uniqueness theorem, it is *the* solution.

To see the form of the field lines we consider two cases, $\epsilon_1 > \epsilon_2$ and $\epsilon_1 < \epsilon_2$; in both cases the field lines for $z < 0$ are those of a point charge, of magnitude q'', at q.

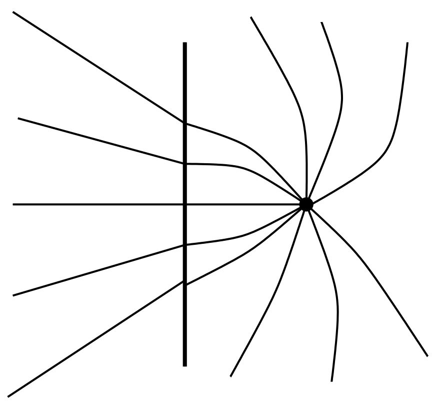
1. $\epsilon_1 > \epsilon_2$.

   Then q' is *same sign* as q.



2. $\epsilon_2 > \epsilon_1$.

   Then q' and q have different signs.



In order to compute the polarisation (bound) charge density, $\sigma_{\text{pol}} = -\underline{\nabla} \cdot \underline{P}$, we observe that $\underline{P}_i = \epsilon_0 \chi_i \underline{E}_i$, $i = 1, 2$, where $\epsilon_i = \epsilon_0(1 + \chi_i)$. Thus we have

$$\underline{P}_i = (\epsilon_i - \epsilon_0)\underline{E}_i.$$

Clearly the polarisation charge density vanishes, except at the point charge q, and at the interface between the two materials. At the interface, there is a discontinuity in $\underline{P}$, and integrating over the discontinuity we obtain

$$\sigma_b = -(\underline{P}_2 - \underline{P}_1) \cdot \underline{n}_{21}, \tag{4.7}$$

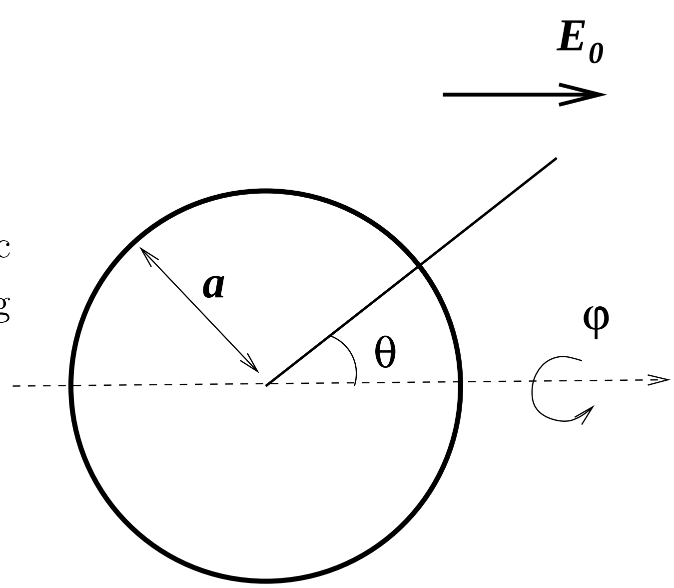
where $\underline{n}_{21}$ is the unit normal from region 1 to region 2, and $\underline{P}_2$ and $\underline{P}_1$ are the polarisations at $z = 0-$ and $z = 0+$ respectively. Thus we have

$$\begin{aligned}
\sigma_b &= -\left\{(\epsilon_1 - \epsilon_0)\frac{1}{4\pi\epsilon_1}\frac{d}{(\rho^2+d^2)^{3/2}}(q'-q) - (\epsilon_2-\epsilon_0)\frac{1}{4\pi\epsilon_2}\frac{d\,q''}{(\rho^2+d^2)^{3/2}}\right\} \\
&= \frac{d}{4\pi(\rho^2+d^2)^{3/2}}\left\{\frac{\epsilon_2-\epsilon_0}{\epsilon_2}q'' - \frac{\epsilon_1-\epsilon_0}{\epsilon_1}(q'-q)\right\} \\
&= \frac{dq}{4\pi(\rho^2+d^2)^{3/2}(\epsilon_1+\epsilon_2)\epsilon_1}\{2(\epsilon_2-\epsilon_0)\epsilon_1 + (\epsilon_1-\epsilon_0)(\epsilon_1+\epsilon_2+\epsilon_2-\epsilon_1)\} \\
&= -\frac{q}{2\pi}\frac{\epsilon_0(\epsilon_2-\epsilon_1)}{\epsilon_1(\epsilon_2+\epsilon_1)}\frac{d}{(\rho^2+d^2)^{3/2}}
\end{aligned}$$

Note that in the limit $\epsilon_2/\epsilon_1 \gg 1$, the electric field in region $z < 0$ becomes very small, and the polarisation charge density approaches the value of the induced surface charge density for a conductor at $z = 0$, up to the factor of ϵ_0/ϵ_1. In that sense, the material in $z < 0$ behaves as a conductor.

Example

Dielectric sphere, radius a, dielectric constant ϵ/ϵ_0, in uniform field along z-axis.



We will work in spherical polar coordinates, and express our solution as an expansion in Legendre polynomials:

$$\phi(r,\theta,\varphi) = \begin{cases} \sum_l A_l r^l P_l(\cos\theta) & r < a \\ \sum_l [B_l r^l + C_l r^{-l-1}] P_l(\cos\theta) & r > a \end{cases},$$

where we have noted that the potential must be finite at $r = 0$.
To determine the coefficients, we impose the boundary conditions. At large distances, the potential is that for a uniform field along the z axis, and thus our boundary condition at infinity is

$$\phi(\rho,\theta,\varphi) \longrightarrow -E_0 r\cos\theta \quad \text{as} \quad r \longrightarrow \infty$$

.

We now impose the boundary conditions at the surface of the sphere

$$\begin{aligned} E_\theta(a-) &= E_\theta(a+) \quad (\textit{tangential condition}) \\ \epsilon_0 E_r(a+) &= \epsilon E_r(a-) \quad (\textit{normal condition}) \end{aligned}$$

The boundary condition at infinity tells us

$$\begin{aligned} B_1 &= -E_0 \\ B_l &= 0 \quad l \neq 1 \end{aligned}$$

To impose the other boundary conditions, we evaluate the components of the electric field, beginning with E_θ:

$$E_\theta = \begin{cases} -\sum\limits_l A_l r^{l-1} \dfrac{d}{d\theta} P_l(\cos\theta) & r < a \\ -\sum\limits_l C_l r^{-l-2} \dfrac{d}{d\theta} P_l(\cos\theta) - B_1 \dfrac{d}{d\theta} P_1(\cos\theta) & r > a \end{cases},$$

From the generalised Rodrigues' formula, we have

$$\begin{aligned} P_l^1(x) &= (-1)^1 (1-x^2)^{1/2} \frac{d}{dx} P_l(x) \\ \Rightarrow P_l^1(\cos\theta) &= -\sin\theta \frac{d}{d\cos\theta} P_l(\cos\theta) \\ &= \frac{d}{d\theta} P_l(\cos\theta), \end{aligned}$$

whence

$$E_\theta = \begin{cases} -\sum_l A_l r^{l-1} P_l^1(\cos\theta) & r < a \\ -\sum_l C_l r^{-l-2} P_l^1(\cos\theta) - B_1 P_1^1(\cos\theta) & r > a \end{cases}.$$

The radial component is straightforward,

$$E_r = \begin{cases} -\sum_l A_l.l.r^{l-1} P_l(\cos\theta) & r < a \\ \sum_l C_l (l+1) r^{-l-2} P_l(\cos\theta) - B_1 P_1(\cos\theta) & r > a \end{cases}.$$

Thus imposing the tangential boundary condition we have

$$\sum_l A_l a^{l-1} P_l^1(\cos\theta) = \sum_l C_l a^{-l-2} P_l^1(\cos\theta) + B_1 P_1^1(\cos\theta).$$

Using the orthogonality property of the Legendre polynomials, we have, for $l \neq 1$,

$$\begin{aligned} A_l a^{l-1} \frac{2}{2l+1} \frac{(l+1)!}{(l-1)!} &= C_l a^{-l-2} \frac{2}{2l+1} \frac{(l+1)!}{(l-1)!} \\ \Rightarrow A_l a^{l-1} &= C_l a^{-l-2} \\ \Rightarrow A_l &= C_l a^{-2l-1}. \end{aligned} \tag{4.8}$$

For the case $l = 1$, we have

$$A_1 = C_1 a^{-3} - E_0. \tag{4.9}$$

The normal boundary condition yields

$$\epsilon_0 \left\{ \sum_l C_l (l+1) a^{-l-2} P_l(\cos\theta) - B_1 P_1(\cos\theta) \right\} = -\epsilon \sum_l A_l \, l \, a^{l-1} P_l(\cos\theta).$$

Once again, there are two cases

$$\epsilon_0 [C_l (l+1) a^{-l-2}] = -\epsilon A_l \, l a^{l-1} \quad l \neq 1 \tag{4.10}$$

$$\epsilon_0 [2C_1 a^{-3} + E_0] = -\epsilon A_1 \quad l = 1 \tag{4.11}$$

Substituting eqn. 4.8 into eqn. 4.10, we find

$$A_l = C_l = 0, \quad l \neq 1.$$

Finally, from eqns. 4.9 and 4.11 we find

$$\begin{aligned} A_1 &= \frac{-3E_0}{2+\epsilon/\epsilon_0} \\ C_1 &= \left(\frac{\epsilon/\epsilon_0 - 1}{2+\epsilon/\epsilon_0}\right) a^3 E_0 \end{aligned}$$

Thus we have

$$\phi(r,\theta,\varphi) = \begin{cases} -\dfrac{3}{2+\epsilon/\epsilon_0} E_0 r\cos\theta & r<a \\ -E_0 r\cos\theta + \left(\dfrac{\epsilon/\epsilon_0 - 1}{2+\epsilon/\epsilon_0}\right)\dfrac{a^3}{r^2}\cos\theta & r>a \end{cases}.$$

- *Inside* the sphere, the field is parallel to the field at infinity,

  $$\underline{E}_{\rm in} = \frac{3}{2+\epsilon/\epsilon_0}\underline{E}_0,$$

  with $|\underline{E}_{\rm in}| < E_0$ if $\epsilon > \epsilon_0$.

- *Outside* the sphere, the field is equivalent to that of the applied field, together with that due to a point dipole at the origin, of moment

  $$p = 4\pi\epsilon_0 \left(\frac{\epsilon/\epsilon_0 - 1}{2+\epsilon/\epsilon_0}\right) a^3 E_0, \tag{4.12}$$

  orientated in the direction of the applied field.

The polarisation $\underline{P} = (\epsilon - \epsilon_0)\underline{E}$ is constant throughout the sphere,

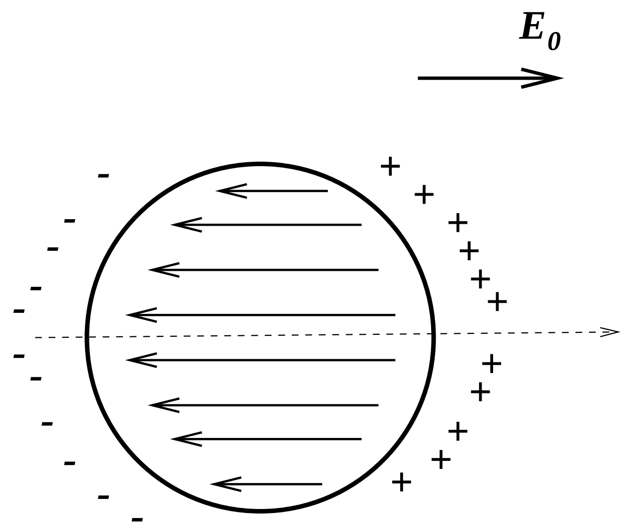
$$\underline{P} = \frac{3(\epsilon-\epsilon_0)}{2+\epsilon/\epsilon_0}\underline{E}_0.$$

We can evaluate the *volume integral* of $\underline{P}$, to obtain

$$\begin{aligned} \int_{r<a} dV\,\underline{P} &= \frac{4}{3}\pi a^3 \frac{3(\epsilon-\epsilon_0)}{2+\epsilon/\epsilon_0}\underline{E}_0 \\ &= 4\pi\epsilon_0 \left(\frac{\epsilon/\epsilon_0 - 1}{2+\epsilon/\epsilon_0}\right) a^3 \underline{E}_0, \end{aligned}$$

which is just the dipole moment we obtained in eqn. 4.12. Thus the dipole moment is just the volume integral of the polarisation.

Because $\underline{P}$ is constant throughout the sphere, the polarisation charge density $-\underline{\nabla} \cdot \underline{P}$ vanishes throughout the interior. However, because of the discontinuity in $\underline{P}$ at the surface, we have a surface polarisation charge density, whose magnitude we can obtain from eqn. 4.7:



$$\begin{aligned} \sigma_b &= \underline{P} \cdot \underline{e}_r \quad (\underline{P} \text{ vanishes outside sphere}) \\ &= 3\epsilon_0 \left(\frac{\epsilon/\epsilon_0 - 1}{2 + \epsilon/\epsilon_0} \right) E_0 \cos\theta \end{aligned}$$

4.6 Electrostatic Energy in Dielectric Media

Back in the introduction, we computed the energy of a system of charges in free space:

$$W = \frac{1}{2}\int d^3x\, \rho(\underline{x})\phi(\underline{x}). \tag{4.13}$$

We obtained this expression by assembling the charges, one-by-one, from infinity under the potential of the charges already assembled. In the case of dielectrics, work is done not only in assembling the charges, but also in polarising the medium. To see how to perform the calculation in this case, consider the change in energy due to a macroscopic charge density $\delta\rho(\underline{x})$,

$$\delta W = \int d^3x\, \delta\rho_f(\underline{x})\phi(\underline{x}).$$

We now use recall that $\underline{\nabla}\cdot\underline{D} = \rho_f$, enabling us to write $\underline{\nabla}\cdot\delta\underline{D} = \delta\rho_f$. Thus we have

$$\begin{aligned}\delta W &= \int d^3x\, \underline{\nabla}\cdot\delta\underline{D}\phi(\underline{x}) \\ &= \int d^3x\underline{E}\cdot\delta\underline{D},\end{aligned}$$

where we have integrated by parts, assuming that the charge is *localised* so that the surface term vanishes. Thus the total energy in constructing the system is

$$W = \int d^3x \int_0^D \underline{E}\cdot\delta\underline{D}.$$

We now make the critical assumption of a *linear, isotropic* constitutive relation between $\underline{E}$ and $\underline{D}$,

$$\underline{D}(\underline{x}) = \epsilon(\underline{x})\underline{E}(\underline{x}).$$

Then we have $\underline{E}\cdot\delta\underline{D} = \frac{1}{2}\delta(\underline{E}\cdot\underline{D})$, and thus

$$W = \int d^3x \int_0^D \frac{1}{2}\delta(\underline{E}\cdot\underline{D})$$

yielding

$$W = \frac{1}{2}\int d^3x\, \underline{E}\cdot\underline{D}.$$

We can recover our expression eqn. 4.13 either by the substitution $\underline{E} = \underline{\nabla}\phi$ and using $\underline{\nabla} \cdot \underline{D} = \rho_f$, or by noting the linear relation between ϕ and ρ. The crucial observation is that the expression eqn. 4.13 is valid *only if the relation between* $\underline{D}$ *and* $\underline{E}$ *is linear.*

4.6.1 Energy of Dielectric in an Electric Field with Fixed Charges

As an important application of this formula, we will consider the case of a dielectric medium introduced into an electric field $\underline{E}_0(\underline{x})$ arising from a **fixed** charge distribution $\rho_f = \rho_0(\underline{x})$. Initially, the energy of the system is

$$W_0 = \frac{1}{2}\int d^3x\, \underline{E}_0 \cdot \underline{D}_0$$

with $\underline{D}_0 = \epsilon_0 \underline{E}_0$; here ϵ_0 is the initial permittivity of the dielectric, not necessarily the permittivity of free space.

We now introduce the medium, of volume V_1, with permittivity

$$\epsilon(\underline{x}) = \begin{cases} \epsilon_1(\underline{x}) & \underline{x} \in V_1 \\ \epsilon_0(\underline{x}) & \underline{x} \notin V_1 \end{cases},$$

noting that the charge distribution is unaltered. Then the new energy is

$$W_1 = \frac{1}{2}\int d^3x\, \underline{E}(\underline{x}) \cdot \underline{D}(\underline{x})$$

and the *change* in energy is

$$W = \frac{1}{2}\int d^3x\, \underline{E} \cdot \underline{D} - \frac{1}{2}\int d^3x\, \underline{E}_0 \cdot \underline{D}_0.$$

With a little juggling, we can write this as

$$W = \frac{1}{2}\int d^3x\, (\underline{E} \cdot \underline{D}_0 - \underline{E}_0 \cdot \underline{D}) + \frac{1}{2}\int d^3x\, (\underline{E} + \underline{E}_0) \cdot (\underline{D} - \underline{D}_0).$$

To evaluate the second term, we note that both $\underline{\nabla} \times \underline{E} = 0$ and $\underline{\nabla} \times \underline{E}_0 = 0$, and thus we may write $\underline{E} + \underline{E}_0 = -\underline{\nabla}\Phi(\underline{x})$. Hence the second integral may be written

$$I = -\frac{1}{2}\int d^3x\, \underline{\nabla}\Phi \cdot (\underline{D} - \underline{D}_0) = \frac{1}{2}\int d^3x\, \Phi \underline{\nabla} \cdot (\underline{D} - \underline{D}_0)$$

where we assume the integrand falls off sufficiently rapidly at infinity.

Now $\underline{\nabla} \cdot (\underline{D} - \underline{D}_0) = \rho_f(\underline{x}) - \rho_{0f}(\underline{x}) = \underline{0}$, since we required that the free charge distribution be unaltered by the introduction of the dielectric. Thus the integral vanishes, and we have

$$W = \frac{1}{2}\int d^3x\,(\underline{E}\cdot\underline{D}_0 - \underline{E}_0\cdot\underline{D}).$$

We now spilt the region of integration into V_1 and the remainder,

$$W = \frac{1}{2}\int_{x\in V_1} d^3x\,(\underline{E}\cdot\underline{D}_0 - \underline{E}_0\cdot\underline{D}) + \frac{1}{2}\int_{x\notin V_1} d^3x\,(\underline{E}\cdot\underline{D}_0 - \underline{E}_0\cdot\underline{D}).$$

For $x \notin V_1$ we have $\underline{D}_0 = \epsilon_0\underline{E}_0$ and $\underline{D} = \epsilon_0\underline{E}$, and the integrand vanishes, so that

$$\begin{aligned} W &= \frac{1}{2}\int_{V_1} d^3x\,(\epsilon_0\underline{E}\cdot\underline{E}_0 - \epsilon_1\underline{E}_0\cdot\underline{E}) \\ &= -\frac{1}{2}\int_{V_1} d^3x\,(\epsilon_1-\epsilon_0)\underline{E}\cdot\underline{E}_0. \end{aligned}$$

We now specialise to the case where the original dielectric is indeed the vacuum, and ϵ_0 the permittivity of free space, and write

$$(\epsilon_1 - \epsilon_0)\underline{E} = \underline{P},$$

yielding

$$\boxed{W = -\frac{1}{2}\int_{V_1} d^3x\,\underline{P}\cdot\underline{E}_0.}$$

We can interpret $w = -\frac{1}{2}\underline{P}\cdot\underline{E}_0$ as the **energy density** of the dielectric. The expression can be likened to that for the energy of a dipole distribution derived at the end of Section 4.3. There we were considering a *permanent* dipole, whilst here energy is expended in polarizing the dielectric, and this is reflected in the factor of 1/2.

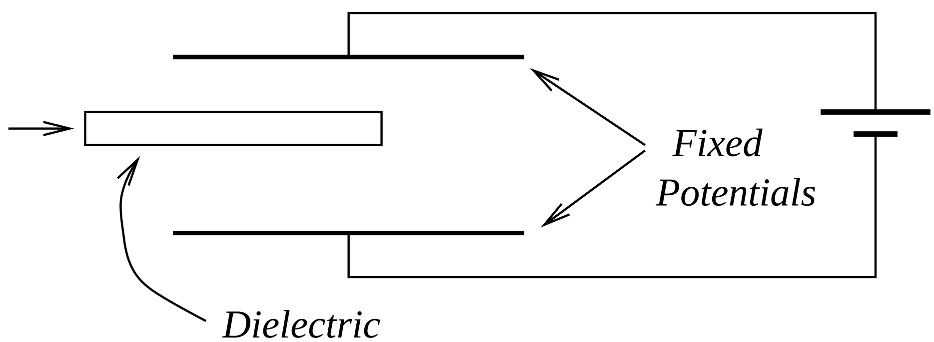
Note that the energy tends to *decrease* if the dielectric moves to a region of increasing $\underline{E}_0$, providing $\epsilon_1 > \epsilon_0$. Since the charges are held fixed, the total energy is *conserved*, and we can interpret the change in field energy W due to a displacement of the dielectric body ξ as producing a corresponding change in potential energy, and hence a force on the body of magnitude

$$F = -\left(\frac{\partial W}{\partial \xi}\right)_Q,$$

where the subscript Q denotes at fixed charge.

4.6.2 Energy of Dielectric Body at Fixed Potentials

We will conclude this section by considering the contrasting case where we introduce a dielectric body into a system where the **potentials**, rather than charges, are kept fixed. A paradigm is the introduction of a dielectric between the plates of a capacitor connected to a battery, and hence at a fixed potential difference.



In this case, charges can flow to or from the conducting plates as the dielectric is introduced to maintain the potentials, and hence the total energy can change. Again, we will assume that the media are **linear**.
It is sufficient to consider *small* changes to the potential $\delta\phi$ and to the charge distribution $\delta\phi$, for which the change in energy δW, from eqn. 4.13, is

$$\delta W = \frac{1}{2}\int d^3x\,(\rho_f \delta\phi + \phi\delta\rho_f).$$

For the case of linear media, these two terms are equal if the dielectric properties are unaltered. However, in the case where the dielectric properties are altered during the change, $\epsilon(\underline{x}) \rightarrow \epsilon(\underline{x}) + \delta\epsilon(\underline{x})$, this is no longer true, because of a polarisation charge density generated in the dielectric. We have already considered this problem for fixed charges, $\delta\rho_f = 0$. In order to compute the change of energy at fixed potentials, we study the problem in two stages;

1. The battery is disconnected, so that the distribution of charges is fixed, $\delta\rho_f = 0$, and the dielectric is introduced. Then there is a change in potential $\delta\phi_1$, and the corresponding change in energy is

$$\delta W_1 = \frac{1}{2}\int d^3x\, \rho_f \delta\phi_1 = -\frac{1}{2}\int (\epsilon_1 - \epsilon_0)\underline{E}\cdot\underline{E}_0,$$

   using the result of the previous subsection.

2. We now reconnect the battery. The potential on the conductors, where the only macroscopic charges reside, must regain its original value, i.e. $\delta\phi_2 = -\delta\phi_1$, and there is a corresponding change in charge density $\delta\rho_{2f}$, yielding

$$\delta W_2 = \frac{1}{2}\int d^3x\, (\rho_f \delta\phi_2 + \phi_2 \delta\rho_{2f}).$$

   In this step, the dielectric properties are unaltered and the two terms are equal, so we have

$$\begin{aligned}\delta W_2 &= \int d^3x\, \rho_f \delta\phi_2 \\ &= -\int d^3x\, \rho_f \delta\phi_1 \\ &= -2\,\delta W_1\end{aligned}$$

Thus the *total* energy change

$$\delta W = \delta W_1 + \delta W_2 = -\delta W_1,$$

which we write as

$$\delta W_V = -\delta W_Q,$$

i.e. the change in energy *at fixed potential* is **minus** the change in energy *at fixed charges*. In this case, if a dielectric with $\epsilon_1 > \epsilon_0$ moves into a region at fixed potentials, the energy **increases**, and a mechanical force

$$F_\xi = +\left(\frac{\partial W}{\partial \xi}\right)_V$$

acts on the body.

Chapter 5

Magnetostatics

5.1 Introduction

The crucial difference between electric and magnetic phenomena is the absence of isolated magnetic charges, or **magnetic monopoles**. Here the basic building blocks are **magnetic dipoles**. For a magnetic field, or flux density, $\underline{B}$, the torque $\underline{\tau}$ acting on a dipole of moment $\underline{\mu}$ is

$$\underline{\tau} = \underline{\mu} \times \underline{B}.$$

The other concept we need in the study of magnetostatics is the electric current $\underline{J}$, defined as the flow of charge per unit time per unit area, with normal in the direction of $\underline{J}$.

$$\underline{J} = \frac{d\underline{I}}{da_\perp} = \frac{dQ}{da_\perp dt}\hat{J}$$

5.1.1 Current Conservation

Current conservation is represented by the **continuity equation**

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{J} = 0,$$

where ρ is the charge density. This statement just states that the rate of change of charge in any volume V is (minus) the flux of charge across the surface of V, as you can see by applying the divergence theorem:

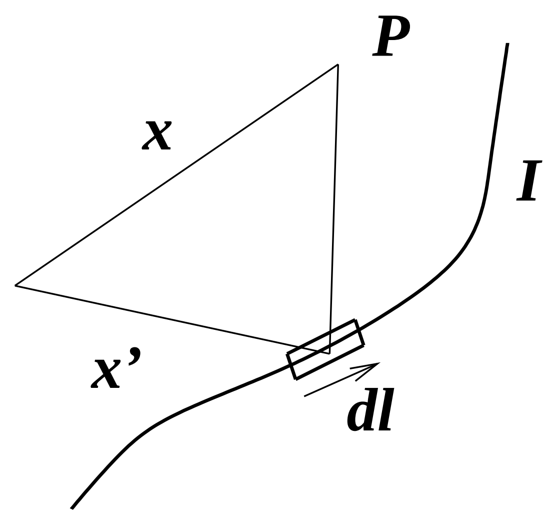
$$\frac{dQ}{dt} = \oint \frac{dQ}{da_\perp dt} da_\perp = \oint J da_\perp = \oint \underline{J} \cdot \underline{da} = \int d^3x \underline{\nabla} \cdot \underline{J}$$

For steady currents we are considering in this chapter

$$\underline{\nabla} \cdot \underline{J} = 0.$$

5.2 Biot-Savart Law

This describes the element of magnetic field $\underline{B}$ at some point $\underline{x}$ due to an element of current flow $I\underline{dl}$ at $\underline{x}'$.



$$d\underline{B} = kI \frac{\underline{dl} \times (\underline{x} - \underline{x}')}{|\underline{x} - \underline{x}'|^3}$$

where, in SI units,

- I is the current (Ampères),
- $\underline{dl}$ is an element of length in the direction of the current flow,
- $k = \mu_0/4\pi$, where μ_0 is the **permeability of free space**.

For a point charge q moving with velocity $\underline{v}$, we can replace $I\underline{dl}$ by $q\underline{v}$, and we have

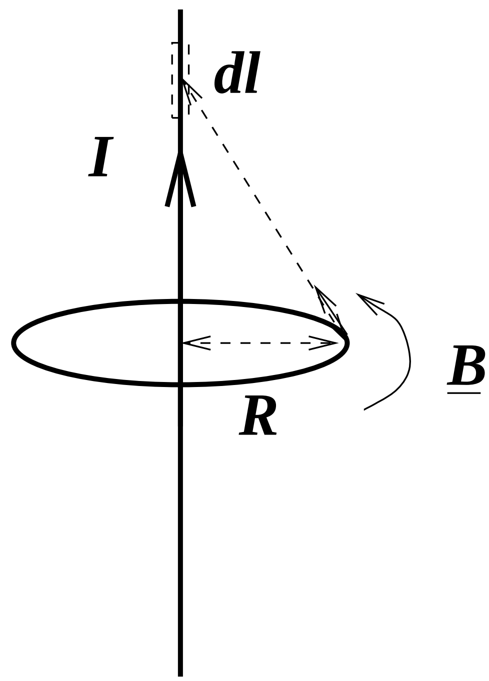
$$\underline{B} = \frac{\mu_0}{4\pi} \frac{q\underline{v} \times (\underline{x} - \underline{x}')}{|\underline{x} - \underline{x}'|^3},$$

providing $\underline{v}$ is constant, and small compared to the velocity of light.

We can apply the superposition principle to the magnetic field, and obtain for a general current density

$$\underline{B}(\underline{x}) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\underline{J}(\underline{x}') \times (\underline{x} - \underline{x}')}{|\underline{x} - \underline{x}'|^3}.$$

Example



Consider the magnetic field due to straight wire carrying current I. Then the field a distance R from the wire is tangential, and can be written

$$\begin{aligned}
\underline{B} &= \frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty} \frac{dl}{(l^2+R^2)}\frac{R}{\sqrt{l^2+R^2}}\underline{e_\theta} \\
&= \frac{\mu_0 I}{4\pi} R \int \frac{dl}{(l^2+R^2)^{3/2}}\underline{e_\theta} \\
&= \frac{\mu_0 I}{4\pi} R \int_{-\pi/2}^{\pi/2} \frac{R\sec^2\theta d\theta}{\sec^3\theta}\underline{e_\theta} = \frac{\mu_0 I}{2\pi R}\underline{e_\theta}
\end{aligned}$$

5.2.1 Force on a Current in Presence of Magnetic Field

The element of force on a current element $I\underline{dl}$ at $\underline{x}$ in a magnetic field $\underline{B}(\underline{x})$ is

$$\underline{dF} = I\underline{dl} \times \underline{B}.$$

Thus the force on a closed loop of current l_1 due to magnetic field from closed loop l_2 is

$$\begin{aligned}
\underline{F}_{12} &= \frac{\mu_0}{4\pi} I_1 I_2 \oint \underline{dl}_1 \times \left\{\oint \frac{\underline{dl}_2 \times (\underline{x}_1 - \underline{x}_2)}{|\underline{x}_1 - \underline{x}_2|^3}\right\} \\
&= \frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \frac{\underline{dl}_1 \times [\underline{dl}_2 \times (\underline{x}_1 - \underline{x}_2)]}{|\underline{x}_1 - \underline{x}_2|^3}.
\end{aligned}$$

We can put this expression in a more symmetric form by writing

$$\underline{dl}_1 \times (\underline{dl}_2 \times \underline{x}_{12}) = (\underline{dl}_1 \cdot \underline{x}_{12})\underline{dl}_2 - (\underline{dl}_1 \cdot \underline{dl}_2)\underline{x}_{12},$$

yielding

$$\underline{F}_{12} = \frac{\mu_0 I_1 I_2}{4\pi} \oint \oint \left\{ -\frac{\underline{dl}_1 \cdot \underline{dl}_2}{|\underline{x}_{12}|^3}\underline{x}_{12} + \underline{dl}_2 \frac{\underline{dl}_1 \cdot \underline{x}_{12}}{|\underline{x}_{12}|^3} \right\}. \tag{5.1}$$

We will now show that the second term vanishes. Consider the integration around loop 1, for fixed $\underline{x}_2$. Then under a change $x_1 \to x_1 + \underline{dl}_1$, we have

$$\underline{x}_{12} \to \underline{x}_{12} + \underline{dl}_1.$$

Now consider the change in $1/|\underline{x}_{12}|$:

$$\begin{aligned} \delta\left(\frac{1}{|\underline{x}_{12}|}\right) &= \frac{1}{|\underline{x}_{12} + \underline{dl}_1|} - \frac{1}{|\underline{x}_{12}|} \\ &= \frac{1}{|\underline{x}_{12}|}\left\{1 - \frac{\underline{x}_{12} \cdot \underline{dl}_1}{|\underline{x}_{12}|^2} - 1\right\} \\ &= -\frac{\underline{x}_{12} \cdot \underline{dl}_1}{|\underline{x}_{12}|^3}. \end{aligned}$$

Thus the integrand in the second terms of eqn. 5.1 is an exact differential, and therefore the integrand around the closed loop vanishes, and we have

$$\underline{F}_{12} = \frac{\mu_0 I_1 I_2}{4\pi} \int \frac{\underline{dl}_1 \cdot \underline{dl}_2}{|\underline{x}_{12}|^3}\underline{x}_{12}.$$

Now Newton's third law is satisfied explicity, and we have

$$\underline{F}_{12} = -\underline{F}_{21}.$$

For a general current density $\underline{J}(\underline{x})$ in a magnetic field $\underline{B}(\underline{x})$, we have

$$\begin{aligned} \underline{F} &= \int d^3x\, \underline{J}(\underline{x}) \times \underline{B}(\underline{x}) \\ \underline{\tau} &= \int d^3x\, \underline{x} \times (\underline{J} \times \underline{B}). \end{aligned}$$

5.3 Laws of Magnetostatics in Differential Form

In analogy with electrostatics, our starting point is the expression for $\underline{B}$ due to general current density

$$\underline{B}(\underline{x}) = \frac{\mu_0}{4\pi} \int d^3x' \, \frac{\underline{J}(\underline{x}') \times (\underline{x} - \underline{x}')}{|\underline{x} - \underline{x}'|^3}.$$

We begin by recalling that $\underline{\nabla} \times (\varphi \underline{a}) = \underline{\nabla}\varphi \times \underline{a}$, where $\underline{a}$ is a constant vector. Thus

$$\begin{aligned} \underline{\nabla}_x \times \left(\underline{J}(\underline{x}') \frac{1}{|\underline{x} - \underline{x}'|} \right) &= \underline{\nabla}_x \left(\frac{1}{|\underline{x} - \underline{x}'|} \right) \times \underline{J}(\underline{x}') \\ &= -\frac{(\underline{x} - \underline{x}')}{|\underline{x} - \underline{x}'|^3} \times \underline{J}(\underline{x}') \end{aligned}$$

Thus we can write

$$\boxed{\underline{B}(\underline{x}) = \frac{\mu_0}{4\pi} \underline{\nabla} \times \int d^3x' \, \frac{1}{|\underline{x} - \underline{x}'|} \underline{J}(\underline{x}') \qquad (5.2)}$$

From eqn. 5.2, we immediately see that

$$\boxed{\underline{\nabla} \cdot \underline{B} = 0. \qquad (5.3)}$$

This is another of Maxwell's equations, and is just another statement that you cannot have isolated magnetic charges, and that the total flux of $\underline{B}$ through any closed surface vanishes

$$\int_{S=\partial V} \underline{dS} \cdot \underline{B} = 0.$$

To obtain another differential equation, we evaluate $\underline{\nabla} \times \underline{B}$. We begin by recalling the vector identity

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{A}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{A}) - \underline{\nabla}^2 \underline{A},$$

so that

$$\underline{\nabla} \times \underline{B} = \frac{\mu_0}{4\pi} \underline{\nabla} \int d^3x' \, \underline{\nabla}_x \cdot \frac{\underline{J}(\underline{x}')}{|\underline{x} - \underline{x}'|} - \frac{\mu_0}{4\pi} \int d^3x' \, \underline{J}(\underline{x}') \underline{\nabla}_x^2 \left(\frac{1}{|\underline{x} - \underline{x}'|} \right).$$

Now

$$\begin{aligned}\underline{\nabla}_x \cdot \frac{\underline{J}(\underline{x}')}{|\underline{x}-\underline{x}'|} &= -\underline{J}(\underline{x}') \cdot \underline{\nabla}_{x'} \left(\frac{1}{|\underline{x}-\underline{x}'|}\right) \\ \underline{\nabla}_x^2 \left(\frac{1}{|\underline{x}-\underline{x}'|}\right) &= -4\pi\delta(\underline{x}-\underline{x}'),\end{aligned}$$

and thus

$$\begin{aligned}\underline{\nabla} \times \underline{B} &= -\frac{\mu_0}{4\pi}\underline{\nabla} \int d^3x'\, \underline{J}(\underline{x}') \cdot \underline{\nabla}_{x'} \left(\frac{1}{|\underline{x}-\underline{x}'|}\right) + \mu_0 \int d^3x'\, \underline{J}(\underline{x}')\delta(\underline{x}-\underline{x}') \\ &= \frac{\mu_0}{4\pi}\underline{\nabla} \int d^3x'\, \frac{1}{|\underline{x}-\underline{x}'|}\underline{\nabla}_{x'} \cdot \underline{J} + \mu_0 \underline{J}(\underline{x})\end{aligned}$$

For, for magnetostatics, we have $\underline{\nabla} \cdot \underline{J} = 0$, and thus

$$\boxed{\underline{\nabla} \times \underline{B} = \mu_0 \underline{J} \qquad (5.4)}$$

This is the second fundamental differential equation. We can apply Stoke's theorem to a closed curve C spanned by a surface S to obtain

$$\int \underline{\nabla} \times \underline{B} \cdot \underline{dS} = \int \underline{B} \cdot \underline{dl} = \mu_0 \int \underline{J} \cdot \underline{dS}.$$

5.4 Vector Potential

For static fields, the governing equations of magnetostatics are

$$\begin{aligned}\underline{\nabla} \cdot \underline{B} &= 0 \\ \underline{\nabla} \times \underline{B} &= \mu_0 \underline{J}\end{aligned}$$

For the case $\underline{J} \equiv \underline{0}$, we have $\underline{\nabla} \times \underline{B} = 0$, and we can introduce a magnetic scalar potential ϕ_M.
Much more interesting is the general case $\underline{J} \neq \underline{0}$. We can show that if $\underline{\nabla} \cdot \underline{B} = 0$ is a **star-shaped** region,[1] then a **vector potential** $\underline{A}$ can be found such that

$$\underline{B} = \underline{\nabla} \times \underline{A}.$$

In the case where $\underline{B}$ is the magnetic field, we call $\underline{A}$ the **magnetic vector potential**.

5.4.1 Uniqueness of $\underline{A}$ and Gauge Transformations

If $\underline{A}$ is a solution of $\underline{B} = \underline{\nabla} \times \underline{A}$, then $\underline{A}' = \underline{A} + \underline{\nabla} f$, where f is an arbitrary, continuously differentiable scalar field, is also a solution, because

$$\underline{\nabla} \times (\underline{\nabla} f) = 0.$$

Transformation of this form are called **Gauge Transformations**; we say that $\underline{B}$ is invariant under gauge transformations. To simplify calculations, we often make a specific choice of gauge.

Examples

1. We could require $A_1(\underline{x}) = 0 \quad \forall \underline{x}$.

[1] A star shaped region is one in which there exists a point which can be connected to every other point by a straight line

2. We could require

$$\underline{\nabla} \cdot \underline{A} = 0 \quad \forall \underline{x}.$$

This is the **Coulomb Gauge**.

Choosing, or fixing, the gauge reduces the number of degrees of freedom, clear in example (1) above. All the fundamental forces of nature are described by **Gauge Theories**, having the property of a gauge, or local, symmetry.

5.4.2 Solutions for the Vector Potential in Free Space

We will specify that we work in the Coulomb gauge, $\underline{\nabla} \cdot \underline{A} = 0$. Then the second of our governing equation becomes

$$\underline{\nabla} \times \underline{B} = \underline{\nabla} \times (\underline{\nabla} \times \underline{A}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{A}) - \nabla^2 \underline{A} = \mu_0 \underline{J}$$

and thus

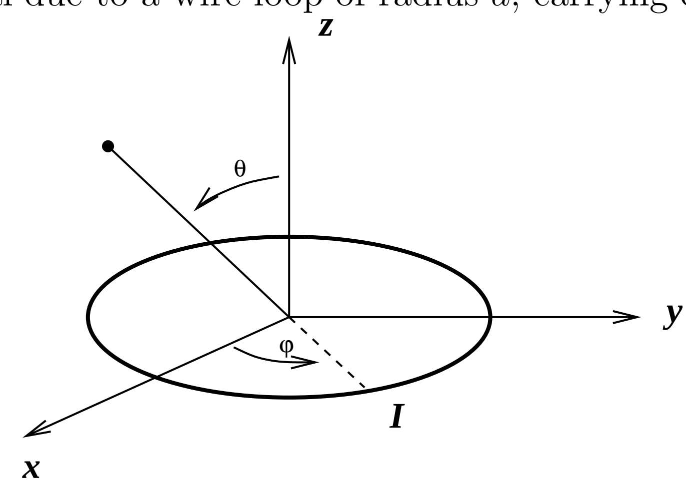
$$\nabla^2 \underline{A} = -\mu_0 \underline{J}.$$

This is just Poisson's equation, applied to each of the Cartesian components of $\underline{A}$, and from our investigation of electrostatics has the solution

$$\underline{A}(\underline{x}) = \frac{\mu_0}{4\pi} \int d^3x' \, \frac{\underline{J}(\underline{x}')}{|\underline{x} - \underline{x}'|}. \tag{5.5}$$

Example

Potential due to a wire loop of radius a, carrying current I.



The current is purely in the azimuthal direction, and in spherical polars, we can write the current density as

$$J_\varphi = I \sin\theta' \delta(\cos\theta') \frac{\delta(r'-a)}{a}.$$

You should convince yourself that this expression is correct. W.l.o.g. we will consider the case where the observation point is in the $x - z$ plane, so that, in Cartesian coordinates, the current density is

$$\underline{J} = -J_\varphi \sin\varphi' \underline{i} + J_\varphi \cos\varphi' \underline{j}.$$

Thus the vector potential, from eqn. 5.5, is given by

$$\underline{A}(\underline{x}) = \frac{\mu_0}{4\pi} \int d\Omega' \, r'^2 dr' \, \{-J_\varphi \sin\varphi' \underline{i} + J_\varphi \cos\varphi' \underline{j}\} \times \frac{1}{|\underline{x} - \underline{x}'|}.$$

The x component of $\underline{A}$ will vanish, since the expansion of $1/|\underline{x} - \underline{x}'|$ is symmetric under $\varphi' \leftrightarrow -\varphi'$. Thus the only non-vanishing component of $\underline{A}$ is in the y-direction, which coincides with $\underline{e}_\varphi$. Thus we have

$$A_\varphi = \frac{\mu_0}{4\pi} I \int d\Omega' \, dr' r'^2 \sin\theta' \delta(\cos\theta') \frac{\delta(r'-a)}{a} \cos\varphi' \, \frac{1}{|\underline{x} - \underline{x}'|}.$$

Performing the integrations over r' and θ' yields

$$A_\varphi = \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} d\varphi' \, \cos\varphi' \, \{a^2 + r^2 - 2ar \sin\theta \cos\varphi'\}^{-1/2}.$$

This is an eliptic integral, and its expression in eliptic functions is not particularly illuminating. Instead, we will perform an expansion in spherical harmonics:

$$A_\varphi = \frac{\mu_0 I}{4\pi} \Re \int d\Omega' \, dr' \, r'^2 \sin\theta' \delta(\cos\theta') \frac{\delta(r'-a)}{a} e^{i\varphi'}$$
$$\times 4\pi \sum_{l,m} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi),$$

where we write

$$\cos\varphi' = \Re e^{i\varphi'}.$$

Performing the delta-function integrations, we arrive at

$$A_\varphi = \mu_0 I a \Re \sum_{l,m} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta, 0) \frac{1}{2l+1} \int d\varphi' \, e^{i\varphi'} Y_{lm}^*(\pi/2, \varphi').$$

We now use the orthogonality properties of the functions $\exp im\varphi$ to write (you see why we expressed $\cos\varphi'$ this way...):

$$\int d\varphi' \, e^{i\varphi'} Y_{lm}^*(\pi/2, \varphi') = \begin{cases} 2\pi Y_{l1}(\pi/2, 0) & m = 1 \\ 0 & \text{otherwise} \end{cases},$$

and thus

$$A_\varphi = 2\pi \mu_0 I a \sum_{l=1}^{\infty} \frac{r_<^l}{r_>^{l+1}} Y_{l1}(\theta, 0) Y_{l1}(\pi/2, 0) \frac{1}{2l+1}.$$

Now we have that

$$Y_{l1}(\pi/2, 0) = \sqrt{\frac{(l-1)!(2l+1)}{4\pi(l+1)!}} P_l^1(0)$$

which vanishes if l is even, since $P_l^1(0)$ has the opposite parity to $P_l(0)$. The explicit evaluation of these integrals is performed in *Jackson*, so I leave it for you to look them up there. However, the important feature is that even when we have azimuthal symmetry, the vector potential and magnetic fields involve the P_l^1 Legendre polynomials; this reflects the vector nature of the source in magnetostatics, as opposed to the scalar nature of the source in electrostatics.

