Contents

Chapter 6

Time-dependent Phenomena and Maxwell's Equations

6.1 Maxwell's Equations

6.1.1 Faraday's Law of Magnetic Induction

So far we have studied static (time-independent) behavior of electric and magnetic fields. The governing equations are

$$\begin{aligned} \nabla \cdot \mathbf{D} =& \rho \\ \nabla \times \mathbf{E} =& 0 \end{aligned} \tag{6.1.1}$$

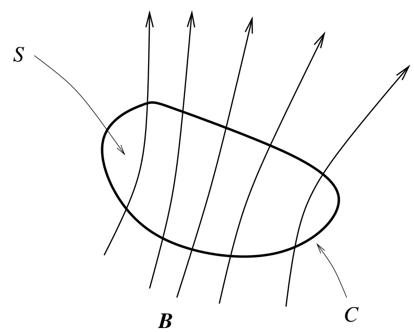
and

$$\begin{aligned} \nabla \times \mathbf{H} =& \mathbf{J} \\ \nabla \cdot \mathbf{B} =& 0. \end{aligned} \tag{6.1.2}$$

Electric and magnetic phenomena are completely separate, except for the fact that current density is associated with the motion of charges.
Faraday (1831) observed that a current could be induced in a closed loop of wire by varying the **flux** of magnetic field through a surface spanning the loop.
We define the *flux* ϕ of the magnetic field through the loop by

$$\phi = \int_S \mathbf{B} \cdot \mathbf{dS}, \tag{6.1.3}$$

where S is any surface spanning C.
N.B. Since $\nabla \cdot \mathbf{B} = 0$, ϕ is independent of the precise surface.



The **electromotive force**, or voltage, across the curve C is

$$\mathcal{E} = \oint_C \mathbf{E} \cdot \mathbf{dl}. \tag{6.1.4}$$

Then Faraday's law, in integral form, may be written

$$\boxed{\mathcal{E} = -k\frac{d\phi}{dt}\ ,}$$

where, in SI units, $k = 1$. Note that the sign here is a consequence of Lenz's law: the induced current is in such a direction as to oppose the change of flux producing it.

One could argue that the whole application of electricity in the modern world rests on Faraday's law; the observation that a changing magnetic field can produce an electric current. We can generalize this integral equation as applying to *any* closed curve in space, spanned by a surface,

$$\oint_C \mathbf{E} \cdot \mathbf{dl} = -\frac{d}{dt}\int_S \mathbf{B} \cdot \mathbf{dS}. \tag{6.1.5}$$

We now apply Stokes' theorem to the l.h.s.,

$$\oint \mathbf{E} \cdot \mathbf{dl} = \int_S (\nabla \times \mathbf{E}) \cdot \mathbf{dS}. \tag{6.1.6}$$

Specializing to the case where both C and S are fixed in time, we have

$$\frac{d}{dt}\int_S \mathbf{B} \cdot \mathbf{dS} = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS}, \tag{6.1.7}$$

and thus

$$\int_S (\nabla \times \mathbf{E}) \cdot \mathbf{dS} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS}, \tag{6.1.8}$$

yielding

$$\int_S (\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}) \cdot \mathbf{dS} = 0. \tag{6.1.9}$$

Since both C, S are arbitrary, we obtain the differential form of Faraday's law,

$$\boxed{\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0}$$

This equation replaces the second equation in Eq. (6.1.1).

6.1.2 Maxwell's modification of Ampere's law

Eqs. (6.1.1) and (6.1.2) reveal an immediate inconsistency when applied to time-dependent phenomena. Let us take the divergence of both parts of the first equation in (6.1.2). This gives

$$\nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot \mathbf{J}. \tag{6.1.10}$$

The l.h.s. is identically zero, whilst the r.h.s. vanishes *only* for *time-independent* problems. Recall that, in general, we have the *continuity equation*

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \,. \tag{6.1.11}$$

To see how to resolve this inconsistency, let us return to Coulomb's law

$$\nabla \cdot \mathbf{D} = \rho, \tag{6.1.12}$$

and substitute into the continuity equation, to obtain

$$\nabla \cdot \mathbf{J} + \nabla \cdot \frac{\partial \mathbf{D}}{\partial t} = 0. \tag{6.1.13}$$

We can make Ampere's law ($\nabla \times \mathbf{H} = \mathbf{J}$) consistent with the continuity equation simply by making the substitution

$$\mathbf{J} \to \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{6.1.14}$$

that results in a modified equation

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \tag{6.1.15}$$

6.1.3 Set of Maxwell's Equations

With this final modification of Ampere's law, and Faraday's law, we have completed the construction of Maxwell's equations

$$\begin{aligned}
\nabla\cdot\mathbf{D} &= \rho \quad \text{(ME1)} \ \textit{Coulomb's Law}\\
\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t} &= 0 \quad \text{(ME2)} \ \textit{Faraday's Law}\\
\nabla\times\mathbf{H} &= \mathbf{J}+\frac{\partial\mathbf{D}}{\partial t} \quad \text{(ME3)} \ \textit{Ampere's Law + Maxwell}\\
\nabla\cdot\mathbf{B} &= 0 \quad \text{(ME4)}
\end{aligned}$$

The unification of electrical and magnetic phenomena through these equations represents the crowning achievement of classical, 19$^{\text{th}}$ century physics. The addition of the electric displacement to the r.h.s. of Ampere's law was essential to showing that the solutions admit wave propagation at the speed of light.

6.1.4 Vector and Scalar Potentials

Maxwell's equations comprise a set of coupled, first-order PDE's. In particularly simple cases, they can be solved directly, but in the case of both electrostatics and magnetostatics we have seen the efficacy of introducing vector and scalar potentials. We will now do likewise for the time-dependent case.

We introduce potentials so that the two homogenous equations (Faraday's law and the solenoidal condition) are satisfied automatically. Since

$$\nabla\cdot\mathbf{B}=0\ , \tag{6.1.1}$$

we can introduce a **vector potential A** such that

$$\mathbf{B}=\nabla\times\mathbf{A}\ . \tag{6.1.2}$$

Substituting into Faraday's law (ME2), we obtain

$$\begin{aligned}
\nabla\times\mathbf{E}+\frac{\partial}{\partial t}\left[\nabla\times\mathbf{A}\right] &= 0\\
\Longrightarrow \nabla\times\left[\mathbf{E}+\frac{\partial\mathbf{A}}{\partial t}\right] &= 0.
\end{aligned}$$

We can now introduce a **scalar potential** Φ such that

$$\mathbf{E}+\frac{\partial\mathbf{A}}{\partial t}=-\nabla\Phi. \tag{6.1.3}$$

Thus the electric and magnetic fields can be written

$$\boxed{\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A} && (6.1.4)\\ \mathbf{E} &= -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} && (6.1.5) \end{aligned}}$$

and ME2 and ME4 are automatically satisfied.

