HW1 solutions

Problem1

$$\vec{\nabla} \times \vec{v}^a = \hat{e}_i \epsilon_{ijk} \frac{\partial}{\partial x_j} \vec{v}^a_k = -6xz\hat{e}_1 + 2z\hat{e}_2 + 3z^2\hat{e}_3$$

$$\vec{\nabla} \times \vec{v}^b = \hat{e}_i \epsilon_{ijk} \frac{\partial}{\partial x_j} \vec{v}^b_k = -2y\hat{e}_1 - 3z\hat{e}_2 - x\hat{e}_3$$

Problem 2.

$T = x^2 + 4xy + 2yz^3$

$$T(1,1,1) - T(,0,0,0) = 7, \qquad \vec{\nabla} T = (2x+4y)\hat{e}_1 + (4x+2z^3)\hat{e}_2 + 6yz^2\hat{e}_3$$

(a)

$$\int_0^1 dz\ (\nabla T)_3\Big|_{x=y=0} + \int_0^1 dy\ (\nabla T)_2\Big|_{x=0,\ z=1} + \int_0^1 dx\ (\nabla T)_1\Big|_{y=z=1}$$
$$= 2\int_0^1 dy\ + \int_0^1 dx\ (2x+4) = 2+5 = 7$$

(b) To calculate a line integral $\int_{\vec{a}}^{\vec{b}} \vec{dl} \cdot \vec{v}$ we take a parametrization $x = x(t), \quad y = y(t), \quad z = z(t)$ so that

$$\vec{dl} = \frac{dx(t)}{dt}\hat{e}_1 + \frac{dy(t)}{dt}\hat{e}_1 + \frac{dz(t)}{dt}\hat{e}_3$$

and calculate

$$\int_{\vec{a}}^{\vec{b}} \vec{dl} \cdot \vec{v} = \int_{t_a}^{t_b} dt\ \Big(\frac{dx(t)}{dt}\hat{e}_1 + \frac{dy(t)}{dt}\hat{e}_2 + \frac{dz(t)}{dt}\hat{e}_3\Big) \cdot \vec{v}(x(t), y(t), z(t))$$

In our case it is convenient to take $t = x$ so $x = y = t$, $z = t^2$ and therefore

$$\int_{(0,0,0)}^{(1,1,1)} \vec{dl} \cdot \vec{\nabla} T = \int_0^1 dt\ \Big(\hat{e}_1 + \hat{e}_2 + 2t\hat{e}_3\Big) \cdot (6t\hat{e}_1 + (4t + 2t^6)\hat{e}_2 + 6t^5\hat{e}_3)$$
$$= \int_0^1 dt\ [6t + (4t+2t^6) + 12t^6] = \int_0^1 dt\ [10t + 14t^6] = 7$$

Problem 3.

$$\vec{v} \ = \ xy\hat{e}_1 + 2yz\hat{e}_2 + 3xz\hat{e}_3, \ \ \vec{\nabla}\times\vec{v} \ = \ -2y\hat{e}_1 - 3z\hat{e}_2 - x\hat{e}_3$$

The l.h.s. of Stokes theorem is

$$\int_S dydz\ \hat{e}_1\cdot\vec{\nabla}\times\vec{v} \ = \ \int_0^2 dy\int_0^{2-y} dz\ (-2y) \ = \ -\int_0^2 dy\ 2y(2-y) \ = \ -\frac{8}{3}$$

The r.h.s is a sum of three integrals (for the second integral take the parametrization $t = z$)

$$\int_{\partial S} \vec{dl}\cdot\vec{v} \ = \ \int_0^2 dy\ v_2(0,y,0) \ + \int_0^2 dt\ \Big(\frac{dx(t)}{dt}\hat{e}_1 + \frac{dy(t)}{dt}\hat{e}_2 + \frac{dz(t)}{dt}\hat{e}_3\Big)\cdot\vec{v}(x(t),y(t),z(t))$$
$$+ \int_2^0 dz\ v_3(0,0,z) \ = \ \int_0^2 dt\ \Big(-\hat{e}_2+\hat{e}_3\Big)\cdot\vec{v}(0,2-t,t) \ = \ -2\int_0^2 dt\ (2-t)t \ = \ -\frac{8}{3}$$