# Contents

# Chapter 9

# Radiating Systems

## 9.1 Preliminaries

In this chapter, we will study radiation of varying current distributions. We will begin by working in Lorentz gauge, where the equation for the *vector potential* is

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} \ . \tag{9.1.1}$$

From Chapter 6, we recall that this equation has the **retarded** solution

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int d^3x'\, dt'\, G^{(+)}(\mathbf{x}, t; \mathbf{x}', t')\, \mathbf{J}(\mathbf{x}', t') \tag{9.1.2}$$

where

$$G^{(+)}(\mathbf{x}, t; \mathbf{x}', t') = \frac{1}{|\mathbf{x} - \mathbf{x}'|}\delta(t' - t + \frac{|\mathbf{x} - \mathbf{x}'|}{c}) \ . \tag{9.1.3}$$

We now consider the case where the fields arise from a current with harmonic time variation

$$\mathbf{J}(\mathbf{x}', t') = \mathbf{J}(\mathbf{x}')e^{-i\omega t'} \ . \tag{9.1.4}$$

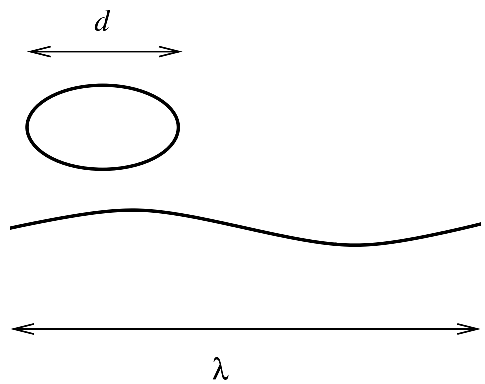
More general time dependence can be studied simply by taking the Fourier transform. The potential corresponding to this current is then

$$\begin{aligned}\mathbf{A}(\mathbf{x}, t) &= \frac{\mu_0}{4\pi}\int d^3x'\, dt' \mathbf{J}(\mathbf{x}')e^{-i\omega t'} G^{(+)}(\mathbf{x}, t; \mathbf{x}', t') \\ &= \mathbf{A}(\mathbf{x})e^{-i\omega t},\end{aligned}$$

with

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x'\, \mathbf{J}(\mathbf{x}')\frac{1}{|\mathbf{x} - \mathbf{x}'|}e^{ik|\mathbf{x}-\mathbf{x}'|} \ , \tag{9.1.5}$$

where $k \equiv \omega/c$ is the wave number




We will now consider the form of the field a distance $r = |\mathbf{x}|$ away from a time-varying source of extent $d$ localized near the coordinate origin, i.e., $|\mathbf{x}'| \sim d$. We begin by introducing the wavelength

$$\lambda = \frac{2\pi}{k} \equiv \frac{2\pi c}{\omega} \ , \tag{9.1.6}$$

where $\lambda \gg d$ (or $kd \ll 1$).
We now consider the form of the potential in three different regions:

1. $d \ll r \ll \lambda$ (or $kd \ll kr \ll 1$) – the **near zone**
   Then $\exp(ik|\mathbf{x}-\mathbf{x}'|) \sim \exp(ikr) \sim 1$, and we have

   $$\mathbf{A}(\mathbf{x}) \simeq \frac{\mu_0}{4\pi} \int d^3x' \, \mathbf{J}(\mathbf{x}') \frac{1}{|\mathbf{x}-\mathbf{x}'|}. \tag{9.1.7}$$

   The field is of the familiar form which we can expand as a series in, say, Legendre polynomials.

2. $r \gg \lambda \gg d$ (or $kr \gg 1 \gg kd$) – the **radiation zone**
   Then the exponent is rapidly oscillating, and we can write

   $$|\mathbf{x}-\mathbf{x}'| = \left[x^2 - 2\mathbf{x}\cdot\mathbf{x}' + x'^2\right]^{1/2} = \left[r^2(1 - 2\mathbf{x}\cdot\mathbf{x}'/r^2 + x'^2/r^2)\right]^{1/2} \tag{9.1.8}$$

   $$\simeq r\,\left(1 - \mathbf{n}\cdot\mathbf{x}'/r + \mathcal{O}(|\mathbf{x}'|^2/r^2)\right) = \ r - \mathbf{n}\cdot\mathbf{x}' + \mathcal{O}\left(\frac{|\mathbf{x}'|^2}{r}\right) \ , \tag{9.1.9}$$

   where $\mathbf{n} = \mathbf{x}/r$ is the unit vector in the direction of the observation point $\mathbf{x}$. Thus, to leading order in $1/r$ we have

   $$\mathbf{A}(\mathbf{x}) \xrightarrow{kr\gg 1} \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r} \int d^3x' \, \mathbf{J}(\mathbf{x}') \exp[-ik\,\mathbf{n}\cdot\mathbf{x}'] \ . \tag{9.1.10}$$

Thus we have an outgoing spherical wave. We can compute the magnetic and electric fields using $\mathbf{B} = \nabla \times A$ and a Maxwell curl equation

$$\nabla \times \mathbf{H} = -i\omega\epsilon_0 \mathbf{E}$$

accompanied by $\mathbf{B} = \mu_0 \mathbf{H}$ which gives

$$\begin{aligned} \mathbf{H} &= \frac{1}{\mu_0} \nabla \times \mathbf{A} \,, \qquad (9.1.11) \\ \mathbf{E} &= \frac{i}{\omega\epsilon_0} \nabla \times \mathbf{H} = \frac{iZ_0}{k} \nabla \times \mathbf{H} \,, \end{aligned}$$

which also fall off as $1/r$, corresponding to **radiation**.
(Hereafter $Z_0 \equiv \sqrt{\frac{\mu_0}{\epsilon_0}} = \mu_0 c = 1/\epsilon_0 c$).
Since $k\mathbf{n} \cdot \mathbf{x}' \ll 1$ – recall that $kd \ll 1$ - we can expand the exponent in Eq. (9.1.10) yielding

$$\mathbf{A}(\mathbf{x}) \simeq \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \sum_N \frac{(-ik)^N}{N!} \int d^3x' \, \mathbf{J}(\mathbf{x}')(\mathbf{n} \cdot \mathbf{x}')^N \,. \qquad (9.1.12)$$

Successive terms are $\mathcal{O}((kd)^N)$, which dies off with increasing $N$.