The two remaining equations (ME1 and ME3) determine the dynamical behavior, i.e. the dependence of $\mathbf{A}$ and Φ on t and $\mathbf{x}$. To solve them, we need some constitutive relation between $(\mathbf{D}, \mathbf{H})$ and $(\mathbf{E}, \mathbf{B})$. We will initially restrict ourselves to the case of the vacuum, where we have

$$\begin{aligned} \mathbf{D} &= \epsilon_0 \mathbf{E} \,, \\ \mathbf{H} &= \frac{1}{\mu_0} \mathbf{B} \,. \end{aligned}$$

Coulomb's law, ME1, is thus

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \,, \tag{6.1.6}$$

whilst Ampère's law, ME3, is

$$\frac{1}{\mu_0} \nabla \times \mathbf{B} = \mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{6.1.7}$$

Thus, in terms of the potential $(\Phi, \mathbf{A})$, ME1 becomes

$$\boxed{\nabla^2 \Phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\epsilon_0} \qquad (6.1.8)}$$

Substituting for the potential in ME3, we have

$$\begin{aligned} \frac{1}{\mu_0} \nabla \times (\nabla \times \mathbf{A}) &= \mathbf{J} + \epsilon_0 \left\{ -\nabla \frac{\partial \Phi}{\partial t} - \frac{\partial^2 \mathbf{A}}{\partial t^2} \right\} \\ \Longrightarrow \nabla[\nabla \cdot \mathbf{A}] - \nabla^2 \mathbf{A} &= \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \left\{ -\nabla \frac{\partial \Phi}{\partial t} - \frac{\partial^2 \mathbf{A}}{\partial t^2} \right\}. \end{aligned}$$

We now write $\epsilon_0 \mu_0 = 1/c^2$ (we of course all know what c will be!), and write

$$\boxed{\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla \left[\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \right] = -\mu_0 \mathbf{J} \qquad (6.1.9)}$$

Thus we have derived two coupled second-order PDE's that are, with the definitions of the potentials in Eqs. (6.1.4) and (6.1.5), equivalent to the original four Maxwell equations.

6.2 Gauge Transformations Revisited

Is it possible to decouple these two equations? One way to do this is through a clever choice of *gauge transformation*. A gauge transformation exploits the redundant degrees of freedom in the problem to simplify the problem.

Recall that the physical fields are not $(\mathbf{A}, \Phi)$, but rather $(\mathbf{B}, \mathbf{E})$. A gauge transformation is a transformation of the $(\mathbf{A}, \Phi)$ that leaves the physics unaltered. In this section, we will derive gauge transformations for the complete Maxwell equations.

We have already encountered gauge transformations in the context of magnetostatics; the substitution

$$\mathbf{A} \longrightarrow \mathbf{A}' = \mathbf{A} + \nabla\Lambda \tag{6.2.1}$$

leaves $\mathbf{B} = \nabla \times \mathbf{A}$ invariant. In this case, however, $\mathbf{E}$ *also* depends on $\mathbf{A}$, and the above transformation will change $\mathbf{E}$ unless we make a suitable change $\Phi \longrightarrow \Phi'$. In terms of the transformed potentials $(\mathbf{A}', \Phi')$, we have

$$\begin{aligned} \mathbf{E} &= -\nabla\Phi' - \frac{\partial \mathbf{A}'}{\partial t} \\ &= -\nabla\Phi' - \frac{\partial}{\partial t}\left[\mathbf{A} + \nabla\Lambda\right]. \end{aligned}$$

But we have

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}, \tag{6.2.2}$$

and thus equating the two expressions gives

$$\begin{aligned} \nabla\left[\Phi' + \frac{\partial\Lambda}{\partial t} - \Phi\right] &= 0 \\ \Longrightarrow \Phi' &= \Phi - \frac{\partial\Lambda}{\partial t} \end{aligned}$$

where we have noted that the potential is only defined up to an additive constant.

Thus the gauge transformation of Maxwell's equations takes the form

$$\boxed{\begin{aligned} \mathbf{A} &\longrightarrow \mathbf{A}' = \mathbf{A} + \nabla\Lambda \qquad (6.2.3) \\ \Phi &\longrightarrow \Phi' = \Phi - \frac{\partial\Lambda}{\partial t} \qquad (6.2.4) \end{aligned}}$$

We will now discuss some particular choice of gauges.

6.2.1 Lorentz Condition

Suppose we can find a gauge transformation such that

$$\boxed{\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0. \qquad (6.2.5)}$$

This is known as the **Lorentz condition**, and the dynamical equations assume the form

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \qquad (6.2.6)$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}. \qquad (6.2.7)$$

The $\mathbf{A}$ and Φ fields have become decoupled, and the simplified equations are just the **wave equations**, with a inhomogeneous source. But is it actually possible to *find* a gauge transformation that satisfies Eq. (6.2.5)?
Let $(\mathbf{A}, \Phi)$ be potentials satisfying Eqs. (6.1.8) and (6.1.9), and let Λ be a gauge transformation such that the transformed fields satisfy Eq. (6.2.5). Then we have

$$\begin{aligned} \nabla \cdot \mathbf{A}' + \frac{1}{c^2}\frac{\partial \Phi'}{\partial t} &= 0 \\ \Longrightarrow \nabla \cdot \mathbf{A} + \nabla^2 \Lambda + \frac{1}{c^2}\left[\frac{\partial \Phi}{\partial t} - \frac{\partial^2 \Lambda}{\partial t^2}\right] &= 0 \end{aligned}$$

Thus we need to find Λ satisfying

$$\nabla^2 \Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} = -\left[\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right]. \qquad (6.2.8)$$

If such a function is found, then potentials $\mathbf{A}'$ and Φ' satisfy the Lorentz condition.
Note, however, that the solution of this differential equation is not unique. One can always add to Λ a function $\delta\Lambda$ that satisfies the homogeneous equation

$$\nabla^2(\delta\Lambda) - \frac{1}{c^2}\frac{\partial^2(\delta\Lambda)}{\partial t^2} = 0\ , \qquad (6.2.9)$$

and the new function $\Lambda' = \Lambda + \delta\Lambda$ would also satisfy the inhomogeneous equation (6.2.8).
In other words, the Lorentz condition does not specify a gauge uniquely. Indeed, let $(\mathbf{A}, \Phi)$ satisfy the Lorentz condition. Now consider the transformation

$$\begin{aligned} \mathbf{A} &\longrightarrow \mathbf{A}' = \mathbf{A} + \nabla(\delta\Lambda) \\ \Phi &\longrightarrow \Phi' = \Phi - \frac{\partial(\delta\Lambda)}{\partial t}. \end{aligned}$$

Then the combination entering the Lorentz condition transforms as

$$\nabla \cdot \mathbf{A}' + \frac{1}{c^2}\frac{\partial \Phi'}{\partial t} = \nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} + \nabla^2(\delta\Lambda) - \frac{1}{c^2}\frac{\partial^2(\delta\Lambda)}{\partial t^2} \; . \tag{6.2.10}$$

By assumption, the fields $(\mathbf{A}, \Phi)$ satisfy the Lorentz condition $\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0$. Thus, the fields $(\mathbf{A}', \Phi')$ will also satisfy the Lorentz condition if

$$\nabla^2(\delta\Lambda) - \frac{1}{c^2}\frac{\partial^2(\delta\Lambda)}{\partial t^2} = 0 \; . \tag{6.2.11}$$

The Lorentz gauge is important because:

- The wave equation is manifest explicitly,
- $(\mathbf{A}, \Phi)$ are treated on equal footing and, when we discuss *Special Relativity*, we will see that the Lorentz condition is **Lorentz covariant**, i.e. independent of the choice of the 4-dimensional $(ct, \mathbf{x})$ coordinate system.