5.5 Magnetic Field Far from Current Distribution

Consider a localized current distribution $\underline{J}(\underline{x}')$, and the magnetic vector potential produced at a point $P(\underline{x})$ where $|\underline{x}| \gg |\underline{x}'|$. Then we can write

$$\frac{1}{|\underline{x} - \underline{x}'|} = \frac{1}{|\underline{x}|} + \frac{\underline{x} \cdot \underline{x}'}{|\underline{x}|^3} + \ldots,$$

so that, in the Coulomb gauge

$$A_i(\underline{x}) = \frac{\mu_0}{4\pi} \left\{ \frac{1}{|\underline{x}|} \int d^3x' \, J_i(\underline{x}') + \frac{\underline{x}}{|\underline{x}|^3} \cdot \int d^3x' \, J_i(\underline{x}') \underline{x}' + \ldots \right\} \tag{5.6}$$

To study this expression further, we begin by proving a small identity. Recall that for magnetostatics, $\underline{\nabla} \cdot \underline{J} \equiv 0$, and thus for any two scalar functions $f(\underline{x}')$ and $g(\underline{x}')$ we have

$$0 = -\int d^3x'\, f(\underline{x}')g(\underline{x}')\underline{\nabla}' \cdot \underline{J} = \int d^3x'\, \underline{\nabla}'[f(\underline{x}')g(\underline{x}')] \cdot \underline{J}(\underline{x}')$$

where in the second step we have integrated by parts, using the fact that the surface integral vanishes for a localised current distribution. Thus we have

$$\int d^3x'[f\underline{J} \cdot \underline{\nabla}'g + g\underline{J} \cdot \underline{\nabla}'f] = 0 \tag{5.7}$$

We now consider the first term in eqn. 5.6. Applying eqn. 5.7 for the case $f(\underline{x}') = 1$, $g(\underline{x}') = x_i'$, we have

$$\begin{aligned} \int d^3x'\, [J_j\delta_{ij} + x_i'J_j \cdot 0] &= 0 \\ \Rightarrow \int d^3x'\, J_i &= 0. \end{aligned}$$

Thus the first term vanishes. This is just a further restatement that there is no "monopole" contribution to the multipole expansion for magnetic fields.
We now applying the identity to the case $f = x_i'$, and $g = x_j'$. Then from eqn. 5.7, we have

$$\begin{aligned} \int d^3x'\, [x_i'J_k\frac{\partial x_j'}{\partial x_k'} + x_j'J_k\frac{\partial x_i'}{\partial x_k'}] &= 0 \\ \Rightarrow \int d^3x'\, [x_i'J_j + x_j'J_i] &= 0. \end{aligned}$$

Thus, going back to eqn. 5.6, we may write

$$\begin{aligned} A_i(\underline{x}) &= \frac{\mu_0}{4\pi}\frac{1}{|\underline{x}|^3}x_j \int d^3x'\, J_i x_j' \\ &= -\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\underline{x}|^3}x_j \int d^3x'[x_i'J_j - x_j'J_i]. \end{aligned}$$

Levi-Civita Tensor

To take the discussion further, we recall the definition of the **Levi-Civita tensor**

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if any two if } i, j, k \text{ are equal} \\ 1 & \text{if } (ijk) \text{ is an } \textit{even} \text{ permutation of } (123) \\ -1 & \text{if } (ijk) \text{ is an } \textit{odd} \text{ permutation of } (123) \end{cases}$$

This tensor is *isotropic*, and *totally anti-symmetric*. In particular, we have

$$\underline{A} \times \underline{B}|_i = \epsilon_{ijk} A_j B_k.$$

There is the following well-known and easily shown identity

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl},$$

which we will now use to write

$$\begin{aligned} x'_i J_j - x'_j J_i &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) x'_l J_m \\ &= \epsilon_{kij}\epsilon_{klm} x'_l J_m \\ &= \epsilon_{ijk} (\underline{x}' \times \underline{J})_k. \end{aligned}$$

Thus we have

$$\boxed{A_i(\underline{x}) = -\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\underline{x}|^3}\left[\underline{x} \times \int d^3x'\, \underline{x}' \times \underline{J}\right]_i}$$

The vector

$$\underline{m} = \frac{1}{2}\int d^3x'\, \underline{x}' \times \underline{J}$$

is the **magnetic moment**, whilst

$$\underline{\mu} = \frac{1}{2}\underline{x}' \times \underline{J}$$

is the **magnetic moment density**. Thus we can write

$$\boxed{\underline{A}(\underline{x}) = \frac{\mu_0}{4\pi}\frac{1}{|\underline{x}|^3}\underline{m} \times \underline{x}}$$

This is the lowest, non-vanishing term in the multipole expansion of the magnetic vector potential for a localised current density. Applying $\underline{B} = \underline{\nabla} \times \underline{A}$, we have

$$\underline{B} = \frac{\mu_0}{4\pi}\left[\frac{3(\underline{x}\cdot\underline{m})\underline{x} - r^2\underline{m}}{r^5}\right],$$

exactly analogous to the electrostatic field due to a point dipole.[2]

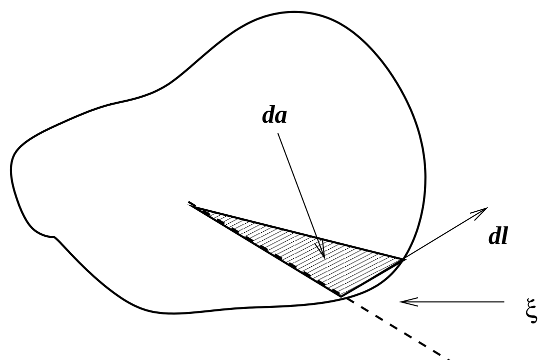
Example

For the case of a current confined to a loop, we have

$$\underline{m} = \frac{I}{2}\oint \underline{x} \times \underline{dl}.$$

Furthermore, if we have a planar loop, $\underline{x} \times \underline{dl}$ is normal to the plane of the loop, and we have

$$\begin{aligned} \frac{1}{2}\underline{x} \times \underline{dl} &= \frac{1}{2}x\,dl\,\sin\xi \\ &= da \end{aligned}$$



so that

$$\underline{m} = IA\underline{n}$$

where $\underline{n}$ is a normal to the plane of the loop, and A is the total area of the loop.

Example

We conclude this section by considering the case where the current distribution arises from the motion of a number of charged point-like particles:

$$\underline{J} = \sum_i q_i\underline{v}_i\delta(\underline{x} - \underline{x}_i),$$

[2]It is possible to introduce a vector potential to describe electric dipole fields

where $\underline{v}_i$ is the velocity of the i^{th} particle, which we assume is much less than the velocity of light.
Then we have

$$\underline{m} = \frac{1}{2}\sum_i q_i \underline{x}_i \times \underline{v}_i.$$

Now the orbital angular momentum of a particle is given by

$$\underline{L}_i = M_i \underline{x}_i \times \underline{v}_i,$$

where M_i is the mass of the i^{th} particle. Thus we may write

$$\underline{m} = \sum_i \frac{q_i}{2M_i}\underline{L}_i.$$

In the case where all the particles have equal mass, we see that the magnetic moment is proportional to the **total angular momentum**.

5.6 Magnetostatics of Matter

5.6.1 Torques and forces on magnetic dipoles

First, consider a magnetic dipole in the uniform magnetic field $\underline{B}$. Let us visualize magnetic dipole $\underline{m}$ as a wire loop with area a carrying current I such as $m = Ia$. The total force acting on the loop is zero:

$$\underline{F} \;=\; I\oint \underline{dl} \times \underline{B} \;=\; -I\underline{B} \times \oint \underline{dl} = 0$$

The torque acting on the loop is $\underline{m} \times \underline{B}$:

$$\underline{N} \;= \oint \underline{x}' \times d\underline{F} \;= \oint \underline{x}' \times (I\underline{dx}' \times \underline{B}) \;= I\oint \underline{dx}'(\underline{x}' \cdot \underline{B}) - \underline{B}I\oint dx'^2 \;= I\oint \underline{dx}'(\underline{x}' \cdot \underline{B})$$

It is easy to prove that for an arbitrary constant vector $\underline{a}$

$$\oint \underline{dx}'(\underline{x}' \cdot \underline{a}) = -\frac{1}{2}\underline{a} \times \oint(\underline{x}' \times \underline{dx}') \tag{5.8}$$

Indeed,

$$\underline{a} \times \oint(\underline{x}' \times \underline{dx}') = \oint[\underline{x}'(\underline{a} \cdot \underline{dx}') - \underline{dx}'(\underline{a} \cdot \underline{x}')]$$
$$\oint \underline{x}'(\underline{a} \cdot \underline{dx}') = \oint[d(\underline{x}'(\underline{a} \cdot \underline{x}')) - \underline{dx}'(\underline{a} \cdot \underline{x}')] = -\oint \underline{dx}'(\underline{a} \cdot \underline{x}')$$

and therefore $\underline{a} \times \oint(\underline{x}' \times \underline{dx}') = -2\oint\underline{dx}'(\underline{a} \cdot \underline{x}')$. Taking $\underline{a} = \underline{B}$ we get

$$\underline{N} = -\frac{I}{2}\underline{B} \times \oint(\underline{x}' \times \underline{dx}') \;=\; (\frac{I}{2}\oint \underline{x}' \times \underline{dx}') \times \underline{B} \;=\; \underline{m} \times \underline{B}$$

so the torque in a uniform external field is a cross product of the magnetic moment and the field.

Let us now consider a small dipole in the non-uniform external field (the size of the dipole $\ll$ characteristic size of the field). The formula for the torque remains the same: $\underline{N} = \underline{m} \times \underline{B}$ where the magnetic field should be taken at the position of the dipole. However, the total force is no longer zero.

$$\underline{F} \;=\; I\oint \underline{dl} \times \underline{B} \;\neq 0$$

Since our dipole is small we can expand $\underline{B}(\underline{x}')$ in powers of $\underline{x}'$. For simplicity, suppose that the dipole is located at the origin. We get

$$\underline{B}(\underline{x}') = \underline{B}(0) + (\underline{x}' \cdot \underline{\nabla})B(0) + ...$$

and therefore ($\underline{dx}' \equiv \underline{dl}'$)

$$\underline{F} \;=\; I\oint \underline{dl}' \times \underline{B}(0) + I\oint \underline{dl}' \times (\underline{x}' \cdot \underline{\nabla})\underline{B} + O(x'^2) \;=\; I\oint \underline{dx}'(\underline{x}' \cdot \underline{\nabla}) \times \underline{B}$$

Next we use formula (5.8) with $\underline{a} = \underline{\nabla}$ and obtain

$$I\oint \underline{dx}'(\underline{x}' \cdot \underline{\nabla}) \;=\; \frac{I}{2}\oint(\underline{x}' \times \underline{dx}') \times \underline{\nabla} \;=\; \underline{m} \times \underline{\nabla}$$

so finally

$$\underline{F} \;=\; (\underline{m} \times \underline{\nabla}) \times B \;=\; \underline{\nabla}(\underline{m} \cdot \underline{B}) - \underline{m}(\underline{\nabla} \cdot \underline{B}) \;=\; \underline{\nabla}(\underline{m} \cdot \underline{B})$$

because $\underline{\nabla} \cdot \underline{B} = 0$.
Since $\underline{F} = -\underline{\nabla}U$ we see that the potential energy of a (small) magnetic dipole in the external magnetic field is

$$U \;=\; -\,\underline{m} \cdot \underline{B}$$

(similarly to $U = -\underline{p} \cdot \underline{E}$ for the electric dipole).

5.6.2 Maxwell equations in matter

We could, in principle, attempt to describe the magnetostatics of a material in terms of the microscopic, or “vacuum”, fields. As in the case of electrostatics, this approach is neither feasible nor desirable. At the microscopic level, the individual atoms have magnetic moments and eddy currents are generated that we cannot account for exactly. Rather, we discuss macroscopic quantities, including that part of the magnetic field arising from these microscopic currents. In the following, we will use the subscript *micro* to denote microscopic properties, with the remaining variables denoting macroscopic quantities.

At the microscopic level, we have $\underline{\nabla} \cdot \underline{B}_{\text{micro}} = 0$. We can average this to obtain

$$\underline{\nabla} \cdot \underline{B} = 0$$

and hence we know that we can write the *macroscopic* magnetic field in terms of a vector potential

$$\underline{B} = \underline{\nabla} \times \underline{A}.$$

Suppose now that we have a collection of atoms of various types i, with magnetisation $\underline{m}_i$. Then the macroscopic magnetisation

$$\underline{M} = \sum_i N_i \langle \underline{m}_i \rangle,$$

where N_i is the numer of atoms of type i/unit volume, and $\langle \underline{m}_i \rangle$ is their average magnetic moment. Note the $\underline{M}$ is analogous to the polarisation density of electrostatics.

We will now consider the contribution to the vector potential at $\underline{x}$ due to an infinitessimal volume ΔV at $\underline{x}'$. There are two contributions

$$\Delta \underline{A}(\underline{x}) = \frac{\mu_0}{4\pi} \frac{\underline{J}_f(\underline{x}')}{\mid \underline{x} - \underline{x}' \mid} \Delta V + \frac{\mu_0}{4\pi} \frac{1}{\mid \underline{x} - \underline{x}' \mid} \underline{M} \times (\underline{x} - \underline{x}') \Delta V,$$

where the first term arises from the "free" macroscopic current densities and the second is due to the macroscopic magnetisation described above. We now sum over the volume elements ΔV and get

$$\underline{A}(\underline{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\underline{J}_f(\underline{x}')}{\mid \underline{x} - \underline{x}' \mid} + \frac{\mu_0}{4\pi} \int d^3x' \frac{1}{\mid \underline{x} - \underline{x}' \mid} \underline{M} \times (\underline{x} - \underline{x}').$$

There is a way to rewrite the second term in a more illuminating way. First, note that

$$\int d^3x' \frac{\underline{M} \times (\underline{x} - \underline{x}')}{\mid \underline{x} - \underline{x}' \mid^3} = \int d^3x' \, \underline{M} \times \underline{\nabla}' \left(\frac{1}{\mid \underline{x} - \underline{x}' \mid} \right)$$

From the formula $\underline{\nabla} \times (f\underline{M}) = f(\underline{\nabla} \times \underline{M}) - \underline{M} \times (\underline{\nabla} f)$ (see the cover of *Jackson*) for $f = \frac{1}{|\underline{x} - \underline{x}'|}$ we obtain

$$\int d^3x' \, (\underline{\nabla}' \times \underline{M}(\underline{x}') \frac{1}{\mid \underline{x} - \underline{x}' \mid} + \int d^3x' \, \underline{\nabla}' \times \left(\frac{\underline{M}(\underline{x}')}{\mid \underline{x} - \underline{x}' \mid} \right),$$

Using the divergence theorem for vector fields (again, see the cover)

$$\int_V d^3x' \, (\underline{\nabla}' \times \underline{A}(\underline{x}') \;=\; \int_{S=\partial V} \underline{n} \times \underline{A} \; dS \tag{5.9}$$

the second term can be rewritten as a surface integral

$$\frac{\mu_0}{4\pi} \int_{S=\partial V} \frac{\underline{\underline{M}}(\underline{x}') \times \underline{n}}{|\underline{x} - \underline{x}'|} \, dS$$

Finally, we get

$$\int d^3x' \frac{\underline{\underline{M}} \times (\underline{x} - \underline{x}')}{|\, \underline{x} - \underline{x}' \,|^3} = \frac{\mu_0}{4\pi} \int d^3x' \, \frac{\underline{J_b}}{|\, \underline{x} - \underline{x}' \,|} + \frac{\mu_0}{4\pi} \int_{S=\partial V} \frac{\underline{K_b}(\underline{x}')}{|\underline{x} - \underline{x}'|} \, dS$$

where $J_b \equiv \underline{\nabla} \times \underline{M}$ is called a bound volume current density and $K_b \equiv \underline{M} \times \underline{n}$ a bound surface current density.

If we take the surface to be an infinitely large sphere and assume that $\sim K$ vanishes at infinity ,we get

$$\underline{A}(\underline{x}) = \frac{\mu_0}{4\pi} \int \frac{dV'}{|\, \underline{x} - \underline{x}' \,|} \left\{ \underline{J}_f(\underline{x}') + \underline{\nabla}' \times \underline{M}' \right\}.$$

Comparing with the fundamental equation of magnetostatics *in vacua*,

$$\underline{\nabla} \times \underline{B} = \mu_0 \underline{J}_f,$$

we have

$$\underline{\nabla} \times \underline{B} = \mu_0 \{ \underline{J}_f + \underline{\nabla} \times \underline{M} \}.$$

It is now conventional to introduce the **magnetic field** $\underline{H}$, where

$$\underline{H} = \frac{1}{\mu_0} \underline{B} - \underline{M}.$$

In the context of media, the field $\underline{B}$ is known as the **magnetic induction** or **magnetic flux density**. In terms of $\underline{H}$ and $\underline{B}$, the fundamental equations of magnetostatics in matter are

$$\boxed{\begin{aligned} \underline{\nabla} \cdot \underline{B} &= 0 \\ \underline{\nabla} \times \underline{H} &= \underline{J}_f \end{aligned}}$$

Note that $\underline{H}$ is analogous to $\underline{D}$ in electrostatics; $\underline{E}$ and $\underline{B}$ are the fundamental fields, whilst $\underline{H}$ and $\underline{D}$ depend on the medium.

5.6.3 Constitutive relation

In the case of (isotropic) diamagnetic and paramagnetic materials, where the magnetic moment arises solely from the applied magnetic field, there is a simple linear relation between $\underline{H}$ and $\underline{B}$

$$\underline{M} = \chi_m \underline{H},$$

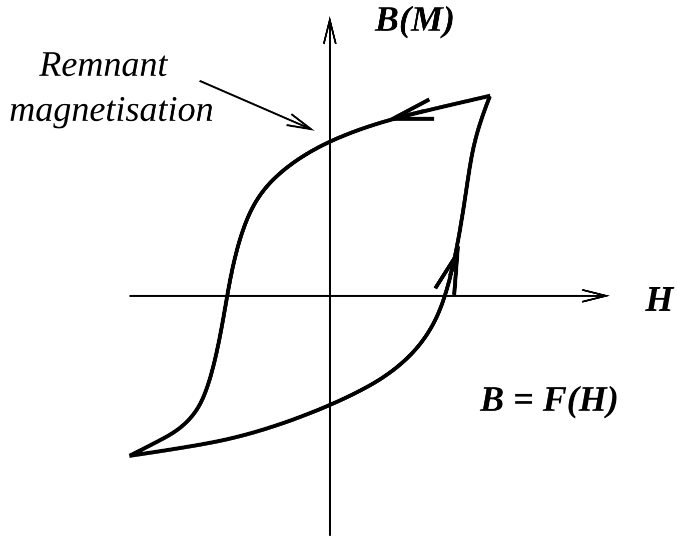
where χ_m is the **magnetic susceptibility**. Then we may write

$$\underline{H} = \frac{1}{\mu_0}(\underline{B} - \mu_0 \chi_m \underline{H})$$

yielding

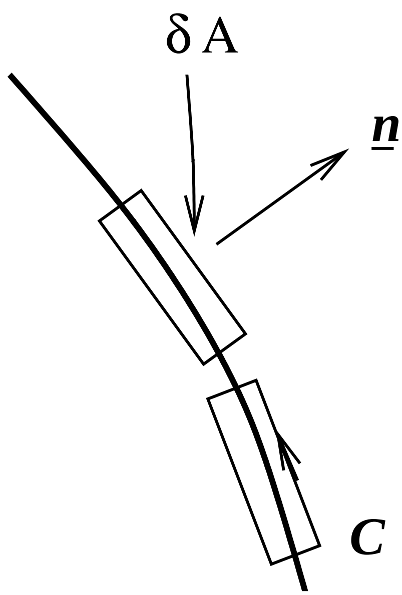
$$\underline{B} = \mu \underline{H}$$

where $\mu \equiv \mu_0(1 + \chi_m)$ is the **magnetic permeability**.
For *ferromagnets*, the corresponding relation is **non-linear** and exhibits *hysterisis*, i.e. the material retains a *memory* of its preparation.



5.6.4 Boundary Conditions at Surface Between Media

We will now obtain boundary conditions for the normal and tangential components of the field at the boundary between two materials. Note that the followign discussion is independent of whether or not there is a linear relation between the $\underline{H}$ and $\underline{B}$.



Normal Condition

Apply Gauss' Law to the pillbox shown

$$0 = \int dV\, \underline{\nabla} \cdot \underline{B} = \int \underline{B} \cdot \underline{n}\, dS = (\underline{B}_2 - \underline{B}_1) \cdot \underline{n}\, \delta A$$

where $\underline{n}$ is a unit normal from medium 1 to medium 2, and δA is the surface area of the pillbox. Thus we have

$$\boxed{\underline{B}_1^{\perp} = \underline{B}_2^{\perp}}$$

Tangential Condition

We now apply Stoke's theorem to get the boundary conditions on the tangential components:

$$\oint_c \underline{H} \cdot \underline{dl} = \int_S (\underline{\nabla} \times \underline{H}) \cdot \underline{da} = \int_S \underline{J}_f \cdot \underline{da},$$

where S is a surface spanning C.
Thus we have the tangential boundary condition

$$\boxed{\underline{H}_2 - \underline{H}_1 = \underline{K} \times \underline{n} \quad \Rightarrow \quad \underline{n} \times (\underline{H}_2 - \underline{H}_1) = \underline{K}}$$

where $\underline{K}$ is the **surface current density**.

5.7 Methods of Solving Boundary Value Problems

We will now look at various methods of solving boundary value problems between different media. The method depends on nature of the constitutive relation between $\underline{B}$ and $\underline{H}$, and on whether there is non-zero current density.

5.7.1 Vector Potential

The magnetic field is always solenoidal, and therefore we can essentially *always* introduce a vector potential $\underline{A}$ such that $\underline{B} = \underline{\nabla} \times \underline{A}$.
The dynamical information for the magnetostatics of media is provided by the equation

$$\underline{\nabla} \times \underline{H} = \underline{J}_f.$$

We will now specialise to the case where we have a **linear** constitutive relation, $\underline{B} = \mu \underline{H}$, enabling us to write

$$\underline{\nabla} \times \left[\frac{\underline{\nabla} \times \underline{A}}{\mu} \right] = \underline{J}_f.$$

This can be written

$$\nabla^2\underline{A} - \underline{\nabla}[\underline{\nabla} \cdot \underline{A}] = -\mu \underline{J}_f,$$

which in Coulomb gauge ($\underline{\nabla} \cdot \underline{A} = 0$) becomes

$$\nabla^2\underline{A} = -\mu \underline{J}_f.$$

This is analogous to the case discussed in Section 5.4.2, and the solution is that of eqn. 5.5, with μ_0 replaced by μ.

5.7.2 Solution when $\underline{J_f} \equiv 0$

In this case we have $\underline{\nabla} \times \underline{H} = 0$, and therefore we may admit introduce a **scalar potential** ϕ_M such that

$$\underline{H} = -\underline{\nabla}\phi_M.$$

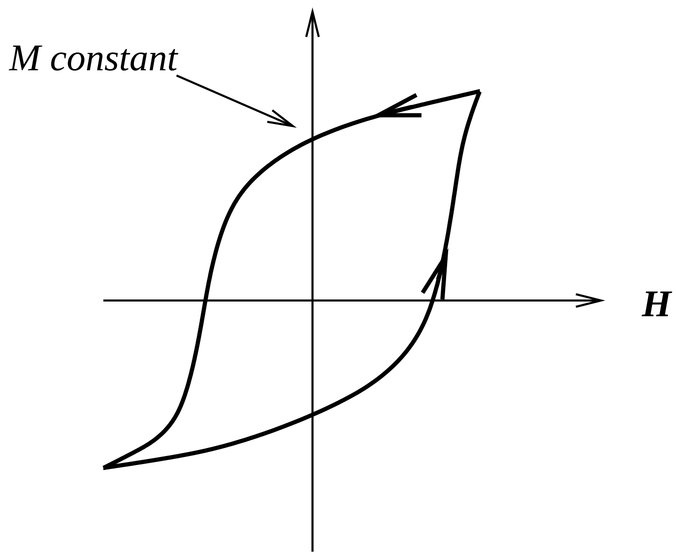
Once again, we will consider *linear* media, so that $\underline{B} = \mu\underline{H}$. Then we find that the scalar potential satisfies Laplace's equation

$$\nabla^2\phi_M = 0,$$

where we assume that μ is piecewise constant, i.e has a constant value in each of the different media we are considering.

5.7.3 Hard Ferromagnetic

In the case of a hard ferromagnet, we have $\underline{J}_f \equiv 0$, and the magnetisation is non-zero, and essentially independent of the magnetic field $\underline{H}$ providing it is sufficiently small.



Since $\underline{J}_f \equiv 0$, we can solve this problem using either a scalar or a vector potential.

Solution using Scalar Potential

The governing equations are

$$\begin{aligned} \underline{\nabla} \cdot \underline{B} &= 0 \quad (5.10) \\ \underline{\nabla} \times \underline{H} &= 0 \quad (5.11) \\ \underline{H} &= \frac{1}{\mu_0}\underline{B} - \underline{M} \quad (5.12) \end{aligned}$$

We will introduce a scalar potential for the magnetic field,

$$\underline{H} = -\underline{\nabla}\phi_M.$$

Then from eqns. (5.10) and (5.12), we have

$$\nabla^2\phi_M = -\rho_M,$$

where

$$\rho_M = -\underline{\nabla} \cdot \underline{M}.$$

In the case where there are no boundaries, this equation has the solution

$$\phi_M = \frac{1}{4\pi}\int d^3x' \, \frac{\rho_M}{|\underline{x} - \underline{x}'|}$$

$$= -\frac{1}{4\pi}\int d^3x' \, \underline{\nabla}' \left(\frac{1}{|\underline{x}-\underline{x}'|}\right) \underline{M}(\underline{x}') \qquad (\textit{integration by parts})$$
$$= -\frac{1}{4\pi}\underline{\nabla}\cdot\int d^3x' \, \frac{1}{|\underline{x}-\underline{x}'|}\underline{M}(\underline{x}').$$

Note that if we are far away from a non-zero $\underline{M}$, i.e. $f \gg r'$, then we have

$$\phi_M \simeq -\frac{1}{4\pi}\underline{\nabla}\left(\frac{1}{r}\right)\cdot\int d^3x' \, \underline{M}(\underline{x}') = \frac{1}{4\pi r^3}\underline{m}\cdot\underline{x}.$$

where

$$\underline{m} = \int d^3x' \, \underline{M}(\underline{x}').$$

Suppose now that we had a hard ferrormagnet confined to a volume V, with surface S. Then there is a contribution arising from the discontinuity in $\underline{M}$ at the surface, which we can express as a *surface magnetisation density*,

$$\sigma_M = \underline{n}\cdot\underline{M},$$

and apply Gauss' Law to obtain its contribution

$$\phi_M = -\frac{1}{4\pi}\int_V d^3x' \frac{\underline{\nabla}'\cdot\underline{M}(\underline{x}')}{|\underline{x}-\underline{x}'|} + \frac{1}{4\pi}\oint_S dS \frac{\sigma_M}{|\underline{x}-\underline{x}'|}. \tag{5.13}$$

Note that for a *uniform* magnetisation, the bulk volume integral vanishes, and the only contribution arises from the surface term.

Solution using Vector Potential

We now write $\underline{B} = \underline{\nabla}\times\underline{A}$, so that we have

$$\underline{H} = \frac{1}{\mu_0}\underline{\nabla}\times\underline{A} - \underline{M}.$$

Thus eqn. (5.11) becomes

$$0 = \underline{\nabla}\times\underline{H} = \frac{1}{\mu_0}\underline{\nabla}\times(\underline{\nabla}\times\underline{A}) - \underline{\nabla}\times\underline{M}.$$

Introducing an effective magnetisation current

$$\underline{J}_{\mathrm{M}} = \underline{\nabla} \times \underline{M},$$

we have, in Coulomb gauge,

$$\nabla^2 \underline{A} = -\mu_0 \underline{J}_M.$$

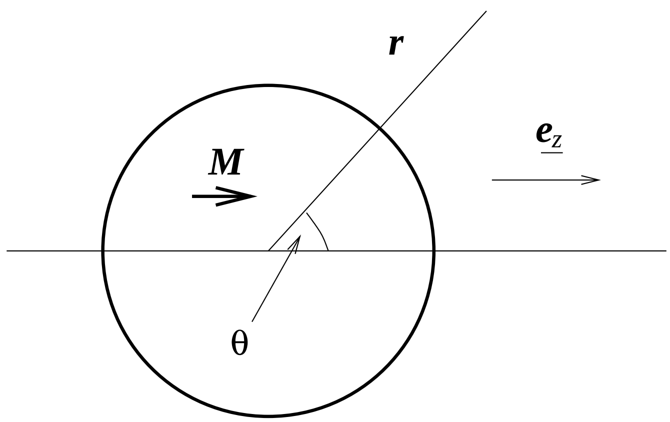
Thus again each component of $\underline{A}$ satisfies Poisson's equation, with solution

$$\underline{A} = \frac{\mu_0}{4\pi} \int d^3x' \, \frac{\underline{J}_M}{|\underline{x} - \underline{x}'|}.$$

In the case where there is a sharp boundary between two media, we again have a surface contribution which we treat as for the case of a scalar potential, yielding

$$\underline{A}(\underline{x}) = \frac{\mu_0}{4\pi} \int_V d^3x' \, \frac{\underline{\nabla}' \times \underline{M}}{|\underline{x} - \underline{x}'|} + \frac{\mu_0}{4\pi} \oint_S dS \frac{\underline{M}(\underline{x}') \times \underline{n}'}{|\underline{x} - \underline{x}'|}.$$

Example: uniformly magnetised sphere in a vacuum



Consider a sphere of radius a, with uniform magnetisation $\underline{M} = M_0 \underline{e}_z$. We will consider the solution using a scalar potential.

Since the magnetisation is constant throughout the body of the sphere, only the surface integral contributes in eqn. (5.13), and we have

$$\begin{aligned} \phi_M &= \frac{1}{4\pi} \oint_S dS' \, \frac{\underline{n}' \cdot \underline{M}(\underline{x}')}{|\underline{x} - \underline{x}'|} \\ &= \frac{M_0 a^2}{4\pi} \int d\Omega' \, \frac{\cos\theta'}{|\underline{x} - \underline{x}'|}. \end{aligned}$$

To proceed further, we expand $\frac{1}{|\underline{x}-\underline{x}'|}$ in terms of spherical harmonics

$$\frac{1}{|\underline{x}-\underline{x}'|} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi).$$

Noting that $\cos\theta' = P_1(\cos\theta') = \sqrt{4\pi}Y_{10}(\theta',\varphi')$, and using orthogonality, we can write

$$\phi_M(r,\theta) = \frac{1}{3}M_0 a^2 \frac{r_<}{r_>^2}\cos\theta,$$

where $r_{<\{>\}} = \min\{\max\}(r,a)$.
Inside the sphere, we have $r_< = r$ and $r_> = a$. Thus

$$\phi_M = \frac{1}{3}M_0 r\cos\theta = \frac{1}{3}M_0 z.$$

Thus we have

$$\left.\begin{array}{r} \underline{H}_{\text{in}} = -\underline{\nabla}\phi_M = -\frac{1}{3}\underline{M} \\ \underline{B}_{\text{in}} = \mu_0(\underline{H}+\underline{M}) = \frac{2}{3}\mu_0\underline{M} \end{array}\right\},$$

and we have that $\underline{H}$ ($\underline{B}$) is anti-parallel (parallel) to $\underline{M}$.
Outside the sphere,

$$\phi_M = \frac{1}{3}M_0\frac{a^3}{r^2}\cos\theta.$$

Since $\underline{M}$ is uniform inside the sphere, we can associate this with the potential due to a magnetic dipole of moment

$$\underline{m} = \frac{4\pi a^3}{3}\underline{M}.$$

Both the magnetic induction and the magnetic field are parallel to the magnetisation

$$\underline{B}_{\text{out}} = \mu_0\underline{H}_{\text{out}} = -\mu_0\underline{\nabla}\phi_M = \frac{2}{3}M_0\mu_0\frac{a^3}{r^3}(\cos\theta\underline{e_r} + \frac{1}{2}\sin\theta\underline{e_\theta})$$

Sphere in External Field

Suppose now we add a uniform magnetic induction $\underline{B_0} = \mu_0\underline{H_0}$. Then by the principle of linear superposition, the resulting field inside the sphere is just the sum of the two solutions

$$\underline{B}_{\text{in}} = \underline{B}_0 + \frac{2\mu_0}{3}\underline{M} \qquad (5.14)$$

$$\underline{H}_{\text{in}} = \frac{1}{\mu_0}\underline{B}_0 - \frac{1}{3}\underline{M} \qquad (5.15)$$

Suppose now that the field is *not* permanently magnetised, but rather has a linear relation between $\underline{B}$ and $\underline{H}$,

$$\underline{B}_{\text{in}} = \mu \underline{H}_{\text{in}}$$

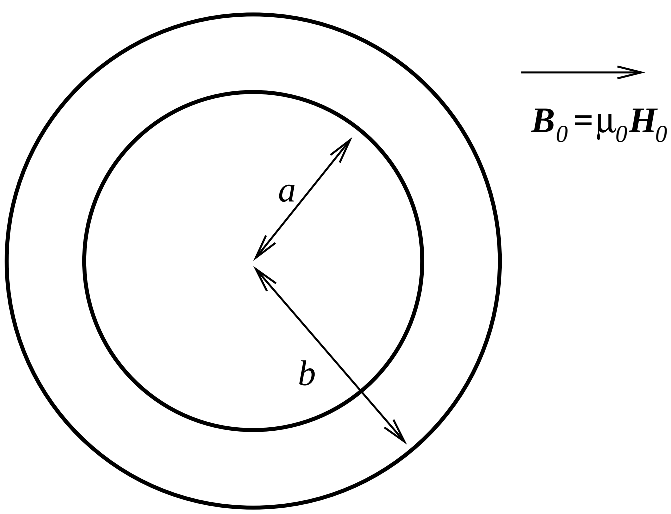
Then $\underline{M}$ is also linearly related, and from eqns. (5.14) and (5.15) we have

$$\underline{M} = \frac{3}{\mu_0}\left(\frac{\mu - \mu_0}{\mu + 2\mu_0}\right)\underline{B}_0.$$

For the case of ferromagnets described earlier, we do not have such a linear relation; indeed we have non-zero $\underline{M}$ for zero applied magnetic field. We can obtain one relation between $\underline{B}_{\text{in}}$ and $\underline{H}_{\text{in}}$ by eliminating $\underline{M}$ in eqns. (5.14) and (5.15), whilst obtaining another from the hysterisis curve.

Example: spherical shell in uniform field

Consider a shell of permeability μ in a vacuum, as shown below.



Since the current density is zero, we can once again write $\underline{H} = -\underline{\nabla}\phi_M$. Furthermore, $\underline{B} = \mu\underline{H}$, and thus $\underline{\nabla} \cdot \underline{H} = 0$ so that the scalar potential satisfies

$$\nabla^2\phi_M = 0,$$

subject to the boundary conditions at $r = a$ and $r = b$. We are now experts at writing down the solution in terms of Legendre polynomials.

$$\begin{aligned}
\phi_M &= -H_0 r\cos\theta + \sum_{l=0}^{\infty}\frac{\alpha_l}{r^{l+1}}P_l(\cos\theta) \qquad r > b \\
\phi_M &= \sum_{l=0}^{\infty}\left[\beta_l r^l + \frac{\gamma_l}{r^{l+1}}\right]P_l(\cos\theta) \qquad a < r < b \\
\phi_M &= \sum_{l=0}^{\infty}\delta_l r^l P_l(\cos\theta) \qquad r < a
\end{aligned}$$

where we have imposed that there be a uniform field at infinity for the case $r > b$, and that the solution is regular as $r \to 0$.
We now impose the boundary conditions at the interfaces $r = a$ and $r = b$

$$\begin{aligned}
&\underline{B^\perp} \text{ is continuous} \\
&\underline{H^\parallel} \text{ is continuous}
\end{aligned}$$

which become:

$$\begin{aligned}
\frac{\partial\phi_M}{\partial\theta}(b_+) &= \frac{\partial\phi_M}{\partial\theta}(b_-) \\
\mu_0\frac{\partial\phi_M}{\partial r}(b_+) &= \mu\frac{\partial\phi_M}{\partial r}(b_-) \\
\frac{\partial\phi_M}{\partial\theta}(a_+) &= \frac{\partial\phi_M}{\partial\theta}(a_-) \\
\mu_0\frac{\partial\phi_M}{\partial r}(a_-) &= \mu\frac{\partial\phi_M}{\partial r}(a_+)
\end{aligned}$$

We now use these equations to determine the coefficients $\alpha_l, \beta_l, \gamma_l$, noting that

$$\frac{\partial}{\partial\theta}P_l(\cos\theta) = P_l^1(\cos\theta).$$

All the coefficients vanish for $l > 1$ (*exercise*), and we have (see *Jackson*)

$$\begin{aligned}\alpha_1 &= \left[\frac{(2\mu'+1)(\mu'-1)}{(2\mu'+1)(\mu'+2)-\frac{2a^3}{b^3}(\mu'-1)^2}\right](b^3-a^3)H_0 \\ \delta_1 &= -\left[\frac{9\mu'}{(2\mu'+1)(\mu'+2)-\frac{2a^3}{b^3}(\mu'-1)^2}\right]H_0, \end{aligned} \tag{5.16}$$

where $\mu' = \mu/\mu_0$.

For $r > b$, we have the uniform field together with a dipole of moment α_1, parallel to H_1:

$$\phi_M = -H_0 r\cos\theta + \frac{\alpha_1}{r^2}\cos\theta.$$

For $r < a$, there is a uniform magnetic field parallel to H_0, of magnitude $-\delta_1$:

$$\phi_M = -(-\delta_1)r\cos\theta.$$

From eqn. (5.16), we see that $\delta_1 \simeq 1/\mu'$ as $\mu' \to \infty$: the effect of a shell of high permeability is to shield the interior from the magnetic field.

Chapter 6

Time-dependent Phenomena and Maxwell's Equations

So far we have studied static (time-independent) behaviour of electric and magnetic fields. The governing equations are

$$\begin{aligned} \underline{\nabla} \cdot \underline{D} &= \rho \\ \underline{\nabla} \times \underline{E} &= 0 \end{aligned} \tag{6.1}$$

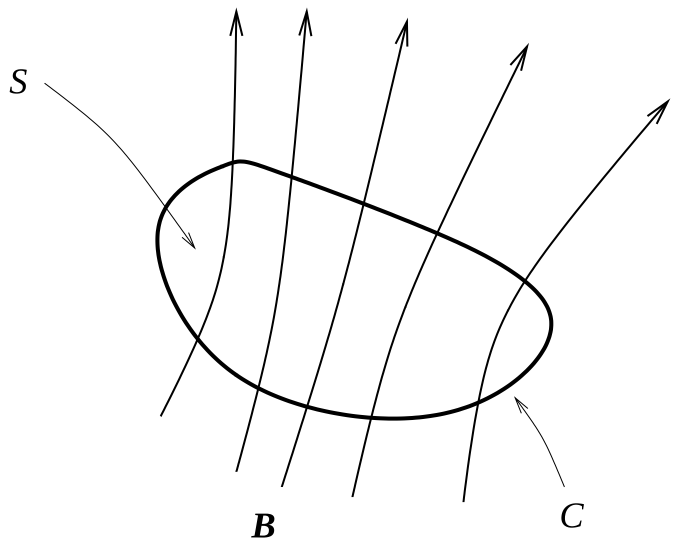
and

$$\begin{aligned} \underline{\nabla} \times \underline{H} &= \underline{J} \\ \underline{\nabla} \cdot \underline{B} &= 0. \end{aligned} \tag{6.2}$$

Electric and magnetic phenomena are completely separate, except for the fact that current density is associated with the motion of charges.

6.1 Faraday's Law of Magnetic Induction

Faraday (1831) observed that a current could be induced in a closed loop of wire by varying the **flux** of magnetic field through a surface spanning the loop.



We define the *flux* ϕ of the magnetic field through the loop by

$$\phi = \int_S \underline{B} \cdot \underline{dS},$$

where S is any surface spanning C.
N.B. Since $\underline{\nabla} \cdot \underline{B} = 0$, ϕ is independent of the precise surface.
The **electromotive force**, or voltage, across the curve C is

$$\mathcal{E} = \oint_C \underline{E} \cdot \underline{dl}.$$

Then Faraday's law, in integral form, may be written

$$\boxed{\mathcal{E} = -k\frac{d\phi}{dt},}$$

where, in SI units, $k = 1$. Note that the sign here is a consequence of Lenz's law: the induced current is in such a direction as to oppose the change of flux producing it. You could argue that the whole application of electricity in the modern world dates rests on Faraday's law; the observation that a changing magnetic field can produce an electric current.

We can can generalise this integral equation as applying to *any* closed curve in space, spanned by a surface,

$$\oint_C \underline{E} \cdot \underline{dl} = -\frac{d}{dt} \int_S \underline{B} \cdot \underline{dS}.$$

We now apply Stokes' theorem to the l.h.s.,

$$\oint \underline{E} \cdot \underline{dl} = \int_S (\underline{\nabla} \times \underline{E}) \cdot \underline{dS}.$$

Specialising to the case where both C and S are fixed in time, we have

$$\frac{d}{dt} \int_S \underline{B} \cdot \underline{dS} = \int_S \frac{\partial \underline{B}}{\partial t} \cdot \underline{dS},$$

and thus

$$\int_S (\underline{\nabla} \times \underline{E}) \cdot \underline{dS} = -\int_S \frac{\partial \underline{B}}{\partial t} \cdot \underline{dS},$$

yielding

$$\int_S (\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t}) \cdot \underline{dS} = 0.$$

Since both C, S are arbitrary, we obtain the differential form of Faradays law,

$$\boxed{\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = 0}$$

This equation replaces the second equation in eqn. (6.1).