3. $r \sim \lambda$. Here we need to expand the solution in terms of the *vector multipole expansion*, discussed in detail in *Jackson, 9.6*.

An analogous analysis for the scalar potential yields

$$\Phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0} \int d^3x' \int dt' \frac{\rho(\mathbf{x}', t')}{|\mathbf{x} - \mathbf{x}'|} \delta \left( t' + \frac{|\mathbf{x} - \mathbf{x}'|}{c} - t \right) . \qquad (9.1.13)$$

For large $r$, i.e., when $|\mathbf{x}| = r \gg d \sim |\mathbf{x}'|$, keeping the leading term yields

$$\Phi(\mathbf{x}, t) \simeq \frac{q(t' = t - r/c)}{4\pi\epsilon_0 r} \,. \qquad (9.1.14)$$

where $q$ is the total charge of the source. If the source is localized, and isolated, no charge can flow in and out, and thus the total charge is constant in time – *the monopole part of the potential is static*, i.e. has no time dependence.

## 9.2 Electric Dipole Fields

If we keep only the leading term in Eq. (9.1.12), we have

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\,\mathbf{J}(\mathbf{x}'). \tag{9.2.1}$$

In fact, as discussed in *Jackson*, this is the leading $l = 0$ term in the vector multipole expansion of the vector potential, and as such is valid everywhere outside the source as part of the multipole expansion. We will now show that this corresponds to a dipole term. We begin by recalling the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0 \tag{9.2.2}$$

which with our assumed time dependence becomes

$$-i\omega\rho + \nabla\cdot\mathbf{J} = 0. \tag{9.2.3}$$

We now use integration by parts to write

$$\begin{aligned}\int d^3x'\,\mathbf{J} = \int d^3x'\,(\mathbf{J}\cdot\nabla')\mathbf{x}' \;&=\; -\int d^3x'\mathbf{x}'(\nabla'\cdot\mathbf{J})\\ &=\; -i\omega\int d^3x'\,\mathbf{x}'\rho(\mathbf{x}') \;=\; -i\omega\mathbf{p}\end{aligned}$$

enabling the potential to be expressed as

$$\mathbf{A}(\mathbf{x}) = -\frac{i\mu_0\omega}{4\pi}\frac{e^{ikr}}{r}\mathbf{p} \tag{9.2.4}$$

where

$$\mathbf{p} \equiv \int d^3x'\,\mathbf{x}'\rho(\mathbf{x}') \tag{9.2.5}$$

is the electric dipole moment.
The magnetic and electric fields are simply obtained from Eq. (9.1.11):

$$\begin{aligned}\mathbf{H} &= \frac{1}{\mu_0}\nabla\times\mathbf{A}\ ,\\ \mathbf{E} &= \frac{i}{\omega\epsilon_0}\nabla\times\mathbf{H} = \frac{iZ_0}{k}\nabla\times\mathbf{H}\ .\end{aligned}$$

To proceed, we will need the formulas

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r} = n_i\ . \tag{9.2.6}$$

and

$$\nabla_i f(r) = \frac{df(r)}{dr}\,\nabla_i r = \hat{\mathbf{n}}_i \frac{df(r)}{dr} \tag{9.2.7}$$

Thus, we get

$$\begin{aligned}\mathbf{H} = \frac{1}{\mu_0}\nabla\times\mathbf{A} &= \frac{1}{\mu_0}\left(-\frac{i\mu_0\omega}{4\pi}\right)(\mathbf{n}\times\mathbf{p})\frac{\partial}{\partial r}\,\frac{e^{ikr}}{r}\\ &= -\frac{i\omega}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{e^{ikr}}{r}\Big(ik-\frac{1}{r}\Big)\\ &= \frac{ck^2}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{e^{ikr}}{r}\Big(1-\frac{1}{ikr}\Big)\ .\end{aligned} \tag{9.2.8}$$

Analogously (using $ck^2 = \omega k$),

$$\mathbf{E} = \frac{i}{\omega\epsilon_0}\nabla\times\mathbf{H} \;=\; \frac{ik}{4\pi\epsilon_0}\Big\{\mathbf{n}\times(\mathbf{n}\times\mathbf{p})\frac{\partial}{\partial r}\left[\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\right] + \frac{e^{ikr}}{r}\Big(1-\frac{1}{ikr}\Big)\nabla\times(\mathbf{n}\times\mathbf{p})\Big\}\ . \tag{9.2.9}$$

Here we should take into account that $\hat{\mathbf{n}} = \mathbf{r}/r$ is a function of $\mathbf{r}$:

$$\frac{\partial n_i}{\partial x_j} = \frac{\partial}{\partial x_j}\frac{x_i}{r} = \frac{r^2\delta_{ij} - x_i x_j}{r^3}\ , \tag{9.2.10}$$

hence,

$$\nabla\cdot\mathbf{n} = \frac{2}{r}$$

and

$$p_j\frac{\partial n_i}{\partial x_j} = \frac{r^2 p_i - (\mathbf{p}\cdot\mathbf{x})x_i}{r^3} = \frac{p_i - (\mathbf{p}\cdot\mathbf{n})n_i}{r} \Rightarrow (\mathbf{p}\cdot\nabla)\mathbf{n} = \frac{\mathbf{p} - (\mathbf{p}\cdot\mathbf{n})\mathbf{n}}{r}\ .$$

Thus,

$$\nabla\times(\mathbf{n}\times\mathbf{p}) = (\mathbf{p}\cdot\nabla)\mathbf{n} - \mathbf{p}(\nabla\cdot\mathbf{n}) = (\mathbf{p}\cdot\nabla)\mathbf{n} - \frac{2}{r}\,\mathbf{p} = -\frac{\mathbf{n}(\mathbf{n}\cdot\mathbf{p}) + \mathbf{p}}{r}\ . \tag{9.2.11}$$

The leading $1/r$ term comes from differentiating the exponential in the first term,

$$\begin{aligned}\frac{\partial}{\partial r}\left[\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\right] &= ik\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right) - \frac{e^{ikr}}{r^2}\left(1-\frac{1}{ikr}\right) + \frac{e^{ikr}}{ikr^3}\\ &= ik\frac{e^{ikr}}{r} - 2\,\frac{e^{ikr}}{r^2}\left(1-\frac{1}{ikr}\right)\ .\end{aligned} \tag{9.2.12}$$