6.2.2 Coulomb Gauge

We have introduced the gauge

$$\nabla \cdot \mathbf{A} = 0 \tag{6.2.12}$$

in the discussion of magnetostatics. It is not manifestly Lorentz covariant, but has the property that the scalar potential satisfies Poisson's equation (Coulomb's law!),

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0} \; , \tag{6.2.13}$$

with solution

$$\Phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\rho(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} \; . \tag{6.2.14}$$

The vector potential satisfies the **inhomogeneous wave equation**

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} + \frac{1}{c^2}\nabla\frac{\partial \Phi}{\partial t}. \tag{6.2.15}$$

Note that the scalar potential $\Phi(\mathbf{x}, t)$ is the **instantaneous** Coulomb potential due to a charge density $\rho(\mathbf{x}, t)$, i.e. we do not take account of "causality" through the use of a retarded potential.

The equation for the vector potential contains a gradient operator, $\nabla \partial\Phi/\partial t$ arising from the solution of Poisson's equation for the scalar potential, and this term is *irrotational*,

$$\nabla \times \left[\nabla \frac{\partial \Phi}{\partial t}\right] = 0. \tag{6.2.16}$$

It would be useful to completely decouple the equations governing the vector and scalar potentials, as in the case of the Lorentz gauge. To accomplish this, we will separate the current into an *irrotational*, or **longitudinal**, piece and a *solenoidal*, or **transverse**, piece,

$$\mathbf{J} = \mathbf{J^l} + \mathbf{J^t} \tag{6.2.17}$$

with

$$\begin{aligned} \nabla \times \mathbf{J^l} &= 0 \\ \nabla \cdot \mathbf{J^t} &= 0. \end{aligned}$$

We can always perform this separation, as will now be demonstrated. At first, we do it for the Fourier transforms

$$\mathbf{J}(\mathbf{k}) = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \mathbf{J}(\mathbf{x}) = \mathbf{J}^t(\mathbf{k}) + \mathbf{J}^l(\mathbf{k}) \ . \tag{6.2.18}$$

The components $\mathbf{J}^t(\mathbf{k})$ and $\mathbf{J}^l(\mathbf{k})$ should satisfy

$$\begin{aligned} \mathbf{k} \times \mathbf{J}^l(\mathbf{k}) &= 0 \\ \mathbf{k} \cdot \mathbf{J^t}(\mathbf{k}) &= 0. \end{aligned}$$

This is achieved if we take $\mathbf{J}^l(\mathbf{k})$ proportional to $\mathbf{k}$ (i.e., "longitudinal")

$$\mathbf{J}^l(\mathbf{k}) = \mathbf{k}\frac{\mathbf{k}\cdot\mathbf{J}(\mathbf{k})}{\mathbf{k}^2}, \qquad \mathbf{J}^t(\mathbf{k}) = \mathbf{J}(\mathbf{k}) - \mathbf{k}\frac{\mathbf{k}\cdot\mathbf{J}(\mathbf{k})}{\mathbf{k}^2} \ , \tag{6.2.19}$$

or, in components,

$$J_i^l(\mathbf{k}) = \frac{k_i k_j}{\mathbf{k}^2} J_j(\mathbf{k}), \qquad J_i^t(\mathbf{k}) = \Big(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\Big) J_j(\mathbf{k}) \ . \tag{6.2.20}$$

As usual, the summation over repeated indices is assumed. Going back to the coordinate space we obtain

$$\begin{aligned} J_i^t(\mathbf{x}) &= \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \Big(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\Big) \int d^3y \ J_j(\mathbf{y}) e^{-i\mathbf{k}\cdot\mathbf{y}} \\ &= \int d^3y \ J_j(\mathbf{y}) \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \Big(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\Big) \\ &= \Big(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} - \delta_{ij}\nabla^2\Big) \int d^3y J_j(\mathbf{y}) \int \frac{d^3k}{(2\pi)^3} e^{i\ \mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \frac{1}{\mathbf{k}^2} \end{aligned}$$

Using the formula

$$\int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \frac{1}{\mathbf{k}^2} = \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} \ , \tag{6.2.21}$$

that is consistent with the equation

$$\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right) = -4\pi\delta(\mathbf{x}-\mathbf{y}) \ , \tag{6.2.22}$$

we obtain

$$J_i^t(\mathbf{x}) = \Big(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} - \delta_{ij}\nabla^2\Big)\int d^3y \frac{J_j(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|} \ ,$$

and similarly

$$J_i^l(\mathbf{x}) = -\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\int d^3y \frac{J_j(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|} \ .$$

These results allow to easily check that $J_i^t(\mathbf{x}) + J_i^l(\mathbf{x}) = J_i(\mathbf{x})$.
Now we want to transform

$$\frac{\partial}{\partial x_j}\int d^3y \frac{J_j(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|} = -\int d^3y \ J_j(\mathbf{y})\frac{\partial}{\partial y_j}\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} = \int d^3y \ \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\frac{\partial}{\partial y_j}J_j(\mathbf{y}) \ .$$

At the last step, we used integration by parts, and discarded the surface term which vanishes for $y \to \infty$. Thus we can write the longitudinal part as

$$\mathbf{J}^{\mathbf{l}} \ = \ -\frac{1}{4\pi}\nabla\int d^3x' \ \frac{\nabla'\cdot\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \ . \tag{6.2.23}$$

From the continuity equation, we have

$$\nabla\cdot\mathbf{J} + \frac{\partial\rho}{\partial t} = 0 \ . \tag{6.2.24}$$

and substituting in Eq. (6.2.23) we obtain

$$\mathbf{J}^{\mathbf{l}} = \frac{1}{4\pi}\nabla\int d^3x' \ \frac{1}{|\mathbf{x}-\mathbf{x}'|}\frac{\partial\rho}{\partial t}. \tag{6.2.25}$$

We now identify the r.h.s. of this equation with our expression for the scalar potential of Eq. (6.2.14) and observe that

$$\begin{aligned} \mathbf{J}^{\mathbf{l}} \ &= \ \epsilon_0\nabla\frac{\partial\Phi}{\partial t} \\ \Longrightarrow \mu_0\mathbf{J}^{\mathbf{l}} \ &= \ \frac{1}{c^2}\nabla\frac{\partial\Phi}{\partial t}, \end{aligned}$$

where we have used $\mu_0\epsilon_0 = 1/c^2$. Returning now to the original equation for the vector potential, Eq. (6.2.15),

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J} + \frac{1}{c^2}\nabla\frac{\partial\Phi}{\partial t} \ , \tag{6.2.26}$$

we see that it reduces to

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}^t \ . \tag{6.2.27}$$

Only the **transverse** part of the current is a source for $\mathbf{A}$.
Thus this gauge is also known as the **transverse** or **radiation** gauge, and once again we have decoupled the scalar and vector potentials.
Observing that the equality $\nabla \times [\nabla \times \mathbf{F}] = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$ can be written as

$$(\nabla \times [\nabla \times \mathbf{F}])_i = \nabla_i(\nabla_j F_j) - \nabla^2 F_i = (\nabla_i \nabla_j - \nabla^2 \delta_{ij}) F_j \ , \tag{6.2.28}$$

we may write the transverse part of the current also as

$$\mathbf{J^t}(\mathbf{x}) \ = \ \frac{1}{4\pi} \nabla \times \left[\nabla \times \int d^3x' \, \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \right] \equiv \nabla \times [\nabla \times \mathbf{j}(\mathbf{x})] \ . \tag{6.2.29}$$

Note, that the function $\mathbf{j}(\mathbf{x})$

$$\mathbf{j}(\mathbf{x}) \ \equiv \ \frac{1}{4\pi} \int d^3x' \, \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \tag{6.2.30}$$

(and hence also $\mathbf{J^t}(\mathbf{x})$) is non-zero everywhere in space, even if the current $\mathbf{J}(\mathbf{x}')$ is localized in some finite volume V'.