The eqns. (6.1) and (6.2) reveal an immediate inconsistency when applied to time-dependent phenomena. Let us apply the divergence theorem to the first equation in eqn. (6.2),

$$\underline{\nabla} \cdot (\underline{\nabla} \times \underline{H}) = \underline{\nabla} \cdot \underline{J}.$$

The l.h.s. is identically zero, whilst the r.h.s. vanishes *only* for *time-independent* problems; in general, we have the *continuity equation*

$$\underline{\nabla} \cdot \underline{J} = -\frac{\partial \rho}{\partial t}.$$

To see how to resolve this inconsistency, let us return to Coulomb's law

$$\underline{\nabla} \cdot \underline{D} = \rho,$$

and substitute into the continuity equation, to obtain

$$\underline{\nabla} \cdot \underline{J} + \underline{\nabla} \cdot \frac{\partial \underline{D}}{\partial t} = 0.$$

We can make Ampere's law ($\underline{\nabla} \times \underline{H} = \underline{J}$) consistent with the continuity equation simply by modifying through the substitution

$$\underline{J} \rightarrow \underline{J} + \frac{\partial \underline{D}}{\partial t}$$

giving

$$\underline{\nabla} \times \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t}.$$

6.2 Maxwell's Equations

With this final modification of Ampere's law, and Faraday's law, we have the completed the construction of Maxwell's equations

$$\begin{aligned}
\underline{\nabla} \cdot \underline{D} &= \rho \quad \text{(ME1) } \textit{Coulomb's Law} \\
\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} &= 0 \quad \text{(ME2) } \textit{Faraday's Law} \\
\underline{\nabla} \times \underline{H} &= \underline{J} + \frac{\partial \underline{D}}{\partial t} \quad \text{(ME3) } \textit{Ampere's Law + Maxwell} \\
\underline{\nabla} \cdot \underline{B} &= 0 \quad \text{(ME4)}
\end{aligned}$$

The unification of electrical and magnetic phenomena through these equations represents the crowning achievement of classical, 19th. century physics. The addition of the electric displacement to the r.h.s. of Ampere's law was essential to showing that the solutions admit wave propagation at the speed of light.

6.3 Vector and Scalar Potentials

Maxwell's equations comprise a set of coupled, first-order PDE's. In particularly simple cses, they can be solved directly, but in the case of both electrostatics and magnetostatics we have seen the efficacy of introducing vector and scalar potentials. We will now do likewise for the time-dependent case.
We introduce potentials so that the two homogenous equations (Faraday's law and the solenoidal condition) are satisfied automatically. Since

$$\underline{\nabla} \cdot \underline{B} = 0$$

we have seen we can introduce a **vector potential** $\underline{A}$ such that

$$\underline{B} = \underline{\nabla} \times \underline{A}.$$

Substituting into Faraday's law (ME2), we obtain

$$\begin{aligned} \underline{\nabla} \times \underline{E} + \frac{\partial}{\partial t}\left[\underline{\nabla} \times \underline{A}\right] &= 0 \\ \Longrightarrow \underline{\nabla} \times \left[\underline{E} + \frac{\partial \underline{A}}{\partial t}\right] &= 0. \end{aligned}$$

We can now introduce a **scalar potential** ϕ such that

$$\underline{E} + \frac{\partial \underline{A}}{\partial t} = -\underline{\nabla}\phi.$$

Thus the electric and magnetic fields can be written

$$\underline{B} = \underline{\nabla} \times \underline{A} \qquad (6.3)$$

$$\underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t} \qquad (6.4)$$

and ME2 and ME4 are automatically satisfied.

The two remaining equations (ME1 and ME3) determine the dynamical behaviour, i.e. the dependence of $\underline{A}$ and ϕ on t and $\underline{x}$. To solve them, we need some constitutive relation between $(\underline{D}, \underline{H})$ and $(\underline{E}, \underline{B})$. We will initially restrict ourselves to the case of the vacuum, where we have

$$\begin{aligned} \underline{D} &= \epsilon_0 \underline{E} \\ \underline{H} &= \frac{1}{\mu_0} \underline{B}. \end{aligned}$$

Coulomb's law, ME1, is thus

$$\underline{\nabla} \cdot \underline{E} = \rho / \epsilon_0$$

whilst Ampère's law, ME3, is

$$\frac{1}{\mu_0} \underline{\nabla} \times \underline{B} = \underline{J} + \epsilon_0 \frac{\partial \underline{E}}{\partial t}.$$

Thus, in terms of the potential $(\phi, \underline{A})$, ME1 becomes

$$\boxed{\nabla^2 \phi + \frac{\partial}{\partial t} (\underline{\nabla} \cdot \underline{A}) = -\rho / \epsilon_0 \qquad (6.5)}$$

Substituting for the potential in ME3, we have

$$\begin{aligned} \frac{1}{\mu_0} \underline{\nabla} \times (\underline{\nabla} \times \underline{A}) &= \underline{J} + \epsilon_0 \left\{ -\underline{\nabla} \frac{\partial \phi}{\partial t} - \frac{\partial^2 \underline{A}}{\partial t^2} \right\} \\ \Longrightarrow \underline{\nabla} [\underline{\nabla} \cdot \underline{A}] - \nabla^2 \underline{A} &= \mu_0 \underline{J} + \mu_0 \epsilon_0 \left\{ -\underline{\nabla} \frac{\partial \phi}{\partial t} - \frac{\partial^2 \underline{A}}{\partial t^2} \right\}. \end{aligned}$$

We now write $\epsilon_0 \mu_0 = 1/c^2$ (we of course all no what c will be!), and write

$$\boxed{\nabla^2 \underline{A} - \frac{1}{c^2} \frac{\partial^2 \underline{A}}{\partial t^2} - \underline{\nabla} \left[\underline{\nabla} \cdot \underline{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} \right] = -\mu_0 \underline{J} \qquad (6.6)}$$

Thus we have derived two, coupled second-order PDE's that are, with the definitions of the potentials in eqns (6.3) and (6.4), equivalent to the original four Maxwell equations.

6.4 Gauge Transformations Revisited

Is it possible to decouple these two equations? One way to do this is through a clever choice of *gauge transformation*. A gauge transformation exploits the redundant degrees of freedom in the problem to simplify the problem.
Recall that the physical fields are not $(\underline{A}, \phi)$, but rather $(\underline{B}, \underline{E})$. A gauge transformation is a transformation of the $(\underline{A}, \phi)$ that leaves the physics unaltered. In this section, we will derive gauge transformations for the complete Maxwell equations. We have already encoutered gauge transformations in the context of magnetostatics; the substition

$$\underline{A} \longrightarrow \underline{A}' = \underline{A} + \underline{\nabla}\Lambda$$

leaves $\underline{B} = \underline{\nabla} \times \underline{A}$ invariant. In this case, however, $\underline{E}$ *also* depends on $\underline{A}$, and the above transformation will change $\underline{E}$ unless we make a suitable change $\phi \longrightarrow \phi'$. In terms of the transformed potentials $(\underline{A}', \phi')$, we have

$$\begin{aligned} \underline{E} &= -\underline{\nabla}\phi' - \frac{\partial \underline{A}'}{\partial t} \\ &= \underline{\nabla}\phi' - \frac{\partial}{\partial t}\left[\underline{A} + \underline{\nabla}\Lambda\right]. \end{aligned}$$

But we have

$$\underline{E} = \underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t},$$

and thus equating the two expressions gives

$$\begin{aligned} \underline{\nabla}\left[\phi' + \frac{\partial \Lambda}{\partial t} - \phi\right] &= 0 \\ \Longrightarrow \phi' &= \phi - \frac{\partial \Lambda}{\partial t} \end{aligned}$$

where we have noted that the potential is only defined up to an additive constant. Thus the gauge transformation of Maxwell's equations takes the form

$$\underline{A} \longrightarrow \underline{A}' = \underline{A} + \underline{\nabla}\Lambda \quad (6.7)$$
$$\phi \longrightarrow \phi' = \phi - \frac{\partial \Lambda}{\partial t} \quad (6.8)$$

We will now discuss some particular choice of gauges.

6.4.1 Lorentz Condition

Suppose we can find a gauge transformation such that

$$\underline{\nabla} \cdot \underline{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0. \quad (6.9)$$

This is known as the **Lorentz condition**, and the dynamical equations assume the form

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\rho/\epsilon_0 \quad (6.10)$$
$$\nabla^2 \underline{A} - \frac{1}{c^2}\frac{\partial^2 \underline{A}}{\partial t^2} = -\mu_0 \underline{J}. \quad (6.11)$$

The $\underline{A}$ and ϕ fields have become decoupled, and the simplified equations are just the **wave equations**, with a inhomogeneous source. But is it actually possible to *find* a gauge transformation that satisfies eqn. (6.9)?

Let $(\underline{A}, \phi)$ be potentials satisfying eqns. (6.6) and (6.5), and let Λ be a gauge transformation such that the transformed fields satisfy eqn. (6.9). Then we have

$$\underline{\nabla} \cdot \underline{A}' + \frac{1}{c^2}\frac{\partial \phi'}{\partial t} = 0$$
$$\Longrightarrow \underline{\nabla} \cdot \underline{A} + \nabla^2 \Lambda + \frac{1}{c^2}\left[\frac{\partial \phi}{\partial t} - \frac{\partial^2 \Lambda}{\partial t^2}\right] = 0$$

Thus we need to find Λ satisfying

$$\nabla^2 \Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} = -\left[\underline{\nabla} \cdot \underline{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}\right]. \quad (6.12)$$

Note that the Lorentz condition does not specify a gauge uniquely. Let $(\underline{A}, \phi)$ satisfy the Lorentz condition. Now consider the transformation

$$\begin{aligned} \underline{A} &\longrightarrow \underline{A}' = \underline{A} + \underline{\nabla}\Lambda \\ \phi &\longrightarrow \phi' = \phi - \frac{\partial \Lambda}{\partial t}. \end{aligned}$$

Then the Lorentz condition transforms as

$$\underline{\nabla} \cdot \underline{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} \longrightarrow \underline{\nabla} \cdot \underline{A}' + \frac{1}{c^2}\frac{\partial \phi'}{\partial t} = \nabla^2 \Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2}$$

Thus the new gauge also satisfies the Lorentz condition, providing

$$\nabla^2 \Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} = 0.$$

The Lorentz gauge is important because:

- The wave equation is manifest explicitly,
- $(\underline{A}, \phi)$ are treated on an equal footing and, when we discuss *Special Relativity*, we will see that the Lorentz condition is **Lorentz covariant**, i.e. independent of the choice of coordinate system.

6.4.2 Coulomb Gauge

We have introduced this gauge,

$$\underline{\nabla} \cdot \underline{A}$$

in the discussion of magnetostatics. It is not manifestly Lorentz covariant, but has the property that the scalar potential satisfies Poisson's equation (Coulomb's law!),

$$\nabla^2 \phi = -\rho/\epsilon_0,$$

with solution

$$\phi(\underline{x}, t) = \frac{1}{4\pi\epsilon_0} \int d^3x' \, \frac{\rho(\underline{x}', t)}{|\underline{x} - \underline{x}'|}. \tag{6.13}$$

The vector potential satisfies the **inhomogeneous wave equation**

$$\nabla^2 \underline{A} - \frac{1}{c^2}\frac{\partial^2 \underline{A}}{\partial t^2} = -\mu_0 \underline{J} + \frac{1}{c^2}\underline{\nabla}\frac{\partial \phi}{\partial t}. \tag{6.14}$$

Note that the scalar potential $\phi(\underline{x}, t)$ is the **instantaneous** Coulomb potential due to a charge density $\rho(\underline{x}, t)$, i.e. we do not take account of "causality" through the use of a retarded potential.

The equation for the vector potential contains a gradient operator, $\underline{\nabla}\partial\phi/\partial t$ arising from the solution of Poisson's equation for the scalar potential, and this term is *irrotational*,

$$\underline{\nabla} \times \left[\underline{\nabla}\frac{\partial \phi}{\partial t}\right] = 0.$$

It would be useful to completely decouple the equations governing the vector and scalar potentials, as in the case of the Lorentz gauge. To accomplish this, we will separate the current into an *irrotational*, or **longitudinal**, piece and a *solenoidal*, or **transverse**, piece,

$$\underline{J} = \underline{J_l} + \underline{J_t} \tag{6.15}$$

with

$$\begin{aligned} \underline{\nabla} \times \underline{J_l} &= 0 \\ \underline{\nabla} \cdot \underline{J_t} &= 0. \end{aligned}$$

We can always perform this separation, as will now be demonstrated. At first, we do it for the Fourier transforms

$$\underline{J}(\underline{k}) = \int d^3x e^{-i\underline{k}\cdot\underline{x}} \underline{J}(\underline{x}) \tag{6.16}$$

$$\underline{J}(\underline{k}) = \underline{J}^t(\underline{k}) + \underline{J}^l(\underline{k}), \qquad J_i^t(\underline{k}) = (\delta_{ij} - \frac{k_i k_j}{\underline{k}^2}) J_j(\underline{k}), \qquad J_i^l(\underline{k}) = \frac{k_i k_j}{\underline{k}^2} J_j(\underline{k})$$

Going back to the coordinate space we obtain

$$J_i^t(\underline{x}) = \int \frac{d^3k}{8\pi^3} e^{i\underline{k}\cdot\underline{x}} (\delta_{ij} - \frac{k_i k_j}{\underline{k}^2}) \int d^3y \underline{J}(\underline{y}) e^{-i\underline{k}\cdot\underline{y}}$$

$$= \int d^3y \underline{J}(\underline{y}) \int \frac{d^3k}{8\pi^3} e^{i\underline{k}\cdot(\underline{x}-\underline{y})} (\delta_{ij} - \frac{k_i k_j}{\underline{k}^2}) = (\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} - \delta_{ij}\nabla^2) \int d^3y \underline{J}(\underline{y}) \int \frac{d^3k}{8\pi^3} e^{i\underline{k}\cdot(\underline{x}-\underline{y})}$$

$$= (\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} - \delta_{ij}\nabla^2) \int d^3y \frac{\underline{J}(\underline{y})}{4\pi|\underline{x}-\underline{y}|}$$

and similarly

$$J_i^l(\underline{x}) = -\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} \int d^3y \frac{\underline{J}(\underline{y})}{4\pi|\underline{x}-\underline{y}|}$$

Using the formula

$$\nabla^2 \left(\frac{1}{|\underline{x}-\underline{y}|} \right) = -4\pi\delta(\underline{x}-\underline{y})$$

it is easy to check the self-consistency $J_i^t(\underline{x}) + J_i^l(\underline{x}) = J_i(\underline{x})$.
Thus we have performed the decomposition of eqn. (6.15) with

$$\underline{J_l} = -\frac{1}{4\pi}\underline{\nabla} \int d^3x' \, \frac{\underline{\nabla}' \cdot \underline{J}(\underline{x}')}{|\underline{x}-\underline{x}'|} \quad (6.17)$$

$$\underline{J_t} = \frac{1}{4\pi}\underline{\nabla} \times \left[\underline{\nabla} \times \int d^3x' \, \frac{\underline{J}(\underline{x}')}{|\underline{x}-\underline{x}'|} \right] \quad (6.18)$$

Now, from the continuity equation, we have

$$\underline{\nabla} \cdot \underline{J_l} + \frac{\partial \rho}{\partial t} = 0.$$

and substituting in eqn. (6.17) we obtain

$$\underline{J_l} = \frac{1}{4\pi}\underline{\nabla} \int d^3x' \, \frac{1}{|\underline{x}-\underline{x}'|}\frac{\partial \rho}{\partial t}.$$

We now identify the r.h.s. of this equation with our expression for the scalar potential of eqn. (6.13) and observe that

$$\underline{J_l} = \epsilon_0 \underline{\nabla} \frac{\partial \phi}{\partial t}$$

$$\Longrightarrow \mu_0 \underline{J_l} = \frac{1}{c^2}\underline{\nabla}\frac{\partial \phi}{\partial t},$$

where we have used $\mu_0\epsilon_0 = 1/c^2$. Thus, returning to the equation for the vector potential, eqn. (6.14), we find

$$\nabla^2\underline{A} - \frac{1}{c^2}\frac{\partial^2\underline{A}}{\partial t^2} = -\mu_0\underline{J_t}. \tag{6.19}$$

Only the **transverse** part of the current is a source for $\underline{A}$. Thus this gauge is also known as the **transverse** or **radiative** gauge, and once again we have decoupled the scalar and vector potentials.

6.5 Green Function for the Wave Equation

In both the *Lorentz* and *Coulomb* gauges, we have reduced the problem of finding the potentials to the solution of the **wave equation**

$$\underline{\nabla}^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -4\pi f(\underline{x}, t), \qquad (6.20)$$

where f is some *known* source, and c, as we have intimated earlier, is the velocity of wave propagation.

Such a hyperbolic equation, like the elliptic equations encountered in electrostatics, can be solved by means of *Green functions.* In particular, we will find the Green function $G(\underline{x}, t; \underline{x}', t')$ satsifying

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] G(\underline{x}, t; \underline{x}', t') = -4\pi\delta(\underline{x} - \underline{x}')\delta(t - t'). \qquad (6.21)$$

The solution to the inhomogeneous wave equation, eqn. (6.20), for a general source is then

$$\psi(\underline{x}, t) = \psi_0(\underline{x}, t) + \int d^3x'\, dt'\, G(\underline{x}, t; \underline{x}', t') f(\underline{x}', t')$$

where ψ_0 is a **solution** of the **homogeneous equation**. Note that this is essentially an *initial-value problem*, rather than the boundary-value problem encountered with elliptic equations.

To obtain the Green function, we take the Fourier transform with respect to t:

$$\begin{aligned} G(\underline{x}, t; \underline{x}', t') &= \frac{1}{2\pi}\int d\omega e^{-i\omega t} g(\underline{x}, \omega; \underline{x}', t') \\ g(\underline{x}, \omega; \underline{x}', t') &= \int dt\, e^{i\omega t}\, G(\underline{x}, t; \underline{x}', t') \end{aligned}$$

Then taking the F.T. of eqn. (6.21), we find

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right) g(\underline{x}, \omega; \underline{x}', t') = -4\pi\delta(\underline{x} - \underline{x}')e^{i\omega t'}.$$

We now introduce the *spatial* Fourier transform,

$$\tilde{g}(\underline{q}, \omega; \underline{x}', t') = \int d^3x e^{-i\underline{q}\cdot\underline{x}} g(\underline{x}, \omega; \underline{x}', t'),$$

yielding

$$\begin{aligned}(-q^2+k^2)\tilde{g}(\underline{q},\omega;\underline{x}',t') &= -4\pi e^{-i\underline{q}\cdot\underline{x}'}e^{i\omega t'} \\ \Longrightarrow \tilde{g}(\underline{q},\omega;\underline{x}',t') &= 4\pi\frac{e^{-i\underline{q}\cdot\underline{x}'}e^{i\omega t}}{q^2-k^2}\end{aligned}$$

where $k \equiv \omega/c$ is the **wave number**. We can invert this expression to obtain

$$g(\underline{x},\omega;\underline{x}',t') = \frac{4\pi}{(2\pi)^3}e^{i\omega t'}\int d^3q\,\frac{e^{i\underline{q}\cdot(\underline{x}-\underline{x}')}}{q^2-k^2}.$$

In order to exhibit the behaviour of this integral, we consider a oordinate system in which the z-axis is aligned with $\underline{x}-\underline{x}'$, and let θ be the angle between $\underline{q}$ and $\underline{x}-\underline{x}'$. Thus

$$\begin{aligned}g(\underline{x},\omega;\underline{x}',t') &= \frac{4\pi}{(2\pi)^3}e^{i\omega t'}\int_0^\infty dq\,q^2\int_0^{2\pi}d\psi\int_{-1}^{1}d(\cos\theta)\frac{e^{iq|\underline{x}-\underline{x}'|\cos\theta}}{q^2-k^2} \\ &= \frac{4\pi}{(2\pi)^2}e^{i\omega t'}\int_0^\infty dq\,\frac{q^2}{q^2-k^2}\left\{\frac{e^{iq|\underline{x}-\underline{x}'|}}{iq|\underline{x}-\underline{x}'|}-\frac{e^{-iq|\underline{x}-\underline{x}'|}}{iq|\underline{x}-\underline{x}'|}\right\} \\ &= \frac{4\pi}{(2\pi)^2}\frac{e^{i\omega t'}}{i|\underline{x}-\underline{x}'|}\int_{-\infty}^{\infty}\frac{dq\,q}{q^2-k^2}e^{iq|\underline{x}-\underline{x}'|}\end{aligned}$$

The integrand has poles at $q=\pm k$, and therefore we have to specify how to treat the poles in order to evaluate the integrals. We will do this by displacing the poles off the real axis as follows:

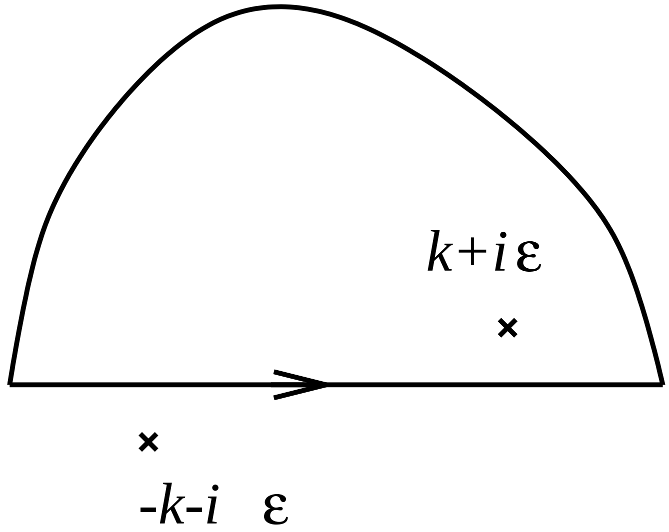
$$g^{(\pm)}(\underline{x},\omega;\underline{x}',t') = \frac{4\pi}{(2\pi)^2}\frac{e^{i\omega t}}{i|\underline{x}-\underline{x}'|}\int_{-\infty}^{\infty}\frac{dq\,q}{q^2-k^2\mp i\eta}e^{iq|\underline{x}-\underline{x}'|},$$

where η is small. We now write

$$q^2-k^2\mp i\eta = (q-k\mp i\epsilon)(q+k\pm i\epsilon),$$

with $\epsilon = \eta/2k$ $(\eta, k>0)$.

We first consider the case of $g^{(+)}$, which has a pole in the *upper half plane* at $q=k+i\epsilon$, and in the *lower half plane* at $q=-k-i\epsilon$.



We can complete the contour in the upper-half plane, where the contribution from the semi-circle at infinity vanishes, and obtain

$$g^{(+)}(\underline{x}, \omega; \underline{x}', t') = \frac{1}{|\underline{x} - \underline{x}'|} e^{i\omega t' + ik|\underline{x} - \underline{x}'|}.$$

Similarly, in the case of $g^{(-)}$, we have a pole in the upper half plane at $q = -k + i\epsilon$, and performing the contour integration we obtain,

$$g^{(\pm)}(\underline{x}, \omega; \underline{x}', t') = \frac{1}{|\underline{x} - \underline{x}'|} e^{i\omega t' \pm ik|\underline{x} - \underline{x}'|}.$$

We now invert the temporal Fourier Transform

$$G^{(\pm)}(\underline{x}, t; \underline{x}', t') = \frac{1}{2\pi} \int d\omega \, e^{-i\omega t} \frac{1}{|\underline{x} - \underline{x}'|} e^{i\omega t' \pm k|\underline{x} - \underline{x}'|}.$$

The ω integration is straightforward, and we find

$$G^{(\pm)}(\underline{x}, t; \underline{x}', t') = \frac{1}{|\underline{x} - \underline{x}'|} \delta \left[(t' - t) \pm \frac{1}{c} |\underline{x} - \underline{x}'| \right] \qquad (6.22)$$

The Green function $G^{(+)}$ is known as the **retarded Green function**, because a change at time t arises from an effect at an earlier time

$$t' = t - \frac{1}{c} |\underline{x} - \underline{x}'|.$$

It manifestly exhibits causality. $G^{(-)}$ is known as the **advanced Green function**. We now construct the complete solutions as follows:

1. *Retarded Solution.* We imagine that, as $t \longrightarrow -\infty$, we have a wave $\psi_{\rm in}(\underline{x}, t)$ satisfying the *homogeneous* equation. The source $f(\underline{x}, t)$ then turns on, and the complete solution is

$$\psi(\underline{x}, t) = \psi_{\rm in}(\underline{x}, t) + \int d^3x'\, dt\, G^{(+)}(\underline{x}, t; \underline{x}', t') f(\underline{x}', t').$$

The use of the *retarded* Green function ensures that the observer only feels the effect of the source **after** it is turned on.

2. *Advanced Solution* Here we measure a wave $\psi_{\rm out}(\underline{x}, t)$ as $t \longrightarrow \infty$,

$$\psi(\underline{x}, t) = \psi_{\rm out}(\underline{x}, t) + \int d^3x'\, dt' G^{(-)}(\underline{x}, t; \underline{x}', t') f(\underline{x}', t').$$

The use of $G^{(-)}$ means that, once the source ceases, the effects from the source are no longer felt, or more precisely they are contained within $\psi_{\rm out}$.

Case 1 above is the more commonly encountered, for example in the case $\psi_{\rm in} \equiv 0$ so that there is no wave in the distant past, and a source $f(\underline{x}, t)$ switches on at some time. Then inserting our explicit expression for the Green function, we obtain

$$\psi(\underline{x}, t) = \int d^3x' \frac{f(\underline{x}', t'_{\rm ret})}{|\underline{x} - \underline{x}'|}$$

where the subscript *ret* denotes that the function f is evaluate at time

$$t'_{\rm ret} = t - \frac{1}{c}|\underline{x} - \underline{x}'|.$$

6.6 Conservation of Energy and Momentum and Poynting Vector

In this section, we will derive laws expressing conservation of energy and momentum for electric and magnetic fields.

The force acting on a particle carrying charge q, and moving with velocity v is

$$\underline{F} = q(\underline{E} + \underline{v} \times \underline{B}).$$

The work done/unit time, or rate of change of *mechanical* energy, is then

$$\begin{aligned} \frac{d}{dt} E_{\rm mech} &= \underline{v} \cdot [q(\underline{E} + \underline{v} \times \underline{B})] \\ &= q\underline{v} \cdot \underline{E}, \end{aligned}$$

since the second term vanishes. Thus generalising to a **current density** $\underline{J}$ we have

$$\frac{d}{dt} E_{\rm mech} = \int d^3x \, \underline{J} \cdot \underline{E}. \tag{6.23}$$

We will now relate the rate of change of mechanical energy to the change of energy in the electric and magnetic fields. The starting point is Maxwell-Ampère's law (ME3), which gives

$$\int_V d^3x \, \underline{J} \cdot \underline{E} = \int d^3x \, \underline{E} \cdot \left[\underline{\nabla} \times \underline{H} - \frac{\partial \underline{D}}{\partial t} \right].$$

We can use the vector identity $\underline{\nabla} \cdot (\underline{E} \times \underline{H}) = \underline{H} \cdot \underline{\nabla} \times \underline{E} - \underline{E} \cdot (\underline{\nabla} \times \underline{H})$ to write

$$\int_V d^3x \, \underline{J} \cdot \underline{E} = \int d^3x \left\{ \underline{H} \cdot (\underline{\nabla} \times \underline{E}) - \underline{\nabla} \cdot [\underline{E} \times \underline{H}] - \underline{E} \cdot \frac{\partial \underline{D}}{\partial t} \right\}.$$

Identifying the l.h.s. of this equation with the rate of change of mechanical energy in eqn. (6.23), and using Faraday's law (ME2) on the r.h.s., we obtain

$$\frac{d}{dt} E_{\rm mech} = - \int d^3x \left\{ \underline{H} \cdot \frac{\partial \underline{B}}{\partial t} + \underline{\nabla} \cdot (\underline{E} \times \underline{H}) + \underline{E} \cdot \frac{\partial \underline{D}}{\partial t} \right\}.$$

We will now assume that the medium is **linear**, allowing us to write

$$\begin{aligned} \underline{H} \cdot \frac{\partial \underline{B}}{\partial t} &= \frac{1}{2} \frac{\partial}{\partial t} [\underline{H} \cdot \underline{B}] \\ \underline{E} \cdot \frac{\partial \underline{D}}{\partial t} &= \frac{1}{2} \frac{\partial}{\partial t} (\underline{E} \cdot \underline{D}), \end{aligned}$$

and thus

$$\frac{d}{dt} E_{\rm mech} = - \int d^3x \left\{ \underline{\nabla} \cdot (\underline{E} \times \underline{H}) + \frac{\partial}{\partial t} \left[\frac{1}{2} (\underline{H} \cdot \underline{B} + \underline{E} \cdot \underline{D}) \right] \right\}$$

We have already, in Chapter 4.6, interpreted $\frac{1}{2}\epsilon_0|\underline{E}|^2 \equiv \frac{1}{2}\underline{E}\cdot\underline{D}$ as the energy density of an electric field. Likewise we will identify $\frac{1}{2}\underline{H}\cdot\underline{B}$ as the magnetic energy density and hence their sum

$$u = \frac{1}{2}(\underline{H}\cdot\underline{B} + \underline{E}\cdot\underline{D}) \tag{6.24}$$

as the **electromagnetic energy density**. With this identification, we now have **Poynting's Theorem** expressing conservation of energy

$$-\int_V d^3x\, \underline{J}\cdot\underline{E} = \int_V d^3x \left[\frac{\partial u}{\partial t} + \underline{\nabla}\cdot(\underline{E}\times\underline{H})\right] \tag{6.25}$$

Since this applies for any volume V, we have a differential energy continuity equation

$$\frac{\partial u}{\partial t} + \underline{\nabla}\cdot(\underline{E}\times\underline{H}) = \underline{J}\cdot\underline{E} \tag{6.26}$$

The vector

$$\underline{S} = \underline{E}\times\underline{H}$$

is the **Poynting Vector**. It only enters through a divergence in the above expressions but, when we come to consider its properties under Lorentz transformations later in the course, we will discover that it is essentially unique.
We can reduce the integral over the Poynting vector in eqn. (6.25) to a surface integral using the divergence theory. Thus we can interpret the Poynting vector as the energy flux across a surface, and the Poynting theorem in essence says: *"The rate of change of electromagnetic energy in a volume together with energy flux across the boundary is equal to minus the total work done by sources within the volume"*.

6.6.1 Energy Conservation in terms of the Fundamental Microscopic Fields

The field energy density of eqn. (6.24) contains not only the fundamental fields, but also the *"derived"* fields $\underline{H}$ and $\underline{D}$. Thus they include contributions associated with the *polarization* and *magnetisation* of the medium which are in essence *mechanical*, and should be associated with the $\underline{J} \cdot \underline{E}$ term.

Let $E_{\rm mech}$ be the **mechanical energy** in some fixed volume V. We have seen that the work done per unit time per unit volume $\underline{J} \cdot \underline{E}$ is the rate of increase of mechanical energy,

$$\frac{dE_{\rm mech}}{dt} = \int_V d^3x\, \underline{J} \cdot \underline{E}.$$

In the case of a vacuum, we have

$$\begin{aligned} \int_V d^3x\, u &= \frac{1}{2}\int d^3x(\underline{H} \cdot \underline{B} + \underline{E} \cdot \underline{D}) \\ &= \frac{\epsilon_0}{2}\int_V d^3x(\underline{E}^2 + c^2\underline{B}^2) \\ &= E_{\rm field} \end{aligned}$$

where now we have expressed the field energy solely in terms of the fundamental fields. It is this expression that is more naturally associated with the field energy, and Poynting's theorem reads

$$\frac{d}{dt}(E_{\rm mech} + E_{\rm field}) = -\oint \underline{dA} \cdot \underline{S} \qquad (6.27)$$

6.6.2 Conservation of Linear Momentum

Again we work with the **microscopic fields**. The force on a particle of charge q is

$$\underline{F} = q(\underline{E} + \underline{v} \times \underline{B}).$$

Thus Newton's second law may be expressed as

$$\frac{d}{dt}\underline{P}_{\mathrm{mech}} = \int d^3x \, [\rho \underline{E} + \underline{J} \times \underline{B}]$$

where $\underline{P}_{\mathrm{mech}}$ is the total momentum of the particles in a volume V. To evaluate this expression, we once again use Coulomb's law (ME1) and Ampère's law (ME3), yielding for the integrand

$$\rho \underline{E} + \underline{J} \times \underline{B} = \epsilon_0 \underline{E}(\underline{\nabla} \cdot \underline{E}) - \underline{B} \times \left[\frac{1}{\mu_0} \underline{\nabla} \times \underline{B} - \epsilon_0 \frac{\partial \underline{E}}{\partial t} \right].$$

We now use

$$\begin{aligned} \frac{\partial}{\partial t} \underline{E} \times \underline{B} &= \frac{\partial \underline{E}}{\partial t} \times \underline{B} + \underline{E} \times \frac{\partial \underline{B}}{\partial t} \\ \underline{\nabla} \cdot \underline{B} &= 0 \end{aligned}$$

to write

$$\begin{aligned} &\rho \underline{E} + \underline{J} \times \underline{B} = \\ &\quad \epsilon_0 [\underline{E}(\underline{\nabla} \cdot \underline{E}) + c^2 \underline{B}(\underline{\nabla} \cdot \underline{B}) - c^2 \underline{B} \times (\underline{\nabla} \times \underline{B}) + \underline{E} \times \frac{\partial \underline{B}}{\partial t} - \frac{\partial}{\partial t}(\underline{E} \times \underline{B})]. \end{aligned}$$

We now use Faraday's law (ME2) to write

$$\begin{aligned} &\frac{d}{dt}\underline{P}_{\mathrm{mech}} + \frac{d}{dt}\epsilon_0 \int_V d^3x \, \underline{E} \times \underline{B} = \\ &\quad \epsilon_0 \int d^3x \, [\underline{E \nabla} \cdot \underline{E} + c^2 \underline{B \nabla} \cdot \underline{B} - \underline{E} \times (\underline{\nabla} \times \underline{E}) - c^2 \underline{B} \times (\underline{\nabla} \times \underline{B})], \quad (6.28) \end{aligned}$$

where we assume that the volume V is fixed. The second term on the l.h.s. we associate with the momentum carried by the field

$$\underline{P}_{\mathrm{field}} = \epsilon_0 \int d^3x \, \underline{E} \times \underline{B}, \qquad (6.29)$$

which we can rewrite as

$$\underline{P}_{\text{field}} \;=\; ` \int d^3x\, \frac{1}{c^2} \underline{E} \times \underline{H} \;=\; \int d^3x \; g, \tag{6.30}$$

where $\underline{g}$ is the **electromagnetic momentum density** given, up to a constant, by the *Poynting Vector*,

$$\underline{g} = \frac{1}{c^2} \underline{S}. \tag{6.31}$$

To proceed further, let us consider the r.h.s. of the momentum conservation law, eqn. (6.28). Using index notation, we may write

$$\begin{aligned}
[\underline{E}(\underline{\nabla} \cdot \underline{E}) - \underline{E} \times (\underline{\nabla} \times \underline{E})]_i \;&=\; E_i \frac{\partial E_j}{\partial x_j} - \epsilon_{ijk}\epsilon_{klm} E_j \frac{\partial E_m}{\partial x_l} \\
&=\; E_i \frac{\partial E_j}{\partial x_j} - E_j \frac{\partial E_j}{\partial x_i} + E_j \frac{\partial E_i}{\partial x_j} \\
&=\; \frac{\partial}{\partial x_j} [E_i E_j - \frac{1}{2} E^2 \delta_{ij}].
\end{aligned}$$

What we have done is to write the electric part of the integrand as a derivative. We may treat the magnetic term similarly, and now introduce the **Maxwell Stress Tensor**

$$\boxed{T_{ij} = \epsilon_0 \left[E_i E_j + c^2 B_i B_j - \frac{1}{2}(E^2 + c^2 B^2)\delta_{ij} \right]} \tag{6.32}$$

Note that this tensor is **symmetric**.

We can thus write the momentum conservation law as

$$\frac{d}{dt} \left[\underline{P}_{\text{mech}} + \underline{P}_{\text{field}} \right] = \int_V d^3x \, \frac{\partial T_{ij}}{\partial x_j}$$

which, after applying the divergenceb theorem, becomes

$$\boxed{\frac{d}{dt} [\underline{P}_{\text{mech}} + \underline{P}_{\text{field}}] = \oint_{S=\partial V} dA \, T_{ij} n_j} \tag{6.33}$$

where $\underline{n}$ is the outward normal to the surface enclosing V.

Note that $T_{ij} n_j$ is the flow of momentum per unit area across surface S into the volume V, i.e. it is the force per unit area acting on the combined system of particles and fields within volume V.

Chapter 7

Plane Electromagnetic Waves and Wave Propagation

We begin by the considering the propagation of waves in a non-conducting medium. Thus $\underline{J} \equiv 0$, we assume $\rho \equiv 0$ and Maxwell's equations reduce to

$$\begin{aligned} \underline{\nabla} \cdot \underline{B} &= 0 \\ \underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} &= 0 \\ \underline{\nabla} \cdot \underline{D} &= 0 \\ \underline{\nabla} \times \underline{H} - \frac{\partial \underline{D}}{\partial t} &= 0. \end{aligned}$$

In the case of plane waves, it is sufficient to consider those propagating with a definite frequency ω, and hence time dependence $\exp -i\omega t$; essentially this is equivalent to taking the Fourier Transform. We have a set of linear, homogeneous equations and hence *all* fields have the same harmonic behaviour. Thus we may write Maxwell's equations as

$$\begin{aligned} \underline{\nabla} \cdot \underline{B} &= 0 \\ \underline{\nabla} \cdot \underline{D} &= 0 \\ \underline{\nabla} \times \underline{E} - i\omega \underline{B} &= 0 \\ \underline{\nabla} \times \underline{H} + i\omega \underline{D} &= 0. \end{aligned}$$

We will now specialise to the case of a linear constitutive relation between the fields: $\underline{D} = \epsilon\underline{E}$ and $\underline{B} = \mu\underline{H}$. We will also assume ϵ, μ are **real**. Note that later we will consider the complex case; taking them to be real corresponds to there being no energy losses. Then the last two equations of eqn. (7.1) become

$$\begin{aligned} \underline{\nabla} \times \underline{E} - i\omega\underline{B} &= 0 \\ \underline{\nabla} \times \underline{B} + i\omega\epsilon\mu\underline{E} &= 0, \end{aligned}$$

which, with the remaing two Maxwell equations, yield

$$\begin{aligned} \nabla^2\underline{E} + \omega^2\epsilon\mu\underline{E} &= 0 \\ \nabla^2\underline{B} + \omega^2\epsilon\mu\underline{B} &= 0 \end{aligned} \qquad (7.1)$$

These are known as the **Helmholtz wave equations**. As is well known, they support the plane-wave solutions

$$\begin{pmatrix} \underline{E} \\ \underline{B} \end{pmatrix} = \begin{pmatrix} \underline{E_0} \\ \underline{B_0} \end{pmatrix} e^{i\underline{k}\cdot\underline{x}-i\omega t}, \qquad (7.2)$$

where $k = \omega\sqrt{\mu\epsilon}$, and

$$v = \omega/k = 1/\sqrt{\mu\epsilon}$$

is the **phase velocity**.

We now recall the velocity of light in a vacuum is given by

$$c = 1/\sqrt{\mu_0\epsilon_0}.$$

Thus we can write

$$v = c/n$$

where

$$n = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} \qquad (7.3)$$

is the **index of refraction**. It is usually a function of the frequency, e.g. a prism, and therefore the *phase velocity* is likewise *frequency dependent* - hence the name.

7.1 Propagation of Monochromatic Plane Wave

We will now consider in greater detail a monochromatic plane wave of frequency ω, propagating in the direction $\underline{n}$ with wave number k. Note that complex $\underline{n}$ corresponds to dissipation. We have seen that the solution of the Helmholtz equations are

$$\begin{aligned} \underline{E}(\underline{x},t) &= \underline{E}_0 e^{ik\underline{n}\cdot\underline{x}-i\omega t} \\ \underline{B}(\underline{x},t) &= \underline{B}_0 e^{ik\underline{n}\cdot\underline{x}-i\omega t} \end{aligned} \tag{7.4}$$

with

$$k^2 = \mu\epsilon\omega^2.$$

Thisis actually shorthand for

$$\underline{E}(\underline{x},t) = \Re\left\{\underline{E}_0 e^{ik\underline{n}\cdot\underline{x}-i\omega t}\right\}.$$

The imaginary part contains no physical information. It is important to remember this when considering quantities that are quadratic or higher in the fields, such as the energy density.

7.1.1 Energy Density for Monochromatic Plane Wave

Recall the expression for the energy density

$$u = \frac{1}{2}\left[\epsilon\underline{E}^2 + \frac{1}{\mu}\underline{B}^2\right].$$

The *real* parts of the fields $\underline{B}$ and $\underline{E}$ must be taken **before** evaluating the quadratic terms. In the case of the *time-averaged* energy density, we have the particularly simple result

$$\langle u\rangle = \frac{1}{4}\left[\epsilon\underline{E}\cdot\underline{E}^* + \frac{1}{\mu}\underline{B}\cdot\underline{B}^*\right] \tag{7.5}$$

where we use $\langle\ldots\rangle$ to denote that the time average has been taken, and the additional factor of one half arises from the observation

$$\langle\cos^2\omega t\rangle = 1/2.$$

Likewise, the time-averaged Poynting vector is

$$\langle \underline{S} \rangle = \frac{1}{2} \underline{E} \times \underline{H}^* = \frac{\epsilon v}{2} \underline{E} \cdot \underline{E}^* \underline{n}. \tag{7.6}$$

This quantity is called the intensity of the wave.

7.2 Polarisation of a Monochromatic Plane Wave

Applying $\underline{\nabla} \cdot \underline{B} = 0$ and $\underline{\nabla} \cdot \underline{E} = 0$ to the solutions of eqn. (7.4), we find

$$\begin{aligned} \underline{n} \cdot \underline{B}_0 &= 0 \\ \underline{n} \cdot \underline{E}_0 &= 0. \end{aligned} \tag{7.7}$$

Thus both $\underline{E}$ and $\underline{B}$ are perpendicular to the direction of propagation. We say they are **transverse wave**.

We now apply the remaining Maxwell equations

$$\begin{aligned} \underline{\nabla} \times \underline{E} - i\omega \underline{B} &= 0 \\ \underline{\nabla} \times \underline{B} + i\omega\mu\epsilon \underline{E} &= 0, \end{aligned}$$

to yield

$$\underline{B}_0 = \sqrt{\mu\epsilon}\, \underline{n} \times \underline{E}_0. \tag{7.8}$$

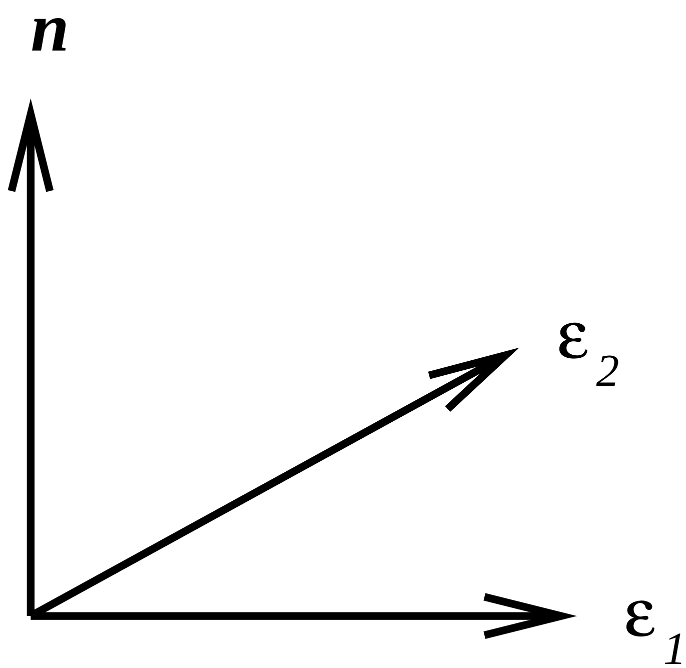
Setting $c = 1/\sqrt{\mu\epsilon}$ to be the volocity of light *in the medium*, we see that both $c\underline{B}$ and $\underline{E}$ have the same magnitude.