We will leave the leading term in the original $\mathbf{n}\times(\mathbf{n}\times\mathbf{p}) = -(\mathbf{n}\times\mathbf{p})\times\mathbf{n}$ form, and use

$$\mathbf{n}\times(\mathbf{n}\times\mathbf{p}) = \mathbf{n}(\mathbf{n}\cdot\mathbf{p}) - \mathbf{p}$$

for the remaining contribution.
This gives

$$\begin{aligned}\mathbf{E} =& \frac{1}{4\pi\epsilon_0}\Big[k^2(\mathbf{n}\times\mathbf{p})\times\mathbf{n}\frac{e^{ikr}}{r} - 2\,\frac{e^{ikr}}{r^2}\Big(ik - \frac{1}{r}\Big)[\mathbf{n}(\mathbf{n}\cdot\mathbf{p}) - \mathbf{p}] - \frac{e^{ikr}}{r}\Big(ik - \frac{1}{r}\Big)\frac{(\mathbf{n}\cdot\mathbf{p})\mathbf{n}+\mathbf{p}}{r}\Big] \\ =& \frac{1}{4\pi\epsilon_0}\Big[k^2(\mathbf{n}\times\mathbf{p})\times\mathbf{n}\frac{e^{ikr}}{r} + (3(\mathbf{n}\cdot\mathbf{p})\mathbf{n} - \mathbf{p})\Big(\frac{1}{r^3} - \frac{ik}{r^2}\Big)e^{ikr}\Big]\ , \qquad (9.2.13)\\ \mathbf{H} =& \frac{ck^2}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{e^{ikr}}{r}\Big(1 - \frac{1}{ikr}\Big)\ . \qquad (9.2.14)\end{aligned}$$

Let us choose now the spherical coordinates in which the $z$-axis is along $\mathbf{p}$, and $\theta$ is the polar angle of $\mathbf{n}$. Then $\mathbf{p} = p(\mathbf{n}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta)$ and $\mathbf{n}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\varphi}}$, so that we have $\mathbf{n}\times\mathbf{p} = -\hat{\boldsymbol{\varphi}}\,p\sin\theta$. Using $\hat{\boldsymbol{\varphi}}\times\mathbf{n} = \hat{\boldsymbol{\theta}}$, we find that expressions for the fields take the form

$$\begin{aligned}\mathbf{H} &= -\hat{\boldsymbol{\varphi}}\,\frac{pck^2}{4\pi}\sin\theta\,\frac{e^{ikr}}{r}\Big(1 - \frac{1}{ikr}\Big)\ , \qquad (9.2.15)\\ \mathbf{E} &= \frac{p}{4\pi\epsilon_0}\Big[-\hat{\boldsymbol{\theta}}\,k^2\frac{e^{ikr}}{r}\sin\theta + (2\hat{\boldsymbol{n}}\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta)\Big(\frac{1}{r^3} - \frac{ik}{r^2}\Big)e^{ikr}\Big]\ .\end{aligned}$$

It is interesting to examine their limiting forms

- **Radiation Zone**: $r \gg \lambda \gg d$ (or $kr \gg 1 \gg kd$):

$$\begin{aligned}\mathbf{H} &= \frac{ck^2}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{e^{ikr}}{r} = -\frac{\omega^2 p}{4\pi c}\hat{\boldsymbol{\varphi}}\sin\theta\frac{e^{ikr}}{r}\ .\\ \mathbf{E} &= -\frac{\mu_0}{4\pi}\omega^2 p\,\hat{\boldsymbol{\theta}}\,\sin\theta\frac{e^{ikr}}{r} = Z_0\mathbf{H}\times\mathbf{n}\ .\end{aligned}$$

  Both these fields manifest clearly the characteristic properties of radiation:

  – The fields fall off as $1/r$.

  – The electric and magnetic fields are normal to the direction of propagation $\mathbf{n}$.

- **Near Zone**: $\lambda \gg r \gg d$ (or $1 \gg kr \gg kd$) :
  Here the leading behaviour of the fields is given by

$$\begin{aligned}\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\Big[3\mathbf{n}(\mathbf{n}\cdot\mathbf{p}) - \mathbf{p}\Big]\frac{1}{r^3}\\ \mathbf{H} &= \frac{1}{4\pi\epsilon_0}\frac{i}{Z_0}(\mathbf{n}\times\mathbf{p})\frac{k}{r^2}.\end{aligned}$$

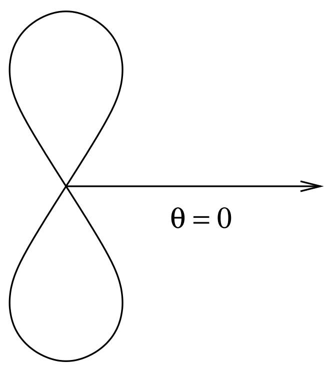
  Thus at very short distances, there is essentially an electric dipole field with time dependence $\exp(-i\omega t)$, and a magnetic field suppressed by $kr/Z_0$ that vanishes as $k \to 0$

In order to show that this solution does indeed correspond to radiation, we will look at the **time-averaged power flux** in the *radiation zone*. This, of course, is just given by the **Poynting Vector**, and we have

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{1}{2}r^2\mathrm{Re}\left[\mathbf{n}\cdot\mathbf{E}\times\mathbf{H}^*\right] = \frac{Z_0}{2}r^2\mathrm{Re}\left[((\mathbf{H}\times\mathbf{n})\times\mathbf{H}^*)\cdot\mathbf{n}\right] \\ &= \frac{Z_0}{2}r^2\mathrm{Re}\left[(\mathbf{H}\times\mathbf{n})\cdot(\mathbf{H}^*\times\mathbf{n})\right] = \frac{c^2Z_0}{32\pi^2}k^4|(\mathbf{n}\times\mathbf{p})\times\mathbf{n}|^2\end{aligned} \quad (9.2.16)$$