6.3 Wave Equation

6.3.1 Green Function for the Wave Equation

In both the *Lorentz* and *Coulomb* gauges, we have reduced the problem of finding the potentials to the solution of the **wave equation**

$$\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = -4\pi f(\mathbf{x}, t), \tag{6.3.1}$$

where f is some *known* source, and c, as we have intimated earlier, is the velocity of wave propagation.
Such a hyperbolic equation, like the elliptic equations encountered in electrostatics, can be solved by means of *Green functions*. In particular, we will find the Green function $G(\mathbf{x}, t; \mathbf{x}', t')$ satsifying

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right] G(\mathbf{x}, t; \mathbf{x}', t') = -4\pi \delta(\mathbf{x} - \mathbf{x}') \delta(t - t'). \tag{6.3.2}$$

The solution to the inhomogeneous wave equation, Eq. (6.3.1), for a general source is then

$$\psi(\mathbf{x},t) = \psi_0(\mathbf{x},t) + \int d^3x'\, dt'\, G(\mathbf{x},t;\mathbf{x}',t') f(\mathbf{x}',t') \tag{6.3.3}$$

where ψ_0 is a **solution** of the **homogeneous equation**. Note that this is essentially an *initial-value problem*, rather than the boundary-value problem encountered with elliptic equations.

To obtain the Green function, we take the Fourier transform with respect to t:

$$\begin{aligned} G(\mathbf{x},t;\mathbf{x}',t') &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega t} g(\mathbf{x},\omega;\mathbf{x}',t')\ , \\ g(\mathbf{x},\omega;\mathbf{x}',t') &= \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, G(\mathbf{x},t;\mathbf{x}',t')\ . \end{aligned}$$

Note the opposite signs in exponentials for the direct and inverse Fourier transforms. Then taking the Fourier transform of Eq. (6.3.6), we find

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right) g(\mathbf{x},\omega;\mathbf{x}',t') = -4\pi\delta(\mathbf{x}-\mathbf{x}')e^{i\omega t'}. \tag{6.3.4}$$

We now introduce the *spatial* Fourier transform,

$$\begin{aligned} g(\mathbf{x},\omega;\mathbf{x}',t') =& \frac{1}{(2\pi)^3}\int d^3q e^{i\mathbf{q}\cdot\mathbf{x}} \tilde{g}(\mathbf{q},\omega;\mathbf{x}',t')\ , \\ \tilde{g}(\mathbf{q},\omega;\mathbf{x}',t') =& \int d^3x e^{-i\mathbf{q}\cdot\mathbf{x}} g(\mathbf{x},\omega;\mathbf{x}',t')\ . \end{aligned} \tag{6.3.5}$$

Note that the sign convention for the exponential in the spatial Fourier transform, $e^{i\mathbf{q}\cdot\mathbf{x}}$, differs from that in the time Fourier transform, where we had $e^{-i\omega t}$.

Let us find the Green's function $G(\mathbf{x},t;\mathbf{x}',t')$ for the wave equation. It satsifies

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] G(\mathbf{x},t;\mathbf{x}',t') = -4\pi\delta(\mathbf{x}-\mathbf{x}')\delta(t-t'). \tag{6.3.6}$$

To solve this equation, we use combined *4-dimensional* Fourier transfom

$$G(\mathbf{x},t;\mathbf{x}',t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}} \tilde{g}(\mathbf{q},\omega;\mathbf{x}',t')\ ,$$

in which the exponential $e^{-i\omega t + i\mathbf{q}\cdot\mathbf{x}}$ contains time t and space $\mathbf{x}$ coordinates in the $\omega t - \mathbf{q}\cdot\mathbf{x}$ combination. The Fourier transform $\tilde{g}(\mathbf{q},\omega;\mathbf{x}',t')$ given by

$$\tilde{g}(\mathbf{q},\omega;\mathbf{x}',t') = \int_{-\infty}^{\infty} dt\, e^{i\omega t} \int d^3x\, e^{-i\mathbf{q}\cdot\mathbf{x}} G(\mathbf{x},t;\mathbf{x}',t')\ ,$$

satisfies an algebraic equation

$$\begin{aligned}(-\mathbf{q}^2 + k^2)\tilde{g}(\mathbf{q},\omega;\mathbf{x}',t') &= -4\pi e^{-i\mathbf{q}\cdot\mathbf{x}'}e^{i\omega t'} \\ \Longrightarrow \tilde{g}(\mathbf{q},\omega;\mathbf{x}',t') &= 4\pi\frac{e^{-i\mathbf{q}\cdot\mathbf{x}'}e^{i\omega t'}}{q^2-k^2}\ ,\end{aligned}$$

where $k \equiv \omega/c$ is the **wave number**. Thus

$$G(\mathbf{x},t;\mathbf{x}',t') = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}\underbrace{e^{i\omega t'}\int\frac{d^3q}{(2\pi)^3}e^{i\mathbf{q}\cdot\mathbf{x}}4\pi\frac{e^{-i\mathbf{q}\cdot\mathbf{x}'}}{q^2-k^2}}_{g(\mathbf{x},\omega;\mathbf{x}',t')}\ .$$

First, let us take the integral over q

$$I_q = 4\pi\int\frac{d^3q}{(2\pi)^3}\frac{e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}}{q^2-k^2}\ .$$

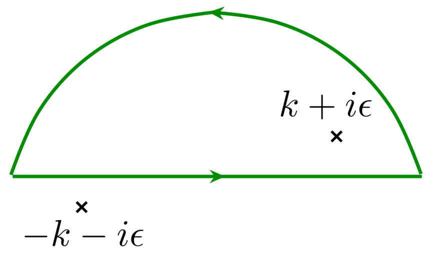
In order to exhibit the behavior of this integral, we consider a coordinate system in which the z-axis is aligned with $\mathbf{x}-\mathbf{x}'$, and let θ be the angle between $\mathbf{q}$ and $\mathbf{x}-\mathbf{x}'$. Thus

$$\begin{aligned}I_q &= \frac{4\pi}{(2\pi)^3}\int_0^{\infty}dq\,q^2\underbrace{\int_0^{2\pi}d\varphi}_{2\pi}\int_{-1}^{1}d(\cos\theta)\frac{e^{iq|\mathbf{x}-\mathbf{x}'|\cos\theta}}{q^2-k^2} \\ &= \frac{1}{\pi}\int_0^{\infty}dq\,\frac{q^2}{q^2-k^2}\left\{\frac{e^{iq|\mathbf{x}-\mathbf{x}'|}}{iq|\mathbf{x}-\mathbf{x}'|}-\frac{e^{-iq|\mathbf{x}-\mathbf{x}'|}}{iq|\mathbf{x}-\mathbf{x}'|}\right\} \\ &= \frac{1}{\pi}\frac{1}{i|\mathbf{x}-\mathbf{x}'|}\int_0^{\infty}\frac{dq}{q^2-k^2}\left[qe^{iq|\mathbf{x}-\mathbf{x}'|}+(-q)e^{-iq|\mathbf{x}-\mathbf{x}'|}\right] \\ &= \frac{1}{\pi}\frac{1}{i|\mathbf{x}-\mathbf{x}'|}\int_{-\infty}^{\infty}\frac{dq\,q}{q^2-k^2}e^{iq|\mathbf{x}-\mathbf{x}'|}\end{aligned}$$

