$R+T=1$ revisited

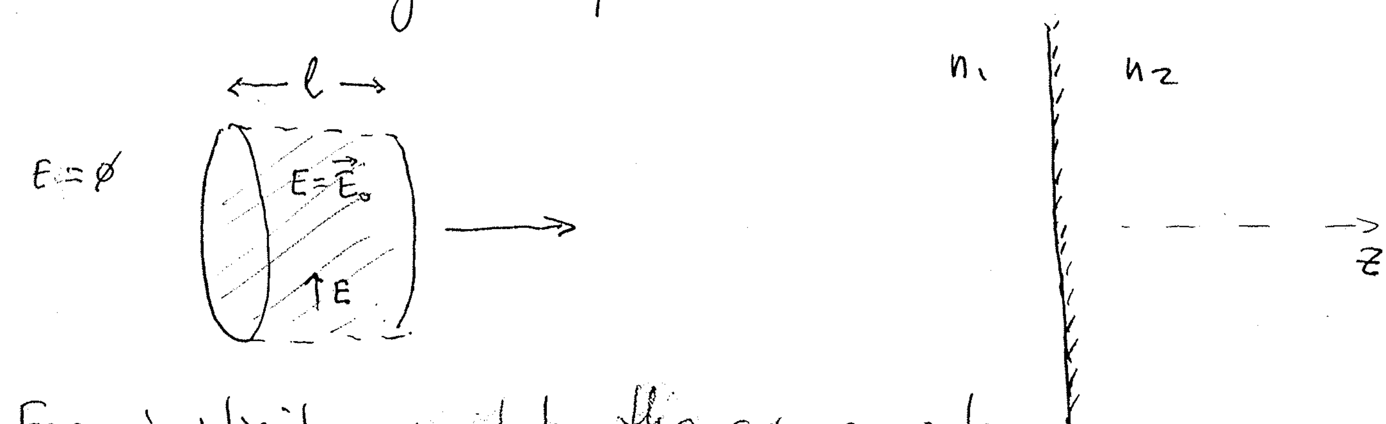
$v_{1,2} = \dfrac{c}{n_{1,2}}$

For simplicity, consider the case of incident wave normal to the interface

$$\frac{E_0^R}{E_i^0} = \frac{n_1-n_2}{n_1+n_2} \qquad \frac{E_0^T}{E_0^i} = \frac{2n_1}{n_1+n_2} \quad \Rightarrow \quad R = \frac{(n_1-n_2)^2}{(n_1+n_2)^2} \qquad T = \frac{4n_1n_2}{(n_1+n_2)^2}$$

$\mu_1=\mu_2$ for simplicity

To avoid confusion due to the interference between incident and reflected waves, consider the scattering of an electromagnetic pulse



For simplicity, we take the square pulse

$$\vec{E}_I(z-v_1t) = E_0\hat{e}_1 S_\ell(z-v_1t) \qquad S_\ell(x) = \begin{cases} 1 & \text{if } 0<x<\ell \\ \emptyset & \text{otherwise} \end{cases}$$

Recall that a general solution of the 1 dim wave eqn $\frac{\partial^2}{\partial z^2}f(z,t) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}f(z,t) = 0$ is $\underbrace{f_1(z-v_1t)}_{\text{right-moving}} + \underbrace{f_2(z+v_1t)}_{\text{left-moving}}$

How to solve the scattering problem for a pulse?
Idea: formally decompose a pulse into the sum of plane waves, solve the scattering problem for each plane wave separately, and reassemble the scattered pulse from the plane waves.

$$E_i(v_1 t - z) = \int_{-\infty}^{\infty} dk\, \tilde{E}_i(k) e^{-ik(v_1 t - z)} \equiv \int_{-\infty}^{\infty} dk\, \tilde{E}_i(k) e^{-i\omega t + ikz}$$

Maxwell's eqns are linear $\Rightarrow$ superposition principle (if $\vec{E}_{(1)}, \vec{B}_{(1)}$ and $\vec{E}_{(2)}, \vec{B}_{(2)}$ are solutions of Maxwell's eqns, so is $\vec{E}_{(1)} + \vec{E}_{(2)}$ and $\vec{B}_{(1)} + \vec{B}_{(2)}$) $\Rightarrow$ each $\tilde{E}_i(k) e^{-i\omega t + ikz}$ is a plane wave which is transmitted (and reflected) independently of other constituents with different k's