There is a net flux of power away from the charge distribution, independent of $r$, i.e., **radiation**. For the case where all components of $\mathbf{p}$ have the same phase, we have the characteristic expression for dipole radiation,

$$\frac{dP}{d\Omega} = \frac{c^2Z_0}{32\pi^2}k^4|\mathbf{p}|^2\sin^2\theta\ . \quad (9.2.17)$$




The total power transmitted is just obtained by integrating Eq. (9.2.16) over the unit sphere, and is independent of the phases of $\mathbf{p}$:

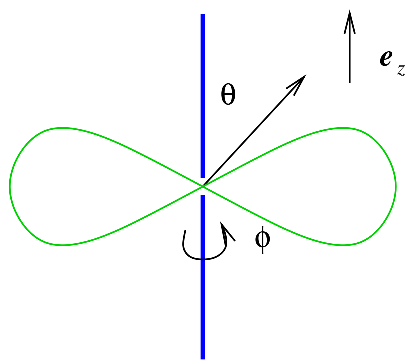
$$P = \frac{c^2Z_0}{32\pi^2}k^4|\mathbf{p}|^2\times 2\pi\times\int_0^\pi \sin^2\theta\,\sin\theta d\theta = \frac{c^2Z_0k^4}{12\pi}|\mathbf{p}|^2. \quad (9.2.18)$$

**Center-fed Linear Antenna**

Once again we assume that the dimensions of the antenna are much smaller than the wavelength. The antenna consists of two conductors of length $d/2$, along the $z$ axis. The linear current density in the wires is

$$I(z) = I_0\left(1 - \frac{2|z|}{d}\right) \tag{9.2.19}$$

where we again suppress the time dependence.




This current flow gives rise to a *line charge density* $\Lambda$ through the continuity equation

$$i\omega\Lambda(z) = \frac{\partial I}{\partial z}. \tag{9.2.20}$$

yielding

$$\Lambda(z) = \frac{2iI_0}{\omega d}\mathrm{sgn}(z). \tag{9.2.21}$$

This charge density has a non-zero dipole moment

$$\begin{aligned}\mathbf{p} &= \int_{-d/2}^{d/2} dz\,|z|\frac{2iI_0}{\omega d}\mathbf{e_z} \\ &= \frac{iI_0 d}{2\omega}\mathbf{e_z}.\end{aligned}$$

*N.B.* if we had current flowing in **opposite** directions in the two arms of the antenna, there would have been no dipole radiation term.
Thus, from Eq. (9.2.16), we see that this apparatus gives dipole radiation, with power distribution

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{Z_0 I_0^2}{128\pi^2}(kd)^2\sin^2\theta \\ P &= \frac{Z_0 I_0^2 (kd)^2}{48\pi}.\end{aligned} \tag{9.2.22}$$

If we identify the power radiated with energy dissipation through an effective resistance, the coefficient of $I_0^2/2$ is Eq. (9.2.22) is the **radiation resistance** - the factor of 2 arises from time-averaging, in the usual way.

## 9.3 Dipole Fields Revisited

In this section we will derive the formulas for the dipole radiation again – this time without Fourier transformation $\int d\omega e^{-i\omega t}$ implied.
The general formulas for vector and scalar potentials due to an arbitrary source are:

$$\Phi(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\rho(\mathbf{x}',t_r)}{|\mathbf{x}-\mathbf{x}'|} \ ,$$
$$\mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\mathbf{J}(\mathbf{x}',t_r)}{|\mathbf{x}-\mathbf{x}'|} \ , \tag{9.3.1}$$

where $t_r = t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}$ is the retarded time.
To study the behavior of these expressions in the radiation zone $|\mathbf{x}| \gg |\mathbf{x}'|$, we choose the origin somewhere inside the radiating body and expand the denominators in a usual way:

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{r}\Big(1 + \frac{\hat{\mathbf{n}}\cdot\mathbf{x}'}{r} + ...\Big) \tag{9.3.2}$$

where $r \equiv |\mathbf{x}|$ and $\hat{\mathbf{n}} \equiv \hat{\mathbf{r}}$ is the propagation vector for our would-be spherical wave. We need also to expand the retarded time in powers of $r'/r$:

$$t_r = t - \frac{|\mathbf{x}-\mathbf{x}'|}{c} \simeq t - \frac{r}{c} + \frac{\hat{\mathbf{n}}\cdot\mathbf{x}'}{c}$$

so that

$$\rho(\mathbf{x}',t_r) = \rho(\mathbf{x}',t_0) + \frac{\hat{\mathbf{n}}\cdot\mathbf{x}'}{c}\dot{\rho}(\mathbf{x}',t_0) + ... \tag{9.3.3}$$

where $t_0 \equiv t - r/c$ is the retarded time for our origin. The parameter of the expansion (9.3.3) is $d/\lambda \ll 1$ (see previous Section). Indeed, $\dot{\rho} \sim \omega_{\text{char}}\rho$ where $\omega_{\text{char}}$ are the characteristic frequencies of the emitted radiation, hence $\frac{d\dot{\rho}}{c\rho} \sim \frac{d\omega}{c} = \frac{d}{\lambda} \ll 1.$ ) Substituting the expansions (9.3.2) and (9.3.3) in the expression (9.3.1), one obtains:

$$\Phi(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0 r}\int d^3x' \left[\rho(\mathbf{x}',t_0) + \frac{\hat{\mathbf{n}}\cdot\mathbf{x}'}{c}\dot{\rho}(\mathbf{x}',t_0)\right]\Big(1 + \frac{\hat{\mathbf{n}}\cdot\mathbf{x}'}{r} + \ldots\Big)$$
$$= \frac{Q}{4\pi\epsilon_0 r} + \frac{\hat{\mathbf{n}}\cdot\mathbf{p}(t_0)}{4\pi\epsilon_0 r^2} + \frac{\hat{\mathbf{n}}\cdot\dot{\mathbf{p}}(t_0)}{4\pi\epsilon_0 rc} + ...$$

For the vector potential in Eq. (9.5.1), the first term in the expansions (9.3.2) and (9.3.3) is sufficient:

$$\mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\mathbf{J}(\mathbf{x}',t_r)}{|\mathbf{x}-\mathbf{x}'|} \simeq \frac{\mu_0}{4\pi r}\int d^3x' \mathbf{J}(\mathbf{x}',t_0) \ .$$