On the last step, we combined two terms into one integral with $(-\infty,\infty)$ limits.
The integrand has poles at $q=\pm k$, and therefore we have to specify how to treat the poles in order to evaluate the integrals. We will do this by displacing the poles off the real axis as follows:

$$g^{(\pm)}(\mathbf{x},\omega;\mathbf{x}',t') = \frac{1}{i\pi}\frac{e^{i\omega t'}}{|\mathbf{x}-\mathbf{x}'|}\int_{-\infty}^{\infty}\frac{dq\,q}{q^2-(k\pm i\epsilon)^2}e^{iq|\mathbf{x}-\mathbf{x}'|}, \qquad (6.3.7)$$

where $\epsilon>0$ is very small, $\epsilon\to 0$. The two possibilities in $q^2-(k\pm i\epsilon)^2$ correspond to two types of Green functions: *retarded* $g^{(+)}$ for which the poles are shifted from the real axis to $q=\pm(k+i\epsilon)$ and *advanced* $g^{(-)}$ with poles for $q=\pm(k-i\epsilon)$.
We first consider the case of $g^{(+)}$, which has a pole in the *upper half plane* at $q=k+i\epsilon$, and in the *lower half plane* at $q=-k-i\epsilon$.



We can complete the contour in the upper-half plane, where the contribution from the semi-circle at infinity vanishes, and obtain

$$g^{(+)}(\mathbf{x}, \omega; \mathbf{x}', t') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} e^{i\omega t' + ik|\mathbf{x} - \mathbf{x}'|}. \tag{6.3.8}$$

Similarly, in the case of $g^{(-)}$, we have a pole in the upper half plane at $q = -k + i\epsilon$, and performing the contour integration we obtain,

$$g^{(-)}(\mathbf{x}, \omega; \mathbf{x}', t') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} e^{i\omega t' - ik|\mathbf{x} - \mathbf{x}'|}. \tag{6.3.9}$$

We now invert the temporal Fourier Transform

$$G^{(\pm)}(\mathbf{x}, t; \mathbf{x}', t') = \frac{1}{2\pi} \int d\omega \, e^{-i\omega t} \, \frac{1}{|\mathbf{x} - \mathbf{x}'|} e^{i\omega t' \pm \omega|\mathbf{x} - \mathbf{x}'|/c} \tag{6.3.10}$$

(we also substituted $k = \omega/c$). The ω integration is straightforward, and we find

$$\boxed{G^{(\pm)}(\mathbf{x}, t; \mathbf{x}', t') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} \delta \left[(t' - t) \pm \frac{1}{c} |\mathbf{x} - \mathbf{x}'| \right] \qquad (6.3.11)}$$

The Green function $G^{(+)}$ is known as the **retarded Green function**, because a change at time t arises from an effect at an earlier time

$$t' = t - \frac{1}{c} |\mathbf{x} - \mathbf{x}'|. \tag{6.3.12}$$

It manifestly exhibits causality. $G^{(-)}$ is known as the **advanced Green function**.

We now construct the complete solutions as follows:

1. *Retarded Solution*. We imagine that, as $t \to -\infty$, we have a wave $\psi_{\text{in}}(\mathbf{x}, t)$ satisfying the *homogeneous* equation. The source $f(\mathbf{x}, t)$ then turns on, and the complete solution is

$$\psi(\mathbf{x}, t) = \psi_{\text{in}}(\mathbf{x}, t) + \int d^3x' \, dt' \, G^{(+)}(\mathbf{x}, t; \mathbf{x}', t') f(\mathbf{x}', t'). \tag{6.3.13}$$

The use of the *retarded* Green function ensures that the observer only feels the effect of the source **after** it is turned on.

2. *Advanced Solution* Here we measure a wave $\psi_{\rm out}(\mathbf{x},t)$ as $t\to\infty$,

$$\psi(\mathbf{x},t)=\psi_{\rm out}(\mathbf{x},t)+\int d^3x'\,dt'G^{(-)}(\mathbf{x},t;\mathbf{x}',t')f(\mathbf{x}',t'). \tag{6.3.14}$$

The use of $G^{(-)}$ means that, once the source ceases, the effects from the source are no longer felt, or more precisely they are contained within $\psi_{\rm out}$.

Case 1 above is the more commonly encountered, for example in the case $\psi_{\rm in}\equiv 0$ so that there is no wave in the distant past, and a source $f(\mathbf{x},t)$ switches on at some time. Then inserting our explicit expression for the Green function, we obtain

$$\psi(\mathbf{x},t)=\int d^3x'\frac{f(\mathbf{x}',t'_{\rm ret})}{|\mathbf{x}-\mathbf{x}'|}\ , \tag{6.3.15}$$

where the subscript `ret` denotes that the function f is evaluated at time

$$t'_{\rm ret}=t-\frac{1}{c}|\mathbf{x}-\mathbf{x}'|\ . \tag{6.3.16}$$

6.3.2 Retarded Solutions for the Fields

Since in the Lorentz gauge both scalar Φ and vector potential $\mathbf{A}$ satisfy wave equations

$$\nabla^2\Phi-\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \tag{6.3.1}$$

$$\nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J}\ , \tag{6.3.2}$$

we can write them as

$$\Phi(\mathbf{x},t)=\frac{1}{4\pi\epsilon_0}\int d^3x'\frac{\rho(\mathbf{x}',t_{\rm ret})}{R}\ , \tag{6.3.3}$$

and

$$\mathbf{A}(\mathbf{x},t)=\frac{\mu_0}{4\pi}\int d^3x'\frac{\mathbf{J}(\mathbf{x}',t_{\rm ret})}{R}\ , \tag{6.3.4}$$

where $R\equiv|\mathbf{x}-\mathbf{x}'|$ and

$$t_{\rm ret}=t-\frac{1}{c}|\mathbf{x}-\mathbf{x}'|=t-\frac{R}{c}\ . \tag{6.3.5}$$

In other words, the subscript `ret` denotes that the functions $\rho,\mathbf{J}$ are evaluated at an earlier time, the difference being equal to the time interval necessary for propagation over the distance $R=|\mathbf{x}-\mathbf{x}'|$ with the velocity of light c. Mathematically, $t_{\rm ret}$ is a function of t, $\mathbf{x}$ and $\mathbf{x}'$.