N.B. Had we chosen to work with $\underline{H}$, rather than $\underline{B}$, then we would have

$$\underline{H}_0 = \frac{\underline{n} \times \underline{E}_0}{Z}$$

where $Z = \sqrt{\mu/\epsilon}$ is the **impedence**

We will now specialise to the case where $\underline{n}$ is indeed **real**. Then $\underline{B}_0$ is perpendicular to $\underline{E}_0$, and has the same phase.



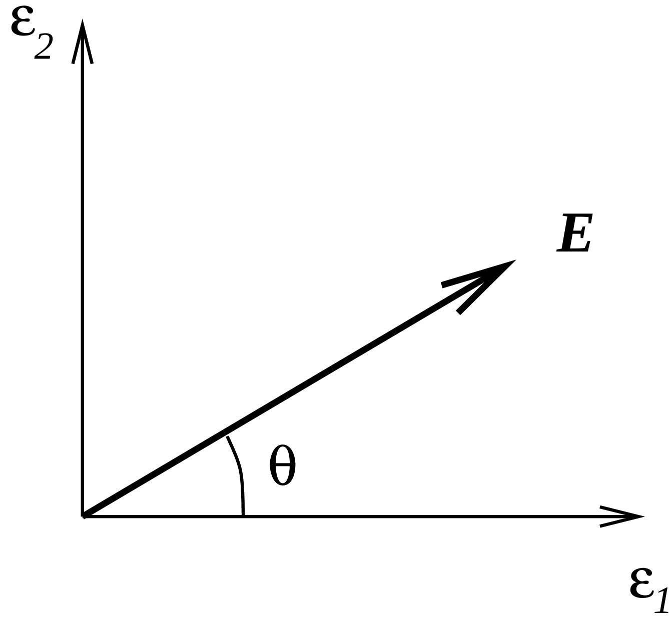
The vectors $\underline{E}$, $\underline{B}$ and $\underline{n}$ form an orthogonal triad, and it is usual to introduce three mutually-orthogonal basis vectors $\underline{\epsilon_1}, \underline{\epsilon_2}$ and $\underline{n}$ and to write the electromagnetic field as

$$\underline{E_1}(\underline{x}, t) = \underline{\epsilon_1} E_1 e^{i(\underline{k}\cdot\underline{x} - \omega t)} \quad ; \quad c\underline{B_1} = \underline{\epsilon_2} E_1 e^{i(\underline{k}\cdot\underline{x} - \omega t)}$$
$$\underline{E_2}(\underline{x}, t) = \underline{\epsilon_2} E_2 e^{i(\underline{k}\cdot\underline{x} - \omega t)} \quad ; \quad c\underline{B_2} = -\underline{\epsilon_1} E_2 e^{i(\underline{k}\cdot\underline{x} - \omega t)} \tag{7.9}$$

Note that E_1 and E_2 can be complex to account for a phase shift between the two plane waves.
The general solution for the wave equation is

$$\underline{E}(\underline{x}, t) = (\underline{\epsilon_1} E_1 + \underline{\epsilon_2} E_2) e^{i(\underline{k}\cdot\underline{x} - \omega t)}.$$

Linear Polarization



If E_1 and E_2 have the **same phase** we talk about a **linearly polarized wave**; the direction of the $\underline{E}$ field is constant, with the angle given by

$$\theta = \tan^{-1} E_2/E_1.$$

Elliptical and Circular Polarization

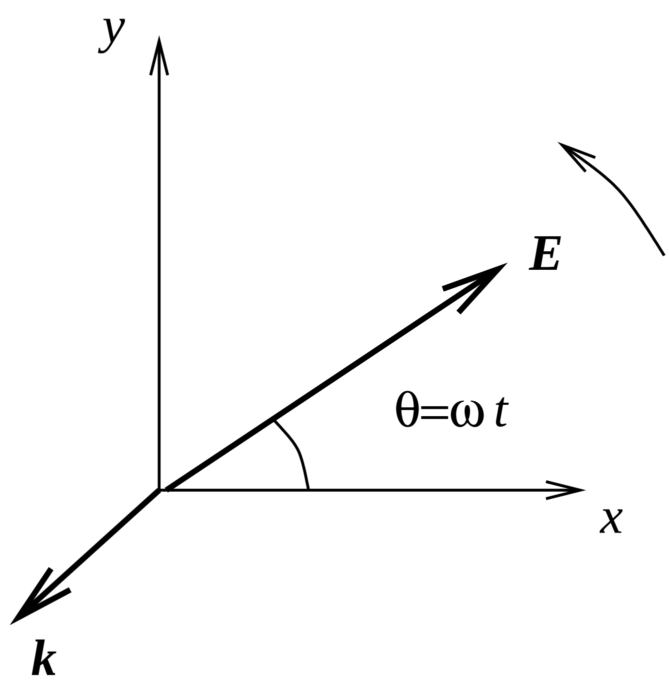
If E_1 and E_2 have **different phases**, we say the wave is **elliptically polarized**. The direction of $\underline{E}$ is no longer constant.
A special case is that of **circularly polarized waves**. Here E_1 and E_2 have the same magnitude, but differ by a phase of $\pm\pi/2$. Thus we can write

$$\underline{E}(\underline{x}, t) = E_0(\underline{\epsilon_1} \pm i\underline{\epsilon_2})e^{i(\underline{k}\cdot\underline{x} - \omega t)}$$

where E_0 is real. W.l.o.g., we tak $\underline{\epsilon_1}$ and $\underline{\epsilon_2}$ in the x and y directions respectively. Thus taking the real (physical) part, we find

$$\begin{aligned} E_x &= E_0 \cos(kz - \omega t) = E_0 \cos(\omega t - kz) \\ E_y &= \mp E_0 \sin(kz - \omega t) = \pm E_0 \sin(\omega t - kz). \end{aligned}$$

At fixed z, this is just the equation of a circle.



The different signs correspond to rotating to the *left* or rotating to the *right*; these are more commonly known as **positive** and **negative** helicities.
Since it is possible to use *any* two mutually orthogonal vectors as polarization vectors, it is usually for circularly polarized waves to introduce

$$\underline{\epsilon}^{\pm} = \frac{1}{\sqrt{2}}(\underline{\epsilon_1} \pm i\underline{\epsilon_2}) \tag{7.10}$$

with the properties

$$\underline{\epsilon}^{\pm *} \cdot \underline{\epsilon}^{\pm} = 1, \; \underline{\epsilon}^{\pm *} \cdot \underline{\epsilon}^{\mp} = 0, \; \underline{\epsilon}^{\pm *} \cdot \underline{n} = 0,$$

so that a general plane-wave solution is

$$\underline{E}(\underline{x}, t) = (E_+\underline{\epsilon}^+ + E_-\underline{\epsilon}^-)e^{i(\underline{k}\cdot\underline{x} - \omega t)}.$$

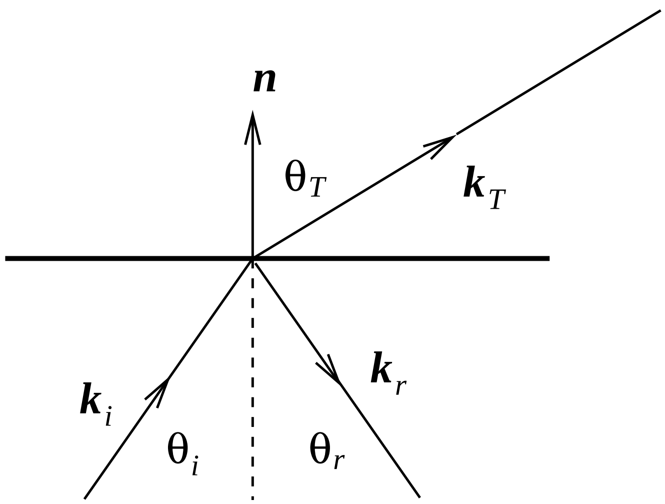
An important question is, given an electric field $\underline{E}(\underline{x}, t)$, how can one determine its polarization properties; one way of specifying the relative importance of the different components is through the **Stokes Parameters**. This is described in *Jackson 7.2.*

7.3 Reflection and Refraction at Plane Interface between Dielectrics

The laws describing the behaviour of a wave at the interface between two media are well known:

1. Angle of reflection = Angle of incidence
2. $\sin\theta_i/\sin\theta_t = n'/n$ (Snells's law) where n', n are the refractive indices of the final and initial media respectively.

These are simple kinematic laws; we would like to determine dynamic properties - intensities and phase changes.



We begin by writing

$$\text{Incident wave:} \quad \underline{E_i} = \underline{E_0^i}\, e^{i(\underline{k_i}\cdot\underline{x}-\omega t)}$$
$$\underline{B_i} = \sqrt{\mu\epsilon}\frac{1}{k_i}\underline{k_i}\times\underline{E_i}$$

$$\text{Reflected wave:} \quad \underline{E_r} = \underline{E_0^r}\, e^{i(\underline{k_r}\cdot\underline{x}-\omega t)}$$
$$\underline{B_r} = \sqrt{\mu\epsilon}\frac{1}{k_r}\underline{k_r}\times\underline{E_r}$$

$$\text{Refracted wave:} \qquad \underline{E_T} = \underline{E_0^T}\, e^{i(\underline{k_T}\cdot\underline{x}-\omega t)}$$

$$\underline{B_T} = \sqrt{\mu'\epsilon'}\frac{1}{k_T}\underline{k_T}\times\underline{E_T}$$

where $k_i^2 = k_r^2 = \mu\epsilon\omega^2$ and $k_t^2 = \mu'\epsilon'\omega^2$.

Boundary Conditions at Interface

We first observe that the boundary conditions must be satisfied $\forall x, y$ at all times t. Thus all fields must have the same phase factor at $z = 0$. N.B.: We have implicitly assumed this in saying that the frequency in $z > 0$ must be the same as that in $z < 0$.

Thus $\underline{k_i}\cdot\underline{x} = \underline{k_r}\cdot\underline{x} = \underline{k_T}\cdot\underline{x}$ at $z = 0$. The k's lie in a plane - **plane of incidence**. From the figure, we see that

$$\begin{aligned}\underline{k_i}\cdot\underline{x} &= |\underline{x}||\underline{k_i}|\cos(\pi/2+\theta_i) = -|\underline{x}||\underline{k_i}|\sin\theta_i \\ \underline{k_r}\cdot\underline{x} &= |\underline{x}||\underline{k_r}|\cos(\pi/2+\theta_r) = -|\underline{x}||\underline{k_r}|\sin\theta_r\end{aligned}$$

and thus we have

$$\boxed{\theta_i = \theta_r}$$

Similarly,

$$\begin{aligned}|\underline{k_i}|\sin\theta_i &= |\underline{k_T}|\sin\theta_T \\ \Longrightarrow \mu\epsilon\sin\theta_i &= \mu'\epsilon'\sin\theta_T.\end{aligned}$$

and thus

$$\boxed{\frac{\sin\theta_i}{\sin\theta_T} = \sqrt{\frac{\mu'\epsilon'}{\mu\epsilon}} = \frac{n'}{n}}$$

Thus both laws are purely kinematic properties.

The boundary conditions themselves are

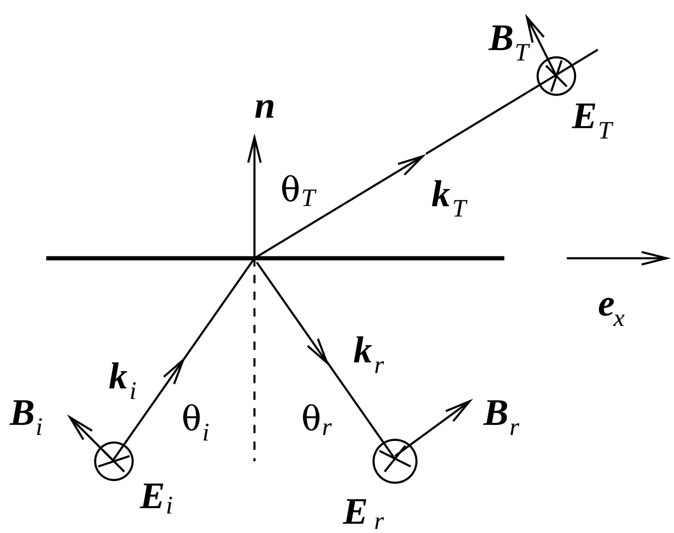
$$\underline{E}^{\parallel} \quad \text{is continuous}$$
$$\underline{H}^{\parallel} \quad \text{is continuous}$$
$$\underline{D}^{\perp} \quad \text{is continuous}$$
$$\underline{B}^{\perp} \quad \text{is continuous}$$

Applying to the fields at the interface, we have

$$(\underline{E}_0^i + \underline{E}_0^R - \underline{E}_0^T) \times \underline{n} = 0 \tag{7.11}$$
$$\left[\frac{1}{\mu}(\underline{k_i} \times \underline{E}_0^i + \underline{k_r} \times \underline{E}_0^r) - \frac{1}{\mu'}\underline{k_T} \times \underline{E}_0^T\right] \times \underline{n} = 0 \tag{7.12}$$
$$\left[\epsilon(\underline{E}_0^i + \underline{E}_0^r) - \epsilon' \underline{E}_0^T\right] \cdot \underline{n} = 0 \tag{7.13}$$
$$\left[\underline{k_i} \times \underline{E}_0^i + \underline{k_T} \times \underline{E}_0^r - \underline{k_T} \times \underline{E}_0^T\right] \cdot \underline{n} = 0. \tag{7.14}$$

We we now consider two cases; where the electric polarization vector is *normal to plane of incidence* and where it is *parallel to plane of incidence*.

7.3.1 Normal to Plane of Incidence



The z axis is normal to the interface, and we choose the x axis to be in the plane of incidence as shown. Thus the electric field is along the y axis. The first boundary condition eqn. (7.11) yields

$$\underline{E_0^i} + \underline{E_0^r} - \underline{E_0^T} = 0. \tag{7.15}$$

We now turn to the second second boundary condition eqn. (7.12). The first term yields

$$\begin{aligned}\frac{1}{\mu}\left[\underline{k_i} \times \underline{E_0^i}\right] \times \underline{n} &= \frac{1}{\mu}\left[\underline{E_0^i}(\underline{n}\cdot\underline{k_i}) - \underline{k_i}(\underline{n}\cdot\underline{E_0^i})\right] \\ &= \frac{1}{\mu}\underline{E_0^i}|\underline{k_i}|\cos\theta_i = \omega\sqrt{\frac{\epsilon}{\mu}}\cos\theta_i\underline{E_0^i}.\end{aligned}$$

We treat the other two terms similarly, and we find

$$(\underline{E_0^i} - \underline{E_0^r})\omega\sqrt{\frac{\epsilon}{\mu}}\cos\theta_i - \omega\sqrt{\frac{\epsilon'}{\mu'}}\cos\theta_T\underline{E_0^T} = 0,$$

yielding

$$\cos\theta_i\sqrt{\frac{\epsilon}{\mu}}\left(E_0^i - E_0^r\right) - \sqrt{\frac{\epsilon'}{\mu'}}\cos\theta_T E_0^T = 0 \tag{7.16}$$

The remaining boundary conditions yield no new information, so combining eqns. (7.15) and (7.16) we find

$$\frac{E_0^r}{E_0^i} = \frac{1-\sqrt{\frac{\epsilon'\mu}{\epsilon\mu'}}\frac{\cos\theta_T}{\cos\theta_i}}{1+\sqrt{\frac{\epsilon'\mu}{\epsilon\mu'}}\frac{\cos\theta_T}{\cos\theta_i}} = \frac{1-\frac{\mu}{\mu'}\frac{\tan\theta_i}{\tan\theta_T}}{1+\frac{\mu}{\mu'}\frac{\tan\theta_i}{\tan\theta_T}} \tag{7.17}$$

$$\frac{E_0^T}{E_0^i} = \frac{2}{1+\sqrt{\frac{\epsilon'\mu}{\epsilon\mu'}}\frac{\cos\theta_T}{\cos\theta_i}} = \frac{2}{1+\frac{\mu}{\mu'}\frac{\tan\theta_i}{\tan\theta_T}} \tag{7.18}$$

For visible light, we can usually put $\mu = \mu'$, giving

$$\begin{aligned}\frac{E_0^r}{E_0^i} &= \frac{\sin(\theta_T - \theta_i)}{\sin(\theta_i + \theta_T)} \\ \frac{E_0^T}{E_0^i} &= \frac{2\sin\theta_T\cos\theta_i}{\sin(\theta_T + \theta_i)}.\end{aligned}$$

This is just **Fresnel's formula** for light polarized perpendicular to plane of incidence.

7.3.2 Electric Field in Plane of Incidence

Here we use boundary conditions eqns (7.11) and (7.13) to yield

$$\frac{E_0^r}{E_0^i} = \frac{1-\sqrt{\frac{\epsilon\mu'}{\epsilon'\mu}}\frac{\cos\theta_T}{\cos\theta_i}}{1+\sqrt{\frac{\epsilon\mu'}{\epsilon'\mu}}\frac{\cos\theta_T}{\cos\theta_i}} = \frac{1-\frac{\epsilon}{\epsilon'}\frac{\tan\theta_i}{\tan\theta_T}}{1+\frac{\epsilon}{\epsilon'}\frac{\tan\theta_i}{\tan\theta_T}} \tag{7.19}$$

$$\frac{E_0^T}{E_0^i} = \frac{2\sqrt{\frac{\epsilon\mu'}{\epsilon'\mu}}}{1+\sqrt{\frac{\epsilon\mu'}{\epsilon'\mu}}\frac{\cos\theta_T}{\cos\theta_i}} = \frac{2\frac{n'}{n}\frac{\epsilon}{\epsilon'}}{1+\frac{\epsilon}{\epsilon'}\frac{\tan\theta_i}{\tan\theta_T}} \tag{7.20}$$

If $\mu = \mu'$, then $\epsilon/\epsilon' = \sin^2\theta_T/\sin^2\theta_i = n^2/n'^2$, and we have

$$\frac{E_0^r}{E_0^i} = \frac{\tan(\theta_i-\theta_T)}{\tan(\theta_i+\theta_T)}$$

$$\frac{E_0^T}{E_0^i} = \frac{2\sin\theta_T\cos\theta_i}{\sin(\theta_i+\theta_T)\cos(\theta_i-\theta_T)}$$

Incident Wave Normal to Interface

In this particular case, we find

$$\frac{E_0^r}{E_0^i} = \frac{1-\sqrt{\frac{\epsilon'\mu}{\epsilon\mu'}}}{1+\sqrt{\frac{\epsilon'\mu}{\epsilon\mu'}}} \longrightarrow \frac{n-n'}{n+n'} \quad \text{if } \mu=\mu'$$

$$\frac{E_0^T}{E_0^i} = \frac{2}{1+\sqrt{\frac{\epsilon'\mu}{\epsilon\mu'}}} \longrightarrow \frac{2}{1+n'/n} \quad \text{if } \mu=\mu'$$

In this formula we assume that the directions of E_0^r and E_0^i are the same. (contrary to Eq. (7.42) from *Jackson* where the directions of E_0^r and E_0^i are assumed to be opposite).

Thus we see that, if both refractive indices are equal

$$E_0^r = 0$$

$$E_0^T = E_0^r$$

as expected. If the second media is a conductor, $n' \longrightarrow 0$, so all of the wave is reflected, with

$$\underline{E}_0^r = -\underline{E}_0^i \tag{7.21}$$

7.4 Brewster's Angle and Total Internal Reflection

7.4.1 Brewster's Angle

In the case of polarization in the plane of incidence, we have

$$\frac{E_0^r}{E_0^i} = \frac{1 - \frac{\epsilon}{\epsilon'}\frac{\tan\theta_i}{\tan\theta_T}}{1 + \frac{\epsilon}{\epsilon'}\frac{\tan\theta_i}{\tan\theta_T}}.$$

There is an angle for which *no* wave is reflected, given by

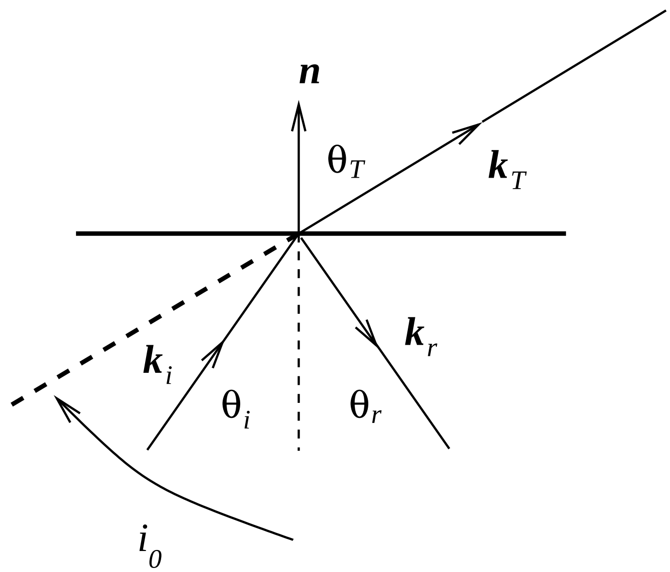
$$\frac{\epsilon}{\epsilon'}\frac{\tan\theta_i}{\tan\theta_T} = 1.$$

Setting $\mu = \mu' = 1$, so that $\epsilon/\epsilon' = n^2/n'^2$, we find

$$\theta_i = \tan^{-1}\left(\frac{n'}{n}\right). \tag{7.22}$$

This is **Brewster's Angle**. If we have a plane wave of mixed polarization incident at this angle, the reflected radiation only has a polarization component perpendicular to the plane of incidence. It is a simple way to produce plane-polarized light.

7.4.2 Total Internal Reflection



If light passes from a medium of higher optical density to one of lower optical density, the **angle of refraction** is **greater** than the **angle of incidence**. Hence there is a θ_i for which $\theta_T = \pi/2$, given by

$$\sin\theta_i = \sin i_0 = n'/n \tag{7.23}$$

From Snell's law, we have in general

$$\begin{aligned}\cos\theta_T = \sqrt{1-\sin^2\theta_T} &= \sqrt{1-\frac{n^2}{n'^2}\sin^2\theta_i} \\ &= \sqrt{1-\left(\frac{\sin\theta_i}{\sin i_0}\right)^2}.\end{aligned}$$

For $\theta_i > i_0$, $\cos\theta_T$ becomes purely imaginary. Thus the refractive wave has a phase factor

$$\begin{aligned} e^{i\underline{k_T}\cdot\underline{x}} &= e^{i(k_T x\sin\theta_T + k_T z\cos\theta_T)} \\ &= e^{ik_T x(n/n')\sin\theta_i} e^{-k_T z\sqrt{(\sin\theta_i/\sin i_0)^2-1}}.\end{aligned}$$

We see that the refracted wave propagates **parallel to the surface**, and is **exponentially attenuated** with increasing z. The attenuation occurs over only a few wavelengths *unless* $\theta_i \approx i_0$.

Note that the time-averaged energy flux across the interface is

$$\langle \underline{S} \cdot \underline{n} \rangle = \frac{1}{2} \Re \left[\underline{n} \cdot (\underline{E_T} \times \underline{H_T^*}) \right].$$

Now $\underline{H_T} = (\underline{k_T} \times \underline{E_T})/\mu' \omega$, and thus

$$\begin{aligned} \underline{n} \cdot (\underline{E_T} \times \underline{H_T^*}) &= \underline{n} \cdot \left[\underline{E_T} \times (\underline{k_T} \times \underline{E_T^*}) \right] / \mu' \omega \\ &= |\underline{E_T}|^2 \underline{n} \cdot \underline{k_T} / \mu' \omega, \end{aligned}$$

whence

$$\begin{aligned} \langle \underline{S} \cdot \underline{n} \rangle &= \frac{1}{2} \Re \left[|\underline{E_T}|^2 \underline{n} \cdot \underline{k_T} \right] / \mu' \omega \\ &= \frac{1}{2} \Re \left[|\underline{E_T}|^2 k_T \cos \theta_T \right] / \mu' \omega \\ &= 0, \end{aligned}$$

since $\cos \theta_T$ is purely imaginary; there is no *time-averaged* energy flux across the interface.

The principle of total internal reflection has many applications, most notably in fibre-optic cables. The analysis presented here assumes, of course, that the material is wide compared to the wave length of light.

7.5 Dispersion

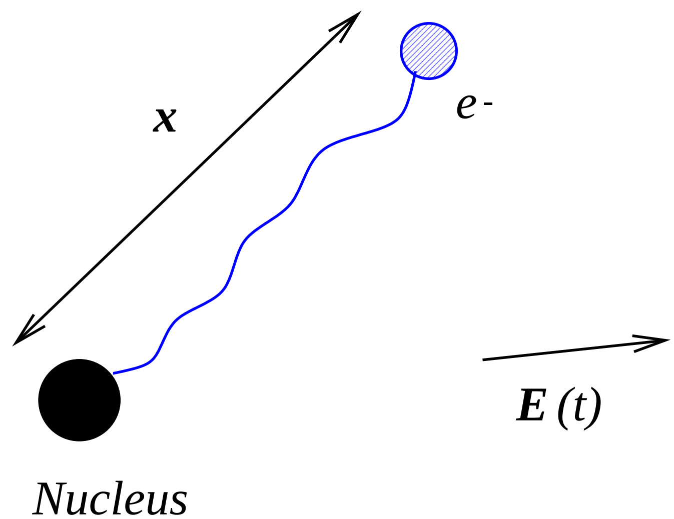
So far, we have been investigating the propagation of waves of a fixed frequency. The wave number is related by

$$k^2 = \mu\epsilon\omega^2.$$

Suppose now we consider a wave having a spread of frequencies. In general, the values of μ and, in particular, ϵ are frequency dependent, and thus different frequencies have different propagation properties. This is call **dispersion**

7.5.1 Simple Model for Dispersion

Consider an electron of mass m and charge $-e$, bound to a (fixed) nucleus by a harmonic potential with resonant frequency ω_0, and a damping term with damping constant γ. In the absence of an external electric field, the electron will undergo damped simple-harmonic motion about an equlibrium.



We now apply an external electromagnetic field $(\underline{E}, \underline{B})$. Then the force on the electron is

$$\underline{F}(t) = -e(\underline{E}(t) + \underline{v} \times \underline{B}(t)).$$

Providing the velocity is small compared to that of light, the magnetic force will be negligible; recall that $c|\underline{B}| \approx |\underline{E}|$. Thus the equation of motion of the electron is

$$m(\frac{d^2}{dt^2}\underline{x} + \gamma\frac{d}{dt}\underline{x} + \omega_0^2\underline{x}) = -e\underline{E}(t).$$

The dipole moment of the system is just $\underline{p} = -e\underline{x}$. We now assume that the external field has frequency ω, so that the time dependence is

$$\underline{E} = \underline{E_0}e^{-i\omega t}.$$

Thus the displacement will have the same frequency dependence, and we have an equation of motion

$$m(-\omega^2 - i\omega\gamma + \omega_0^2)\underline{x} = \underline{E_0},$$

yielding a dipole moment

$$\underline{p} = \frac{e^2}{m}(\omega_0^2 - \omega^2 - i\omega\gamma)^{-1}.$$

We now consider the case of N atoms/unit volume, each having Z electrons of which f_j electrons have resonant frequency ω_j. We will take this as a model for a linear medium, in which the polarization $\underline{P}$ arises solely from the applied external field. Thus, recalling that

$$\underline{P} = \epsilon_0\chi_e\underline{E}$$

we find

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \chi_e = 1 + \frac{Ne^2}{\epsilon_0 m}\sum_j f_j(\omega_j^2 - \omega^2 - i\omega\gamma_j)^{-1}.$$

with $\sum_j f_j = Z$. We will rewrite this expression as

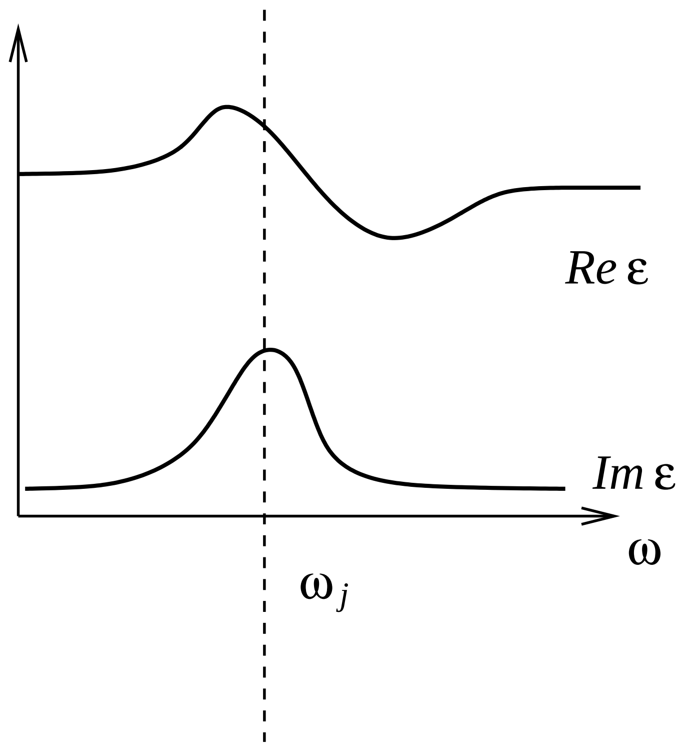
$$\frac{\epsilon}{\epsilon_0} = 1 + \frac{Ne^2}{\epsilon_0 m}\sum_j f_j\frac{(\omega_j^2 - \omega^2) + i\omega\gamma_j}{(\omega_j^2 - \omega^2)^2 + \omega^2\gamma_j^2}. \tag{7.24}$$

We have thus seen how even a simple model gives a frequency-dependent permittivity.

7.5.2 Permittivity in Resonance Region

In general, we can assume that the damping factor γ is small. From the form of eqn. (7.24), it is clear that at very log frequencies, the susceptibility is positive and the relative permittivity greater than one. As successive resonant frequencies are passed, more negative terms contribute and eventually the relative permittivity is less than one.

Particularly interesting is the behaviour in the neighbourhood of a resonance.



Here the real part of $\epsilon(\omega)$ is peaked around ω_j, and furthermore displays *anomaloous dispersion* in which light of higher frequency is less refracted than light of lower frequency.

The presence of an appreciable imaginary part of $\epsilon(\omega)$ near $\omega = \omega_j$ represents **absorption**; energy dissipated in the medium. To see how this arises, consider a wave propagating in the z-direction. We will write the wave number as

$$k = \beta + i\alpha/2; \qquad \text{amplitude} \approx e^{-\alpha z/2}.$$

Thus α clearly represents absorption of the wave. Setting $\mu = \mu_0$, and recalling $k^2 = \sqrt{\mu\epsilon}\omega$, we have

$$(\beta^2 - \alpha^2/4) + i\alpha\beta = (\sqrt{\mu_0\epsilon_0})^2\omega^2\epsilon/\epsilon_0$$

which gives

$$\left.\begin{array}{rcl} \beta^2 - \alpha^2/4 &=& \frac{\omega^2}{c^2}\Re\,\epsilon/\epsilon_0 \\ \alpha\beta &=& \frac{\omega^2}{c^2}\Im\,\epsilon/\epsilon_0 \end{array}\right\}.$$

Note that if $\alpha \ll \beta$, we have

$$\alpha = \frac{\Im\,\epsilon(\omega)}{\Re\,\epsilon(\omega)}\beta$$

where

$$\beta = \frac{\omega}{c}\sqrt{\Re\,\epsilon/\epsilon_0}$$

7.5.3 Low Frequency Behaviour and Electrical Conductivity

In a conductor, some of the electrons can move freely. Thus there are some electrons with resonant frequency $\omega_0 = 0$, whose contribution to the permittivity is

$$\epsilon(\omega) = \tilde{\epsilon}(\omega) + i\frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)},$$

where $\tilde{\epsilon}$ represents the background permittivity coming from all the other modes. We see from this that $\epsilon(\omega)$ is singular as $\omega \longrightarrow 0$, and we will now relate this property to electrical conductivity.

Our starting point is the Maxwell-Ampère law (ME3):

$$\underline{\nabla} \times \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t}. \tag{7.25}$$

We will now impose that $\underline{J}$ and $\underline{E}$ are related through Ohm's law

$$\underline{J} = \sigma\underline{E}$$

where σ is the conductivity. If we assume the usual frequency behaviour $\exp -i\omega t$, and assume the background permittivity is a constant $\epsilon_b = \tilde{\epsilon}(\omega)$, eqn. (7.25) becomes

$$\underline{\nabla} \times \underline{H} = -i\omega \left[\epsilon_b + i\frac{\sigma}{\omega} \right] \underline{E}. \tag{7.26}$$

An alternative procedure is to ascribe all properties, **including current flow**, to the dielectric properties of the medium. In that case we have

$$\underline{\nabla} \times \underline{H} = -i\omega \underline{D} = -i\omega \left[\epsilon_b + \frac{Ne^2 f_0}{m\omega(\gamma - i\omega)} \right] \underline{E}. \tag{7.27}$$

Comparing eqns. (7.26) and (7.27), we find

$$i\sigma/\omega = i\frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)}$$

i.e.

$$\boxed{\sigma = \frac{Ne^2 f_0}{m(\gamma_0 - i\omega)}. \qquad (7.28)}$$

Note that we can rewrite this expression as

$$\sigma = \frac{\sigma_0}{1 - i\omega\tau}$$

where

$$\sigma_0 = \frac{N f_0 e^2}{m\gamma_0},$$

and $\tau = \gamma^{-1}$. Essentially, we have

- $N f_0$ is *number of free electrons per unit volume.*
- γ_0/f_0 is *damping constant*, determined experimentally.

For good conductors $\gamma_0/f_0 \simeq 4 \times 10^{13}\,\mathrm{s}^{-1}$. If we assume $f_0 \simeq 1$, then $\omega\tau$ is small rights up to the microwave region $\omega \simeq 10^{11}\,\mathrm{s}^{-1}$; σ **is real**.
Note that if $\omega\gamma_0 \gg 1$, then σ is purely imaginary, and we have a phase shift between $\underline{E}$ and $\underline{J}$.

7.6 High-Frequency Behaviour and Plasma Frequency

Suppost that ω is much larger than the highest resonance frequency. Then we have

$$\begin{aligned} \epsilon/\epsilon_0 &= 1 + \frac{Ne^2}{\epsilon_0 m} \sum_j f_j \frac{(\omega_j^2 - \omega^2) + i\omega\gamma_j}{(\omega_j^2 - \omega^2)^2 + \omega^2\gamma_j^2} \\ &\overset{\omega/\omega_j \gg 1}{\longrightarrow} 1 - \frac{Ne^2}{\epsilon_0 m} \sum_j f_j \omega^2/\omega^4 \\ &= 1 - \omega_P^2/\omega^2 \end{aligned} \tag{7.29}$$

where

$$\omega_P^2 = \frac{NZe^2}{\epsilon_0 m} \tag{7.30}$$

is the **plasma frequency**, so called because all the electrons essentially behave as if free.

Recalling that

$$k = \sqrt{\mu\epsilon}\omega = \frac{1}{c}\sqrt{\frac{\epsilon}{\epsilon_0}}\omega,$$

where c is the velocity of light *in vacua*, we have

$$ck = \sqrt{\omega^2 - \omega_P^2}$$

whence

$$\omega(k)^2 = \omega_P^2 + c^2 k^2. \tag{7.31}$$

Such an expression, describing the relationship between wave number and frequency, is known as a **dispersion relation**. Similar expressions occur in many places in physics, including special relativity and sound propagation.

In a typical dielectric, when $\omega^2 \gg \omega_P^2$, the dielectric constant is slightly less than, but close to, unity.

In a true plasma, such as the ionosphere, all the electrons are essentially free, and the expression eqn. (7.29) is valid for a range of frequencies, *including* $\omega < \omega_P$. The wave number k is purely imaginary for frequencies less than the plasma frequency. Thus a wave incident on a plasma are attenuated in the direction of propagation, with intensity

$$I \propto e^{-2\sqrt{\omega_P^2-\omega^2}z/c} \xrightarrow{\omega\to 0} e^{-2\omega_P z/c}.$$

7.6.1 Model of Wave Propagation in the Atmosphere

The above plasma model for the ionosphere is modified considerably through the presence of the earth's magnetic field. In the model we now construct, we assume propagation parallel to the earth's field $\underline{B_0}$. We assume that there is a force acting on the charges due to a propagating *electric* field, but that the only magnetic force is that arising from the earth's field; recall once again that $c|\underline{B}| \simeq |\underline{E}|$.
Thus the equation of motion for an electron of charge $-e$ and mass m is

$$m\frac{d^2\underline{x}}{dt^2} = -e\underline{v} \times \underline{B_0} - e\underline{E}.$$

Once again, we consider a monochromatic plane wave with time dependence

$$e^{-i\omega t}.$$

It is convenient to consider the case of *circularly polarized* waves, for which we introduce the complex polarization vectors

$$\begin{aligned} \underline{\epsilon_\pm} &= \frac{1}{\sqrt{2}}(\underline{\epsilon_1} \pm i\underline{\epsilon_2}) \\ \underline{\epsilon_3} &= \underline{\hat{k}} \qquad \text{(Normal in direction of } \underline{k}). \end{aligned}$$

Thus we have

$$\underline{x} = x_+\underline{\epsilon_+} + x_-\underline{\epsilon_-} + x_3\underline{\epsilon_3},$$

so that the equation of motion becomes

$$m[\frac{d^2x_+}{dt^2}\underline{\epsilon_+}+\frac{d^2x_-}{dt^2}\underline{\epsilon_-}+\frac{d^2x_3}{dt^2}\underline{\epsilon_3}]-eB_0\underline{\epsilon_3}\times[\frac{dx_+}{dt}\underline{\epsilon_+}+\frac{dx_-}{dt}\underline{\epsilon_-}+\frac{dx_3}{dt}\underline{\epsilon_3}] = -e[E_+\underline{\epsilon_+}+E_-\underline{\epsilon_-}]e^{-i\omega t}.$$

First, it is easy to see that since $\underline{\epsilon_3}\times\underline{\epsilon_3} = 0$, the motion along the Z direction is free: $x_3 = x_{30} + v_3 t$. Since the forces acting in XY plane are periodic, the motion of the charges in the XY plane will be periodic, too: $x_+(t) = x_+e^{-i\omega t}, \quad x_-(t) = x_-e^{-i\omega t}$.

Now

$$\begin{aligned} \underline{\epsilon_3} \times \underline{\epsilon_+} &= \frac{1}{\sqrt{2}}(\underline{\epsilon_3} \times \underline{\epsilon_1} + i\underline{\epsilon_3} \times \underline{\epsilon_2}) \\ &= -\frac{i}{\sqrt{2}}(\underline{\epsilon_1} + i\underline{\epsilon_2}), \end{aligned}$$

yielding

$$\begin{aligned} \underline{\epsilon_3} \times \underline{\epsilon_+} &= -i\underline{\epsilon_+} \\ \underline{\epsilon_3} \times \underline{\epsilon_-} &= i\underline{\epsilon_-} \end{aligned} \tag{7.32}$$

and thus

$$-\omega^2 m[x_+\underline{\epsilon_+} + x_-\underline{\epsilon_-}] + i\omega e B_0[-ix_+\underline{\epsilon_+} + ix_-\underline{\epsilon_-}] = -e[E_+\underline{\epsilon_+} + E_-\underline{\epsilon_-}].$$

Looking at the individual components, we find

$$\begin{aligned} -\omega^2 m x_+ + \omega e B_0 x_+ &= -eE_+ \\ -\omega^2 m x_- - \omega e B_0 x_- &= -eE_- \end{aligned}$$

which we may write

$$x_\pm = \frac{e}{m\omega(\omega \pm eB_0/m)} E_\pm.$$

We now introduce

$$\omega_B \equiv eB_0/m,$$

the frequency of precession of a charged particle in a magnetic field.
Recalling that $\underline{p} = -e\underline{x}$, we have a dipole moment of the particle

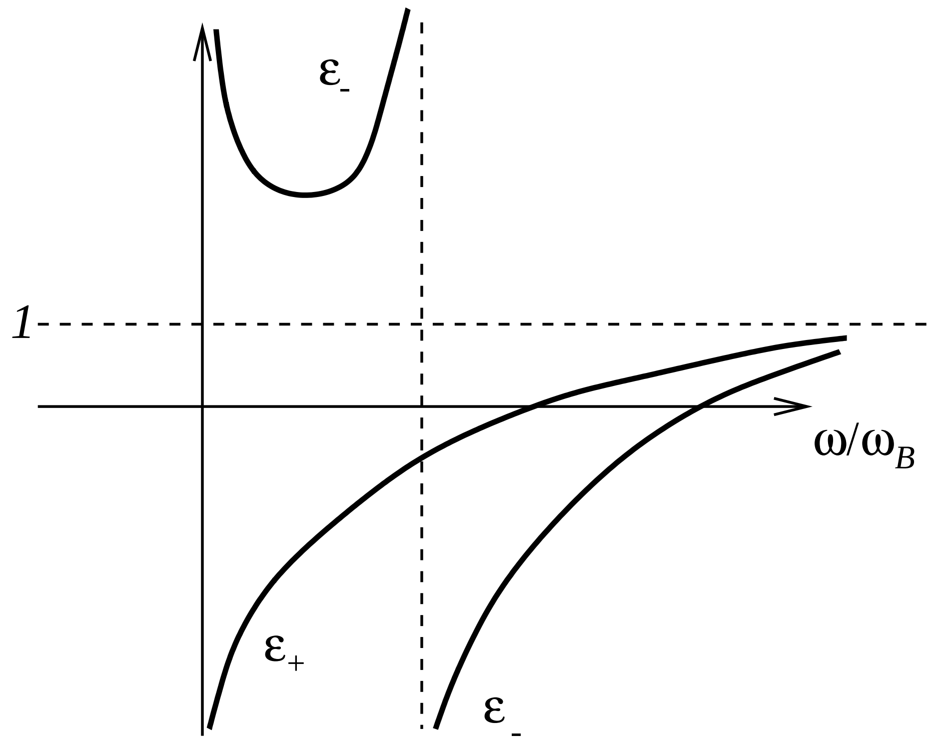
$$p_\pm = \frac{-e^2}{m\omega(\omega \pm \omega_B)} E_\pm.$$

Thus, recalling the expression for the plasma frequency eqn (7.30), the polarizability may be written

$$P_\pm = -\epsilon_0 \frac{\omega_P^2}{\omega(\omega \pm \omega_B)} E_\pm$$

whence

$$\epsilon_\pm/\epsilon_0 = 1 - \frac{\omega_P^2}{\omega(\omega \pm \omega_B)} \qquad (7.33)$$



Thus, in this highly simplified model, we see that the permittivity depends on the polarization of the incident wave. Indeed, for certain ranges of ω we find that the permittivity can be negative, and hence one or both polarizations no longer propagate.

7.7 Superposition of Waves and Group Velocity

So far we have considered monochromatic waves, but have seen that, if the medium is dispersive, different frequencies will travel with different velocities. In the section, we will describe how, for a general plane wave, the rate of energy flow is in general different from the phase velocity, or velocity of propagation of a particular frequency component. To simplify the discussion, we will consider the problem in one dimension.