Just like in the previous Section, we can incorporate the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0 \tag{9.3.4}$$

to get

$$\begin{aligned}\int d^3x'\,\mathbf{J}(\mathbf{x}',t_0) &= \int d^3x'\,(\mathbf{J}(\mathbf{x}',t_0)\cdot\nabla')\mathbf{x}' = -\int d^3x'\mathbf{x}'(\nabla'\cdot\mathbf{J}(\mathbf{x}',t_0)) \\ &= \int d^3x'\,\mathbf{x}'\,\frac{\partial\rho(\mathbf{x}',t_0)}{\partial t_0} = \frac{d}{dt_0}\int d^3x'\,\mathbf{x}'\rho(\mathbf{x}',t_0) = \dot{\mathbf{p}}(t_0)\ ,\end{aligned}$$

where

$$\mathbf{p}(t_0) \equiv \int d^3x'\,\mathbf{x}'\rho(\mathbf{x}',t_0) \tag{9.3.5}$$

is the electric dipole moment.
So, the dipole potentials in the radiation zone take the form

$$\begin{aligned}\Phi(\mathbf{x},t) &= \frac{1}{4\pi\epsilon_0 r}\int d^3x'\left[\rho(\mathbf{x}',t_0) + \frac{\hat{\mathbf{n}}\cdot\mathbf{x}'}{c}\dot{\rho}(\mathbf{x}',t_0)\right]\left(1+\frac{\hat{\mathbf{n}}\cdot\mathbf{x}'}{r}+...\right) \\ &= \frac{Q}{4\pi\epsilon_0 r} + \frac{\hat{\mathbf{n}}\cdot\mathbf{p}(t_0)}{4\pi\epsilon_0 r^2} + \frac{\hat{\mathbf{n}}\cdot\dot{\mathbf{p}}(t_0)}{4\pi\epsilon_0 rc} + \dots\ , \\ \mathbf{A}(\mathbf{x},t) &= \frac{\mu_0}{4\pi}\int d^3x'\frac{\mathbf{J}(\mathbf{x}',t_r)}{|\mathbf{x}-\mathbf{x}'|} = \frac{\mu_0\dot{\mathbf{p}}(t_0)}{4\pi r} + \dots\ .\end{aligned} \tag{9.3.6}$$

Next we calculate the electric and magnetic field in the radiation zone. Discarding terms $\sim \frac{1}{r^2}$, one obtains after some algebra (note that $\nabla f(t_0) = \dot{f}(t_0)\nabla t_0$ and $\nabla t_0 = -\frac{\hat{\mathbf{n}}}{c}$, i.e., $\nabla f(t_0) = -\dot{f}(t_0)\hat{\mathbf{n}}/c$):

$$\begin{aligned}\nabla\Phi(\mathbf{x},t) &= -\frac{\hat{\mathbf{n}}}{4\pi\epsilon_0 rc^2}(\hat{\mathbf{n}}\cdot\ddot{\mathbf{p}}(t_0)) = -\frac{\mu_0\hat{\mathbf{n}}}{4\pi r}(\hat{\mathbf{n}}\cdot\ddot{\mathbf{p}}(t_0))\ , \\ \frac{\partial}{\partial t}\mathbf{A}(\mathbf{x},t) &= \frac{\mu_0\ddot{\mathbf{p}}(t_0)}{4\pi r}, \quad \nabla\times\mathbf{A} = -\frac{\mu_0}{4\pi rc}\mathbf{n}\times\ddot{\mathbf{p}}(t_0)\ .\end{aligned}$$

Thus, the dipole fields in the radiation zone are

$$\begin{aligned}\mathbf{E}(\mathbf{x},t) &= \frac{\mu_0}{4\pi r}[\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\ddot{\mathbf{p}}(t_0)) - \ddot{\mathbf{p}}(t_0)] = \frac{\mu_0}{4\pi r}\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\ddot{\mathbf{p}}(t_0)) \\ \mathbf{B}(\mathbf{x},t) &= -\frac{\mu_0}{4\pi cr}\hat{\mathbf{n}}\times\ddot{\mathbf{p}}(t_0) = \frac{\hat{\mathbf{n}}}{c}\times\mathbf{E}(\mathbf{x},t)\end{aligned} \tag{9.3.7}$$

If we choose the frame with $OZ$ axis collinear to $\ddot{\mathbf{p}}(t_0)$, the fields take the form

$$\mathbf{E}(r,\theta,\varphi) \;=\; \frac{\mu_0 \ddot{p}(t_0)}{4\pi}\frac{\sin\theta}{r}\hat{\boldsymbol{\theta}}, \qquad \mathbf{B}(r,\theta,\varphi) \;=\; \frac{\mu_0 \ddot{p}(t_0)}{4\pi c}\frac{\sin\theta}{r}\hat{\boldsymbol{\varphi}}, \tag{9.3.8}$$

The Poynting vector is then

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \;=\; \frac{\mu_0}{16\pi^2 c}(\ddot{p}(t_0))^2\frac{\sin^2\theta}{r^2}\hat{\mathbf{n}}$$

$\Rightarrow$ the total radiated power takes the form

$$P \;=\; \int \mathbf{S}\cdot\hat{\mathbf{n}}\,dA = \frac{\mu_0}{16\pi^2 c}(\ddot{p}(t_0))^2 \underbrace{\int_0^{2\pi} d\varphi}_{2\pi}\,\underbrace{\int_0^{\pi} d\theta\,\sin^3\theta}_{4/3} \;=\; \frac{\mu_0}{6\pi c}(\ddot{p}(t_0))^2 \tag{9.3.9}$$

For a single point charge $q$ we have $\mathbf{p}(t) = q\mathbf{x}(t)$, so we get the Larmor formula

$$P \;=\; \frac{\mu_0 q^2 a^2}{6\pi c} \tag{9.3.10}$$

The Larmor formula can be also obtained using the Liénard-Wiechert potentials of the moving point charge.

# 9.4 Liénard-Wiechert Potentials

## 9.4.1 Potentials of a moving charge

Consider a point charge moving along the trajectory $\mathbf{r} = \mathbf{w}(t)$. What are the electric and magnetic fields due to this charge?