6.3.3 Direct check of solution through retarded potentials

It is instructive to check directly that these expressions for Φ and $\mathbf{A}$ satisfy the wave equations (6.3.1), (6.3.2). To calculate the Laplacian of $\Phi(\mathbf{x}, t)$ (or $\mathbf{A}(\mathbf{x}, t)$), the important point to notice is that the integrand of the $\mathbf{x}'$-integral depends on $\mathbf{x}$ in two places: *explicitly*, in the denominator ($R = |\mathbf{x} - \mathbf{x}'|$), and *implicitly*, through $t_{\rm ret} = t - R/c$, in the numerator. As a result,

$$\nabla\Phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \left[\rho\nabla\left(\frac{1}{R}\right) + (\nabla\rho)\,\frac{1}{R}\right] . \tag{6.3.6}$$

Using chain rule, we have

$$\nabla\rho = \dot\rho\nabla t_{\rm ret} = -\frac{\dot\rho}{c}\,\nabla R , \tag{6.3.7}$$

where the dot denotes differentiation with respect to time. Now, $\nabla R = \hat{\mathbf{e}}_R \equiv \mathbf{R}/R$ and $\nabla(1/R) = -\hat{\mathbf{e}}_R/R^2$. Hence,

$$\nabla\rho = -\frac{\dot\rho}{c}\,\nabla R = -\frac{\dot\rho}{c}\,\hat{\mathbf{e}}_R , \tag{6.3.8}$$

and

$$\nabla\Phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \left[-\rho\frac{\hat{\mathbf{e}}_R}{R^2} - \frac{\dot\rho}{c}\frac{\hat{\mathbf{e}}_R}{R}\right] . \tag{6.3.9}$$

Taking the divergence,

$$\begin{aligned}\nabla^2\Phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \Bigg\{ &-\left[\rho\nabla\cdot\left(\frac{\hat{\mathbf{e}}_R}{R^2}\right) + \frac{\hat{\mathbf{e}}_R}{R^2}\cdot(\nabla\rho)\right] \\ &-\frac{1}{c}\left[\dot\rho\nabla\cdot\left(\frac{\hat{\mathbf{e}}_R}{R}\right) + \frac{\hat{\mathbf{e}}_R}{R}\cdot(\nabla\dot\rho)\right]\Bigg\} .\end{aligned} \tag{6.3.10}$$

Now, using

$$\nabla\cdot\left(\frac{\hat{\mathbf{e}}_R}{R^2}\right) = -\nabla^2\left(\frac{1}{R}\right) = 4\pi\delta^3(\mathbf{R}) , \tag{6.3.11}$$

and (note that $\nabla\cdot\mathbf{R} = 3$)

$$\nabla\cdot\left(\frac{\hat{\mathbf{e}}_R}{R}\right) = \nabla\cdot\left(\frac{\mathbf{R}}{R^2}\right) = \frac{\nabla\cdot\mathbf{R}}{R^2} - 2\mathbf{R}\cdot\frac{\nabla R}{R^3} = \frac{3}{R^2} - 2\mathbf{R}\cdot\frac{\hat{\mathbf{e}}_R}{R^3} = \frac{1}{R^2}, \tag{6.3.12}$$

and also

$$\nabla\dot\rho = \ddot\rho\nabla t_{\rm ret} = -\frac{1}{c}\ddot\rho\nabla R = -\frac{1}{c}\,\ddot\rho\,\hat{\mathbf{e}}_R , \tag{6.3.13}$$

we derive

$$\begin{aligned}\nabla^2\Phi(\mathbf{x},t) =& \frac{1}{4\pi\epsilon_0}\int d^3x'\left\{-\left[\rho(4\pi\delta^3(\mathbf{R})) + \frac{\hat{\mathbf{e}}_R}{R^2}\cdot\left(-\frac{1}{c}\dot{\rho}\,\hat{\mathbf{e}}_R\right)\right] - \frac{1}{c}\left[\dot{\rho}\frac{1}{R^2} + \frac{\hat{\mathbf{e}}_R}{R}\cdot\left(-\frac{1}{c}\,\ddot{\rho}\,\hat{\mathbf{e}}_R\right)\right]\right\} \\ =& \frac{1}{4\pi\epsilon_0}\int d^3x'\left[-4\pi\rho\,\delta^3(\mathbf{R}) + \frac{1}{c^2}\frac{\ddot{\rho}}{R}\right] = -\frac{1}{\epsilon_0}\rho(\mathbf{x},t) + \frac{\partial^2}{c^2\partial t^2}\left(\frac{1}{4\pi\epsilon_0}\int d^3x'\frac{\rho}{R}\right) \\ =& -\frac{1}{\epsilon_0}\rho(\mathbf{x},t) + \frac{\partial^2}{c^2\partial t^2}\,\Phi(\mathbf{x},t)\ ,\end{aligned} \tag{6.3.14}$$

confirming that the retarded potential (6.3.3) satisfies the inhomogenious wave equation (6.3.1).

6.3.4 Generalization of Coulomb and Biot-Savart Laws

Given the retarded potentials

$$\Phi(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3x'\frac{\rho(\mathbf{x}',t_{\rm ret})}{R}\ , \tag{6.3.15}$$

and

$$\mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int d^3x'\frac{\mathbf{J}(\mathbf{x}',t_{\rm ret})}{R}\ , \tag{6.3.16}$$

it is, in principle, a straightforward matter to determine the fields:

$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t} \qquad , \qquad \mathbf{B} = \nabla\times\mathbf{A}\ . \tag{6.3.17}$$

However, taking derivatives with respect to $\mathbf{x}$, we should remember again that the integrands depend on $\mathbf{x}$ both *explicitly*, through $R = |\mathbf{x}-\mathbf{x}'|$ in the denominator, and *implicitly*, through the retarded time $t_{\rm ret} = t - R/c$ in the argument of numerator. We already have expression

$$\nabla\Phi(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3x'\left[-\rho\frac{\hat{\mathbf{e}}_R}{R^2} - \frac{\dot{\rho}}{c}\frac{\hat{\mathbf{e}}_R}{R}\right] \tag{6.3.18}$$

for the gradient of Φ. Since

$$\frac{\partial\mathbf{A}}{\partial t} = \frac{\mu_0}{4\pi}\int d^3x'\frac{\dot{\mathbf{J}}(\mathbf{x}',t_{\rm ret})}{R} = \frac{1}{4\pi\epsilon_0 c^2}\int d^3x'\frac{\dot{\mathbf{J}}(\mathbf{x}',t_{\rm ret})}{R}\ , \tag{6.3.19}$$

we have

$$\boxed{\mathbf{E}(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3x'\left[\frac{\rho(\mathbf{x}',t_{\rm ret})}{R^2}\hat{\mathbf{e}}_R + \frac{\dot{\rho}(\mathbf{x}',t_{\rm ret})}{cR}\,\hat{\mathbf{e}}_R - \frac{\dot{\mathbf{J}}(\mathbf{x}',t_{\rm ret})}{c^2R}\right]\ . \quad (6.3.20)}$$

This is the time-dependent generalization of Coulomb's law, to which it reduces in the static case (then the second and third terms drop out and the first term loses its dependence on time $t_{\rm ret}$). For $\mathbf{B}$, the curl of $\mathbf{A}$ contains two terms

$$\nabla \times \mathbf{A} = \frac{\mu_0}{4\pi} \int d^3x' \left[-\mathbf{J} \times \nabla \left(\frac{1}{R} \right) + \frac{1}{R} (\nabla \times \mathbf{J}) \right] \ , \tag{6.3.21}$$