We will write a general wave in terms of its physical components. The dispersive properties are encompassed in the dispersion relation

$$\omega \equiv \omega(k)$$

where $\omega(-k) = \omega(k)$. The general solution is then

$$u(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k) e^{ikx - i\omega(k)t},$$

where the amplitudes $A(k)$ are given by

$$A(k) = \int_{\infty}^{\infty} dx u(x,0) e^{-ikx}.$$

For a monochromatic wave, of wave number k_0, we have

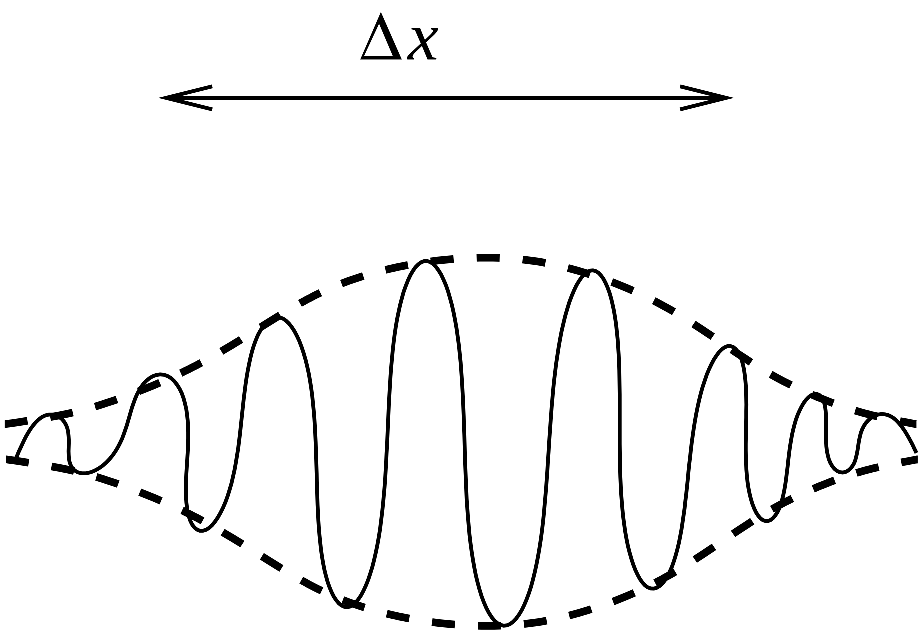
$$u(x,0) = e^{ik_0 x}$$

yielding

$$A(k) = 2\pi\delta(k - k_0).$$

In practice we virtually never deal with pure monochromatic plane waves of fixed frequency k_0, but rather with pulses, centred about a frequency k_0. In particular, we will consider the propagation of a Gaussian wave packet, of width Δx, centred at $x = 0$. Then

$$u(x,0) = \left(\frac{1}{2\pi\Delta x^2}\right)^{1/4} e^{-x^2/4\Delta x^2} e^{ik_0 x}.$$



This satisfies

$$\begin{aligned}
\langle x^2 \rangle &= \int dx\, |u(x,0)|^2 x^2 \\
&= \left(\frac{1}{2\pi\Delta x^2}\right)^{1/2} \int dx e^{-x^2/2\Delta x^2} x^2 \\
&= \left(\frac{1}{2\pi\Delta x^2}\right)^{1/2} (-2) \frac{d}{d(1/\Delta x^2)} \int dx\, e^{-x^2/2\Delta x^2} \\
&= \Delta x^2,
\end{aligned}$$

showing that the width is indeed Δx.
The amplitudes of the various componenents are given by

$$\begin{aligned}
A(k) &= \left(\frac{1}{2\pi\Delta x^2}\right)^{1/2} \int dx\, e^{-x^2/4\Delta x^2} e^{i(k_0-k)x} \\
&\simeq e^{-(k_0-k)^2/4(1/2\Delta x)^2}. \qquad \textit{(exercise)}
\end{aligned}$$

By analogy with the width of the wave packet, we see that the amplitude $A(k)$ is centred at $k = k_0$, with width

$$\Delta k = \frac{1}{2\Delta x}.$$

In fact, more generally we have

$$\Delta x \Delta k \geq 1/2 \tag{7.34}$$

Thus we have the important observation that a short pulse, even of "fixed" frequency k_0, contains a spread of monochromatic components. This expression, of course, is more familiar from *Heisenberg's Uncertainty Principle.*

7.7.1 Group Velocity

To see how this spread of frequencies effects the propagation of a wave, we consider the simple case of two monochromatic waves, of the same amplitude and of neighbouring frequencies (k_1, ω_1) and (k_2, ω_2), where $k_1, k_2 \sim k_0$. Then the resulting "wave packet" propagates as

$$\begin{aligned} U(x,t) &= A\left[e^{i(k_1 x - \omega_1 t)} + e^{i(k_2 x - \omega_2 t)}\right] \\ &= A e^{i[(k_1+k_2)x/2 - (\omega_1+\omega_2)t/2]}\left\{e^{i[(k_1-k_2)x/2-(\omega_1-\omega_2)t/2]} + e^{i[(k_2+k_1)x/2-(\omega_2+\omega_1)t/2]}\right\} \\ &= 2A\cos\left[\frac{k_1-k_2}{2}x - \frac{\omega_1-\omega_2}{2}t\right] e^{i[(k_1+k_2)x/2-(\omega_1+\omega_2)t/2]} \end{aligned}$$

We have written the wave as a *slowly moving amplitude factor* with velocity

$$v_g = \frac{\omega_1 - \omega_2}{k_1 - k_2} \longrightarrow \left.\frac{d\omega}{dk}\right|_{k_0} \quad \text{as } k_2 \to k_1, \tag{7.35}$$

known as the **group velocity**, and a rapidly moving "phase" with velocity

$$v_p \longrightarrow \frac{\omega_1 + \omega_2}{k_1 + k_2} = \frac{\omega}{k} \quad \text{as } k_2 \to k_1. \tag{7.36}$$

Since the energy density is associated with the amplitude of the wave, we see that, in this approximation, energy is transmitted with the group velocity, given by eqn. (7.35) with k_0 the central value of the wave number.

We now recall the relationship between ω and k

$$\omega = \frac{ck}{n(k)}, \tag{7.37}$$

where $n(k)$ is the index of refraction, and c is the velocity of light in a vacuum. The phase velocity can then be written

$$v_p = \frac{\omega(k)}{k} = \frac{c}{nk}. \tag{7.38}$$

This can be either **less than** or **greater than** the speed of light; for most media at optical frequencies, $n(k) > 1$. We can rewrite the group velocity using eqn. (7.37), regarding $k = k(\omega)$, and find

$$\begin{aligned} n(\omega) + \omega\frac{dn}{d\omega} &= c\frac{dk}{d\omega} \\ \Longrightarrow v_g = \frac{d\omega}{dk}\Big|_{k_0} &= \frac{c}{n(\omega) + \omega\frac{dn}{d\omega}}. \end{aligned}$$

Providing $dn/d\omega > 0$, we have $v_g < c$. However, if $dn/d\omega < 0$ (anomalous refraction), v_g can be greater than c.

7.7.2 Propagation of a Gaussian wave packet in the dispersive medium

First, let us recall the propagation of a Gaussian pulse in a linear medium without dispersion

$$u_0(x,t) = (\frac{1}{\pi L^2})^{1/4} \exp\left\{ - \frac{(x-vt)^2}{2L^2} + ik_0(x-vt)\right\} \tag{7.39}$$

where $L = \Delta x\sqrt{2}$ is the width of the Gaussian wave packet.
Suppose at $t = 0$ we switch on the dispersion so that $\omega = \omega(k)$ (some non-linear function). What will happen with the pulse?
Starting from $t = 0$, the solution of the wave eqn is

$$u(x,t) = \Re\int\frac{dk}{2\pi}\, A(k)\, e^{-i\omega(k)t+ikx}$$

A solution of the second-order duifferential eqn is specified if we know both $u(x,0)$ and $\dot{u}(x,0)$. It is easy to prove that

$$A(k) = \int dx\, e^{-ikx}[u(x,0) + \frac{i}{\omega_k}\dot{u}(x,0)] \tag{7.40}$$

The initial condition should be taken from the form of a non-dispersive Gaussian pulse (7.39) at $t = 0$:

$$\begin{aligned} u(x,0) &= u_0(x,t) = \left(\frac{1}{\pi L^2}\right)^{1/4} \exp\left\{-\frac{x^2}{2L^2} + ik_0 x\right\} \\ i\dot{u}(x,0) &= i\dot{u}_0(x,t) = \left(\frac{1}{\pi L^2}\right)^{1/4} \left(k_0 v + \frac{ivx}{L^2}\right) \exp\left\{-\frac{x^2}{2L^2} + ik_0 x\right\} \end{aligned}$$

From Eq. (7.40) we obtain

$$A(k) = \left(4\pi L^2\right)^{1/4} \left[1 + \frac{kv}{\omega_k}\right] \exp\left\{-\frac{(k-k_0)^2 L^2}{2}\right\} \tag{7.41}$$

A typical behavior of $\omega(k)$ is given by eq. (7.31). For simplicity, we will consider an approximate model of the behavior of frequency in the vicinity of ω_0 in the form

$$\omega(k) = \omega_0 \left(1 + \frac{a^2 k^2}{2}\right) \tag{7.42}$$

where $\omega_0 = vk_0$ is the center of our Gaussian wave packet.
We obtain

$$u(x,t) = \Re (4\pi L^2)^{1/4} \int \frac{dk}{2\pi} \left(1 + \frac{k\omega_0}{k_0 \omega_k}\right) e^{-\frac{(k-k_0)^2 L^2}{2}} e^{-i\omega_k t + ikx}$$

The term $\frac{k\omega_0}{k_0\omega_k}$ is approximately 1 in the vicinity of k_0 so

$$u(x,t) = \Re \frac{2(4\pi L^2)^{1/4}}{\sqrt{L^2 + i\omega_0 a^2 t}} e^{-i\omega_0 t\left(1 + \frac{a^2 k_0^2}{2}\right) + ik_0 x} \exp\left\{-\frac{(x - \omega_0 a^2 k_0 t)^2}{2L^2(1 + i\omega_0 \frac{a^2 t}{L^2})}\right\} \tag{7.43}$$

The peak of the pulse (7.43) is located at $x = \omega_0 a^2 k_0 t \Rightarrow$ it moves with the it group velocity $\omega_0 a^2 k_0 = \left.\frac{\partial \omega_k}{\partial k}\right|_{k=k_0}$.
The wave packet spreads as it moves:

$$\sqrt{2}\Delta x(t) \equiv L(t) = \sqrt{L^2 + \frac{a^4 \omega_0^2 t^2}{L^2}}$$

(for a proof, see *Jackson*). This is a general feature of non-linear Gaussian wave packets: for the same reason ($\omega_k = \sqrt{(m^2c^4/\hbar^2) + k^2}$) wave packets corresponding to relativistic particles broaden with time.

7.8 Causality between $\underline{E}$ and $\underline{D}$ and Kramers-Kronig Relations

When $\epsilon(\omega)$ is frequency dependent, there is a non-local temporal relation between $\underline{D}$ and $\underline{E}$. To exhibit this, we write $\underline{D}$ and $\underline{E}$ in terms of their temporal Fourier components

$$\underline{D}(\underline{x},t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \underline{\tilde{D}}(\underline{x},\omega)e^{-i\omega t}.$$

For a linear medium

$$\underline{\tilde{D}}(\underline{x},\omega) = \epsilon(\omega)\underline{\tilde{E}}(\underline{x},\omega),$$

and thus

$$\underline{D}(\underline{x},t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \epsilon(\omega)\underline{\tilde{E}}(\underline{x},\omega)e^{-i\omega t}.$$

We now use

$$\underline{\tilde{E}}(\underline{x},\omega) = \int_{-\infty}^{\infty} dt'\, \underline{E}(\underline{x},t')e^{+i\omega t'}.$$

to write

$$\underline{D}(\underline{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \epsilon(\omega)e^{-i\omega t}\int_{-\infty}^{\infty} dt'\, e^{i\omega t'}\underline{E}(\underline{x},t').$$

To display the non-locality, we write

$$\epsilon(\omega) = \epsilon_0\left[(\epsilon(\omega)/\epsilon_0 - 1) + 1\right] = \epsilon_0[\chi_e + 1]$$

and thus

$$\underline{D}(\underline{x},t) = \epsilon_0\left\{\underline{E}(\underline{x},t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega dt' e^{i\omega(t'-t)}\chi_e(\omega)\underline{E}(\underline{x},t')\right\}.$$

By a change of variable $\tau = t - t'$, we can rewrite this as

$$\underline{D}(\underline{x},t) = \epsilon_0\left\{\underline{E}(\underline{x},t) + \int_{\infty}^{\infty} d\tau G(\tau)\underline{E}(\underline{x},t-\tau)\right\} \tag{7.44}$$

where

$$G(\tau) = \frac{1}{2\pi}\int d\omega\, \chi_e(\omega)e^{-i\omega\tau}. \tag{7.45}$$

We have essentially just used the convolution theorem of Fourier transforms, and have exhibited the non-local connection between $\underline{D}$ and $\underline{E}$.

To explore the nature of this connection, we consider a simple one-resonance model for $\chi_e(\omega)$

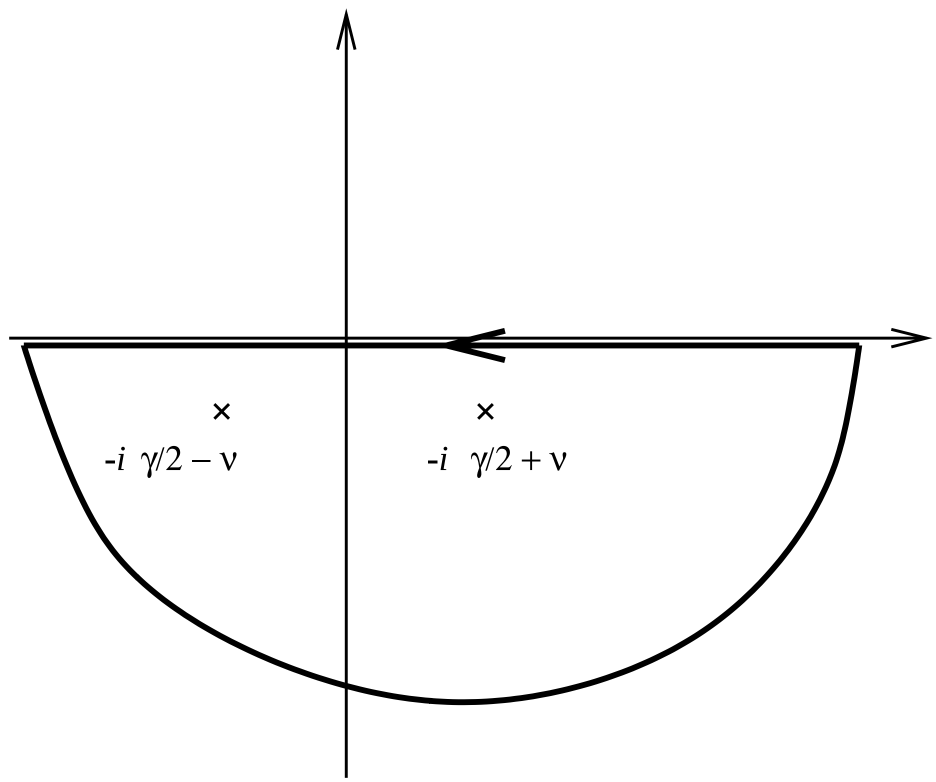
$$\chi_e(\omega) = \frac{\omega_P^2}{\omega_0^2 - \omega^2 - i\gamma\omega},$$

where ω_P is the plasma frequency. This has poles in the l.h.p. at

$$\omega = -i\frac{\gamma}{2} \pm \nu_0$$

where

$$\nu_0^2 = \omega_0^2 - \gamma^2/4.$$



To evaluate $G(\tau)$ we use contour integration, noting that there are two cases

1. $\tau < 0$: circle at ∞ vanishes in lower half plane.

2. $\tau > 0$: circle at ∞ vanishes in upper half plane.

Thus $G(\tau)$ vanishes for $\tau < 0$. By the residue theorem

$$G(\tau > 0) \;=\; \frac{\omega_P^2}{2\pi} \times 2\pi i \times \sum_{\text{residues}}$$

$$= \omega_P^2 e^{-\gamma\tau/2} \frac{\sin \nu_0 \tau}{\nu_0},$$

and thus

$$G(\tau) = \omega_P^2 e^{-\gamma\tau/2} \frac{\sin \nu_0 \tau}{\nu_0} \theta(\tau). \tag{7.46}$$

We can make two observations

- There is an oscillatory frequency $\approx \omega_0$.
- The damping factor $1/\gamma$ is that of the oscillators.

Thus non-locality is confined to a region $\tau \approx \gamma^{-1}$.

7.8.1 Causality

Because $G(\tau)$ vanishes for $\tau < 0$, $\underline{D}$ only depends on the values of $\underline{E}$ at earlier times, i.e.

$$\underline{D}(\underline{x}, t) = \epsilon_0 \left[\underline{E}(\underline{x}, t) + \int_0^\infty d\tau \, G(\tau) \underline{E}(\underline{x}, t - \tau) \right].$$

We can thus write the dielectric constant as

$$\epsilon(\omega)/\epsilon_0 = 1 + \int_0^\infty d\tau G(\tau) e^{i\omega\tau}.$$

Since $G(\tau)$ is real, we have

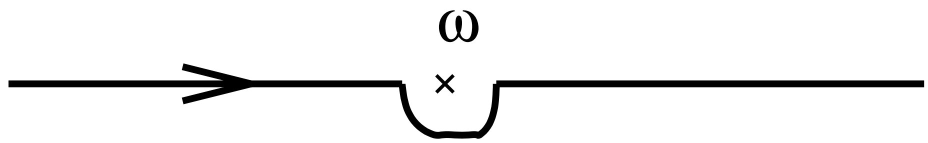
$$\epsilon(-\omega) = \epsilon^*(\omega^*).$$

Furthermore, if $G(\tau)$ is finite $\forall \tau$, $\epsilon(\omega)/\epsilon_0$ is analytic in the upper half plane, since integral is convergent there. We can therefore apply Cauchy's theorem for any z in the upper half plane

$$\epsilon(z)/\epsilon_0 = 1 + \frac{1}{2\pi i} \oint d\omega' \, \frac{\epsilon(\omega')/\epsilon_0 - 1}{\omega' - z}.$$

If we assume that ϵ falls off as fast as $1/\omega^2$, the contribution from the semi-circle at infinity vanishes, and we have

$$\epsilon(z)/\epsilon_0 = 1 + \frac{1}{2\pi i} \int_{-\infty}^\infty d\omega' \, \frac{\epsilon(\omega')/\epsilon_0 - 1}{\omega' - z}.$$



We now consider a pole just *above* the ω-axis, by writing $z = \omega + i\delta$. Then

$$\frac{1}{\omega' - \omega - i\delta} = \mathrm{P}\left(\frac{1}{\omega' - \omega}\right) + i\pi\delta(\omega' - \omega)$$

whence

$$\epsilon(\omega)/\epsilon_0 = 1 + \frac{1}{\pi i}\mathrm{P}\int_{-\infty}^{\infty} d\omega' \, \frac{\epsilon(\omega')/\epsilon_0 - 1}{\omega' - \omega}.$$

Thus taking the real and imaginary parts, we find

$$\begin{aligned} \Re\epsilon/\epsilon_0 &= 1 + \frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty} d\omega' \, \frac{\Im\epsilon(\omega')/\epsilon_0}{\omega' - \omega} \\ \Im\epsilon/\epsilon_0 &= -\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty} d\omega' \, \frac{\Re\epsilon(\omega')/\epsilon_0 - 1}{\omega' - \omega} \end{aligned} \tag{7.47}$$

These are the **Kronig-Kramer** relations; they relate **absorption** (*imaginary* part of ϵ) to **dispersion** (*real* part of ϵ) through analyticity.

Chapter 8

Wave Guides and Cavities

In this chapter we will consider propagation of waves in hollow, metal wave guides and cavities.

- *wave guide*: ends are *open*
- *cavity*: ends are *closed*

8.1 Boundary Conditions at Surface of Conductor

Recall that at the boundary between two media, 1 and 2, we have

$$\begin{aligned} (\underline{H}_2 - \underline{H}_1) \times \underline{n} &= \underline{K} \\ (\underline{B}_2 - \underline{B}_1) \cdot \underline{n} &= 0 \\ (\underline{D}_2 - \underline{D}_1) \cdot \underline{n} &= \sigma \\ (\underline{E}_2 - \underline{E}_1) \times \underline{n} &= 0. \end{aligned}$$

Inside a conductor, the electrons are completely free, with infinitely fast response, such that $\underline{B} = \underline{E} = 0$.
Thus our boundary conditions just below the conducting surface reduce to

$$\begin{aligned} \underline{H} \times \underline{n} &= K \\ \underline{B} \cdot \underline{n} &= 0 \end{aligned}$$

$$\begin{aligned} \underline{D} \cdot \underline{n} &= \sigma \\ \underline{E} \times \underline{n} &= 0. \end{aligned}$$

Thus just outside the surface of the conductor, we have that

- $\underline{B}$ is **tangential** to the surface.
- $\underline{E}$ is **normal** to the surface.

The case where we do not have a **perfect** conductor is discussed in detail in *Jackson*, chapter 8.1. Note that in these cases we have energy losses associated with the absorption at the boundary surface.

8.2 Propagation of Monochromatic Wave

We consider the propagation of monochromatic waves in a hollow cylinder, of arbitrary cross section, which we take to be uniform along, say, the z-direction. We assume a harmonic time dependence $e^{-i\omega t}$, so that Maxwells equations become

$$\begin{aligned} \underline{\nabla} \times \underline{E} &= i\omega \underline{B} \\ \underline{\nabla} \cdot \underline{B} &= 0 \\ \underline{\nabla} \times \underline{B} &= -i\mu\epsilon\omega \underline{E} \\ \underline{\nabla} \cdot \underline{E} &= 0 \end{aligned}$$

Thus, in the usual way, these equations reduce to

$$(\nabla^2 + \mu\epsilon\omega^2) \left\{ \begin{array}{c} \underline{E} \\ \underline{B} \end{array} \right\} = 0 \tag{8.1}$$

Because of the cylindrical symmetry in the problem, we expect to find waves travelling in the positive or negative direction, or standing waves. Therefore we look for solutions of the form

$$\left. \begin{array}{c} \underline{E}(\underline{x}, t) \\ \underline{B}(\underline{x}, t) \end{array} \right\} = \left\{ \begin{array}{c} \underline{E}(x, y) \\ \underline{B}(x, y) \end{array} \right\} e^{\pm ikz - i\omega t}.$$

Note: this *does not mean* that the propagation vector is in the z direction as such.
We now write

$$\nabla^2 = \nabla_T^2 + \nabla_z^2$$

where

$$\begin{aligned} \nabla_T^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \\ \nabla_z^2 &= \frac{\partial^2}{\partial z^2}. \end{aligned}$$

Then our wave equation eqn. (8.1) reduces to

$$[\nabla_T^2 + (\mu\epsilon\omega^2 - k^2)]\underline{E} = 0 \tag{8.2}$$

and similarly for $\underline{B}$.
We now write $\underline{E}$ and $\underline{B}$ in terms of components parallel and transverse to z, i.e. $\underline{E} = \underline{E}_z + \underline{E}_T$ etc., and show that it is only necessary to solve for the longitudinal components E_z and B_z.
We start with two of Maxwell's eqns

$$\begin{aligned} \underline{\nabla} \times \underline{E} &= i\omega\underline{B} \\ \underline{\nabla} \times \underline{B} &= -i\mu\epsilon\omega\underline{E}. \end{aligned} \tag{8.3}$$

Writing the first of these in terms of longitudinal and transverse components, we have

$$(\underline{\nabla}_T + \underline{\nabla}_z) \times (\underline{E}_T + \underline{E}_z) = i\omega(\underline{B}_T + \underline{B}_z).$$

If we now consider the transverse and longitudinal components, we find

$$\underline{\nabla}_T \times \underline{E}_T = i\omega\underline{B}_z \tag{8.4}$$

$$\underline{\nabla}_T \times \underline{E}_z + \underline{\nabla}_z \times \underline{E}_T = i\omega\underline{B}_T. \tag{8.5}$$

From the second of these, we find

$$\begin{aligned} i\omega\underline{\nabla}_z \times \underline{B}_T &= \underline{\nabla}_z \times [\underline{\nabla}_T \times \underline{E}_z + \underline{\nabla}_z \times \underline{E}_T] \\ &= \underline{\nabla}_T[\underline{\nabla}_z \cdot \underline{E}_z] - \underline{\nabla}_z^2\underline{E}_T \end{aligned}$$

THen, using the z-dependence of $\underline{E}, \underline{B}$, we find

$$i\omega \underline{\nabla}_z \times \underline{B}_T = \underline{\nabla}_T[\underline{\nabla}_z \cdot \underline{E}_z] + k^2 \underline{E}_T. \tag{8.6}$$

To proceed further, we use the second equation of (8.3), which becomes

$$\begin{aligned} \underline{\nabla}_T \times \underline{B}_T &= -i\mu\epsilon\omega \underline{E}_z \\ \underline{\nabla}_T \times \underline{B}_z + \underline{\nabla}_z \times \underline{B}_T &= -i\mu\epsilon\omega \underline{E}_T. \end{aligned}$$

Substituting in eqn. (8.6), we find

$$i\omega[-i\mu\epsilon\omega \underline{E}_T - \underline{\nabla}_T \times \underline{B}_z] = k^2 \underline{E}_T + \underline{\nabla}_T[\underline{\nabla}_z \cdot \underline{E}_z],$$

yielding

$$\begin{aligned} \underline{E}_T &= (\mu\epsilon\omega^2 - k^2)^{-1}[\underline{\nabla}_T(\underline{\nabla}_z \cdot \underline{E}_z) - i\omega \underline{e_z} \times \underline{\nabla}_T B_z] \\ \underline{H}_T &= (\mu\epsilon\omega^2 - k^2)^{-1}[\underline{\nabla}_T(\underline{\nabla}_z \cdot \underline{H}_z) + i\epsilon\omega \underline{e_z} \times \underline{\nabla}_T E_z] \end{aligned} \tag{8.7}$$

with an analogous equation for $\underline{B}_T$.
Thus we can see that we have expressed the **transverse** components entirely in terms of **longitudinal** componenets.

8.3 Classification of Modes

We have now shown that the propagation of the waves can be solved solely by solving the two-dimensional wave equation

$$(\underline{\nabla}_T^2 + \mu\epsilon\omega^2 - k^2) \left\{ \begin{array}{c} E_z(x, y) \\ B_z(x, y) \end{array} \right\} = 0, \tag{8.8}$$

subject to suitable boundary conditions. In the case of perfectly conducting walls S, the boundary conditions are

$$\begin{aligned} \underline{n} \times \underline{E}|_S &= 0 \\ \underline{n} \cdot \underline{B}|_S &= 0. \end{aligned}$$

It can be shown that these boundary conditions are equivalent to

$$E_z = 0 \tag{8.9}$$

$$\frac{\partial B_z}{\partial n} = 0. \tag{8.10}$$

Thus we are in principle simultaneously solving two boundary-value equations subject to each of the above conditions. However, in general the eigenvalue equation (8.2) will have *different* eigenvalues for the two different sets of boundary conditions. Hence we cannot satisfy both simultaneously **unless one is trivial**. Thus we classify the solutions as

Transverse Magnetic (TM)

Here $B_z = 0$ everywhere, and $E_z = 0$ on boundary. The differential equation (8.8)a with the above Dirichlet boundary condition determines E_z in the wave guide. If we know E_z, the transverse fields can be obtained from Eq. (8.7):

$$E_T = \frac{ik}{\gamma^2}\vec{\nabla}_T E_z, \qquad H_T = \frac{i\epsilon\omega}{\gamma^2}\hat{e}_3 \times \vec{\nabla}_T E_z \tag{8.11}$$

Transverse Electric (TE)

$E_z = 0$ everywhere, and $\frac{\partial B_z}{\partial n} = 0$ on boundary. Hre we must solve the Eq. (8.8)b with Neumann boundary condition. The transverse fields are

$$E_T = -\frac{i\mu\omega}{\gamma^2}\hat{e}_3 \times \vec{\nabla}_T H_z, \quad H_T = \frac{ik}{\gamma^2}\vec{\nabla}_T H_z \tag{8.12}$$

Finally, we must consider

Transverse Electric Magnetic (TEM)

Here we have $B_z = E_z = 0$ everywhere, so that the only non-trivial components are those in the transverse direction. Then Maxwell's equations reduce to

$$\begin{aligned} \underline{\nabla}_T \times \underline{E}_{\mathrm{TEM}} &= 0 \\ \underline{\nabla}_z \times \underline{E}_{\mathrm{TEM}} &= i\omega \underline{B}_{\mathrm{TEM}}. \end{aligned}$$

In addition, we have

$$\underline{\nabla}_T \cdot \underline{E}_{\text{TEM}} = 0.$$

Combining the first and third of these equations, we find

$$\underline{\nabla}_T^2 \underline{E}_{\text{TEM}} = 0,$$

and comparing with the wave equation (8.2), we find

$$k^2 = \mu\epsilon\omega^2.$$

This is just the **infinite-medium value**. Similarly, we find

$$\underline{B}_{\text{TEM}} = \pm\sqrt{\mu\epsilon}\,\underline{e_Z} \times \underline{E}_{\text{TEM}}.$$

Thus we essentially have **plane-wave propagation**.
We see that $\underline{E}_{\text{TEM}}$ obeys Laplace's equation. Furthermore, the walls of the wave guide are an equipotential. Thus the only solution inside a single, hollow perfect conductor is the trivial one.

TEM modes cannot propagate inside a single conductor

They can, however, propagate inside a coaxial cable.

8.4 Modes of a Waveguide

We begin by discussing TM modes, for which we write

$$E_z = \phi(x, y)e^{\pm ikz - i\omega t}.$$

Then ψ satisfies

$$(\underline{\nabla}_T^2 + \mu\epsilon\omega^2 - k^2)\psi = 0,$$

subject to $\phi = 0$ on the boundary.
We now introduce

$$\gamma^2 = \mu\epsilon\omega^2 - k^2,$$

so that our eigenvalue equation becomes

$$(\underline{\nabla}_T^2 + \gamma^2)\phi = 0.$$

In general, the boundary equations require that γ^2 be positive, yielding a discrete set of eigenvalues $\{\gamma_\lambda\}$, with corresponding wave number

$$k_\lambda^2 = \mu\epsilon\omega^2 - \gamma_\lambda^2. \tag{8.13}$$

If $k_\lambda^2 > 0$, k_λ is real, and the propagation is oscillatory. If it is negative, the wave number is imaginary and the wave will no propagate.
We define the cut-off frequency ω_λ by

$$\omega_\lambda = \frac{\gamma_\lambda}{\sqrt{\mu\epsilon}} \tag{8.14}$$

where

- $\omega < \omega_\lambda$: wave cannot propagate
- $\omega > \omega_\lambda$: wave can propagate

Finally, it is worth noting that the group velocity of the wave in the wave guide is always smaller than the speed of light. We first note that we may write

$$k_\lambda = \sqrt{\mu\epsilon}\sqrt{\omega^2 - \omega_\lambda^2}.$$

We recall that the **phase velocity**

$$\begin{aligned} v_p &= \omega/k \\ &= \frac{1}{\sqrt{\mu\epsilon}}\frac{1}{\sqrt{1-\omega_\lambda^2/\omega^2}} \\ &= \frac{c}{\sqrt{1-\omega_\lambda^2/\omega^2}} \end{aligned}$$

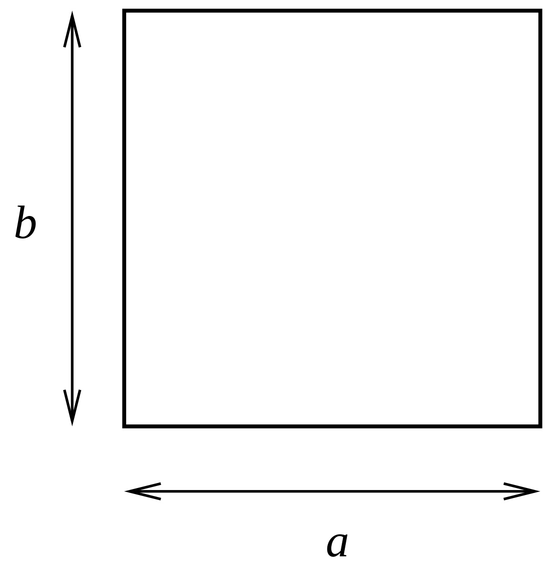
which is always **larger** than the velocity of light, and diverges as $\omega \to \omega_\lambda$.
In contrast, the **group velocity**

$$v_g = \left(\frac{dk}{d\omega}\right)^{-1} = c\sqrt{1-\omega_\lambda^2/\omega^2},$$

which is always **smaller** than the infinite-space velocity of light, and vanishes as $\omega \to \omega_\lambda$. In this limit the *wave no longer propagates.* Note that

$$v_p v_g = c^2.$$

8.5 Modes of a Rectangular Waveguide



For the sake of illustration, we will consider the case of **TE modes**. In Cartesian coordinates, we have to solve the eigenvalue equation

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2\right]\psi = 0$$

subject to

$$\begin{aligned} \frac{\partial \psi(0,y)}{\partial x} = \frac{\partial \psi(a,y)}{\partial x} &= 0, \\ \frac{\partial \psi(x,0)}{\partial y} = \frac{\partial \psi(x,b)}{\partial y} &= 0. \end{aligned}$$

This clearly has eigenfunctions for H_z

$$\psi_{mn}(x,y) = H_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right)$$

with eigenvalues

$$\gamma_{mn}^2 = \pi^2 \left[\frac{m^2}{a^2} + \frac{n^2}{b^2} \right].$$

We denote the modes $\mathrm{TE}_{m,n}$. The lowest non-trivial mode is $\mathrm{TE}_{1,0}$ if $a > b$, with cut-off frequency given by

$$\gamma_{10}^2 = \pi^2/a^2.$$

For this mode, for wave propagating in the positive direction, we have

$$H_z = H_0 \cos\left(\frac{\pi x}{a}\right) e^{ik_{1,0}z - i\omega t}.$$

We can obtain the transverse components of the field from eqn. (8.12)

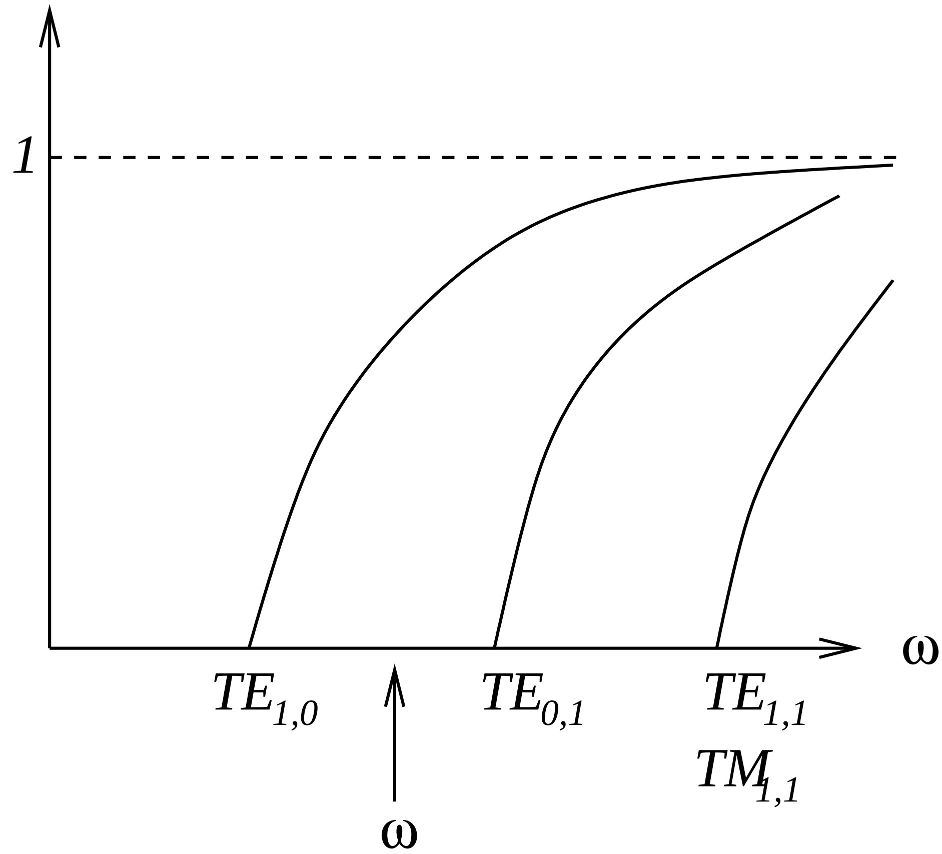
$$\begin{aligned} \underline{H_T} &= -\frac{ika}{\pi} H_0 \sin\left(\frac{\pi x}{a}\right) e^{ikz-i\omega t} \underline{e_x} \\ \underline{E_T} &= \frac{i\omega a\mu}{\pi} H_0 \sin\left(\frac{\pi x}{a}\right) e^{ikz-i\omega t} \underline{e_y}, \end{aligned}$$

with $k = k_{1,0}$.

The analysis of TM modes proceeds likewise. However, here the lowest propagating mode is $\mathrm{TM}_{1,1}$, with a higher cut-off frequency. Wave guides are often constructed such that $\mathrm{TE}_{1,0}$ is the *only* propagating mode. Recalling that

$$k_\lambda = \sqrt{\mu\epsilon\left(\omega^2 - \omega_\lambda^2\right)}$$

we can show $k_\lambda/\sqrt{\mu\epsilon\omega}$ as follows:



8.5.1 Energy Flux along Waveguide

The time-averaged energy flux is given by the *real part* of the **Poynting Vector**

$$\underline{S} = \frac{1}{2}\,\underline{E} \times \underline{H}^*.$$

Let us evaluate this for **TE** modes

$$\underline{S} = \frac{1}{2}\underline{E} \times \underline{H}^* = \frac{1}{2}(\underline{E}_T \times \underline{H}_T^* - H_z^* \hat{e}_3 \times \underline{E}_T).$$

Since $H_z = \psi(x,y)e^{-iwt+ikz}$ we get from Eq. (8.12)

$$\underline{S} = \frac{\omega k \mu}{2\gamma^4}\underline{\nabla}_T H_z \times (\hat{e}_3 \times \underline{\nabla}_T H_z^*) - \frac{i\omega\mu}{\gamma^2} H_z^* \hat{e}_3 \times (\hat{e}_3 \times \underline{\nabla}_T H_z) \ = \ \frac{\omega k \mu}{2\gamma^4}\hat{e}_3 |\underline{\nabla}_t \psi|^2 - i\frac{\omega\mu}{\gamma^2}\psi^* \underline{\nabla}_T \psi$$

Taking the real part, we get

$$\Re \underline{S} = \frac{\omega k \mu}{2\gamma^4}|\underline{\nabla}_T \psi|^2 \hat{e}_3.$$

This is in the z-direction, and we see that energy propagation is along the waveguide.

Similarly, for the **TM** wave $E_z = \phi(x,y)e^{-i\omega t+ikz}$ one obtains

$$\Re \underline{S} = \frac{\omega k \epsilon}{2\gamma^4}|\underline{\nabla}_T\phi|^2\hat{e}_3.$$

The total power transmitted by the **TE** wave is

$$P = \Re\int_A \underline{S}\cdot\underline{e_z}\,dA = \frac{\omega k\mu}{2\gamma^4}\int dA\,(\underline{\nabla}_T\psi)^*\cdot(\underline{\nabla}_T\psi).$$

where A is a cross-section through the wave guide. Recalling Green's identity, we have

$$\int(\psi^*\underline{\nabla}_T^2\psi + \underline{\nabla}_T\psi^*\cdot\underline{\nabla}_T\psi)\,dA = \oint_C \psi^*\frac{\partial\psi}{\partial n}dl.$$

Because of the boundary conditions, either $\frac{\partial\psi}{\partial n}$ or ϕ (for the TM mode) vanish on the surface. Thus

$$\begin{aligned} P &= -\frac{\omega k\mu}{2\gamma^4}\int_A \psi^*\underline{\nabla}_T^2\psi\,dA \\ &= \frac{\omega k\mu}{2\gamma^4}\gamma^2\int_A|\psi|^2\,dA, \end{aligned}$$

using wave equation

$$(\underline{\nabla}_T^2+\gamma^2)\psi = 0.$$

Thus we have

$$P = \frac{\mu}{2\sqrt{\mu\epsilon}}\left(\frac{\omega}{\omega_\lambda}\right)^2\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)^{1/2}\int_A\psi^*\psi\,dA, \qquad (8.15)$$

where we represented k as $\omega\sqrt{\mu\epsilon}\sqrt{1-\frac{\omega_\lambda^2}{\omega^2}}$ and γ^2 as $\mu\epsilon\omega_\lambda^2$).

Similarly, for the **TM** modes we get

$$P = \frac{\mu}{2\sqrt{\mu\epsilon}}\left(\frac{\omega}{\omega_\lambda}\right)^2\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)^{1/2}\int_A\phi^*\phi\,dA, \qquad (8.16)$$

From Chapter 7, we have that the field energy per unit length is given by

$$\begin{aligned} \langle U\rangle &= \frac{1}{4}\int[\epsilon\underline{E}\cdot\underline{E}^* + \mu\underline{H}\cdot\underline{H}^*]\,dA = \frac{1}{4}\int[\epsilon\underline{E}_T\cdot\underline{E}_T^* + \mu\underline{H}_T\cdot\underline{H}_T^* + \mu H_z\cdot H_z^*]\,dA \\ &= \frac{\mu}{4\gamma^4}(\mu\epsilon\omega^2+k^2)\int[|\underline{\nabla}_T\psi|^2+\mu|\psi|^2]\,dA = \frac{\mu}{4\gamma^2}(\mu\epsilon\omega^2+k^2+\gamma^2)\int|\psi|^2\,dA \end{aligned}$$

where we have used the fact that $\int |\underline{\nabla}_T \psi|^2 = \gamma^2 \int |\psi|^2$ since $\underline{\nabla}_T^2 \psi = -\gamma^2 \psi$. Finally, we obtain

$$\langle U \rangle = \frac{\mu^2 \epsilon \omega^2}{2\gamma^2} \int |\psi|^2 dA = \frac{\mu}{2} \frac{\omega^2}{\omega_\lambda^2} \int |\psi|^2 \, dA.$$

Using eqns. (8.15) and (8.5.1), we find

$$\boxed{P/U = \frac{1}{\sqrt{\mu\epsilon}} \left(1 - \frac{\omega_\lambda^2}{\omega^2}\right)^{1/2} \equiv v_g \qquad (8.17)}$$

Thus we see that the **energy propagates** with the **group velocity**. *N.B.* you should convince yourself that this expression has the correct dimension.
For the **TM** wave, we get

$$\langle U \rangle = \frac{\mu}{2} \frac{\omega^2}{\omega_\lambda^2} \int |\phi|^2 \, dA.$$

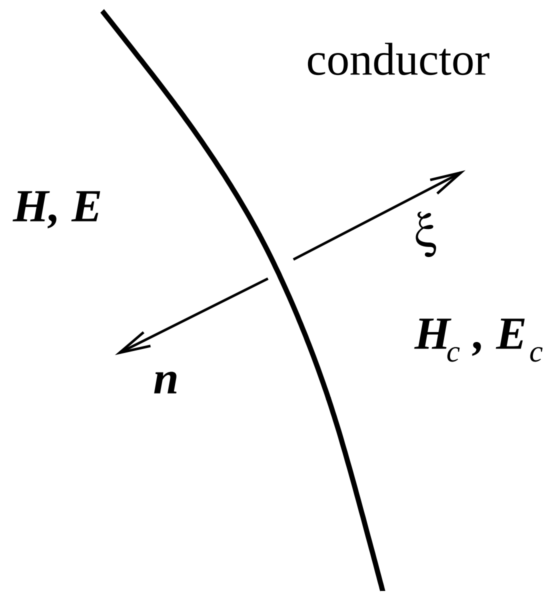
yielding the same result (8.17) for group velocity.