As usual, it is convenient to start with the potentials. In the Lorentz gauge

$$\begin{aligned}\Phi(\mathbf{r},t) \;&=\; \frac{1}{4\pi\epsilon_0}\int d^3x'\int dt'\,\frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta\Big(t'-t+\frac{|\mathbf{r}-\mathbf{r}'|}{c}\Big)\;, \\ \mathbf{A}(\mathbf{r},t) \;&=\; \frac{\mu_0}{4\pi}\int d^3x'\int dt'\,\frac{\mathbf{J}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta\Big(t'-t+\frac{|\mathbf{r}-\mathbf{r}'|}{c}\Big)\;.\end{aligned} \tag{9.4.11}$$

For a point charge

$$\rho(\mathbf{r}',t) \;=\; q\delta(\mathbf{r}'-\mathbf{w}(t)), \qquad \mathbf{J}(\mathbf{r}',t) \;=\; q\mathbf{v}(t)\delta(\mathbf{r}'-\mathbf{w}(t))\;.$$

First, let us find the scalar potential. We have

$$\begin{aligned}\Phi(\mathbf{r},t) &= \frac{q}{4\pi\epsilon_0}\int d^3x'\int dt'\,\frac{\delta(\mathbf{r}'-\mathbf{w}(t'))}{|\mathbf{r}-\mathbf{r}'|}\delta\Big(t'-t+\frac{|\mathbf{r}-\mathbf{r}'|}{c}\Big) \\ &= \frac{q}{4\pi\epsilon_0}\int dt'\frac{\delta\Big(t'-t+\frac{|\mathbf{r}-\mathbf{w}(t')|}{c}\Big)}{|\mathbf{r}-\mathbf{w}(t')|} = \frac{q}{4\pi\epsilon_0}\int dt'\frac{1}{\frac{\partial}{\partial t'}\Big(t'-t+\frac{|\mathbf{r}-\mathbf{w}(t')|}{c}\Big)\Big|_{t'=t_r}}\frac{\delta(t'-t_r)}{|\mathbf{r}-\mathbf{w}(t')|}\;,\end{aligned}$$

where $t_r$ is the solution of the equation $c(t-t_r)=|\mathbf{r}-\mathbf{w}(t_r)|$. Calculating the derivative

$$\frac{\partial}{\partial t'}\Big(t'-t+\frac{|\mathbf{r}-\mathbf{w}(t')|}{c}\Big)=1-\frac{\mathbf{v}(t')\cdot(\mathbf{r}-\mathbf{w}(t'))}{c|\mathbf{r}-\mathbf{w}(t')|}\ , \tag{9.4.12}$$

where $\mathbf{v}(t)\equiv\frac{\partial}{\partial t}\mathbf{w}(t)$ is the velocity of the particle, we obtain

$$\begin{aligned}\Phi(\mathbf{r},t)=&\frac{q}{4\pi\epsilon_0}\int dt'\frac{\delta(t'-t_r)}{|\mathbf{r}-\mathbf{w}(t')|-\frac{1}{c}\mathbf{v}(t')\cdot(\mathbf{r}-\mathbf{w}(t'))}\\ =&\frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{w}(t_r)|-\mathbf{v}(t_r)\cdot(\mathbf{r}-\mathbf{w}(t_r))/c}\ .\end{aligned}$$

Similarly,

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu_0 q}{4\pi}\mathbf{v}(t_r)\frac{1}{|\mathbf{r}-\mathbf{w}(t_r)|-\mathbf{v}(t_r)\cdot(\mathbf{r}-\mathbf{w}(t_r))/c}$$

Introducing the notation $\boldsymbol{\varrho}(t)\equiv\mathbf{r}-\mathbf{w}(t)$, we obtain

$$\begin{aligned}\Phi(\mathbf{r},t)&=\frac{q}{4\pi\epsilon_0}\frac{1}{\varrho(t_r)-\mathbf{v}(t_r)\cdot\boldsymbol{\varrho}(t_r)/c}\equiv\frac{q}{4\pi\epsilon_0}D\\ \mathbf{A}(\mathbf{r},t)&=\frac{\mu_0 q}{4\pi}\frac{\mathbf{v}}{\varrho(t_r)-\mathbf{v}(t_r)\cdot\boldsymbol{\varrho}(t_r)/c}=\frac{\mathbf{v}}{c^2}\Phi(\mathbf{r},t)=\frac{q}{4\pi\epsilon_0}\frac{\mathbf{v}}{c^2}D\ ,\end{aligned} \tag{9.4.13}$$

the Liénard-Wiechert potentials for a point charge.
According to these formulas, the field at the point of observation at time $t$ is determined by the state of motion of the charge at the earlier time $t_r$. Also, $\varrho(t)=\mathbf{r}-\mathbf{w}(t)$ is the radius vector from the charge $q$ to the observation point $P$; like $\mathbf{w}(t)$ it is a given function of the time. Then the time $t_r$ is determined by the equation

$$t_r+\frac{\varrho(t_r)}{c}=t\ .$$

In the system of reference in which the particle is at rest at time $t_r$, the potential at the point of observation at time $t$ is just the Coulomb potential.

## 9.4.2 Electric and magnetic fields of a moving charge

To calculate the intensities of the electric and magnetic fields from the formulas

$$\begin{aligned}\mathbf{E}&=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}\\ \mathbf{B}&=\nabla\times\mathbf{A}\ ,\end{aligned}$$

we must differentiate $\Phi$ and $\mathbf{A}$ with respect to the coordinates $x,y,z$ of the point $P$, and the time $t$ of observation. But our formulas express the potentials as functions of $t_r$, and only through the relation $t_r + \varrho(t_r)/c = t$ are implicit functions of $x, y, z, t$. Therefore to find the required derivatives we must first calculate the derivatives of $t_r$. Differentiating the relation $\varrho(t_r) = c(t - t_r)$ with respect to $t$, we get

$$\frac{\partial \varrho}{\partial t} = c\left(1 - \frac{\partial t_r}{\partial t}\right) = \frac{\partial \varrho}{\partial t_r}\frac{\partial t_r}{\partial t} = -\frac{\boldsymbol{\varrho}\cdot\mathbf{v}}{\varrho}\frac{\partial t_r}{\partial t}$$