$\Rightarrow$ Solution for a certain k_i is $\quad (\omega = v_1 k_i)$

$$\tilde{E}_{0_i} e^{-i\omega t + ik_i z} + \tilde{E}_{0_R} e^{-i\omega t + ik_R z} \qquad z<0 \qquad k_R = -k_i$$

$$\tilde{E}_{0_T} e^{-i\omega t + ik_T z} \qquad z>0 \qquad k_T = \frac{\omega}{v_2} = k_i \frac{v_1}{v_2}$$

$$\tilde{E}_{0_R} = \frac{n_1 - n_2}{n_1 + n_2} E_{0_i}, \quad \tilde{E}_{0_T} = \frac{2n_1}{n_1 + n_2} E_{0_i} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{ll} \tilde{E}_{0_i} e^{-i\omega t + ik_i z} + \frac{n_1 - n_2}{n_1 + n_2} \tilde{E}_{0_i} e^{-i\omega t - ik_i z} & z<0 \\ \tilde{E}_{0_i} \frac{2n_1}{n_1 + n_2} e^{-i\omega t + ik_i z \frac{v_1}{v_2}} & z>0 \end{array}\right\} \text{describes}$$

the result of the scattering of a plane wave with $k = k_i$

$\Rightarrow$ superposition principle $\Rightarrow$

$$E(z,t) \overset{z<0}{=} \int \frac{dk}{2\pi} \left(\tilde{E}_i(k) e^{-ikv_1 t + ikz} + \frac{n_1 - n_2}{n_1 + n_2} \hat{E}_i(k) e^{-iv_1 kt - ikz} \right)$$

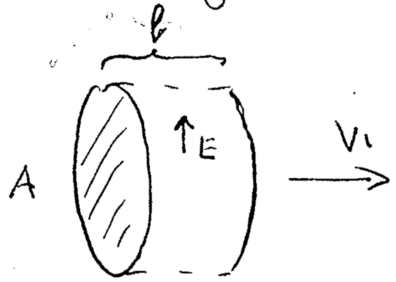
$$E(z,t) \overset{z>0}{=} \int \frac{dk}{2\pi} \tilde{E}_i(k) \frac{2n_1}{n_1 + n_2} e^{-ikv_1 t + ik\frac{v_1}{v_2} z}$$

Performing the integrations over k, we get

$$E(z,t) \overset{z<0}{=} \underset{\substack{\uparrow \\ \text{right-moving} \\ \text{incident wave}}}{E_i(v_1 t - z)} + \frac{n_1 - n_2}{n_1 + n_2} \underset{\substack{\uparrow \\ \text{left-moving} \\ \text{reflected wave}}}{E_i(v_1 t + z)}$$

$$E(z,t) \overset{z>0}{=} \frac{2n_1}{n_1 + n_2} E_i\left(v_1 t - \frac{v_1}{v_2} z\right) \leftarrow \text{right-moving (with } v = v_2\text{) transmitted wave}$$

Scattering: before ($t = -\infty$)



$t \to -\infty$

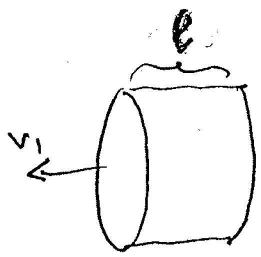
$E_i(v_1 t + z) = \emptyset \Rightarrow$

$\Rightarrow$ only incident pulse

Energy stored in the pulse $\quad W_i = \epsilon_1 \int \vec{E}_i^{\,2}(\vec{r}\,)\, d^3x =$

$$= \epsilon_1 \underbrace{\int E_0^2\, d^3x}_{\text{volume of the pulse}} = \epsilon_1 E_0^2 A \ell$$

after ($t = +\infty$)



reflected pulse

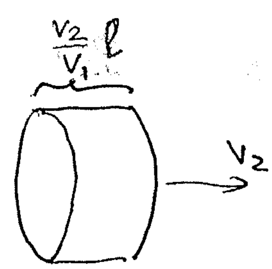
($t \to +\infty$ $E_i(z - vt)$ vanishes)

$$E_R = \frac{n_1 - n_2}{n_1 + n_2} E(v_1 t + z)$$

Energy stored in the reflected pulse

$$W_R = \epsilon_1 \int_{\text{pulse}} E_R^2\, d^3x = \epsilon_1 \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 E_0^2 A \ell$$

$$\Rightarrow R = \frac{W_R}{W_i} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$



transmitted pulse

$$E_T = \frac{2n_1}{n_1 + n_2} E_i\left(v_1 t - \frac{v_1}{v_2} z\right)$$

Energy of the transmitted pulse

$$W_T = \epsilon_2 \int_{\text{pulse}} E_T^2\, d^3x = \epsilon_2 \left(\frac{2n_1}{n_1 + n_2}\right)^2 E_0^2 A \frac{v_2}{v_1} \ell$$

$$W_R + W_T = W_i$$

$$T = \frac{\epsilon_2 v_2}{\epsilon_1 v_1} \left(\frac{2n_1}{n_1 + n_2}\right)^2 = \frac{n_2}{n_1} \left(\frac{2n_1}{n_1 + n_2}\right)^2$$

$$\Rightarrow R + T = 1$$