8.6 Boundary Conditions at Surface of Good Conductor

At surface of infinitely good conductor, we have

$$\begin{aligned} \underline{n} \cdot \underline{B} &= 0 \\ \underline{n} \times \underline{E} &= 0 \\ \underline{n} \cdot \underline{D} &= \Sigma \\ \underline{n} \times \underline{H} &= \underline{K} \end{aligned} \qquad (8.18)$$

where Σ is the surface charge density.



In the case of a conductor of conductivity σ, we have

$$\underline{J} = \sigma \underline{E}.$$

We cannot have a surface current, since that would imply an infinite tangential $\underline{E}$. Instead, we have

$$\underline{n} \times (\underline{H} - \underline{H_c}) = 0.$$

where we use the subscript c to denote fields inside the conductor. (As $\sigma \to \infty$, we recover our surface current as a volume current over the thin layer close to the boundary).

We obtain the results for finite conductivity by successive approximation. We assume that initially $\underline{E}$ is perpendicular, and $\underline{H}$ parallel, to the surface just outside the conductor. Then $\underline{H_c}|_{\text{surface}} \simeq \underline{H_{\|}}$, and Maxwell's equations within the conductor become

$$\begin{aligned} \underline{\nabla} \times \underline{E_c} + \frac{1}{\mu_c}\frac{\partial \underline{H}_c}{\partial t} &= 0 \\ \underline{\nabla} \times \underline{H}_c = \underline{J} + \frac{\partial \underline{D}_c}{\partial t} \end{aligned}$$

. If we assume harmonic time depedence, these reduce to

$$\underline{H_c} = -\frac{i}{\mu_c \omega} \underline{\nabla} \times \underline{E_c}$$

$$\underline{\nabla} \times \underline{H_c} \;=\; \sigma \underline{E_c} - i\omega\epsilon \underline{E_c}.$$

Thus if σ is sufficiently large, these reduce to

$$\begin{aligned} \underline{H_c} &= -\frac{i}{\mu_c\omega}\underline{\nabla} \times \underline{E_c} \\ \underline{E_c} &= \frac{1}{\sigma}\underline{\nabla} \times \underline{H_c}. \end{aligned}$$

We now assume all variation to be normal to the surface. (Spatial variation of the fields on the normal direction is much more rapid than in the parallel direction so we can neglect $\underline{\nabla}_{\|}$ in comparison to $\underline{\nabla}_T$). Then we have

$$\underline{\nabla} = -\underline{n}\frac{\partial}{\partial\xi}$$

and our equations become

$$\begin{aligned} \underline{H_c} &= \frac{i}{\mu_c\omega}\underline{n} \times \frac{\partial \underline{E_c}}{\partial\xi} \\ \underline{E_c} &= -\tfrac{1}{\sigma}\underline{n} \times \tfrac{\partial \underline{H_c}}{\partial\xi}. \end{aligned}$$

We immediately see that $\underline{n} \cdot \underline{H_c} = 0$, consistent with our boundary assumptions. Furthermore, combining these two equations we obtain

$$\underline{H_c} = -\frac{i}{\mu_c\omega\sigma}\underline{n} \times [\underline{n} \times \frac{\partial^2 \underline{H_c}}{\partial\xi^2}],$$

yielding

$$\frac{\partial^2}{\partial\xi^2}\underline{H_c} + \frac{2i}{\delta^2}\underline{H_c} = 0,$$

where

$$\delta \equiv \left(\frac{2}{\mu_c\omega\sigma}\right)^{1/2},$$

is the skin depth. Thus, combining this we the condition $\underline{n} \cdot \underline{H_c} = 0$, we find

$$\underline{H_c} = \underline{H_{\|}} e^{(i-1)\xi/\delta}. \tag{8.19}$$

Thus the magnetic field is **tangential** and falls off **exponentially** as we go into the conductor. We can differentiate this, to obtain

$$\underline{E_c} = \sqrt{\frac{\mu\omega}{2\sigma}}(1-i)(\underline{n} \times \underline{H_\parallel})e^{-\xi/\delta}e^{i\xi/\delta}. \tag{8.20}$$

Thus $\underline{E_c}$ is also tangential to the surface, but of much smaller magnitude.
We now go back to our boundary condition

$$\underline{n} \times (\underline{E} - \underline{E_c}) = 0.$$

Since $\underline{E}_c$ has a small tangential component, so does $\underline{E}$ just outside the conductor.

$$\underline{E}_\parallel = \sqrt{\frac{\mu\omega}{2\sigma}}(1-i)(\underline{n} \times \underline{H_\parallel}).$$

Thus there is a non-zero component of the Poynting vector into the conductor, and hence a net flow of energy, given by

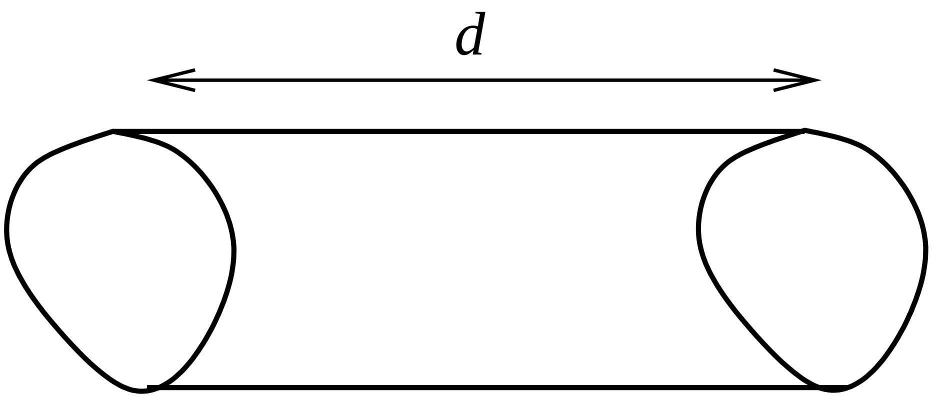
$$\begin{aligned}
\langle \frac{dP}{da} \rangle &= \frac{1}{2}\Re\left[\underline{E} \times \underline{H}^*\right] \cdot (-\underline{n}) \\
&= \frac{\mu_c \omega \delta}{4} |\underline{H}_\parallel|^2 \\
&= \frac{1}{2\sigma\delta} |\underline{H}_\parallel|^2.
\end{aligned}$$

It can be demonstrated that this power is dissipated into heat as ohmic losses in the skin of the conductor.
Applying this to our wave guide, we see that we have an energy loss/unit length given by

$$\begin{aligned}
\frac{dP}{dz} &= -\frac{1}{2\sigma\delta} \oint_C dl\, |\underline{H}_\parallel|^2 = -\frac{1}{2\sigma\delta} \oint_C dl\, |\underline{n} \times \underline{H}|^2 \\
&= \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_\lambda}\right)^2 \oint_C dl \begin{cases} \frac{1}{\mu^2\omega_\lambda^2}\left|\frac{\partial\phi}{\partial n}\right|^2 & \text{(TM)} \\ \frac{1}{\mu\epsilon\omega_\lambda^2}\left(1 - \frac{\omega_\lambda^2}{\omega^2}\right)|\underline{n} \times \underline{\nabla}_T\psi|^2 + \frac{\omega_\lambda^2}{\omega^2}|\psi|^2 & \text{(TE)} \end{cases}
\end{aligned}$$

8.7 Resonant Cavities

A resonant cavity differs from a wave guide in being closed. Thus, rather than having wave propagation, we have standing waves.



As before, we can have both TM and TE fields. However, now the z-dependence is of the form, for the case of **TM modes**,

$$\begin{aligned}\underline{E_z} &= \phi(x,y)[A\sin kz + B\cos kz]\underline{e_z}\\ \underline{H_z} &= 0\end{aligned}$$

Then the transverse part of the wave is

$$\begin{aligned}\underline{E_T} &= \frac{1}{\gamma^2}[\underline{\nabla}_T(\underline{\nabla}_z\cdot\underline{E_z}) - i\omega\underline{e_z}\times\underline{\nabla}_T B_z]\\ &= \frac{k}{\gamma^2}\underline{\nabla}_T\phi(x,y)[A\cos kz - B\sin kz].\end{aligned}$$

Now the boundary condition $\underline{E_T} = 0$ at $z = 0, z = d$ yields $A = 0, k = p\pi/d$ and thus

$$E_z = \phi(x,y)\cos\frac{p\pi}{d} \tag{8.21}$$

$$\underline{E_T} = -\frac{p\pi}{d\gamma^2}\sin\frac{p\pi z}{d}\underline{\nabla}_T\phi. \tag{8.22}$$

We can obtain $\underline{H_T}$ similarly, yielding

$$\underline{H_T} = \frac{i\epsilon\omega}{\gamma^2}\cos\frac{p\pi z}{d}\underline{e_z}\times\underline{\nabla}_T\phi. \tag{8.23}$$

A corresponding analysis for the **TE modes** yields

$$H_z = \psi(x,y)(A\sin kz + B\cos kz)$$

so $E_T = -\frac{i\omega\mu}{\gamma^2}(A\sin kz + B\cos kz)\underline{e}_z \times \underline{\nabla}_T\psi$. From the boundary conditions $\underline{E}_T|_{z=0,d} = 0$ we get

$$\begin{aligned} H_z &= \psi(x,y)\sin\frac{p\pi z}{d} \\ \underline{E_T} &= -\frac{i\omega\mu}{\gamma^2}\sin\frac{p\pi z}{d}\underline{e_z}\times\underline{\nabla}_T\psi \\ \underline{H_T} &= \frac{p\pi}{d\gamma^2}\cos\frac{p\pi z}{d}\underline{\nabla}_T\psi. \end{aligned} \tag{8.24}$$

The function $\psi(x,y)$ now satisfies the wave equation

$$\underline{\nabla}_T^2\psi + [\mu\epsilon\omega^2 - \left[\left(\frac{p\pi}{d}\right)^2\right]\psi = 0$$

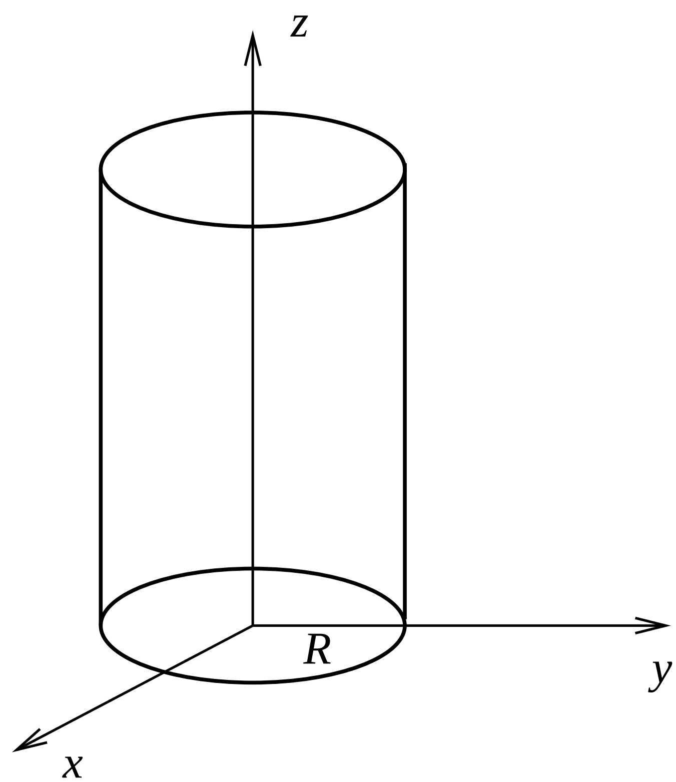
where

$$\gamma^2 = \mu\epsilon\omega^2 - \frac{p^2\pi^2}{d^2}.$$

We can solve this eigenvalue problem as for propatation along a wave guide, but now the eigenvalues γ_λ determine not the cut-off frequencies but the **allowed frequencies**:

$$\omega_{\lambda p}^2 = \frac{1}{\mu\epsilon}\left[\gamma_\lambda^2 + \frac{p^2\pi^2}{d^2}\right] \tag{8.25}$$

Example: cylindrical cavity, radius R



We work in cylindrical polar coords $\psi(s, \varphi)$. Because of cylindrical symmetry, we seek separable solutions to the two-dimensional wave equation of the form

$$\psi(s, \varphi) = \psi(s) e^{\pm im\varphi}$$

where $m = 0, 1, 2, \ldots$. Then we have

$$\left(\frac{\partial^2}{\partial s^2} + \frac{1}{s} \frac{\partial}{\partial s} + \gamma^2 - \frac{m^2}{s^2} \right) \psi(s) = 0.$$

This is just Bessel's equation (see last semester), with solution

$$\psi(s, \varphi) = J_m(\gamma_{mn} s) e^{\pm im\varphi}.$$

In the case of a **TM mode**, where $\psi(s, \varphi) = 0$ at $s = R$, we have

$$\gamma_{mn} R = x_{mn},$$

where x_{mn} is the n^{th} root of $J_m(x) = 0$. Thus the resonant requencies are given by

$$\omega_{mnp}^2 = \frac{1}{\mu\epsilon}\left[\frac{x_{mn}^2}{R^2} + \frac{p\pi^2}{d^2}\right] \qquad \text{(TM mode)}. \tag{8.26}$$

The solution for **TE modes** is similar and the resonant frequencies are given by

$$\omega_{mnp}^2 = \frac{1}{\mu\epsilon}\left[\frac{x_{mn}'^2}{R^2} + \frac{p^2\pi^2}{d^2}\right] \qquad \text{(TE mode)}, \tag{8.27}$$

where x'_{mn} is now the n^{th} root of $J'_m(x) = 0$.

Note that for TM modes we have $p = 0, 1, 2, \ldots$ whilst for TE modes we have $p = 1, 2, 3, \ldots$. Furthermore, the smallest $x'_{mn} < \min(x_{mn})$, and thus for sufficiently large d the dominant mode is

$$\text{TE}_{1,1,1}.$$

We can compute the energy loss is a resonant cavity in a similar manner to that for a wave guide.

Chapter 9

Radiating Systems

In this chapter, we will study radiation of varying current distributions. We will begin by working in Lorentz gauge, where the equation for the *vector potential* is

$$\nabla^2 \underline{A} - \frac{1}{c^2}\frac{\partial^2 \underline{A}}{\partial t^2} = -\mu_0 \underline{J}.$$

From Chapter 6, we recall that this has the **retarded** solution

$$\underline{A} = \frac{\mu_0}{4\pi}\int d^3x'\, dt'\, \underline{J}(\underline{x}', t') \times G^{(+)}(\underline{x}, t; \underline{x}', t'),$$

where

$$G^{(+)}(\underline{x}, t; \underline{x}', t') = \frac{1}{|\underline{x} - \underline{x}'|}\delta(t' - t + \frac{|\underline{x} - \underline{x}'|}{c}).$$

We now consider the case where the fields arise from a current with harmonic time variation

$$\underline{J}(\underline{x}, t) = \underline{J}(\underline{x})e^{-i\omega t}$$

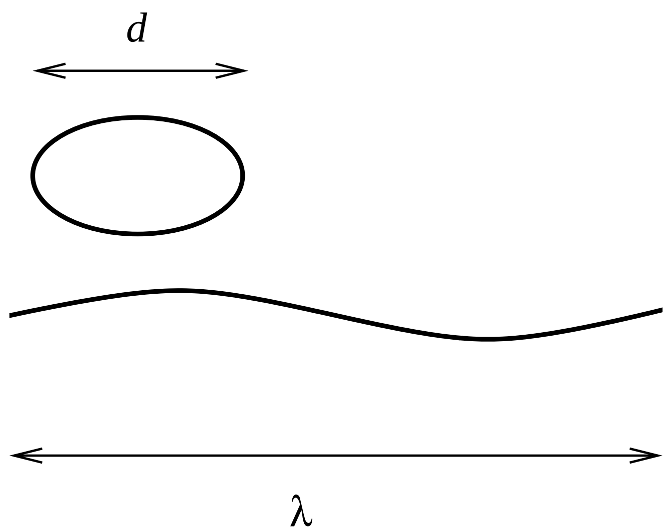
More general time dependence can be studied simply by taking the Fourier transform. The potential corresponding to this current is then

$$\begin{aligned} \underline{A}(\underline{x}, t) &= \frac{\mu_0}{4\pi}\int d^3x'\, dt'\, \underline{J}(\underline{x}')e^{-i\omega t'} G^{(+)}(\underline{x}, t; \underline{x}', t') \\ &= \underline{A}(\underline{x})e^{-i\omega t}, \end{aligned}$$

with

$$\underline{A}(\underline{x}) = \frac{\mu_0}{4\pi}\int d^3x'\, \underline{J}(\underline{x}')\frac{1}{|\underline{x} - \underline{x}'|}e^{ik|\underline{x} - \underline{x}'|}.$$

where $k \equiv \omega/c$ is the wave number.



We will now consider the form of the field a distance r away from a localised, time-varying source of extent d. We begin by introducing the wavelength

$$\lambda = \frac{2\pi}{k} \equiv \frac{2\pi c}{\omega},$$

where $\lambda \gg d$.

We now consider the form of the potential in three different regions:

1. $d \ll r \ll \lambda$ - the **near zone**

   Then $\exp ik|\underline{x} - \underline{x}'| \sim \exp 2\pi i r/\lambda \sim 1$, and we have

   $$\underline{A}(\underline{x}) \simeq \frac{\mu_0}{4\pi} \int d^3x' \, \underline{J}(\underline{x}') \frac{1}{|\underline{x} - \underline{x}'|}.$$

   The field is of the familiar form which we can expand as a series in, say, Legendre polynomials.

2. $r \gg \lambda \gg d$ - the **radiation zone**

   The the exponent is rapidly oscillating, and we can write

   $$\begin{aligned} |\underline{x} - \underline{x}'| &= [x^2 - 2\underline{x} \cdot \underline{x}' + x'^2]^{1/2} \\ &\simeq r - \underline{n} \cdot \underline{x}' + \mathcal{O}\left(\frac{|\underline{x}'|^2}{r}\right). \end{aligned}$$

Thus, to leading order in $1/r$ we have

$$\underline{A}(\underline{x}) \stackrel{kr\to\infty}{\longrightarrow} \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\, \underline{J}(\underline{x}')e^{-ik\underline{n}\cdot\underline{x}'} \tag{9.1}$$

where $\underline{n}$ is a unit vector in the radial direction. Thus we have an outgoing spherical wave. We can compute the magnetic and electric fields through

$$\begin{aligned} \underline{H} &= \frac{1}{\mu_0}\underline{\nabla}\times\underline{A} \\ \underline{E} &= \frac{iZ_0}{k}\underline{\nabla}\times\underline{H} \end{aligned} \tag{9.2}$$

which also fall off as $1/r$, corresponding to **radiation**. (Hereafter $Z_0 \equiv \sqrt{\frac{\mu_0}{\epsilon_0}} = \mu_0 c$).

Since $k\underline{n}\cdot\underline{x}' \ll 1$ - recall that $d \ll \lambda$ - we can expand the exponent in eqn. (9.1) yielding

$$\underline{A}(\underline{x}) \simeq \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\sum_n \frac{(-ik)^n}{n!}\int d^3x'\, \underline{J}(\underline{x}')(\underline{n}\cdot\underline{x}')^n. \tag{9.3}$$

Successive terms are $\mathcal{O}((kd)^n)$, which dies off with increasing n.

3. $r \sim \lambda$ Here we need to expand the solution in terms of the *vector multipole expansion*, discussed in detail in *Jackson, 9.6*.

An analogous analysis for the scalar potential yields

$$\phi(\underline{x},t) = \int d^3x' \int dt'\, \frac{\rho(\underline{x}',t')}{|\underline{x}-\underline{x}'|}\delta(t' + \frac{|\underline{x}-\underline{x}'|}{c} - t).$$

Keeping the leading term yields

$$\phi(\underline{x},t) \simeq \frac{1}{r}q(t' = t - r/c).$$

where q is the toal charge of the source. If the source is localised, and isolated, no charge can flow in and out, and thus the total charge is constant in time- *the monopole part of the potential is static*, i.e. has no time dependence.

9.1 Electric Dipole Fields

If we keep only the leading term in eqn. (9.3), we have

$$\underline{A}(\underline{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\,\underline{J}(\underline{x}'). \qquad (9.4)$$

In fact, as discussed in *Jackson*, this is the leading $l = 0$ term in the vector multipole expansion of the vector potential, and thus valid everywhere outside the source as part of the multipole expansion. We will now show that this corresponds to a dipole term. We begin by recalling the continuity equation

$$\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot\underline{J} = 0$$

which with our assumed time dependence becomes

$$-i\omega\rho + \underline{\nabla}\cdot\underline{J} = 0.$$

We now use integration by parts to write

$$\begin{aligned}\int d^3x'\,\underline{J} = \int d^3x'\,(\underline{J}\cdot\underline{\nabla}')\underline{x}' &= -\int d^3x'\underline{x}'(\underline{\nabla}'\cdot\underline{J}) \\ &= -i\omega\int d^3x'\,\underline{x}'\rho(\underline{x}') \;=\; -\,i\omega\underline{p}\end{aligned}$$

enabling the potential to be expressed as

$$\underline{A}(\underline{x}) = -\frac{i\mu_0\omega}{4\pi}\frac{e^{ikr}}{r}\underline{p}$$

where

$$\underline{p} \equiv \int d^3x'\,\underline{x}'\rho(\underline{x}')$$

is the electric dipole moment.

The magnetic and electric fields are simply obtained from eqn. (9.2).

$$\begin{aligned}\underline{H} &= \frac{1}{\mu_0}\underline{\nabla}\times\underline{A} = \frac{ck^2}{4\pi}(\underline{n}\times\underline{p})\frac{e^{ikr}}{r}(1-\frac{1}{ikr}) \qquad (9.5)\\ \underline{E} &= \frac{iZ_0}{k}\underline{\nabla}\times\underline{H} = \frac{1}{4\pi\epsilon_0}[k^2(\underline{n}\times\underline{p})\times\underline{n}\frac{e^{ikr}}{r} + (3(np)\underline{n}-\underline{p})(\frac{1}{r^3}-\frac{ik}{r^2})e^{ikr}]\end{aligned}$$

In the spherical coordinates it takes the form ($\underline{n} \times \underline{p} = -p \sin\theta \hat{\phi}$)

$$\begin{aligned} \underline{H} &= -\frac{pck^2}{4\pi} \sin\theta \frac{e^{ikr}}{r}(1 - \frac{1}{ikr})\hat{\varphi} \\ \underline{E} &= \frac{p}{4\pi\epsilon_0}[-k^2 \frac{e^{ikr}}{r} \sin\theta\hat{\theta} + (2\cos\theta\hat{r} + \sin\theta\hat{\theta})(\frac{1}{r^3} - \frac{ik}{r^2})e^{ikr}] \end{aligned} \tag{9.6}$$

It is interesting to examine their limiting forms

- **Radiation Zone**: $r \gg \lambda \gg d$:

$$\begin{aligned} \underline{H} &= \frac{ck^2}{4\pi}(\underline{n} \times \underline{p})\frac{e^{ikr}}{r} &=& -\frac{\omega^2 p}{4\pi c}\hat{\varphi}\sin\theta\frac{e^{ikr}}{r} \\ \underline{E} &= Z_0 \underline{H} \times \underline{n} &=& -\frac{\mu_0}{4\pi}\omega^2 p \sin\theta \frac{e^{ikr}}{r}\hat{\theta} \end{aligned}$$

  Both these field manifest clearly the characteristic properties of radiation:

  - The fields fall off as $1/r$.
  - The electric and magnetic fields are normal to the direction of propagation $\underline{n}$.

- **Near Zone**: $\lambda \gg r \gg d$:
  Here the leading behaviour of the fields is given by

$$\begin{aligned} \underline{E} &= \frac{1}{4\pi\epsilon_0}[3\underline{n}(\underline{n} \cdot \underline{p}) - \underline{p}]\frac{1}{r^3} \\ \underline{H} &= \frac{1}{4\pi\epsilon_0}\frac{i}{Z_0}(\underline{n} \times \underline{p})\frac{k}{r^2}. \end{aligned}$$

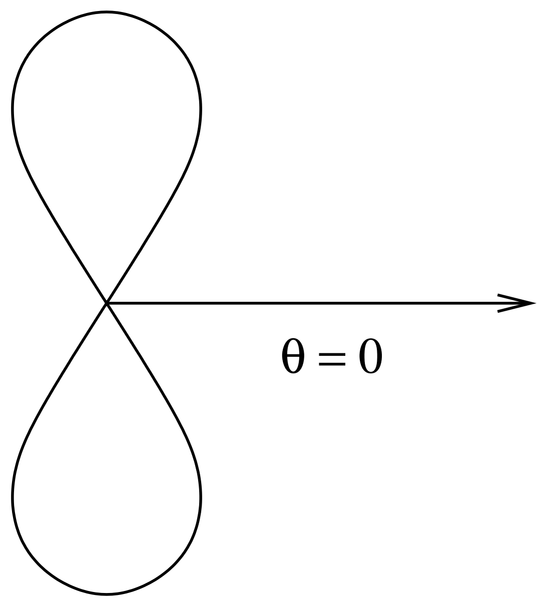
  Thus at very short distances, there is essentially an electric dipole field with time dependence $\exp -i\omega t$, and a magnetic field suppressed by kr/Z_0 that vanishes as $k \to 0$

In order to show that this solution does indeed correspond to radiation, we will look at the **time-averaged power flux** in the *radiation zone*. This, of course, is just given by the **Poynting Vector**, and we have

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{1}{2} r^2 \mathrm{Re}\left[\underline{n} \cdot \underline{E} \times \underline{H}^*\right] \\ &= \frac{c^2 Z_0}{32\pi^2} k^4 |(\underline{n} \times \underline{p}) \times \underline{n}|^2 \end{aligned} \tag{9.7}$$

There is a net flux of power away from the charge distribution, independent of r - **radiation**. For the case where all components of $\underline{p}$ have the same phase, we have the characteristic expression for dipole radiation,

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 |\underline{p}|^2 \sin^2\theta$$



The total power transmitted is just obtained by integrating eqn. (9.7) over the unit sphere, and is independent of the phases of $\underline{p}$:

$$P = \frac{c^2 Z_0 k^4}{12\pi} |\underline{p}|^2.$$

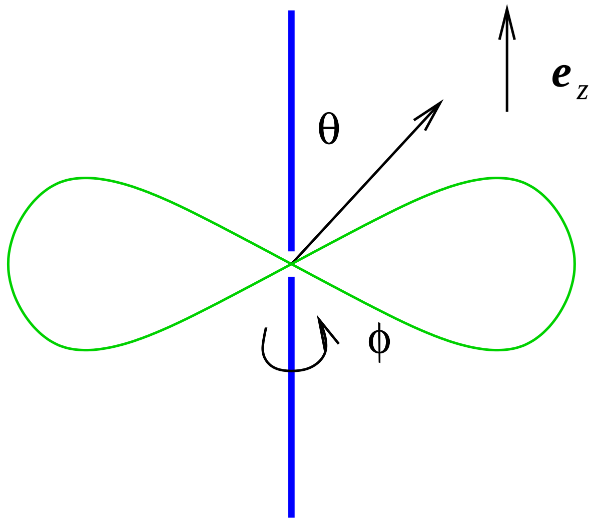
Centre-fed Linear Antenna

Once again we assume that the dimensions of the antenna are much less that the wavelength. The antenna consists of two conductors of length $d/2$, along the z

axis. The linear current density in the wires is

$$I(z) = I_0 \left(1 - \frac{2|z|}{d}\right)$$

where we again suppress the time dependence.



This current flow gives rise to a *line charge density* Λ through the continuity equation

$$i\omega\Lambda(z) = \frac{\partial I}{\partial z}.$$

yielding

$$\Lambda(z) = \frac{2iI_0}{\omega d}\mathrm{sgn}(z).$$

This charge density has a non-zero dipole moment

$$\begin{aligned} \underline{p} &= \int_{-d/2}^{d/2} dz\, z \frac{2iI_0}{\omega d}\underline{e_z} \\ &= \frac{iI_0 d}{2\omega}\underline{e_z}. \end{aligned}$$

N.B. if we had current flowing in **opposite** directions in the two arms of the antenna, there would have been no dipole radiation term.

Thus, from eqn. (9.7), we see that this apparatus gives dipole radiation, with power distribution

$$\frac{dP}{d\Omega} = \frac{Z_0 I_0^2}{128\pi^2}(kd)^2 \sin^2\theta$$

$$P = \frac{Z_0 I_0^2 (kd)^2}{48\pi}. \tag{9.8}$$

If we identify the power radiated with energy dissipation through an effective resistance, the coefficient of $I_0^2/2$ is eqn. (9.8) is the **radiation resistance** - the factor of 2 arises from time-averaging, in the usual way.

9.2 Dipole Fields Revisited

In this section we'll derive the formulas for the dipole radiation again - this time without Fourier transformation $\int d\omega e^{-i\omega t}$ implied.

The general formulas for vector and scalar potentials due to an arbitrary source are:

$$\phi(\underline{x}, t) = \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{\rho(\underline{x}', t_r)}{|\underline{x} - \underline{x}'|}$$
$$\underline{A}(\underline{x}, t) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\underline{J}(\underline{x}', t_r)}{|\underline{x} - \underline{x}'|} \tag{9.9}$$

where $t_r = t - \frac{|\underline{x}-\underline{x}'|}{c}$ is the retarded time.

To study the behavior of these expressions in the radiation zone $|\underline{x}| \gg |\underline{x}'|$, we choose the origin somewhere inside the radiating body and expand the denominators in a usual way:

$$\frac{1}{|\underline{x} - \underline{x}'|} = \frac{1}{r}(1 - \frac{\hat{n} \cdot \underline{x}'}{r} + ...) \tag{9.10}$$

where $r \equiv |\underline{x}|$ and $\hat{n} \equiv \hat{r}$ is the propagation vector for our would-be shperical wave. We need also to expand the retarded time in powers of $\frac{r'}{r}$:

$$t_r = t - \frac{|\underline{x} - \underline{x}'|}{c} \simeq t - \frac{r}{c} + \frac{\hat{n} \cdot \underline{x}'}{c}$$

so that

$$\rho(\underline{x}', t_r) = \rho(\underline{x}', t_0) + \frac{\hat{n} \cdot \underline{x}'}{c} \dot{\rho}(\underline{x}', t_0) + ... \tag{9.11}$$

where $t_0 \equiv t - \frac{r}{c}$ is the retarded time for our origin. The parameter of the expansion (9.11) is $\frac{d}{\lambda} \ll 1$ (see previous Section). Indeed, $\dot{\rho} \sim \omega_{\rm char}\rho$ where $\omega_{\rm char}$ are the characteristic frequencies of the emitted radiation, hence $\frac{d\dot{\rho}}{c\rho} \sim \frac{d\omega}{c} = \frac{d}{\lambda} \ll 1.$) Substituting the expansions (9.10) and (9.11) in the expression (9.9), one obtains:

$$\begin{aligned}\phi(\underline{x},t) &= \frac{1}{4\pi\epsilon_0 r}\int d^3x'[\rho(\underline{x}',t_0) + \frac{\hat{n}\cdot\underline{x}'}{c}\dot{\rho}(\underline{x}',t_0)](1 - \frac{\hat{n}\cdot\underline{x}'}{r} + ...)\\ &= \frac{Q}{4\pi\epsilon_0 r} + \frac{\hat{n}\cdot\underline{p}(t_0)}{4\pi\epsilon_0 r^2} + \frac{\hat{n}\cdot\underline{\dot{p}}(t_0)}{4\pi\epsilon_0 rc} + ...\end{aligned}$$

For the vector potential in Eq. (9.9), the first term in the expansions (9.10) and (9.11) is sufficient:

$$\underline{A}(\underline{x},t) = \frac{\mu_0}{4\pi}\int d^3x'\frac{\underline{J}(\underline{x}',t_r)}{|\underline{x}-\underline{x}'|} \simeq \frac{\mu_0}{4\pi r}\int d^3x'\underline{J}(\underline{x}',t_0)$$

In the previous Section, we demonstrated that

$$\int d^3x'\underline{J}(\underline{x}',t) = \underline{\dot{p}}(t)$$

so the dipole potentials in the radiation zone take the form

$$\begin{aligned}\phi(\underline{x},t) &= \frac{1}{4\pi\epsilon_0 r}\int d^3x'[\rho(\underline{x}',t_0) + \frac{\hat{n}\cdot\underline{x}'}{c}\dot{\rho}(\underline{x}',t_0)](1 - \frac{\hat{n}\cdot\underline{x}'}{r} + ...)\\ &= \frac{Q}{4\pi\epsilon_0 r} + \frac{\hat{n}\cdot\underline{p}(t_0)}{4\pi\epsilon_0 r^2} + \frac{\hat{n}\cdot\underline{\dot{p}}(t_0)}{4\pi\epsilon_0 rc}\\ \underline{A}(\underline{x},t) &= \frac{\mu_0}{4\pi}\int d^3x'\frac{\underline{J}(\underline{x}',t_r)}{|\underline{x}-\underline{x}'|} \simeq \frac{\mu_0\underline{\dot{p}}(t_0)}{4\pi r}\end{aligned} \tag{9.12}$$

Next we calculate the electric and magnetic field in the radiation zone. Discarding terms $\sim \frac{1}{r^2}$, one obtains after some algebra (note that $\underline{\nabla}f(t_0) = \dot{f}(t_0)\underline{\nabla}t_0$ and $\underline{\nabla}t_0 = -\frac{\hat{n}}{c}$:

$$\begin{aligned}\underline{\nabla}\phi(\underline{x},t) &= -\frac{\hat{n}}{4\pi\epsilon_0 rc^2}(\hat{n}\cdot\underline{\ddot{p}}(t_0))\\ \frac{\partial}{\partial t}\underline{A}(\underline{x},t) &= \frac{\mu_0\underline{\ddot{p}}(t_0)}{4\pi r}, \quad \underline{\nabla}\times\underline{A} = -\frac{\mu_0}{4\pi rc}\underline{n}\times\underline{\ddot{p}}(t_0)\end{aligned}$$

Thus, the dipole fields in the radiation zone are

$$\begin{aligned}\underline{E}(\underline{x},t) &= \frac{\mu_0}{4\pi r}[\hat{n}(\hat{n}\cdot\ddot{\underline{p}}(t_0))-\ddot{\underline{p}}(t_0)] = \frac{\mu_0}{4\pi r}\hat{n}\times(\hat{n}\times\ddot{\underline{p}}(t_0)) \\ \underline{B}(\underline{x},t) &= -\frac{\mu_0}{4\pi c r}\ddot{\underline{p}}(t_0) = \frac{\hat{n}}{c}\underline{E}(\underline{x},t)\end{aligned} \quad (9.13)$$

If we choose the frame with OZ axis collinear to $\ddot{\underline{p}}(t_0)$, the fields take the form

$$\underline{E}(r,\theta,\varphi) = \frac{\mu_0\ddot{p}(t_0)}{4\pi}\frac{\sin\theta}{r}\hat{\theta}, \qquad \underline{B}(r,\theta,\varphi) = \frac{\mu_0\ddot{p}(t_0)}{4\pi c}\frac{\sin\theta}{r}\hat{\varphi}, \quad (9.14)$$

The Poynting vector is then

$$\underline{S} = \frac{1}{\mu_0}\underline{E}\times\underline{B} = \frac{\mu_0}{16\pi^2 c}(\ddot{p}(t_0))^2\frac{\sin^2\theta}{r^2}\hat{n}$$

$\Rightarrow$ the total radiated power takes the form

$$P = \int \underline{S}\cdot\hat{n}dA = \frac{\mu_0}{6\pi c}(\ddot{p}(t_0))^2 \quad (9.15)$$

For a single point charge q $\underline{p}(t) = q\underline{x}(t)$ so we get the Larmor formula

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} \quad (9.16)$$

Later, we will reobtain Larmor formula using the Lenard-Wiechert potentials of the moving point charge.

9.3 Magnetic dipole and Electric Quadrupole Radiation

The next term in the multiple expansion is

$$\underline{A}(\underline{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\frac{1}{r}-ik\right)\int d^3x'\,\underline{J}(\underline{x}')\underline{n}\cdot\underline{x}',$$

where the additional term is to ensure the expansion is valid at all distances. To exhibit the form of this potential, we express the integrand as pieces symmetric and anti-symmetric in $\underline{J}$ and $\underline{x}'$, by writing

$$(\underline{n}\cdot\underline{x}')\underline{J} = \frac{1}{2}[(\underline{n}\cdot\underline{x}')\underline{J}+(\underline{n}\cdot\underline{J})\underline{x}'] + \frac{1}{2}(\underline{x}'\times\underline{J})\times\underline{n}. \quad (9.17)$$

We now introduce the magnetisation density

$$\underline{M} = \frac{1}{2}\underline{x} \times \underline{J}.$$

Then the second term gives rise to a vector potential

$$\underline{A}(\underline{x}) = \frac{ik\mu_0}{4\pi}\frac{e^{ikx}}{r}\left(1 - \frac{1}{ikr}\right)\underline{n} \times \underline{m}, \tag{9.18}$$

where $\underline{m}$ is the **magnetic dipole moment**.
As an example of magnetic dipole radiation, consider the circular loop of radius b with current

$$I(t) = I\cos\omega t = \Re I e^{-i\omega t}$$

The magnetic dipole moment of this loop oscillates in time as

$$\underline{m}(t) = m\cos\omega t = \Re\pi b^2 I e^{-i\omega t}$$

Let us calculate the magnetic vector potential due to this setup. W.l.o.g. we can assume that the point $\underline{x}$ lies in the XZ plane. The general formula for the magnetic vector potential has the form

$$\underline{A}(\underline{x}, t) = \frac{\mu_0}{4\pi}\oint dl' \frac{e^{-i\omega_t r'}}{|\underline{x} - \underline{x}'|}\hat{e}_{\phi'} I e^{-i\omega t} \tag{9.19}$$

Expanding $t_{r'} \simeq t_0 - \frac{\underline{x}\cdot\underline{x}'}{c}$ (where $t_0 = t - \frac{r}{c}$) and $\frac{1}{|\underline{x}-\underline{x}'|} \simeq \frac{1}{r}(1 + \frac{\hat{r}\cdot\underline{x}'}{r})$ we get

$$\underline{A}(\underline{x}) = \frac{\mu_0 bI}{4\pi}\frac{e^{ikr}}{r}\int_0^{2\pi} d\phi'(-\hat{e}_1\sin\phi' + \hat{e}_2\cos\phi')(1 + \frac{b}{r}\sin\theta\cos\phi')e^{-ikb\sin\theta\cos\phi'}$$

Since $kb = 2\pi\frac{b}{\lambda} \ll 1$ we can expand the exponential in the r.h.s. of this equation and get

$$\underline{A}(\underline{x}) = \frac{\mu_0 bI}{4\pi}\frac{e^{ikr}}{r}\int_0^{2\pi} d\phi'(-\hat{e}_1\sin\phi' + \hat{e}_2\cos\phi')(1 + \frac{b}{r}\sin\theta\cos\phi' - ikb\sin\theta\cos\phi')$$

Performing integration over ϕ' we obtain

$$\underline{A}(\underline{x}) = \frac{ik\mu_0 I b^2}{4r}\hat{e}_2(1 - \frac{1}{ikr})e^{ikr}\sin\theta \tag{9.20}$$

For our setup $\hat{e}_2 = \hat{e}_\phi$ so the final result for the vector potential takes the form

$$\underline{A}(\underline{x}) = \frac{ik\mu_0 \hat{m}}{4\pi r}\hat{e}_\phi(1 - \frac{1}{ikr})e^{ikr}\sin\theta \tag{9.21}$$

which coincides with Eq. (9.18).

Let us find now electric and magnetic fields of the magnetic dipole radiation. Taking the curl of Eq. (9.18), we find

$$\underline{H} = \frac{1}{4\pi}\left\{k^2(\underline{n}\times\underline{m})\times\underline{n}\frac{e^{ikr}}{r} + [3\underline{n}(\underline{n}\cdot\underline{m}) - \underline{m}]\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)e^{ikr}\right\}. \tag{9.22}$$

The field $\underline{H}$ due to the *magnetic dipole* is of the same form as the field $\underline{E}$ due to the *electric* dipole (see Eq. (9.5)). Similarly we have

$$\underline{E} = -\frac{Z_0}{4\pi}k^2(\underline{n}\times\underline{m})\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right), \tag{9.23}$$

so that the *electric field* due to a *magnetic dipole* is of the same form as the *magnetic field* due to an *electric dipole*:

$$H\text{mag.dipole} = \frac{\epsilon_0 m}{p}E_{\text{el.dipole}}, \qquad E\text{mag.dipole} = \frac{\mu_0 m}{p}H_{\text{el.dipole}},$$

Since the radiated power is proportional to $\underline{n}\cdot(\underline{E}\times\underline{H})$,

$$P_{\text{rad}}^{\text{mag.dipole}} = \frac{m^2}{p^2c^2}P_{\text{rad}}^{\text{el.dipole}} = \frac{\mu_0 m^2\omega^4}{12\pi c^3} \tag{9.24}$$

In order to get an estimate of the relative strength of the electric and magnetic dipole radiation, consider a physical dipole made from two charges q and $-q$ separated by distance d which rotate with angular velocity ω around the center of the dipole. The magnetic moment of this system can be approximated by an oscillating current $I = \frac{q}{T} = \frac{2\pi q}{\omega}$ so we get an oscillating magnetic moment $m = \frac{d^2\omega}{8}$. The ratio of powers for this example is

$$\frac{P_{\text{mag}}}{P_{rmel}} \sim \frac{\omega^2 d^2}{4c^2} = \frac{v^2}{c^2} \tag{9.25}$$

where v is the linear velocity of the rotating charges. We see that for charges moving with non-relativistic velocities the electric dipole radiation is the most important part while the magnetic dipole radiation is of the size of the relativistic corrections.