The value of $\partial\rho/\partial t_r$ is obtained by differentiating the identity $\varrho^2 = \boldsymbol{\varrho}^2$ and substituting $\partial\boldsymbol{\varrho}(t_r)/\partial t_r = -\mathbf{v}(t_r)$. The minus sign is present because $\boldsymbol{\varrho} = \mathbf{r} - \mathbf{w}$. Thus,

$$\frac{\partial t_r}{\partial t} = \frac{1}{1 - (\boldsymbol{\varrho}\cdot\mathbf{v})/c\varrho} = \varrho D \ .$$

$$\frac{\partial \varrho}{\partial t} = -\frac{\boldsymbol{\varrho}\cdot\mathbf{v}}{\varrho}\frac{\partial t_r}{\partial t} = -\boldsymbol{\varrho}\cdot\mathbf{v}D = c(1 - \varrho D)$$

Similarly differentiating the relation $t_r = t - \varrho(t_r)/c$ with respect to the coordinates, we find

$$\nabla t_r = -\frac{1}{c}\nabla\rho(t_r) = -\frac{1}{c}\left(\frac{\partial\varrho}{\partial t_r}\nabla t_r + \frac{\boldsymbol{\varrho}}{\varrho}\right) = -\frac{1}{c}\left(-\frac{\boldsymbol{\varrho}\cdot\mathbf{v}}{\varrho}\nabla t_r + \frac{\boldsymbol{\varrho}}{\varrho}\right) = \frac{1}{c}\left(\frac{\boldsymbol{\varrho}\cdot\mathbf{v}}{\varrho}\nabla t_r - \frac{\boldsymbol{\varrho}}{\varrho}\right) \ ,$$

so that

$$\nabla t_r = -\frac{\boldsymbol{\varrho}}{c(\varrho - (\boldsymbol{\varrho}\cdot\mathbf{v})/c)} = -\frac{\boldsymbol{\varrho}}{c} D \ .$$

With the help of these formulas, one can calculate the fields $\mathbf{E}$ and $\mathbf{B}$. The final results are

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0\left(\varrho - \boldsymbol{\varrho}\cdot\mathbf{v}/c\right)^3}\left\{(1 - v^2/c^2)\left(\boldsymbol{\varrho} - \frac{\mathbf{v}}{c}\varrho\right) + \frac{1}{c^2}\,\boldsymbol{\varrho}\times\left[\left(\boldsymbol{\varrho} - \frac{\mathbf{v}}{c}\varrho\right)\times\mathbf{a}\right]\right\} \tag{9.4.14}$$

$$\mathbf{B} = \frac{1}{c\varrho}\,\boldsymbol{\varrho}\times\mathbf{E} \ . \tag{9.4.15}$$

Here, $\mathbf{a} = \partial\mathbf{v}/\partial t_r$. All quantities on the right sides of the equations refer to the time $t_r$. It is interesting to note that the magnetic field turns out to be everywhere perpendicular to the electric. In the non-relativistic limit, the electric field $\mathbf{E}$ reduces to the Coulomb field

$$\mathbf{E}(\mathbf{r}, t)|_{v/c\to 0} \quad \to \quad \frac{q\boldsymbol{\varrho}}{4\pi\epsilon_0\varrho^3} \ , \tag{9.4.16}$$

while the magnetic field tends to zero as $v/c$.

To derive these formulas, let us first convert them in a simpler form by excluding double vector products. Employing the $BAC - CAB$ formula gives

$$\begin{aligned}\mathbf{E} = \frac{q}{4\pi\epsilon_0\left(\varrho - \boldsymbol{\varrho}\cdot\mathbf{v}/c\right)^3}&\left\{(1 - v^2/c^2)\left(\boldsymbol{\varrho} - \frac{\mathbf{v}}{c}\varrho\right) + \frac{1}{c^2}\left[(\boldsymbol{\varrho}\cdot\mathbf{a})\left(\boldsymbol{\varrho} - \frac{\mathbf{v}}{c}\varrho\right) - \mathbf{a}\varrho\left(\varrho - \frac{\boldsymbol{\varrho}\cdot\mathbf{v}}{c}\right)\right]\right\} \\ &= \frac{q}{4\pi\epsilon_0\left(\varrho - \boldsymbol{\varrho}\cdot\mathbf{v}/c\right)^3}\left\{(1 - v^2/c^2) + \frac{1}{c^2}(\boldsymbol{\varrho}\cdot\mathbf{a})\right\}\left(\boldsymbol{\varrho} - \frac{\mathbf{v}}{c}\varrho\right) \\ &- \frac{q\varrho\mathbf{a}}{4\pi\epsilon_0\left(\varrho c - \boldsymbol{\varrho}\cdot\mathbf{v}\right)^2} \ .\end{aligned}$$

To get this expression, we first we calculate the gradient of the scalar potential

$$\begin{aligned}
-\nabla\Phi = -\frac{q}{4\pi\epsilon_0}\nabla D = & \ \frac{q}{4\pi\epsilon_0}\ \frac{1}{(\varrho - \mathbf{v}\cdot\boldsymbol{\varrho}/c)^2}\nabla\left(\varrho - \mathbf{v}\cdot\boldsymbol{\varrho}/c\right)\\
= & \ \frac{q}{4\pi\epsilon_0}\ D^2\left(-\frac{\mathbf{v}}{c} + \nabla t_r\left[-c - \frac{\partial}{\partial t_r}\mathbf{v}\cdot\boldsymbol{\varrho}/c\right]\right)\\
= & \ \frac{q}{4\pi\epsilon_0}\ D^2\left(-\frac{\mathbf{v}}{c} + \frac{\boldsymbol{\varrho}}{c}\,D\left[c - v^2/c + \mathbf{a}\cdot\boldsymbol{\varrho}/c\right]\right)\\
= & \ \frac{q}{4\pi\epsilon_0}\ \left(-D^2\,\frac{\mathbf{v}}{c} + \boldsymbol{\varrho}\,D^3\left[1 - v^2/c^2 + \mathbf{a}\cdot\boldsymbol{\varrho}/c^2\right]\right)
\end{aligned}$$