The chain rule that gave $\nabla \rho = -\frac{1}{c} \hat{\mathbf{e}}_R \, \dot{\rho}$, in case of curl gives

$$\nabla \times \mathbf{J} = -\frac{1}{c} \, \hat{\mathbf{e}}_R \times \dot{\mathbf{J}} = \frac{1}{c} \, \dot{\mathbf{J}} \times \hat{\mathbf{e}}_R \ . \tag{6.3.22}$$

Recalling $\nabla(1/R) = -\hat{\mathbf{e}}_R / R^2$ gives

$$\boxed{\mathbf{B}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int d^3x' \left[\frac{\mathbf{J}(\mathbf{x}', t_{\rm ret})}{R^2} + \frac{\dot{\mathbf{J}}(\mathbf{x}', t_{\rm ret})}{cR} \right] \times \hat{\mathbf{e}}_R \ . \qquad (6.3.23)}$$

This is the time-dependent generalization of Biot-Savart law, to which it reduces in the static case. Note that in the expressions for the retarded potentials Φ, $\mathbf{A}$, all one should do is to replace t by $t_{\rm ret}$, while the expressions for the fields $\mathbf{E}, \mathbf{B}$ contain completely new terms involving derivatives of ρ and $\mathbf{J}$.

In case of a slowly changing current density, namely, when one can neglect all the higher derivatives in the Taylor expansion

$$\mathbf{J}(t_{\rm ret}) = \mathbf{J}(t) + (t_{\rm ret} - t) \dot{\mathbf{J}}(t) + \ldots \tag{6.3.24}$$

(we suppress here the $\mathbf{x}'$-dependence, which is not an issue) one can write

$$\mathbf{J}(t_{\rm ret}) = \mathbf{J}(t) - \frac{R}{c} \dot{\mathbf{J}}(t) \tag{6.3.25}$$

to obtain

$$B(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int d^3x' \left[\frac{\mathbf{J}(\mathbf{x}', t) \times \hat{\mathbf{e}}_R}{R^2} \right] + \ldots \ , \tag{6.3.26}$$

i.e., the Biot-Savart law with $\mathbf{J}$ evaluated at the *non-retarded* time.

6.4 Energy-Momentum Conservation and Poynting Vector

In this section, we will derive laws expressing conservation of energy and momentum for electric and magnetic fields.

The force acting on a particle carrying charge q and moving with velocity $\mathbf{v}$ is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{6.4.1}$$

The work done per unit time, or rate of change of *mechanical* energy, is then

$$\begin{aligned} \frac{d}{dt} E_{\text{mech}} &= \mathbf{v} \cdot [q(\mathbf{E} + \mathbf{v} \times \mathbf{B})] \\ &= q\mathbf{v} \cdot \mathbf{E}, \end{aligned}$$

since the second term vanishes. Thus generalizing to a **current density J** we have

$$\frac{d}{dt} E_{\text{mech}} = \int d^3x\, \mathbf{J} \cdot \mathbf{E}. \tag{6.4.2}$$

We will now relate the rate of change of mechanical energy to the change of energy in the electric and magnetic fields. The starting point is Maxwell-Ampère's law $\nabla \times \mathbf{H} = \mathbf{J} + \partial \mathbf{D}/\partial t$ (ME3), which gives

$$\int_V d^3x\, \mathbf{J} \cdot \mathbf{E} = \int d^3x\, \mathbf{E} \cdot \left[\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} \right]. \tag{6.4.3}$$

We can use the vector identity $\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H})$
or $\mathbf{E} \cdot (\nabla \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \nabla \cdot (\mathbf{E} \times \mathbf{H})$ to write

$$\int_V d^3x\, \mathbf{J} \cdot \mathbf{E} = \int d^3x \left\{ \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \nabla \cdot [\mathbf{E} \times \mathbf{H}] - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \right\}. \tag{6.4.4}$$

Identifying the l.h.s. of this equation with the rate of change of mechanical energy in Eq. (6.4.2), and using Faraday's law $\nabla \times \mathbf{E} + \partial \mathbf{B}/\partial t = 0$ (ME2) on the r.h.s., we obtain

$$\frac{d}{dt} E_{\text{mech}} = -\int d^3x \left\{ \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{E} \times \mathbf{H}) + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \right\}. \tag{6.4.5}$$

We will now assume that the medium is **linear**, allowing us to write

$$\begin{aligned} \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} &= \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{H} \cdot \mathbf{B}) \ , \\ \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} &= \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{E} \cdot \mathbf{D}), \end{aligned}$$

and thus

$$\frac{d}{dt} E_{\text{mech}} = \int d^3x\, \mathbf{J} \cdot \mathbf{E} = -\int d^3x \left\{ \nabla \cdot (\mathbf{E} \times \mathbf{H}) + \frac{\partial}{\partial t} \left[\frac{1}{2} (\mathbf{H} \cdot \mathbf{B} + \mathbf{E} \cdot \mathbf{D}) \right] \right\} \tag{6.4.6}$$

We have already, in Chapter 4.6, interpreted $\frac{1}{2}\epsilon_0|\mathbf{E}|^2 \equiv \frac{1}{2}\mathbf{E}\cdot\mathbf{D}$ as the energy density of an electric field. Likewise we will identify $\frac{1}{2}\mathbf{H}\cdot\mathbf{B}$ as the magnetic energy density and hence their sum

$$u = \frac{1}{2}(\mathbf{H}\cdot\mathbf{B} + \mathbf{E}\cdot\mathbf{D}) \tag{6.4.7}$$

as the **electromagnetic energy density**. With this identification, we now have **Poynting's Theorem** expressing conservation of energy

$$\boxed{-\int_V d^3x\, \mathbf{J}\cdot\mathbf{E} = \int_V d^3x \left[\frac{\partial u}{\partial t} + \nabla\cdot(\mathbf{E}\times\mathbf{H})\right] \qquad (6.4.8)}$$

Since this applies for any volume V, we have a differential energy continuity equation

$$\boxed{\frac{\partial u}{\partial t} + \nabla\cdot(\mathbf{E}\times\mathbf{H}) = -\mathbf{J}\cdot\mathbf{E} \qquad (6.4.9)}$$

The vector

$$\boxed{\mathbf{S} = \mathbf{E}\times\mathbf{H} \qquad (6.4.10)}$$

is the **Poynting Vector**. It only enters through a divergence in the above expressions but, when we come to consider its properties under Lorentz transformations later in the course, we will discover that it is essentially unique.

We can reduce the integral over the Poynting vector in Eq. (6.4.8) to a surface integral using the divergence theorem.

$$\int_V d^3x\, \nabla\cdot(\mathbf{E}\times\mathbf{H}) \equiv \int_V d^3x\, \nabla\cdot\mathbf{S} = \oint_{A=\partial V} \mathbf{dA}\cdot\mathbf{S}\ , \tag{6.4.11}$$

where A is the surface surrounding the volume V. This gives

$$\boxed{\int_V d^3x\, \frac{\partial u}{\partial t} + \oint_{A=\partial V} \mathbf{dA}\cdot\mathbf{S} = -\frac{d}{dt}E_{\mathrm{mech}} \qquad (6.4.12)}$$

Thus we can interpret the Poynting vector as the energy flux across a surface, and the Poynting theorem in essence says:

"The rate of change of electromagnetic energy in a volume together with energy flux across the boundary is equal to minus the total work done by sources within the volume".