The interesting part is the **quadrupole moment**, obtained from the **symmetric part** of eqn. (9.17). We use

$$\frac{1}{2}\int d^3x' \, \{(\underline{n}\cdot\underline{x}')\underline{J} + (\underline{n}\cdot\underline{J})\underline{x}'\} = -\frac{i\omega}{2}\int d^3x' \, \rho\underline{x}'(\underline{n}\cdot\underline{x}'),$$

using the same tricks we encountered earlier, and write

$$\underline{A}(\underline{x}) = -\frac{\mu_0 ck}{8\pi}\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\int d^3x'\rho(\underline{x}')\underline{x}'(\underline{n}\cdot\underline{x}'). \tag{9.26}$$

In the limit $r \gg \lambda$, we find

$$\begin{aligned} \underline{H} &= ik\underline{n}\times\underline{A}/\mu_0 \\ \underline{E} &= ikZ_0(\underline{n}\times\underline{A})\times\underline{n}/\mu_0. \end{aligned} \tag{9.27}$$

If we now recall our expression for the quadrupole moment

$$Q_{\alpha\beta} = \int d^3x\rho(\underline{x})(3x_\alpha x_\beta - r^2\delta_{\alpha\beta})$$

then we find that $\underline{H}$ can be written

$$\underline{H} = -\frac{ick^3}{24\pi}\frac{e^{ikr}}{r}\underline{n}\times\underline{Q}(\underline{n})$$

where $\underline{Q}(\underline{n})$ is defined by

$$Q_\alpha = \sum_\beta Q_{\alpha\beta}n_\beta.$$

The power dissipation is

$$\frac{dP}{d\Omega} = \frac{c^2Z_0}{1152\pi^2}k^6|[\underline{n}\times\underline{Q}(\underline{n})]\times\underline{n}|^2.$$

We encountered a simple model of a quadrupole moment in the multipole expansion last term:

$$\begin{aligned} Q_{33} &= Q_0 \\ Q_{11} = Q_{22} &= -\frac{1}{2}Q_0, \end{aligned} \tag{9.28}$$

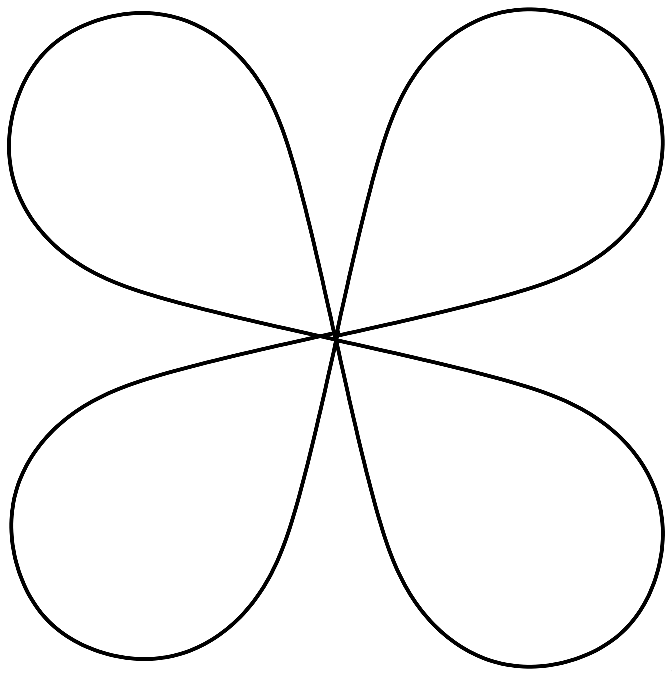
which is clearly traceless. Then the angular power distribution is

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^6}{512\pi^2} Q_0^2 \sin^2\theta \cos^2\theta, \tag{9.29}$$

and the total power radiated is

$$P = \frac{c^2 Z_0 k^6 Q_0^2}{960\pi}. \tag{9.30}$$

For quadrupole radiation, we have a four-lobe pattern of power distribution



The complete description requires the full **multipole expansion** which is beyond what I am going to do in this course.

9.4 Radiation from a moving point charge

9.4.1 Lenard-Wiechert Potentials

Consider a point charge moving along the trajectory $\underline{r} = \vec{w}(t)$. What are the electric and magnetic fields due to this charge?

As usually, it is convenient to start with the potentials. In the Lorentz gauge

$$\begin{aligned}\phi(\underline{r},t) &= \frac{1}{4\pi\epsilon_0}\int d^3x'\frac{\rho(\underline{r}')}{|\underline{r}-\underline{r}'|}\delta(t'-t+\frac{|\underline{r}-\underline{r}'|}{c})\\ \underline{A}(\underline{r},t) &= \frac{\mu_0}{4\pi}\int d^3x'\frac{\underline{J}(\underline{r}')}{|\underline{r}-\underline{r}'|}\delta(t'-t+\frac{|\underline{r}-\underline{r}'|}{c}) \end{aligned} \qquad (9.31)$$

For a point charge

$$\rho(\underline{r},t) \;=\; q\delta(\underline{r}-\vec{w}(t)), \qquad \underline{J}(\underline{r},t) \;=\; q\underline{v}(t)\delta(\underline{r}-\vec{w}(t))$$

At first, let us find the scalar potential

$$\begin{aligned}\phi(\underline{r},t) &= \frac{q}{4\pi\epsilon_0}\int d^3x'\int dt'\,\frac{\delta(\underline{r}'-\vec{w}(t))}{|\underline{r}-\underline{r}'|}\delta(t'-t+\frac{|\underline{r}-\underline{r}'|}{c})\\ &= \frac{q}{4\pi\epsilon_0}\int dt'\frac{\delta(t'-t+\frac{|\underline{r}-\underline{r}'|}{c})}{|\underline{r}-\vec{w}(t')|} = \int dt'\frac{1}{\frac{\partial}{\partial t'}(t'-t+\frac{|\underline{r}-\vec{w}(t')|}{c})\Big|_{t'=t_r}}\frac{\delta(t'-t_r)}{|\underline{r}-\vec{w}(t')|}\\ &= \frac{q}{4\pi\epsilon_0}\int dt'\frac{\delta(t'-t_r)}{|\underline{r}-\vec{w}(t')|-\frac{1}{c}\underline{v}(t')\cdot(\underline{r}-\vec{w}(t')} = \frac{c}{c|\underline{r}-\vec{w}(t_r)|-\underline{v}(t_r)\cdot(\underline{r}-\vec{w}(t_r))}\end{aligned}$$

where $\underline{v}(t) \equiv \frac{\partial}{\partial t}\vec{w}(t)$ is the velocity of the particle and t_r is the solution of the equation $c(t-t_r) = |\underline{r}-\vec{w}(t_r)| = 0$.
Similarly,

$$\underline{A}(\underline{r},t) = \frac{\mu_0 q}{4\pi}\underline{v}(t_r)\frac{c}{c|\underline{r}-\vec{w}(t_r)|-\underline{v}(t_r)\cdot(\underline{r}-\vec{w}(t_r))}$$

The potentials

$$\begin{aligned}\phi(\underline{r},t) &= \frac{q}{4\pi\epsilon_0}\,\frac{c}{c|\underline{r}-\vec{w}(t_r)|-\underline{v}(t_r)\cdot(\underline{r}-\vec{w}(t_r))}\\ \underline{A}(\underline{r},t) &= \frac{\underline{v}}{c^2}V(\underline{r},t)\end{aligned} \qquad (9.32)$$

are called the Lenard-Wiechert potentials for a point charge. The corresponding electric and magnetic fields are (see *Jackson* or *Griffiths*)

$$\begin{aligned}\underline{E}(\underline{r},t) &= \frac{q}{4\pi\epsilon_0}\frac{\varsigma}{(\vec{\varsigma}\cdot\vec{u})^3}[\vec{u}(c^2-v^2)+\vec{\varsigma}\times(\vec{u}\times\underline{a})] \\ \underline{B}(\underline{r},t) &= \frac{\hat{\varsigma}}{c}\times\underline{E}(\underline{r},t)\end{aligned} \tag{9.33}$$

where $\underline{v}=\underline{v}(t_r)$, $\underline{a}=\underline{a}(t_r)$, $\vec{\varsigma}\equiv\underline{r}-\vec{w}(t_r)$, and $\vec{u}\equiv c\hat{\varsigma}-\underline{v}(t_r)$ (as usually, $\hat{\varsigma}\equiv\frac{\vec{\varsigma}}{|\varsigma|}$).

9.4.2 Power radiated by a point charge

The electric and magnetic fields due to a point charge moving along an arbitrary trajectory $\vec{w}(t)$ are given by Eq. (9.33)

$$\begin{aligned}\underline{E}(\underline{r},t) &= \frac{q}{4\pi\epsilon_0\varsigma^2}\frac{\vec{u}(c^2-v^2)}{(\hat{\varsigma}\cdot\vec{u})^3}+\frac{q}{4\pi\epsilon_0\varsigma}\frac{\hat{\varsigma}\times(\vec{u}\times\vec{a})}{(\hat{\varsigma}\cdot\vec{u})^3} \\ \underline{B}(\underline{r},t) &= \frac{\vec{\varsigma}}{c}\times\underline{E}(\underline{r},t).\end{aligned} \tag{9.34}$$

where $\vec{\varsigma}=\vec{r}-\vec{w}(t_r)$, $\vec{u}=c\hat{\varsigma}-\vec{v}$, and t_r is defined as a solution to the equation $c(t-t_r)=\varsigma$. As usuallly, velocity and acceleration in Eq. (9.34) are taken at $t=t_r$. The first term ($\sim\vec{u}$) is called the velocity field and the second ($\sim\vec{a}$) is called the acceleration or the radiation field.

The Poynting vector is

$$\vec{S}=\frac{1}{\mu_0}\vec{E}\times\vec{B}=\frac{1}{\mu_0 c}\vec{E}\times(\hat{\varsigma}\times\vec{E})=\frac{1}{\mu_0 c}[E^2\hat{\varsigma}-(\hat{\varsigma}\cdot\vec{E})\vec{E}] \tag{9.35}$$

Some of the energy is radiation; another part is just a field energy carried along by the particle as it moves. To calculate the power radiated by the particle at time t_*, we draw a large sphere with radius $\varsigma=R$, wait for $t-t_*=\frac{R}{c}$, and integrate Poynting vector over the surface. Since the velocity field is $\sim 1/R^2$ the corresponding $P_{\rm rad}$ is $\sim R^2\frac{1}{R^4}=\frac{1}{R^2}$ so it does not contribute to the radiated power at large R. The power due to the acceleration field ($\sim 1/R$) is finite: $P_{\rm rad}\sim R^2\frac{1}{R^2}=1$. We get

$$\vec{E}_{\rm rad}(\vec{r},t) = \frac{q}{4\pi\epsilon_0\varsigma}\frac{\hat{\varsigma}\times(\vec{u}\times\vec{a})}{(\hat{\varsigma}\cdot\vec{u})^3}$$

$$\rightarrow\ \vec{\varsigma}\times\vec{E}_{\rm rad}(\underline{r},t)=0\ \Rightarrow\ \vec{S}_{\rm rad}=\frac{\hat{\varsigma}}{\mu_0 c}E^2_{\rm rad}. \tag{9.36}$$

For simplicity, consider the charge which is instantaneously at rest at $t=t_*$. Since $\vec{v}(t_*)=0$, $\vec{u}(t_*)=c\hat{\varsigma}$ so the Eq. (9.36) reduces to

$$\vec{S}_{\rm rad}=\frac{\hat{\varsigma}}{\mu_0 c}(\frac{\mu_0 q}{4\pi R})^2[a^2-(\hat{\varsigma}\cdot\vec{a})^2]=\frac{\mu_0 q^2 a^2}{16\pi^2 c}\frac{\sin^2\theta}{R^2}\hat{\varsigma} \tag{9.37}$$

The total power is given by the following Larmor formula

$$P_{\rm rad}=\oint_S \vec{S}_{\rm rad}\cdot\vec{da}=\frac{\mu_0 q^2 a^2}{16\pi^2 cR^2}\int\frac{\sin^2\theta}{R^2}R^2\sin\theta d\theta d\phi\ =\frac{\mu_0 q^2 a^2}{6\pi c} \tag{9.38}$$

which we have already obtained using the electric dipole radiation, see the Eq. (9.16).

We have derived the Larmor formula under the assumption that $v=0$ but one can demonstrate that it holds true as long as $v\ll c$. In the general case of arbitrary velocity, the radiation is given by the Lienard formula

$$P_{\rm rad}=\frac{\mu_0 q^2\gamma^6}{6\pi c}(a^2-\frac{(\vec{v}\cdot\vec{a})^2}{c^2}) \tag{9.39}$$

where $\gamma\equiv 1/\sqrt{1-\frac{v^2}{c^2}}$.

9.4.3 Electromagnetic fields due to a point charge moving with constant velocity.

Potentials

For a point charge moving with constant velocity $\underline{v}$ the trajectory is $\vec{w}=t\underline{v}$ so the retarded time is

$$c(t-t_r)=|\underline{r}-t_r\underline{v}|\quad\Rightarrow\quad r^2-2t_r\underline{r}\cdot\underline{v}+v^2t_r^2=c^2(t^2-2tt_r+t_r^2)$$
$$\Rightarrow\ t_r=\frac{c^2t-\underline{r}\cdot\underline{v}-\sqrt{(c^2t-\underline{r}\cdot\underline{v})^2-(c^2-v^2)(c^2t^2-r^2)}}{c^2-v^2}$$

The Lenard-Wiechert potentials (9.32) take the form

$$\phi(\underline{r},t) = \frac{qc}{4\pi\epsilon_0}\,\frac{1}{c|\underline{r}-t_r\underline{v}|-\underline{v}\cdot(\underline{r}-t_r)} \tag{9.40}$$
$$= \frac{qc}{4\pi\epsilon_0[c^2t-(c^2-v^2)t_r-\underline{r}\cdot\underline{v}]} = \frac{qc}{4\pi\epsilon_0}[(c^2t-vr\cdot\underline{v})^2-(c^2-v^2)(c^2t^2-r^2)]^{-1/2}$$

and

$$\underline{A}(\underline{r},t) = \frac{\underline{v}}{c^2}\phi(\underline{r},t) = \frac{\mu_0 q\underline{v}}{4\pi}[c^2t-(c^2-v^2)t_r-\underline{r}\cdot\underline{v}]^{-1}$$
$$= \frac{\mu_0 qc\underline{v}}{4\pi}[(c^2t-vr\cdot\underline{v})^2-(c^2-v^2)(c^2t^2-r^2)]^{-1/2} \tag{9.41}$$

Let us demonstrate that $\phi(\underline{r},t)$ can be rewritten as

$$\phi(\underline{r},t) = \frac{q}{4\pi\epsilon_0 R}(1-\frac{v^2}{c^2}\sin^2\theta)^{-1/2} \tag{9.42}$$

where $\underline{R}=\underline{r}-t\underline{v}$ and θ is the angle between $\underline{R}$ and $\underline{v}$. (R is the distance to the position of the moving charge at the time of measurement of the fields).
We have

$$(c^2t-\underline{v}\cdot\underline{r})^2-(c^2-v^2)(c^2t^2-r^2) = [c^2t-\underline{v}\cdot(\underline{R}+t\underline{v})]^2-(c^2-v^2)[c^2t^2-(\underline{R}+t\underline{v})^2]$$
$$= [(c^2-v^2)t-\underline{v}\cdot\underline{R}]^2-(c^2-v^2)[(c^2-v^2)t^2-2t\underline{v}\cdot\underline{R}-R^2)^2 = (c^2-v^2)R^2+(\underline{v}\cdot\underline{R})^2$$

and therefore

$$\sqrt{(c^2t-\underline{v}\cdot\underline{r})^2-(c^2-v^2)(c^2t^2-r^2)} = Rc\sqrt{1-\frac{v^2}{c^2}\sin^2\theta} \tag{9.43}$$

Fields

For $\vec{w}=t\underline{v}$ (and $\underline{a}=0$) the electric field in Eq. (9.33) reduces to (recall $\vec{u}\equiv c\hat{\varsigma}-\underline{v}(t_r)=c\hat{\varsigma}-\underline{v}$)

$$\underline{E}(\underline{r},t) = \frac{q}{4\pi\epsilon_0}\,\frac{\varsigma\vec{u}}{(\vec{\varsigma}\cdot\vec{u})^3}(c^2-v^2) = \frac{q(c^2-v^2)}{4\pi\epsilon_0}\,\frac{c\vec{\varsigma}-\varsigma\underline{v}}{(c\varsigma-\underline{v}\cdot\vec{\varsigma})^3} \tag{9.44}$$

and $\underline{B}(\underline{r},t)=\frac{\vec{\varsigma}}{c}\times\underline{E}(\underline{r},t)$.

It is easy to see that ($\vec{\varsigma} \equiv \underline{r} - \vec{w}(t_r) = \underline{r} - \underline{v}t_r = \underline{R} + \underline{v}(t - t_r)$)

$$c\vec{\varsigma} - \underline{v}\varsigma = c(\underline{r} - t_r\underline{v}) - |r - t_r\underline{v}|\underline{v} = c(\underline{r} - t_r\underline{v}) - c(t - t_r)\underline{v} = (\underline{r} - t\underline{v})c = c\underline{R}$$

Similarly, we get

$$c\varsigma - \underline{v} \cdot \vec{\varsigma} = c^2(t - t_r) - \underline{v} \cdot [\underline{R} + (t - t_r)\underline{v}] = (c^2 - v^2)(t - t_r) - \underline{v} \cdot \underline{R}$$

and

$$\begin{aligned} [c\varsigma - \underline{v} \cdot \vec{\varsigma}]^2 &= [(c^2 - v^2)(t - t_r) - \underline{v} \cdot \underline{R}]^2 \\ &= (c^2 - v^2)^2(t - t_r)^2 - 2(c^2 - v^2)(t - t_r)\underline{v} \cdot \underline{R} + (\underline{v} \cdot \underline{R})^2 \\ &= R^2(c^2 - v^2) + (\underline{v} \cdot \underline{R})^2 = R^2c^2(1 - \frac{v^2}{c^2}\sin^2\theta) \qquad (9.45) \end{aligned}$$

so the electic and magnetic fields (9.46) take the form

$$\begin{aligned} \underline{E}(\underline{r}, t) &= \frac{qc}{4\pi\epsilon_0} \frac{\varsigma\vec{u}}{(\vec{\varsigma} \cdot \vec{u})^3}(c^2 - v^2) \qquad (9.46) \\ &= \frac{q(c^2 - v^2)}{4\pi\epsilon_0} \frac{c\underline{R}}{(R^2c^2 - R^2v^2\sin^2\theta)^{3/2}} = \frac{q\hat{R}}{4\pi\epsilon_0 R^2} \frac{1 - \frac{v^2}{c^2}}{(1 - \frac{v^2}{c^2}\sin^2\theta)^{3/2}} \\ \underline{B}(\underline{r}, t) &= = \frac{\hat{\varsigma}}{c} \times \underline{E}(\underline{r}, t) = \frac{1}{c^2}\underline{v} \times \underline{E}(\underline{r}, t) \end{aligned}$$

Let us demonstrate that the fields (9.46) are Lorentz transforms of the usual Coulomb field of a point charge ($\underline{E}(\underline{r}, t) = \frac{q\hat{R}}{4\pi\epsilon_0 R^2}$, $\underline{B} = 0$).

Chapter 11

Special Theory of Relativity and Covariant Electrodynamics

Central to Newtonian Mechanics is the concept of an **inertial frame**; a frame in which a body, acted on by no external forces, moves with a constant velocity. A transformation between two inertial frames is known as a **Galilean Transformation**.

Aside: a practical definition of an inertial frame is one moving with constant velocity relative to the distant stars (Mach's principle).

11.0.4 Galilean Transformations

Consider two inertial frames K, K', moving with a relative constant velocity $\underline{v}$. The coordinates in the two frames are related by

$$\begin{aligned} t' &= t \\ \underline{x}' &= \underline{x} - \underline{v}t \end{aligned} \tag{11.1}$$

Now consider the interactions of an ensemble of N particles at positions $\underline{x}_i; i = 1, \ldots, N$, acting solely under the influence of a central potential $V_{ij}(|\underline{x}_i - \underline{x}_j|)$. Then the eqn. of motion of particle i in K is

$$m_i \frac{d\underline{v}_i}{dt} = -\sum_j \underline{\nabla}_{x_i} V_{ij}(|\underline{x}_i - \underline{x}_j|).$$

Suppose now that we look at the equation of motion in K'. Then we have $\underline{v}'_i = \underline{v}_i - \underline{v}$, and

$$m_i \frac{d\underline{v}_i}{dt} = -\sum_j \underline{\nabla}_{x'_i} V_{ij}(|\underline{x}'_i - \underline{x}'_j|).$$

Now under eqn. (11.1),

$$\frac{\partial}{\partial x'_i} = \frac{\partial}{\partial x_i}$$

and we have

$$|\underline{x}'_i - \underline{x}'_j| = |\underline{x}_i - \underline{x}_j|,$$

and we see that the eqn. of motion in K' is of exactly the same form as that in K - we say that classical Newtonian mechanics transforms **covariantly** under Galilean Transformations.

11.0.5 Maxwellian Mechanics under Galilean Transformations

We have seen that electric and magnetic propagation in a vacuum satisfies the wave equation

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\psi(x, y, z; t) = 0. \tag{11.2}$$

Let us now consider the transformation of this equation under eqn. (11.1). We have

$$\begin{aligned} \frac{\partial}{\partial x_i} &= \frac{\partial x'_j}{\partial x_i}\frac{\partial}{\partial x'_j} + \frac{\partial t'}{\partial x_i}\frac{\partial}{\partial t'} \\ &= \delta_{ij}\frac{\partial}{\partial x'_j} + 0 = \frac{\partial}{\partial x'_i} \\ \frac{\partial}{\partial t} &= \frac{\partial x'_j}{\partial t}\frac{\partial}{\partial x'_j} + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} \\ &= -v_i\frac{\partial}{\partial x'_i} + \frac{\partial}{\partial t'}. \end{aligned}$$

Thus the wave equation (11.2) becomes

$$\left[\nabla'^2 - \frac{1}{c^2}\left(\frac{\partial}{\partial t'} - \underline{v}\cdot\underline{\nabla}'\right)\left(\frac{\partial}{\partial t'} - \underline{v}\cdot\underline{\nabla}'\right)\right]\psi \;=\; 0$$

$$\text{i.e.} \quad \left[\nabla'^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} + \frac{2}{c^2}\underline{v}\cdot\underline{\nabla}'\frac{\partial}{\partial t'} - \frac{1}{c^2}(\underline{v}\cdot\underline{\nabla}')(\underline{v}\cdot\underline{\nabla}')\right]\psi \;=\; 0 \qquad (11.3)$$

This equation is clearly of a different from to equation (11.2). *The wave equation does not transform covariantly under Galilean Transformations.* For sound waves there is no problem; they propagate in a medium, and it is natural to formulate the wave equation in a frame in which the medium is at rest. Thus the natural question arose - *Is there a frame in which the "ether" is at rest"?*. Of course, we all know the answer (Michelson-Morley) that the velocity of light is the same in all frames, and the resolution of this nasty transformation property is the **Special Theory of Relativity**.

11.1 Postulates of Special Theory of Relativity

1. The same laws of nature hold in all systems moving uniformly with respect to one another.
2. The velocity of light has the same value in all systems moving uniformly with respect to each other, independent of velocity of observer relative to the source.

11.2 Lorentz Transformations and Kinematic Results of Special Relativity

We will now derive the relationship between coordinates in two frames K, K' moving with constant velocity $\underline{v}$ relative to one another. W.l.o.g. we will let the origin of the coordinates coincide at $t = t' = 0$.
We suppose that a flashlight is rapidly switched on and off at the origin at $t = t' = 0$. Then, by postulate 2, observers in both K and K' see a spherical shell of radiation expanding with the velocity of light c. The wavefront satisfies

$$\text{In } K: \qquad c^2t^2 - (x^2 + y^2 + z^2) = 0$$

$$\text{In } K': \qquad c^2t'^2 - (x'^2 + y'^2 + z'^2) = 0$$

Thus we see that, under such a transformation, the quantity $c^2t^2-(x^2+y^2+z^2) = 0$ remains invariant. The emission of the light, and its subsequent absorption at some later times, are each **events**. We have considered the case where the events are separated by something travelling at the speed of light. More generally, we have

$$\boxed{\Delta s^2 = c^2t^2 - (x^2 + y^2 + z^2) \qquad (11.4)}$$

is invariant under transformations between inertial frames. This is the **interval** between the two events.

To consider the form of the transformations satisfying eqn. (11.4), we will specialise to the case where the axes in K, K' are parallel, and the frames are moving with a relative velocity $\underline{v} = v\underline{e_3}$. Because the transformations must reduce to Galilean transformations in the limit of small relative velocities, we need consider only the linear relations

$$\begin{aligned} t' &= a_1t + b_1z \\ z' &= a_2t + b_2z \\ x' &= x \\ y' &= y \end{aligned} \qquad (11.5)$$

The transverse dimensions do not change (see the gedanken experiment of Taylor and Wheeler discussed in *Griffiths* textbook).

Because the frames are moving with relative velocity v, we have that the event $z' = 0$ corresponds to $z = vt$, yielding

$$a_2 = -vb_2.$$

We now impose invariance of Δs^2:

$$c^2t^2 - (x^2 + y^2 + z^2) = c^2(a_1t + b_1x)^2 - (a_2t + b_2x)^2 - y^2 - z^2,$$

which we can expand as

$$c^2t^2[1 - a_1^2 + a_2^2/c^2] - z^2[1 + b_1^2c^2 - b_2^2] + 2zt[a_2b_2 - c^2a_1b_1] = 0.$$

This is true $\forall x, t$, so equating the coefficients to zero yields

$$\begin{aligned} a_1^2 - a_2^2/c^2 &= 1 \\ b_2^2 - c^2b_1^2 &= 1 \\ a_2b_2 &= c^2a_1b_1. \end{aligned}$$

Solving these simultaneous equations we find

$$\begin{aligned} ct' &= \gamma[ct - \frac{v}{c}z] \\ z' &= \gamma[z - \frac{v}{c}ct] \\ x' &= x \\ y' &= y \end{aligned}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{11.6}$$

We can write this in an axis-independent form as

$$\left.\begin{aligned} ct' &= \gamma(ct - \beta x_{\parallel}) \\ x'_{\parallel} &= \gamma(x_{\parallel} - \beta ct) \\ \underline{x'}_{\perp} &= \underline{x}_{\perp} \end{aligned}\right\} \tag{11.7}$$

where

$$\begin{aligned} \beta &= v/c \\ \gamma &= (1 - \beta^2)^{-1/2} \\ \underline{x}_{\parallel} &= \frac{\underline{x} \cdot \underline{v}}{|\underline{v}|}. \end{aligned} \tag{11.8}$$

In vector form, this is

$$\begin{aligned} ct' &= \gamma(ct - \underline{\beta} \cdot \underline{x}) \\ \underline{x}' &= \underline{x} + \frac{\gamma - 1}{\beta^2}(\underline{\beta} \cdot \underline{x})\underline{\beta} - \gamma\underline{\beta}ct. \end{aligned} \tag{11.9}$$

Alternative Parametrisation

Introduce $\beta = \tanh\zeta$, so that $\gamma = \cosh\zeta$. Then, for frames moving parallel to the x axis, we have

$$\begin{aligned} ct' &= ct\cosh\zeta - z\sinh\zeta \\ z' &= z\cosh\zeta - ct\sinh\zeta, \end{aligned} \tag{11.10}$$

which has the form of a "rotation" on the complex angle $\phi = i\zeta$

11.3 Kinematical Properties of Lorentz Transformations

Given two events $(ct_1, \underline{x}_1)$ and $(ct_2, \underline{x}_2)$, Lorentz transformations leave the **interval**

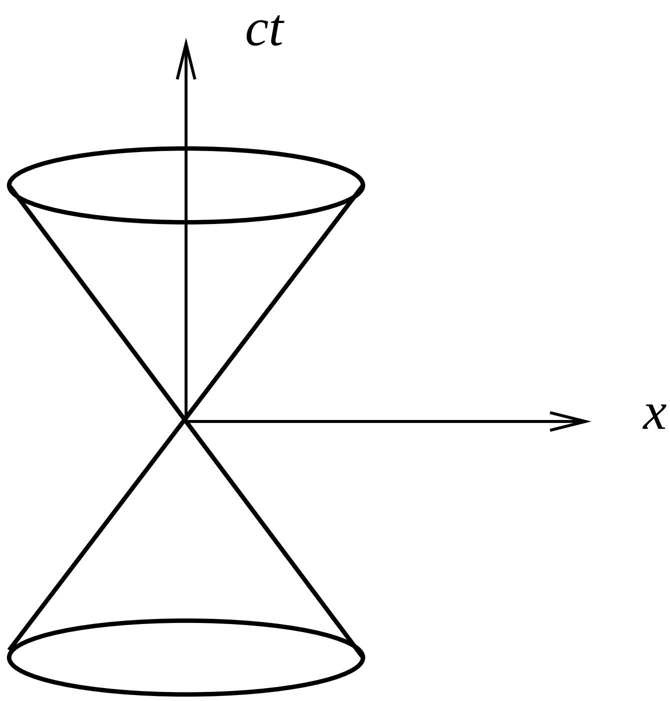
$$\Delta s^2 = c^2(t_2 - t_1)^2 - (\underline{x}_2 - \underline{x}_1)^2$$

invariant. Thus we can classify the interval by the **sign** of δa^2, as follows

- $\Delta s^2 < 0$. This is **timelike** separation. We have $c|t_2 - t_1| > |\underline{x}_2 - \underline{x}_1|$, so that the two points can communicate by a signal travelling at *less than* the speed of light, and indeed a frame can be chosen such that $|\underline{x}_2 - \underline{x}_1| = 0$.
- $\Delta s^2 = 0$. This is **lightlike** separation. We have $c|t_2 - t_1| = |\underline{x}_2 - \underline{x}_1|$, so that the two points can only be connected by a signal travelling *at* the speed of light.
- $\Delta s^2 < 0$. This is **spacelike** separation, with $c|t_2 - t_1| < |\underline{x}_1 - \underline{x}_2|$. The two space-time points cannot communicate, and indeed a frame exists in which $t_1 = t_2$.

11.3.1 Light Cone

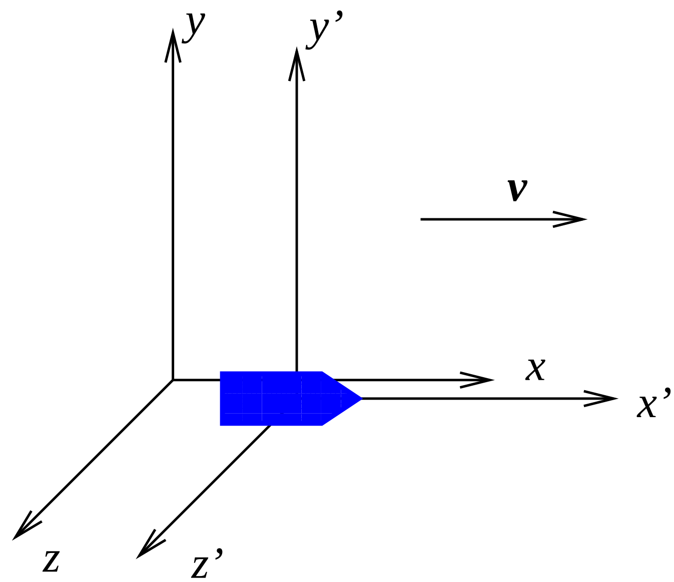
Points that can be connected with the space-time origin by a light signal are said to lie on the **light cone**.



Points within the light cone can be causally connected with the origin, whilst those outside cannot. The forward ($ct > 0$) and backward ($t < 0$) cones define absolute future and absolute past, and the ordering is preserved under Lorentz transformations.

11.3.2 Simultaneity, Length Contraction and Time Dilation

Consider a rocket moving with constant velocity v along the x direction relative to the lab frame K. Let us denote the rest frame of the rocket by K'. We assume that the axes of the frames are parallel, and the origins coincide at $t = 0$.
On the side of a rocket is a meter rule. We also have, in the lab. frame, a high density of observers, each with a very accurate clock synchronised in the frame K.



Simultaneity

At time t, an observer in the lab frame, co-incident with one end of the meter rod, records his position $(ct, \underline{x}_1)$, and an observer coincident with the other end does likewise $(ct, \underline{x}_2)$. Thus $(ct, \underline{x}_1)$ and $(ct, \underline{x}_2)$ denote two events, which are *simultaneous* in the lab. frame.
In the rocket rest frame K' we have

$$\begin{aligned} ct'_1 &= \gamma(ct - \beta x_1) \\ x'_1 &= \gamma(x_1 - \beta ct) \\ ct'_2 &= \gamma(ct - \beta x_2) \\ x'_2 &= \gamma(x_2 - \beta ct) \end{aligned} \tag{11.11}$$

We immediately see that $t'_1 = t'_2$ iff $x_1 = x_2$; in general the points as *not simulataneous* in the rocket rest frame.

Length Contraction

In the rocket frame, our meter rule has length $x_2' - x_1'$. However, from eqn. (11.11), we see that in the laboratory frame the length is given by

$$x_1' - x_2' = \gamma(x_1 - x_2),$$

i.e.

$$\boxed{x_1 - x_2 = \frac{x_1' - x_2'}{\gamma}}$$

Since $\gamma \geq 1$, we have that lengths are **contracted**

Time Dilation

We now imagine that the clocks in K, K' are synchronised at $t_1 = t_1' = 0$ as the rocket passes origin in frame K. An observer at some point x in K records the time t_2 at which rocket passes x, and an observer in K' records time t_2' at which he passes the laboratory observer. The rocket observer is always at $x_2' = 0$, so we have

$$\begin{aligned} 0 &= \gamma(x - \beta c t_2) \\ \Longrightarrow x &= \beta c t_2 \end{aligned}$$

From the third eqn. of (11.11), we have

$$ct_2' = \gamma(ct_2 - \beta x) = \gamma(ct_2 - \beta^2 ct_2)$$

$$\boxed{t_2' = \frac{t_2}{\gamma}.}$$

Thus we see that time is **dilated**.

11.4 Proper Time

We now generalize the discussion to the case where the rocket is moving with a velocity $\underline{v}(t)$ along some path relative to the lab frame K. We will now introduce K' as the **instantaneous rest frame** of the rocket.
Consider two closely separated points on the trajectory, with coordinates in the two frames $\{(ct, \underline{x}), (c[t+dt], \underline{x}+d\underline{x})\}$ and $\{(ct', \underline{x}'), (c[t'+dt'], \underline{x}'+d\underline{x}')\}$ respectively. The interval between the points is the invariant, and we have

$$c^2dt'^2 - \underline{dx}'^2 = c^2dt^2 - \underline{dx}^2.$$

But $\underline{dx}' = 0$ in k', and furthermore $\underline{dx}^2 = \underline{v}^2dt^2$, and thus

$$cdt' = cdt\sqrt{1 - \beta(t)^2},$$

where

$$\beta(t) = \frac{v(t)}{c}.$$

Then the elapsed time in the rocket between two events is

$$t_2' - t_1' = \int_{t_1}^{t_2} dt\sqrt{1 - \beta(t)^2} < t_2 - t_1.$$

The **proper time** τ is the *elapsed time* in the frame in with the object is at rest. Thus

$$cd\tau = ds$$

where ds is the *interval* introduced earlier. In this case we have

$$d\tau = dt\sqrt{1 - \beta(t)^2}. \tag{11.12}$$

Note that proper time can only be defined for *time-like* quantities.

11.5 Addition of Velocities

Suppose now that a projectile is fired with velocity $\underline{u}'$ from the rocket, relative to the rocket. Then the co-ordinates of the projectile in K' satisfies

$$\underline{u}' = \frac{d\underline{x}'}{dt'}.$$

while in K we have

$$\underline{u} = \frac{d\underline{\underline{x}}}{dt}.$$

Using the inverse Lorentz transform we have

$$\begin{aligned} x_\parallel &= \gamma_v[x'_\parallel + \beta ct'] \\ \Longrightarrow u_\parallel &\equiv \frac{dx_\parallel}{dt} = \gamma_v\left[\frac{dx'_\parallel}{dt'}\frac{dt'}{dt} + \beta c\frac{dt'}{dt}\right] \\ &= \gamma_v\left[\frac{dx'_\parallel}{dt'} + \beta c\right]\frac{dt'}{dt}, \end{aligned}$$

where we use $\parallel$ to denote the component along $\underline{v}$. We also have

$$\begin{aligned} ct &= \gamma[ct' + \beta x'_\parallel] \\ \Longrightarrow c &= \gamma_c\left[c\frac{dt'}{dt} + \beta u'_\parallel\frac{dt'}{dt}\right] \\ \Longrightarrow \frac{dt'}{dt} &= \frac{c}{\gamma_v[c + \beta u'_\parallel]} \end{aligned}$$

Combinding these two results, we find

$$u_\parallel = \frac{u'_\parallel + v}{1 + \beta u'_\parallel/c} \tag{11.13}$$

Similarly

$$u_\perp = \frac{dx_\perp}{dt} = \frac{dx'_\perp}{dt'}\cdot\frac{dt'}{dt},$$

yielding

$$u_\perp = \frac{u'_\perp}{\gamma(1 + \beta u'_\parallel/c)}. \tag{11.14}$$

In vector notation, this becomes

$$\begin{aligned} u_\parallel &= \frac{u'_\parallel + v}{1 + \underline{v}\cdot\underline{u}'/c^2} \\ \underline{u_\perp} &= \frac{\underline{u'_\perp}}{\gamma(1 + \underline{v}\cdot\underline{u}'/c^2)} \end{aligned} \tag{11.15}$$

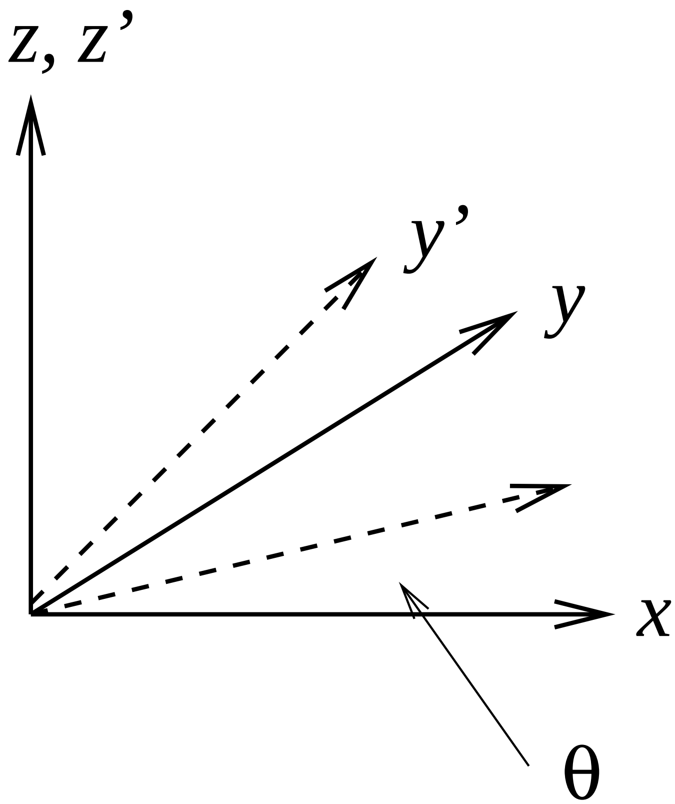
As expected, this reduces to the Galilean result $\underline{u} = \underline{u}' + \underline{v}$ for the case $u', v \ll c$.

11.6 Special Relativity and Four Vectors

We can formulate this picture in a much more convenient fashion through the introduction of *four vectors*. To see how these work, let us return briefly to Galilean transformations, and rotations in Euclidean space.

11.6.1 Vectors, Tensors and Rotations in R^3

Consider two co-ordinate systems P, P' whose origins coincide, but which are related by rotation through an angle θ.



The coordinates of a point in the two systems are related through

$$x'^i = R^i_j x^j, \tag{11.16}$$

where R is a rotation matrix. You will note I have put the indices **upstairs** on the vectors - I will return to this later. For the specific case of a rotation through

θ about he z axis, the rotation matrix is

$$R = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Quantities that transform as

$$A'^i = R^i_j A^j = \frac{\partial x'^i}{\partial x^j} A^j \tag{11.17}$$

are called **vectors**.

A simple example of a vector is $d\underline{x}$, which transforms as

$$dx'^i = \frac{\partial x'^i}{\partial x^j} dx^j,$$

Scalars

A scalar is a quantity which transforms as $f' = f$.

Co-vectors or Forms

Let us now consider how the **gradient** of a function transforms:

$$\underline{\nabla}'_i f = \frac{\partial f}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i} \frac{\partial f}{\partial x^j}.$$

This is an example of the transformation property

$$B'_i = \frac{\partial x^j}{\partial x'^i} B_j, \tag{11.18}$$

which is *different* to that of eqn. (11.17). Quantities that transform in this way are known as **covectors** or **forms**, and we put their indices downstairs.

Summarising, we have

$$\left.\begin{array}{rrcl} \text{Vector:} & A'^i & = & \frac{\partial x'^i}{\partial x^j} A^j \\ \text{Scalar:} & f' & = & f \\ \text{Covector:} & B'_i & = & \frac{\partial x^j}{\partial x'^i} B_j \end{array}\right\} \tag{11.19}$$

Finally, we have that a **tensor** is an object that transforms as a *vector* on each *upstairs* index, and a *covector* on each downstairs index.

$$C'^{i'j'\dots}_{k'l'\dots} = \frac{\partial x'^{i'}}{\partial x^i}\frac{\partial x'^{j'}}{\partial x^j}\dots\frac{\partial x^k}{\partial x'^{k'}}\frac{\partial x^l}{\partial x'^{l'}}\dots C^{i'j'\dots}_{k'l'\dots}$$

11.6.2 Metric Tensor

The **length** of a vector is a bilinear, and independent of the choice of frame. Define the **inner product** of two vectors by

$$X \cdot Y = g_{ij} X^i Y^j.$$

The tensor g_{ij} must be *isotropic*. There is only one isotropic rank-two tensor:

$$g_{ij} = \delta_{ij}.$$

We call g_{ij} the **metric tensor**.
We can use the metric tensor to *raise* or *lower* indices:

$$\begin{aligned} X_i &= g_{ij} X^j \\ X \cdot Y &= X^i Y_i = X_i Y^i. \end{aligned}$$

We only have the luxury of indentifying *vectors* with *covectors* in Cartesian coordinates in Euclidean space, where the components of the two are numerically equal.

Example

Show that in spherical polars

$$g_{ij} = \mathrm{diag}(1, r^2, r^2 \sin^2\theta).$$

11.6.3 Minkowski Space-Time

We will now apply the above ideas to Lorentz transformations of four-dimensional space-time. We will introduce "ct" as the coordinate x_0, and write a **contravariant** four vector as

$$x^{\mu} \equiv (ct, x, y, z) = (x^0, x^1, x^2, x^3) \tag{11.20}$$

The "length" of the vector is the **interval** left invariant under Lorentz transformations. More generally, we define the inner product of two vectors by

$$x \cdot y = g_{\mu\nu} x^{\mu} y^{\nu}, \tag{11.21}$$

and we immediately see that

$$\boxed{g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1) \qquad (11.22)}$$

- Note that it is conventional to use *Greek Letters* for the components of a four-vector. Four vectors are not underlined or printed in bold.

- In some areas of physics, time is introduced as the *fourth* component of the vector. Furthermore, the metric can be defined such that the spatial components are positive, and the temporal component negative. The convention I am using is probably the most widely used, and essentially universal amongst particle physicists.

- The summation convention is as follows - *A index can appear no more than twice. Any index appearing twice must have one upper index and one lower index, and that index is summed over.*

The **covariant four vector** or **form** can be obtained as before by using the raising and lowering properties of the metric tensor

$$x_{\mu} = g_{\mu\nu} x^{\nu}.$$

In our example we have that $x_{\mu} = (ct, -x, -y, -z)$ - *the components of a covector are numerically different to those of the vector.*

11.6.4 Lorentz Transformations and Four Vectors

Let us return to our two frames K and K'. The relation between vectors in the two frames is given by

$$x'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} x^{\nu} = L^{\mu}{}_{\nu} x^{\nu} \tag{11.23}$$

Let us assumed a similar transformation law for the covectors

$$x'_{\mu} = L_{\mu}{}^{\nu} x_{\nu}.$$

Since $x_{\mu} x^{\mu}$ is invariant we have

$$x'^{\mu} x'_{\mu} = L^{\mu}{}_{\nu} L_{\mu}{}^{\sigma} x^{\nu} x_{\sigma},$$

and since this is true for all vectors, we have

$$\boxed{L^{\mu}{}_{\nu} L_{\mu}{}^{\sigma} = \delta_{\nu}{}^{\sigma} \qquad (11.24)}$$

where

$$\delta_{\nu}{}^{\sigma} = \begin{cases} 1 & \text{if} \quad \nu = \sigma \\ 0 & \text{if} \quad \nu \neq \sigma \end{cases} \tag{11.25}$$

Note that

$$L_{\mu}{}^{\sigma} = \frac{\partial x^{\sigma}}{\partial x'^{\mu}}, \tag{11.26}$$

the characteristic transformation property of a form.

Thus the various quantities we will encouter in the remainder of this course are

- **Contravariant Vectors**:
$$A'^{\mu} = L^{\mu}{}_{\nu} A^{\nu}$$

- **Covariant Vectors**:
$$B'_{\mu} = L_{\mu}{}^{\nu} B_{\nu}$$

- **Tensors**:
$$C'^{\mu'\nu'\ldots}{}_{\rho'\sigma'\ldots} = L^{\mu'}{}_{\mu} L^{\nu'}{}_{\nu} \ldots L_{\rho'}{}^{\rho} L_{\sigma}{}^{\sigma} \ldots C^{\mu\nu\ldots}{}_{\rho\sigma\ldots}$$

- **Scalars**:

$$A \cdot B = A_\mu B^\mu = g_{\mu\nu} A^\mu B^\nu$$

Finally we have the relation

$$g_{\mu\nu} g^{\nu\sigma} = \delta_\mu^{\ \sigma}.$$

11.6.5 Derivatives

As we have noted earlier, these transform as *covectors*

$$\begin{aligned} \partial_\alpha &= \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \underline{\nabla}\right) \\ \partial^\alpha &= \frac{\partial}{\partial x_\alpha} = \left(\frac{\partial}{\partial x^0}, -\underline{\nabla}\right). \end{aligned} \tag{11.27}$$

Suppose now that we have a four vector A^μ. Then

$$\partial^\alpha A_\alpha = \partial_\alpha A^\alpha = \frac{\partial A^0}{\partial x^0} + \underline{\nabla} \cdot \underline{A}. \tag{11.28}$$

The Laplacian is defined by

$$\Box = \partial_\alpha \partial^\alpha = \frac{\partial^2}{\partial x^{0 2}} - \nabla^2. \tag{11.29}$$

11.7 Relativistic Dynamics

In our introduction to Lorentz Transformations, we found that the canonical definition of the velocity did not transform as a vector under a Lorentz tranformation, eqn. (11.15). Is it possible to find a definition of a velocity that does indeed transform covariantly under Lorentz transformations, yet reduces to a Galilean transformation for $v \ll c$?

In order to construct a **four velocity**, we need to take the derivative with respect to a *Lorentz Scalar* that can play the role of time. Such a scalar is provided by the **Proper Time** $d\tau$, defined by

$$c^2 d\tau^2 = ds^2,$$

where ds is the Lorentz-invariant *interval*. The proper time is clearly a scalar, and therefore a natural definition of the **four velocity** is

$$\boxed{v^\alpha = \frac{dx^\alpha}{d\tau} \qquad (11.30)}$$

Recalling that the proper time is related to the lab. time by

$$d\tau = dt\,\sqrt{1-\beta(t)^2}$$

we have

$$v^\alpha = \frac{1}{\sqrt{1-\beta^2}}\frac{d}{dt}(ct, \underline{x}) = \gamma(c, \underline{v}),$$

yielding

$$v^\alpha = (\gamma c, \gamma \underline{v}), \tag{11.31}$$

whose spatial components clearly reduce to our familiar definition of velocity in the non-relativistic (NR) limit.

11.7.1 Four Momentum

The definition of a Lorentz-covariant four momentum is now straightforward:

$$p^\mu = m v^\mu = (m\gamma c, m\gamma \underline{v}), \tag{11.32}$$

where m is a **Lorentz scalar** that we will call the **rest mass**.
The spatial components of p^μ clearly reduce to our usual definition of momentum.
To interpret the temporal component, we will look at its NR limit:

$$p^0 = m\gamma c = mc\left\{1 - v^2/c^2\right\}^{-1/2} = \frac{1}{c}\left\{mc^2 + \frac{1}{2}mv^2 + \mathcal{O}(v^4/c^2)\right\}.$$

The second term in braces is clearly the kinetic energy. The first term we identify as the **rest energy**, and write

$$p^0 = E/c$$

where E is the **energy**. Thus the four momentum contains both the energy and the three momentum.

The "length" of p^μ is a Lorentz scalar

$$\begin{aligned} p^\mu p_\mu &= m^2\gamma^2c^2 - m^2\gamma^2v^2 = m^2\gamma^2c^2\left[1 - v^2/c^2\right] \\ &= m^2\gamma^2c^2\gamma^{-2} = m^2c^2. \end{aligned}$$

Thus we have

$$\boxed{p^\mu p_\mu = p^2 = m^2c^2 \qquad (11.33)}$$

confirming that the rest mass is a (frame-independent) scalar.

Finally, if we now go back and write eqn. (11.33) in terms of our old-fashioned three vectors we have

$$\begin{aligned} \frac{1}{c^2}E^2 - \underline{p}^2 &= m^2c^2 \\ \Longrightarrow E^2 &= m^2c^4 + c^2\underline{p}^2. \end{aligned} \tag{11.34}$$

For a particle at rest, we have perhaps the most famous equation in physics.

The use of four-vectors is **essential** to solve problems in special (and general. . .) relativity. Whilst simple kinematical problems can be solved using three vectors, it is very clumsy indeed.

11.8 Covariant Formulation of Maxwell's Equation

Before considering Maxwell's equations in totality, we will return to the charge conservation.

11.8.1 Continuity Equation and Four Current

Charge conservation is expressed through the continuity equation

$$\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot\underline{J} = 0. \tag{11.35}$$

We can write this in a more manifestly covariant form as

$$\frac{1}{c}\frac{\partial}{\partial t}(\rho c) + \underline{\nabla} \cdot \underline{J} = 0.$$

It is therefore tempting to try to introduce a four-current

$$J^\mu = (\rho c, \underline{J}) \tag{11.36}$$

in terms of which eqn. (11.35) can be formally written

$$\partial_\mu J^\mu = 0.$$

However, it remains to be shown that the J^μ thus constructed does indeed transform as a four vector.

Consider J^μ defined through eqn. (11.36) under a transformation to a frame K' moving with velocity v along the x axis. Then, if J^μ we indeed a four vector we would have

$$\begin{aligned} \rho' c &= \gamma\left[\rho c - \frac{v}{c} J_x\right] \\ J'_x &= \gamma\left[J_x - v\rho\right] \\ J'_y &= J_y \\ J'_z &= J_z. \end{aligned}$$

In the NR limit

$$\left.\begin{aligned} \underline{J}' &= \underline{J} - \rho\underline{v} \\ \rho' &= \rho \end{aligned}\right\},$$

as expected.

Consider now the case $J_x = 0$. Then we have

$$\left.\begin{aligned} J'_x &= -\gamma v\rho \\ \rho' &= \gamma\rho \end{aligned}\right\}.$$

The second equation would appear to violate charge conservation. However, let us consider what happens to a volume element under this transformation. In the frame K, we have

$$dV = dx\, dy\, dz.$$

However

$$\begin{aligned} dx &= \gamma(dx' + v\,dt') \\ dt &= \gamma(dt' + \frac{v}{c^2}\,dx') \\ dy &= dy' \\ dz &= dz'. \end{aligned}$$

Thus for measurements made at the same time ($dt' = 0$)

$$dV = dx\,dy\,dz = \gamma dx' dy' dz' = \gamma dV',$$

and the total charge in dV' is

$$\rho' dV' = \rho' \gamma^{-1} dV = \gamma\rho\gamma^{-1} dV = \rho\,dV$$

Thus both the charge densities and volumes are not separately conserved under this Lorentz transformation, but the chage itself is.
There is much experimental evidence that $\rho' = \gamma\rho$, and we will **postulate** that J^μ in eqn. (11.36) is indeed a four vector, and that

$$\partial_\mu J^\mu = 0 \tag{11.37}$$

.

11.8.2 Units

At this point, *Jackson* changes from SI to Gaussian units - the aim being to avoid carrying superfluous factors of c. In my youth I did everything in SI units, and then in units in which $c \equiv 1$ (a huge simplification!). But to avoid confusion (!), I will also make the switch so as to be in keeping with *Jackson*.

Gaussian Units

$$\underline{\nabla} \cdot \underline{D} = 4\pi\rho \tag{11.38}$$

$$\begin{aligned}
\underline{\nabla} \times \underline{H} &= \frac{4\pi}{c}\underline{J} + \frac{1}{c}\frac{\partial \underline{D}}{\partial t} && (11.39) \\
\underline{\nabla} \times \underline{E} + \frac{1}{c}\frac{\partial \underline{B}}{\partial t} &= 0 && (11.40) \\
\underline{\nabla} \cdot \underline{B} &= 0 && (11.41)
\end{aligned}$$

$$\begin{aligned}
\underline{D} &= \epsilon \underline{E} = \underline{E} + 4\pi \underline{P} && (11.42) \\
\underline{H} &= \underline{B}/\mu = \underline{B} - 4\pi \underline{M} && (11.43)
\end{aligned}$$

You will notice that in these units $\partial/\partial t$ has an associated factor of $1/c$, corresponding to our definition of a four vector. Also, ϵ and μ are the *relative* permittivity and permeability respectively.

11.8.3 Potentials as Four Vectors

We introduce vector and scalar potentials so as to satisfy the homogeneous Maxwell equations

$$\begin{aligned}
\underline{B} &= \underline{\nabla} \times \underline{A} \\
\underline{E} &= -\underline{\nabla}\phi - \frac{1}{c}\frac{\partial \underline{A}}{\partial t} && (11.44)
\end{aligned}$$

In a vacuum ($\epsilon = \mu = 1$), the inhomogeneous equations become:

$$\begin{aligned}
\nabla^2\phi + \frac{1}{c}\frac{\partial \underline{\nabla} \cdot \underline{A}}{\partial t} &= -4\pi\rho \\
\nabla^2\underline{A} - \frac{1}{c^2}\frac{\partial^2 \underline{A}}{\partial t^2} - \underline{\nabla}\left[\underline{\nabla} \cdot \underline{A} + \frac{1}{c}\frac{\partial \phi}{\partial t}\right] &= -\frac{4\pi}{c}\underline{J}.
\end{aligned}$$

In the **Lorentz gauge**, we have

$$\underline{\nabla} \cdot \underline{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0,$$

and the dynamical equations become

$$\begin{aligned}
\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} &= -4\pi\rho \\
\nabla^2\underline{A} - \frac{1}{c^2}\frac{\partial^2 \underline{A}}{\partial t^2} &= -\frac{4\pi}{c}\underline{J}. && (11.45)
\end{aligned}$$

We now recognise the operator on the l.h.s. of these equations as the four-dimensional Laplacian introduced in eqn. (11.29), and the r.h.s. as the temporal and spatial components of the current J^μ of eqn. (11.36). We will therefore introduce a four-vector potential

$$\underline{A}^\mu = (\phi, \underline{A}), \tag{11.46}$$

so that both equations in (11.45) can be unified in the manifestly covariant form

$$\Box A^\mu = \frac{4\pi}{c} J^\mu. \tag{11.47}$$

Furthermore, the Lorentz gauge condition is also manifestly covariant:

$$\partial^\mu A_\mu = 0. \tag{11.48}$$

11.8.4 Field-Strength Tensor

In order to formulate the full Maxwell's equations in covariant form, we need to return to the relation between the fields $(\underline{E}, \underline{B})$ and the potentials $(\phi, \underline{A})$ of eqn. (11.44). We need to find a covariant relation between electric and magnetic fields, and the four vector A^μ, and indeed express the fields themselves in covariant form. Let us write out a couple of these components explicitly

$$\begin{aligned} \underline{B}_x &= \frac{\partial \underline{A}_z}{\partial y} - \frac{\partial \underline{A}_y}{\partial z} = \frac{\partial A^3}{\partial x^2} - \frac{\partial A^2}{\partial x^3} = \frac{\partial A^2}{\partial x_3} - \frac{\partial A^3}{\partial x_2} \\ \underline{E}_x &= -\frac{\partial \phi}{\partial x} - \frac{1}{c}\frac{\partial \underline{A}_x}{\partial t} = -\frac{\partial A^0}{\partial x^1} - \frac{\partial A^1}{\partial x^0} = \frac{\partial A^0}{\partial x_1} - \frac{\partial A^1}{\partial x_0} \end{aligned}$$

N.B.: I am using a slightly confusing notation: $\underline{E}_i$ to denote the i component of a **three vector**, where we do not need to distinguish between covariant and contravariant vectors. The equivalent four-vector components are given by

$$\begin{aligned} E^i &= \underline{E}_i \\ E_i &= -\underline{E}_i. \end{aligned}$$

We can see that $(\underline{E}, \underline{B})$ are related to a second-rank tensor, and there are six independent components of the two fields.
For a general second-rank tensor $T^{\mu\nu}$, we can write

$$T^{\mu\nu} = T^{\mu\nu}_{\text{sym}} + T^{\mu\nu}_{\text{anti-sym}}.$$

The symmetric part has ten components, but the anti-symmetric part has the six independent components that we could associate with fields $\underline{E}$ and $\underline{B}$. Thus we introduce the anti-symmetric **Maxwell Field-Strength Tensor**

$$\boxed{F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \qquad (11.49)}$$

Writing out the components of $F^{\mu\nu}$ explicitly, we have

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{11.50}$$

We see that $\underline{E}$ and $\underline{B}$ are not components of four vectors, but rather of an anti-symmetric, second-rank tensor. Note that we can lower the indices in the usual way

$$F_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} F^{\alpha\beta},$$

so that the components corresponding to $\underline{E}$ change sign, whilst those corresponding to $\underline{B}$ are unaltered.

Finally, we will introduce the **dual** field-strength tensor. But as a precursor we will return to the Levi-Civita tensor.

Levi-Civita Tensor

This is the four-dimensional version of the ϵ_{ijk} encountered in 3-D Euclidean space. It is defined by

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} 1 & \text{if } \mu, \nu, \rho, \sigma \text{ is an } \textit{even} \text{ perm of } 0,1,2,3 \\ -1 & \text{if } \mu, \nu, \rho, \sigma \text{ is an } \textit{odd} \text{ perm of } 0,1,2,3 \\ 0 & \text{if any two indices are equal} \end{cases} \tag{11.51}$$

Lowering the indices in the usual way, we immediately see that

$$\epsilon_{\mu\nu\rho\sigma} = -\epsilon^{\mu\nu\rho\sigma}.$$

Note the very useful relation

$$\epsilon^{\alpha\beta\mu\nu} \epsilon_{\alpha\beta\rho\sigma} = -2 \left(\delta^{\mu}{}_{\rho} \delta^{\nu}{}_{\sigma} - \delta_{\sigma}{}^{\mu} \delta_{\rho}{}^{\nu} \right). \tag{11.52}$$

11.8.5 Dual Field-Strength Tensor

The dual field-strength tensor is defined by

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}. \tag{11.53}$$

The elements of $\tilde{F}^{\mu\nu}$ are related to those of $F^{\mu\nu}$ through the substitution

$$\begin{aligned} \underline{E} &\longrightarrow \underline{B} \\ \underline{B} &\longrightarrow -\underline{E}, \end{aligned}$$

so that

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}.$$

Thus $\tilde{F}^{\mu\nu}$ reverses the roles of the electric and magnetic fields.
Finally, using eqn. (11.52), we have

$$\begin{aligned} \tilde{\tilde{F}}^{\mu\nu} &= \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\tilde{F}_{\rho\sigma} \\ &= \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\lambda\tau}F^{\lambda\tau} \\ &= -F^{\mu\nu} \end{aligned} \tag{11.54}$$

11.8.6 Maxwell's Equations

Let us return to Maxwell's equation in a vacuum

$$\underline{\nabla}\cdot\underline{E} = 4\pi\rho \tag{11.55}$$

$$\underline{\nabla}\times\underline{E} + \frac{1}{c}\frac{\partial\underline{\underline{B}}}{\partial t} = 0 \tag{11.56}$$

$$\underline{\nabla}\times\underline{B} = \frac{4\pi}{c}\underline{J} + \frac{1}{c}\frac{\partial\underline{\underline{E}}}{\partial t} \tag{11.57}$$

$$\underline{\nabla}\cdot\underline{B} = 0. \tag{11.58}$$

These are all first-order differential equations expressed in terms of $\underline{E}$ and $\underline{B}$. Thus we might suspect that the covariant form of Maxwell's equations will contain terms of the form

$$\partial_\mu F_{\nu\rho}.$$

Looking at eqn. (11.55), we see that it may be written

$$\frac{\partial}{\partial x^i}E^i = 4\pi\frac{J^0}{c}.$$

Recalling that $E^i = F^{i0}$, and noting that F^{00} vanishes, we can rewrite (11.55) as

$$\partial_\mu F^{\mu 0} = \frac{4\pi}{c}J^0. \tag{11.59}$$

Turning now to the second inhomogeneous equation, eqn. (11.57), we see that it may be written

$$\epsilon^{0ijk}\frac{\partial}{\partial x^j}\tilde{F}^{k0} = \frac{4\pi}{c}J^i + \frac{1}{c}\frac{\partial}{\partial t}E^i,$$

where we use $B^k = \tilde{F}^{k0}$. To put this equation in a form analogous to eqn. (11.59), we perform a clever piece of manipulation:

$$\begin{aligned}
\epsilon^{0ijk}\tilde{F}^{k0} &= \epsilon^{k0ij}\tilde{F}_{k0} \\
&= \frac{1}{2}\epsilon^{\mu\nu ij}\tilde{F}_{\mu\nu} \\
&= \tilde{\tilde{F}}^{ij} \\
&= -F^{ij}
\end{aligned}$$

where in the second line we have used that one of μ or ν must be the temporal component, and the other a spatial component. Thus eqn. (11.57) can be written

$$\begin{aligned}
-\frac{\partial}{\partial x^j}F^{ij} &= \frac{4\pi}{c}J^i + \frac{1}{c}\frac{\partial}{\partial t}F^{i0} \\
\Longrightarrow \frac{\partial}{\partial x^j}F^{ji} + \frac{\partial}{\partial x^0}F^{0i} &= \frac{4\pi}{c}J^i && (11.60) \\
\Longrightarrow \partial_\mu F^{\mu i} &= \frac{4\pi}{c}J^i. && (11.61)
\end{aligned}$$

Thus we see that both the inhomogeneous Maxwell equations can be written in the unified form

$$\boxed{\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu \qquad (11.62)}$$

Turning now to the homogeneous equations, we see that eqn (11.58) can be written

$$\begin{aligned} \frac{\partial}{\partial x^j}\tilde{F}^{i0} &= 0 \\ \Longrightarrow \partial_\mu \tilde{F}^{\mu\nu} &= 0. \end{aligned}$$

Eqn. (11.56) takes the form

$$\begin{aligned} \epsilon^{0ijk}\frac{\partial}{\partial x^j}F^{k0} + \frac{1}{c}\frac{\partial}{\partial t}\tilde{F}^{i0} &= 0 \\ \Longrightarrow \frac{\partial}{\partial x^j}\tilde{F}^{ij} + \frac{\partial}{\partial x^0}\tilde{F}^{i0} &= 0 \\ \Longrightarrow \partial_\mu \tilde{F}^{\mu i} &= 0. \end{aligned}$$

Thus the two homogeneous Maxwell equations can be written in the unified form

$$\boxed{\partial_\mu \tilde{F}^{\mu\nu} = 0 \qquad (11.63)}$$

Eqns. (11.62) and (11.63) constitute the covariant formulation of Maxwell's equations.

Note that we can rewrite eqn. (11.63) as

$$\begin{aligned} \frac{1}{2}\partial_\mu \epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} &= 0 \\ \Longrightarrow \epsilon^{\mu\nu\rho\sigma}\partial_\mu F_{\rho\sigma} &= 0, \end{aligned}$$

which we can express as

$$\partial_\mu F_{\rho\sigma} + \partial_\rho F_{\sigma\mu} + \partial_\sigma F_{\mu\rho} = 0. \tag{11.64}$$

This is known as the **Jacobi Identity**.

11.9 Energy and Momentum Law

The Lorentz force law in Gaussian units is

$$\frac{d\underline{p}}{dt} = q\left\{\underline{E} + \frac{1}{c}\underline{v} \times \underline{B}\right\}.$$

In order to write this in a covariant form, we introduce the proper time

$$d\tau = \gamma^{-1}dt,$$

and write

$$\frac{dp^i}{dt} = \frac{dp^i}{d\tau}\frac{d\tau}{dt} = \frac{1}{\gamma}\frac{dp^i}{d\tau}.$$

Thus the force law may be expressed as

$$\frac{dp^i}{d\tau} = \gamma q\left\{E^i + \frac{1}{c}\epsilon^{0ijk}v^j B^k\right\}.$$

We now introduce the four-velocity $V^\mu = (\gamma c, \gamma \underline{v})$, yielding

$$\begin{aligned}
\frac{dp^i}{d\tau} &= \frac{q}{c}\{V_0 F^{i0} + \epsilon^{0ijk}V^j B^k\} \\
&= \frac{q}{c}\{V_0 F^{i0} + \epsilon^{0ijk}V_j \tilde{F}_{k0}\} \\
&= \frac{q}{c}\{V_0 F^{i0} + F^{ij}V_j\} \qquad \text{(using eqn. (11.54))}.
\end{aligned}$$

Thus the Lorentz force law becomes

$$\frac{dp^i}{d\tau} = \frac{q}{c}V_\mu F^{i\mu}. \tag{11.65}$$

The analogous equation for the energy is

$$\frac{d}{dt}E^{\text{mech}} = \frac{dt}{d\tau}\frac{d}{d\tau}E^{\text{mech}} = q\underline{E}\cdot\underline{v}.$$

Thus we have

$$\begin{aligned}
\frac{dE^{\text{mech}}}{d\tau} &= \gamma q F^{i0}v^i \\
&= qF^{0i}V_i,
\end{aligned}$$

yielding

$$\frac{d}{d\tau}\left(\frac{E^{\text{mech}}}{c}\right) = \frac{q}{c}V_\mu F^{0\mu}.$$

Identifying E^{mech}/c with the component p^0, we see that both this equation and the Lorentz law, eqn. (11.65), can be expressed as

$$\boxed{\frac{dp^\mu}{d\tau} = \frac{q}{c}V_\nu F^{\mu\nu} \qquad (11.66)}$$

and Newton's second law is in a manifestly covariant form.

Lorentz Invariants

There are two invariants we can construct from the field-strength tensor

1.

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} &= F_{0i}F^{0i} + F_{ij}F^{ij} + F_{i0}F^{i0} \\ &= -2(\underline{E}^2 - \underline{B}^2) \end{aligned}$$

Thus

$$\underline{E}^2 - \underline{B}^2 = \frac{1}{2}F_{\mu\nu}F^{\mu\nu} \qquad (11.67)$$

is a **Lorentz Scalar**.

2.

$$\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = 2F_{\mu\nu}\tilde{F}^{\mu\nu} = -8\underline{E}\cdot\underline{B}.$$

Thus

$$-4\underline{E}\cdot\underline{B} = F_{\mu\nu}\tilde{F}^{\mu\nu} \qquad (11.68)$$

is a **Lorentz Scalar**

This are the *only* Lorentz invariants.

11.10 Transformation Properties of EM Field

Since $F^{\mu\nu}$ is a second-rank tensor, we can immediately say it transforms according to

$$F'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} F^{\alpha\beta} \frac{\partial x'^{\nu}}{\partial x^{\beta}},$$

which we can write as

$$F' = \Lambda F \Lambda^{T}, \tag{11.69}$$

where

$$\Lambda^{\mu}{}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}.$$

Specifically, let us consider a boost from K to K' where K' has velocity v in x-direction w.r.t. K, and origins coincide at $t = t' = 0$. Then

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\beta = v/c$ and $\gamma = (1-\beta^2)^{-1/2}$. Using this expression in eqn. (11.69), we find

$$F' = \begin{pmatrix} 0 & -E_1 & -\gamma(E_2 - \beta B_3) & -\gamma(E_3 + \beta B_2) \\ E_1 & 0 & -\gamma(B_3 - \beta E_2) & \gamma(B_2 + \beta E_3) \\ \gamma(E_2 - \beta B_3) & \gamma(B_3 - \beta E_2) & 0 & -B_1 \\ \gamma(E_3 + \beta B_2) & \gamma(B_2 + \beta E_3) & B_1 & 0. \end{pmatrix}$$

Writing out the individual vector components, we find

$$\left.\begin{array}{lllllll} E'_1 & = & E_1 & ; & B'_1 & = & B_1 \\ E'_2 & = & \gamma(E_2 - \beta B_3) & ; & B'_2 & = & \gamma(B_2 + \beta E_3) \\ E'_3 & = & \gamma(E_3 + \beta B_2) & ; & B'_3 & = & \gamma(B_3 - \beta E_2) \end{array}\right\} \tag{11.70}$$

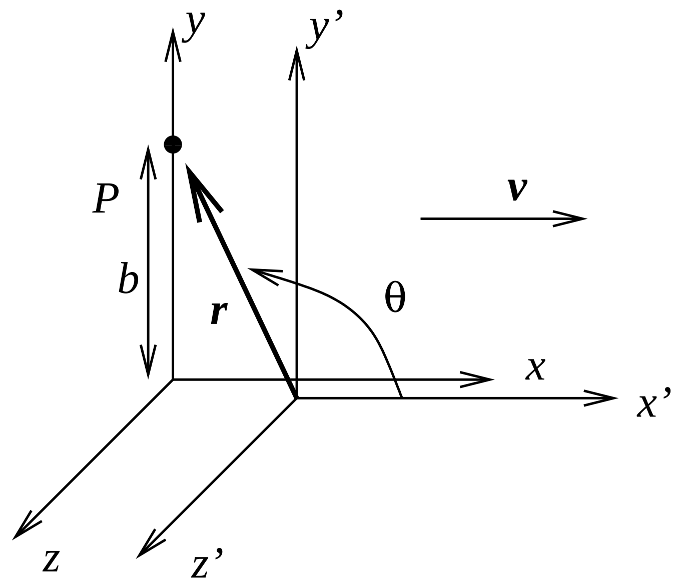
We can express this in (three) vector form as

$$\begin{aligned} \underline{E}' &= \gamma[\underline{E} + \underline{\beta} \times \underline{B}] - \frac{\gamma^2}{\gamma+1}\underline{\beta}(\underline{\beta} \cdot \underline{E}) \\ \underline{B}' &= \gamma(\underline{B} - \underline{\beta} \times \underline{E}) - \frac{\gamma^2}{\gamma+1}\underline{\beta}(\underline{\beta} \cdot \underline{B}), \end{aligned} \tag{11.71}$$

where $\underline{\beta} = \underline{v}/c$. Thus the $\underline{E}$ and $\underline{B}$ fields **mix** under a Lorentz transformation.

11.10.1 Electric and magnetic fields due to relativistically moving point change.

Consider a charge q moving along a line at velocity (in K) $\underline{v} = v\underline{e_1}$. The charge is at rest in the frame K'.



At $t = t' = 0$, the origins of the two frames co-incide. We have an observer P at impact parameter b (i.e. distance of closest approach) as shown above.
We will begin by looking at electric and magnetic fields at point P *in frame* K' at time t'.
P has coordinates

$$\begin{aligned} x' &= -vt' \\ y' &= b \\ z' &= 0. \end{aligned}$$

Thus, from Coulomb's law

$$\begin{aligned} E_1' &= -qvt'/r'^3 \quad ; \quad E_2' = qb/r'^3 \quad ; \quad E_3' = 0 \\ B_1' &= 0 \quad ; \quad B_2' = 0 \quad ; \quad B_3' = 0. \end{aligned}$$

In order to express this in terms of coordinates in K, we note that $r'^2 = b^2 + v^2 t'^2$. But we have

$$ct' = \gamma(ct - \beta x) = \gamma ct.$$

Thus

$$r'^2 = b^2 + v^2\gamma^2 t^2$$

and we have

$$\begin{aligned} E'_1 &= -\frac{q\gamma vt}{(b^2 + v^2\gamma^2 t^2)^{3/2}} \\ E'_2 &= -\frac{q}{(b^2 + v^2\gamma^2 t^2)^{3/2}} \\ E'_3 &= 0. \end{aligned}$$

We now use our transformation laws eqn. (11.70) to write

$$\begin{aligned} E_1 &= E'_1 = -\frac{q\gamma vt}{(b^2 + v^2\gamma^2 t^2)^{3/2}} \\ E_2 &= \gamma E'_2 = \frac{\gamma qb}{(b^2 + v^2\gamma^2 t^2)^{3/2}} \\ E_3 &= \gamma E'_3 = 0 \end{aligned}$$

$$\begin{aligned} B_1 &= 0; B_2 = \gamma B'_2 = 0 \\ B_3 &= \gamma\beta E'_2 = \beta E_2 \end{aligned}$$

Thus in the laboratory frame we see a magnetic induction.

Note that in the limit $v \longrightarrow c$, we have $\beta \longrightarrow 1$ and the magnetic induction equals the transverse electric field. In the Galilean limit $v \longrightarrow 0$,

$$\begin{aligned} B_3 &= \frac{v}{c}\frac{\gamma qb}{(b^2 + v^2\gamma^2 t^2)^{3/2}} \longrightarrow \frac{vqb}{c(b^2 + v^2 t^2)^{3/2}} \\ \Longrightarrow \underline{B} &\sim \frac{q}{c}\frac{\underline{v} \times \underline{r}}{r^3} \end{aligned}$$

where we have used $vb = vr\sin\theta$, which we observe is just the *Biot-Savart Law.*

Finally, let us look at the field lines. We have that

$$\frac{E_2}{E_1} = -\frac{b}{vt},$$

so that the electric field is still a central field in the frame K. If we now look at the *magnitude* of the field, however, we find

$$|\underline{E}| = \frac{\gamma q}{(b^2 + v^2\gamma^2 t^2)^{3/2}}(b^2 + v^2 t^2)^{1/2}.$$

Setting $b = r\sin\theta, vt = -r\cos\theta$, we have

$$|\underline{E}| = \frac{\gamma q r}{r^3(1+\gamma^2\cos^2\theta)^{3/2}} \sim \frac{\gamma q}{r^2}(1+\gamma\cos^2\theta)^{3/2}.$$

So the lines of force, whilst central, are no longer *isotropic* - they are predominantly transverse in strength.

11.11 Plane Electromagnetic Radiation and Doppler Shift

Let us look at the propagation of a plane wave in vacuum. Our starting point is the Jacobi identity eqn. (11.64). Applying ∂^α we find

$$\partial^\alpha\partial_\alpha F_{\beta\gamma} + \partial_\beta\partial^\alpha F_{\gamma\alpha} + \partial_\gamma\partial^\alpha F_{\alpha\beta} = 0. \tag{11.72}$$

In the absence of sources,

$$\partial^\mu F_{\mu\nu} = \frac{4\pi}{c}J_\nu = 0.$$

Thus the last two terms on the r.h.s. of eqn. (11.72) vanish, and we have the plane e.m. waves satisfy

$$\boxed{\Box F_{\mu\nu} = 0 \qquad (11.73)}$$

In complete analogy to the three-dimensional NR formulations, we note that this admits the solution

$$F_{\mu\nu} = f_{\mu\nu}e^{ik_\alpha x^\alpha} \tag{11.74}$$

where

$$k_\alpha k^\alpha = k^2 = 0. \tag{11.75}$$

Writing $k^\alpha = (\omega/c, \underline{k})$, we see that eqn. (11.75) is just

$$\underline{k}^2 = \omega^2/c^2$$

which is our usual relation between wave number and frequency.

We will now look at the transformation properties of the solution. We will let the solution in frame K be

$$F_{\mu\nu} = f_{\mu\nu} e^{ik\cdot x}$$

whilst that in K' be

$$F'_{\mu\nu} = f'_{\mu\nu} e^{ik'\cdot x'}.$$

The solutions in the two frames are related by

$$F'_{\mu\nu} = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma F_{\rho\sigma}.$$

This can be satisfied $\forall x$ iff

$$k' \cdot x' = k \cdot x$$

showing that k and k' are indeed four vectors. Because of this, we know that k_μ and k'_μ are related by

$$\begin{aligned} k'_\parallel &= \gamma[k_\parallel - \beta k_0] \\ k'_0 &= \gamma[k_0 - \beta k_\parallel] \\ \underline{k}'_\perp &= \underline{k}_\perp \end{aligned} \qquad (11.76)$$

Introducing θ as the angle between $\underline{k}$ and $\underline{v}$, we can use the second eqn. of (11.76) to compute the Doppler shift:

$$\frac{\omega'}{c} = \gamma\left[\frac{\omega}{c} - \frac{v}{c}\cos\theta|\underline{k}|\right] = \gamma\left[\frac{\omega}{c} - \frac{v}{c}\cos\theta\,\frac{\omega}{c}\right].$$

Thus we have the Doppler Shift formula

$$\boxed{\omega' = \gamma\omega(1 - \beta\cos\theta) \qquad (11.77)}$$

where $\beta = v/c$. This is modified from the usual Galilean formula through the factor of γ.

11.11.1 Aberation

This is the change in *direction* of a wave vector between the two frames.

We can calculate this from

$$\tan\theta' = \frac{|\underline{k}'_\perp|}{k'_\parallel} = \frac{|\underline{k}_\perp|}{k'_\parallel}.$$

By our Lorentz transformation formula

$$\begin{aligned} k'_\parallel &= \gamma\left[k_\parallel - \beta\frac{\omega}{c}\right] = \gamma\left[\frac{\omega}{c}\cos\theta - \beta\frac{\omega}{c}\right] \\ &= \gamma\frac{\omega}{c}(\cos\theta - \beta) \end{aligned}$$

Also we have

$$\underline{k}_\perp^2 = k_0^2 - k_\parallel^2 = \left(\frac{\omega}{c}\right)^2(1-\cos^2\theta) = \left(\frac{\omega}{c}\sin\theta\right)^2,$$

and thus

$$|\underline{k}_\perp| = \frac{\omega}{c}\sin\theta.$$

Thus, taking the ratio, we find

$$\boxed{\tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - \beta)}} \qquad (11.78)$$

Summary of First Three Chapters

Introduction

There are two founding principles of electrostatics:

- **Coulomb's Law**:
$$\underline{F}_{21} = \frac{1}{4\pi\epsilon_0} q_1 q_2 \frac{\hat{\underline{r}}_{21}}{|\underline{r}_2 - \underline{r}_1|^2}$$
- **Principle of Linear Superposition**

  The resultant force on a test particle due to several charges is the **vector sum** of the forces due to the charges individually.

In the case of a continuous charge distribution, these two principles yield

$$\underline{E}(\underline{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\underline{r}')(\underline{r} - \underline{r}')dV'}{|\ \underline{r} - \underline{r}'\ |^3}.$$

Gauss' Law

If charge density $\rho(\underline{r})$ is the sole source of the electrostatic field $\underline{E}(\underline{r})$, the flux of $\underline{E}$ out of a closed surface S bounding a volume V is given by

$$\int_S \underline{E} \cdot \underline{dS} = 4\pi CQ = \frac{Q}{\epsilon_0} \qquad \text{in SI units}$$

where $Q =$ **total charge** within S. This can be expressed in differential form as **Maxwell's First Equation (ME1)**

$$\underline{\nabla} \cdot \underline{E} = \rho/\epsilon_0.$$

Scalar Potential

It is easy to show $\underline{\nabla} \times \underline{E} = 0$, and for such fields

$$\underline{E}(\underline{x}) = -\underline{\nabla}\phi(\underline{x}),$$

where ϕ is the **Scalar Potential**.

Laplace's and Poisson's Equation

ME1 can be expressed in terms of potential as

$$\nabla^2\phi(\underline{x}) = -\rho(\underline{x})/\epsilon_0$$

Uniqueness Theorem

> The solution $\phi(\underline{x})$ of Poisson's or Laplace's equation inside a volume V bounded by surface S satisfying *either* **Dirichlet** or **Neumann** boundary conditions is **unique** (Dirichlet) or **unique up to a constant** (Neumann).

This is perhaps the most important theorem in the course, which we will use implicitly in much of the following.

Boundary conditions at Surface of Conductor

Throughout the body of a conductor, $\underline{E}$ vanishes and at the surface

$$\phi(\underline{x}) = \text{constant}$$

so that $\underline{E}$ is normal to the surface. The **surface charge density** σ is related to the discontinuity in the normal electrostatic field

$$\underline{E} \cdot \underline{n} = \sigma/\epsilon_0.$$

Boundary-Value Problems in Electrstatics

We explored variety of ways of obtaining *unique* solution.

- **Method of Images**

  Here we introduce an *image charge* outside the region we are seeking a solution such that image system satisfies boundary conditions of original problem. Two crucial things to check

  - Only additional charges are introduced outside the region so that Poisson's equation is unaltered.
  - Image system satisfies correct boundary conditions.

  Then by uniqueness theorem, image potential is the **unique solution** in region of interest.

- **Solution in terms of Green Functions.**

  The Green function $G(\underline{x}, \underline{x}')$ for Laplaces or Poisson's equation in a volume V bounded by surface S satisfies

$$\nabla'^2 G(\underline{x}, \underline{x}') = -4\pi\delta^{(3)}(\underline{x} - \underline{x}')$$

  subject to

$$\begin{aligned} G(\underline{x}, \underline{x}') &= 0 \qquad \text{for } \underline{x}' \in S \text{ - } \textbf{Dirichlet} \\ \frac{\partial G(\underline{x}, \underline{x}')}{\partial n'} &= -\frac{4\pi}{S} \qquad \text{for } \underline{x}' \in S \text{ - } \textbf{Neumann}. \end{aligned}$$

  Poisson's equation has formal solution,,

$$\begin{aligned} \phi(\underline{x}) = \frac{1}{4\pi\epsilon_0} \int_V d^3x' \, G(\underline{x}, \underline{x}')\rho(\underline{x}') + \\ \frac{1}{4\pi} \int_{S=\partial V} dS' \left\{ G(\underline{x}, \underline{x}') \frac{\partial \phi(\underline{x}')}{\partial n'} - \phi(\underline{x}') \frac{\partial G(\underline{x}, \underline{x}')}{\partial n'} \right\} \end{aligned}$$

  Green functions can be obtained using, e.g., *method of images*, or in terms of *orthonormal eigenfunctions*.

- **Expansion of Solution in Orthogonal Functions**.

  Equations of **Sturm-Liouville** type have a set of solutions which are *orthogonal* and *complete*, subject to certain specific boundary conditions. Then any square-integrable function can be expanded in terms of these eigenfunctions.

- Separation of Variables in Cartesian Coordinates

  This provides a powerful method of seeking solutions of Laplace's equation, and indeed often generates a set of useful basis functions.

  Writing $\phi(x, y, z) = X(x)Y(y)Z(z)$, Laplace's equation can be written

  $$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0,$$

  each term of which is separately constant. There are eleven coordinate systems admitting such a separation!

Boundary-value Problems in Curvilinear Coordinates

Here we seek factorisable solutions in *spherical polars* and in *cylindrical polars*

Spherical Polars, Azimuthal Symmetry

We seek solution of form

$$\phi(r,\theta,\psi) = \frac{U(r)}{r} P(\theta) Q(\psi).$$

- In case of **azimuthal** symmetry (no ψ dependence), the general solution assumes the form

$$\phi(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-l-1}\right) P_l(\cos\theta),$$

  where $P_l(\cos\theta)$ are the **Legendre Polynomials**, satisfying **Legendre's equation**

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + l(l+1)P = 0.$$

- $P_l(\cos\theta)$ form a *complete set of orthogonal functions.* Note that they are not *orthonormal.*

- Important expansion for construction of Green functions

$$\frac{1}{|\underline{x} - \underline{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)$$

  where $r_> = \max(r, r')$ and $r_< = \min(r, r')$.

Spherical Harmonics

Where there is no azimuthal symmetry, we have the **Generalized Legendre equation**

$$\frac{d}{dx}\left[(1-x^2)\frac{dP(x)}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right] P(x) = 0.$$

- Convenient to combine ψ and θ functions into *solutions on unit sphere* described by **Spherical Harmonics** $Y_{lm}(\theta, \psi)$, with general solution

$$\phi(r, \theta, \psi) = \sum_{l=}^{\infty} \sum_{m=-l}^{l} \left[A_{lm} r^l + B_{lm} r^{-l-1} \right] Y_{lm}(\theta, \psi)$$

- Important, and very useful, result:

$$\frac{1}{|\underline{x} - \underline{x}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta', \psi') Y_{lm}(\theta, \psi),$$

where we have factorised the (θ, ψ) and (θ', ψ') behaviour.

Laplace Equation in Cylindrical Polars

Here we seek factorisable solutions of form

$$\phi(\rho, \theta, z) = R(\rho) T(\theta) Z(z).$$

- Gives rise to **Bessel's equation**

$$\frac{d^2 R}{dx^2} + \frac{1}{x} \frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right) R = 0$$

- Solutions are the **Bessel Functions** $J_\nu(x)$ and $N_\nu(x)$, which are *linearly independent*. A further set is provided by *Hankel Functions*.

Expansion of Green Function in terms of Orthogonal Functions

- Because any function can be expanded in set of orthogonal functions, we can *expand Green function*.

- **Green Function for Sturm-Liouville Equation**

$$\frac{d}{dx'} \left[p(x') \frac{dg(x, x')}{dx'} \right] + q(x') g(x, x') = -4\pi \delta(x - x'),$$

defined on the interval $x' \in [a, b]$, with homogeneous boundary conditions at a and b.

- Green function

$$g(x,x') = \begin{cases} -\dfrac{4\pi}{p(x)} \dfrac{y_2(x)y_1(x')}{W[y_1(x),y_2(x)]} & a \le x' \le x \\ -\dfrac{4\pi}{p(x)} \dfrac{y_2(x')y_1(x)}{W[y_1(x),y_2(x)]} & x \le x' \le b \end{cases}$$

where

$$W[y_1(x),y_2(x)] = y_1(x)y_2'(x) - y_2(x)y_1'(x).$$

is the **Wronskian**, and y_1 and y_2 are solutions to the homogeneous equation

- **Spectral Representation** Expand Green function in terms of the *eigenfunctions* of some related problem. Consider the solution of

$$\nabla^2\psi(\underline{x}) + [f(\underline{x}) + \lambda]\psi(\underline{x}) = 0,$$

in volume V subject to ψ satisfying certain homogeneous boundary conditions for $x \in S$. Find set of eigenfunctions ψ_n, with eigenvalues λ_n.

Green function satisfies

$$\nabla'^2 G(\underline{x},\underline{x}') + [f(\underline{x}') + \lambda]G(\underline{x},\underline{x}') = -4\pi\delta(\underline{x}-\underline{x}')$$

where λ is, in general, not an eigenvalue, and we can write

$$G(\underline{x},\underline{x}') = 4\pi \sum_n \frac{\psi_n^*(\underline{x})\psi_n(\underline{x}')}{\lambda_n - \lambda}$$