6.4.1 Energy Conservation in terms of the Fundamental Microscopic Fields

The field energy density of Eq. (6.4.7) contains not only the fundamental fields, but also the *"derived"* fields **H** and **D**. Thus they include contributions associated with the *polarization* and *magnetization* of the medium which are in essence *mechanical*, and should be associated with the $\mathbf{J} \cdot \mathbf{E}$ term.

Let $E_{\rm mech}$ be the **mechanical energy** in some fixed volume V. We have seen that the work done per unit time per unit volume $\mathbf{J} \cdot \mathbf{E}$ is the rate of increase of mechanical energy,

$$\frac{dE_{\rm mech}}{dt} = \int_V d^3x\, \mathbf{J} \cdot \mathbf{E}. \tag{6.4.13}$$

In the case of a vacuum, we have

$$\begin{aligned} \int_V d^3x\, u &= \frac{1}{2} \int d^3x (\mathbf{H} \cdot \mathbf{B} + \mathbf{E} \cdot \mathbf{D}) \\ &= \frac{\epsilon_0}{2} \int_V d^3x (\mathbf{E}^2 + c^2 \mathbf{B}^2) = E_{\rm field} \end{aligned}$$

where now we have expressed the field energy solely in terms of the fundamental fields. It is this expression that is more naturally associated with the field energy, and Poynting's theorem reads

$$\frac{d}{dt}(E_{\rm mech} + E_{\rm field}) = -\oint_{A=\partial V} \mathbf{dA} \cdot \mathbf{S} \tag{6.4.14}$$

6.4.2 Conservation of Linear Momentum

Again we work with the **microscopic fields**. The force on a particle of charge q is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{6.4.15}$$

Thus Newton's second law may be expressed as

$$\frac{d}{dt}\mathbf{P}_{\rm mech} = \int d^3x \, [\rho \mathbf{E} + \mathbf{J} \times \mathbf{B}] \tag{6.4.16}$$

where $\mathbf{P}_{\rm mech}$ is the total momentum of the particles in a volume V. To evaluate this expression, we once again use Coulomb's law $\nabla \cdot \mathbf{D} = \rho$ (ME1) and Ampère's law $\nabla \times \mathbf{H} = \mathbf{J} + \partial \mathbf{D}/\partial t$ (ME3), yielding for the integrand

$$\rho\, \mathbf{E} + \mathbf{J} \times \mathbf{B} = \epsilon_0 \mathbf{E}(\nabla \cdot \mathbf{E}) - \mathbf{B} \times \left[\frac{1}{\mu_0} \nabla \times \mathbf{B} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right]. \tag{6.4.17}$$

We now use

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{E}\times\mathbf{B} &= \frac{\partial \mathbf{E}}{\partial t}\times\mathbf{B}+\mathbf{E}\times\frac{\partial \mathbf{B}}{\partial t} \\ \nabla\cdot\mathbf{B} &= 0 \end{aligned}$$

to write

$$\rho\,\mathbf{E}+\mathbf{J}\times\mathbf{B} = \\ \epsilon_0[\mathbf{E}(\nabla\cdot\mathbf{E})+c^2\mathbf{B}(\nabla\cdot\mathbf{B})-c^2\mathbf{B}\times(\nabla\times\mathbf{B})+\mathbf{E}\times\frac{\partial \mathbf{B}}{\partial t}-\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B})].$$

We now use Faraday's law $\nabla\times\mathbf{E}+\partial\mathbf{B}/\partial t=0$ (ME2) to write

$$\begin{aligned} &\frac{d}{dt}\mathbf{P}_{\text{mech}}+\frac{d}{dt}\epsilon_0\int_V d^3x\,\mathbf{E}\times\mathbf{B} \\ &= \epsilon_0\int d^3x\,[\mathbf{E}\nabla\cdot\mathbf{E}+c^2\mathbf{B}\nabla\cdot\mathbf{B}-\mathbf{E}\times(\nabla\times\mathbf{E})-c^2\mathbf{B}\times(\nabla\times\mathbf{B})]\ , \end{aligned} \tag{6.4.18}$$

where we assume that the volume V is fixed. The second term on the l.h.s. we associate with the momentum carried by the field

$$\mathbf{P}_{\text{field}}=\epsilon_0\int d^3x\,\mathbf{E}\times\mathbf{B}, \tag{6.4.19}$$

which we can rewrite as

$$\mathbf{P}_{\text{field}} = \int d^3x\,\frac{1}{c^2}\mathbf{E}\times\mathbf{H} = \int d^3x\ g\ , \tag{6.4.20}$$

where $\mathbf{g}$ is the **electromagnetic momentum density** given, up to a constant factor, by the *Poynting Vector*,

$$\mathbf{g}=\frac{1}{c^2}\mathbf{S}. \tag{6.4.21}$$

To proceed further, let us consider the r.h.s. of the momentum conservation law, Eq. (6.4.18). Using $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$ in the index notation (with the summation over the repeated indices implied),

$$[\mathbf{A}\times(\mathbf{B}\times\mathbf{C})]_i = B_i(A_jC_j)-C_i(A_jB_j)\ = A_jB_iC_j-A_jB_jC_i\ ,$$

we may write (substituting $A\to E, B\to\nabla, C\to E$)

$$[\mathbf{E}\times(\nabla\times\mathbf{E})]_i = E_j\frac{\partial}{\partial x_i}E_j-E_j\frac{\partial}{\partial x_j}E_i = E_j\frac{\partial E_j}{\partial x_i}-E_j\frac{\partial E_i}{\partial x_j}\ ,$$

which gives

$$\begin{aligned}[\mathbf{E}(\nabla\cdot\mathbf{E}) - \mathbf{E}\times(\nabla\times\mathbf{E})]_i &= E_i\frac{\partial E_j}{\partial x_j} - E_j\frac{\partial E_j}{\partial x_i} + E_j\frac{\partial E_i}{\partial x_j} \\ &= \frac{\partial}{\partial x_j}[E_iE_j - \frac{1}{2}E^2\delta_{ij}].\end{aligned}$$

What we have done is to write the electric part of the integrand as a derivative. We may treat the magnetic term similarly, and now introduce the **Maxwell Stress Tensor**

$$\boxed{T_{ij} = \epsilon_0\left[E_iE_j + c^2B_iB_j - \frac{1}{2}(E^2 + c^2B^2)\delta_{ij}\right] \qquad (6.4.22)}$$

Note that this tensor is **symmetric**.

We can thus write the momentum conservation law as

$$\frac{d}{dt}\left[\mathbf{P}_{\rm mech} + \mathbf{P}_{\rm field}\right]_i = \int_V d^3x\,\frac{\partial T_{ij}}{\partial x_j} \qquad (6.4.23)$$

which, after applying the divergence theorem, becomes

$$\boxed{\frac{d}{dt}[\mathbf{P}_{\rm mech} + \mathbf{P}_{\rm field}]_i = \oint_{A=\partial V} dA\,T_{ij}n_j \qquad (6.4.24)}$$

where $\mathbf{n}$ is the outward normal to the surface enclosing V.

Note that $T_{ij}n_j$ is the flow of momentum per unit area across surface A into the volume V, i.e. it is the force per unit area acting on the combined system of particles and fields within volume V.