Consider now the time derivative of the vector potential,

$$\begin{aligned}
\frac{\partial}{\partial t}\mathbf{A} = \frac{\partial t_r}{\partial t}\frac{\partial}{\partial t_r}\mathbf{A} = \frac{q}{4\pi\epsilon_0 c^2}\ \varrho D\frac{\partial}{\partial t_r}\mathbf{v}D = \frac{q}{4\pi\epsilon_0 c^2}\ \varrho D\left[\mathbf{a}D - \mathbf{v}D^2\frac{\partial}{\partial t_r}\left(\varrho - \mathbf{v}\cdot\boldsymbol{\varrho}/c\right)\right]\\
= \frac{q}{4\pi\epsilon_0 c^2}\ \varrho D\left[\mathbf{a}D - \mathbf{v}D^2\left(-\frac{\boldsymbol{\varrho}\cdot\mathbf{v}}{\varrho} + v^2/c - \mathbf{a}\cdot\boldsymbol{\varrho}/c\right)\right]\\
= \frac{q}{4\pi\epsilon_0}\ \left[\frac{1}{c^2}\varrho D^2\mathbf{a} - \frac{\mathbf{v}}{c}D^2 + \frac{\varrho\mathbf{v}}{c}D^3\left(1 - v^2/c^2 + \mathbf{a}\cdot\boldsymbol{\varrho}/c^2\right)\right]
\end{aligned}$$

Combining, we get

$$\mathbf{E} = -\nabla\Phi - \frac{\partial}{\partial t}\mathbf{A} = \frac{q}{4\pi\epsilon_0}\ \left[-\frac{1}{c^2}\varrho D^2\mathbf{a} + \left(\boldsymbol{\varrho} - \frac{\varrho\mathbf{v}}{c}\right)\ D^3\left[1 - v^2/c^2 + \mathbf{a}\cdot\boldsymbol{\varrho}/c^2\right]\right]$$

To get $\mathbf{B}$, we calculate the curl of $\mathbf{A}$,

$$\nabla\times\mathbf{A} = \frac{q}{4\pi\epsilon_0 c^2}\nabla\times\mathbf{v}D = \frac{q}{4\pi\epsilon_0 c^2}\left[D\nabla\times\mathbf{v} - \mathbf{v}\times\nabla D\right]$$

For the first term, we need

$$\nabla\times\mathbf{v} = \nabla t_r\times\frac{\partial}{\partial t_r}\mathbf{v} = -\mathbf{a}\times\nabla t_r = \mathbf{a}\times\boldsymbol{\varrho}\frac{D}{c}\ = -\frac{D}{c}\boldsymbol{\varrho}\times\mathbf{a},$$

which gives

$$\frac{D}{c^2}\nabla\times\mathbf{v} = -\frac{D^2}{c^3}\boldsymbol{\varrho}\times\mathbf{a} = -\frac{\varrho D^2}{c^3\varrho}\boldsymbol{\varrho}\times\mathbf{a}$$

For the second term, we have

$$\begin{aligned}
-\mathbf{v}\times\nabla D = & \mathbf{v}\times\left(-D^2\,\frac{\mathbf{v}}{c} + \boldsymbol{\varrho}\,D^3\left[1 - v^2/c^2 + \mathbf{a}\cdot\boldsymbol{\varrho}/c^2\right]\right)\\
= & \ \mathbf{v}\times\boldsymbol{\varrho}\,D^3\left[1 - v^2/c^2 + \mathbf{a}\cdot\boldsymbol{\varrho}/c^2\right]
\end{aligned}$$

Converting

$$\mathbf{v}\times\boldsymbol{\varrho}\ = -\boldsymbol{\varrho}\times\mathbf{v} = c\boldsymbol{\varrho}\times(\boldsymbol{\varrho} - \varrho\mathbf{v}/c)/\varrho$$

and combining both terms, we get the formula for $\mathbf{B}$

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{q}{4\pi\epsilon_0 c\varrho}\boldsymbol{\varrho}\times\left[-\frac{\varrho D^2}{c^2}\mathbf{a} + \left(\boldsymbol{\varrho} - \varrho\mathbf{v}/c\right)D^3\left[1 - v^2/c^2 + \mathbf{a}\cdot\boldsymbol{\varrho}/c^2\right]\right] = \frac{1}{c\varrho}\,\boldsymbol{\varrho}\times\mathbf{E}\ .$$

### 9.4.3 Power radiated by a point charge

Let us introduce the unit vector $\hat{\boldsymbol{\varrho}}$ through $\hat{\boldsymbol{\varrho}} \equiv \boldsymbol{\varrho}/\varrho$, and also the notation $\mathbf{u} \equiv \hat{\boldsymbol{\varrho}} - \mathbf{v}(t_r)/c$. This vector tends to $\hat{\boldsymbol{\varrho}}$ in the non-relativistic limit $v/c \to 0$. Using $\varrho - \boldsymbol{\varrho} \cdot \mathbf{v}/c = (\boldsymbol{\varrho} \cdot \mathbf{u})$, we can write the electric and magnetic fields due to a point charge moving along an arbitrary trajectory $\mathbf{w}(t)$ as

$$\begin{aligned} \mathbf{E}(\mathbf{r},t) &= \frac{q}{4\pi\epsilon_0} \frac{\varrho}{(\boldsymbol{\varrho}\cdot\mathbf{u})^3}\Big[\mathbf{u}(1-v^2/c^2) + \boldsymbol{\varrho}\times(\mathbf{u}\times\mathbf{a})/c^2\Big] \\ \mathbf{B}(\mathbf{r},t) &= \frac{\hat{\boldsymbol{\varrho}}}{c}\times\mathbf{E}(\mathbf{r},t) \end{aligned} \tag{9.4.17}$$

Recall that $t_r$ is defined as a solution to the equation $c(t - t_r) = \varrho$. As usual, velocity and acceleration in Eq. (9.4.17) are taken at $t = t_r$. The electric field consists of two parts of different type. The first term ($\sim \mathbf{u}$) is called the velocity field and the second ($\sim \mathbf{a}$) is called the acceleration or the radiation field. The first term varies at large distances like $1/\varrho^2$. Since the first term is independent of the acceleration it corresponds to the field produced by a uniformly moving charge.

The second term in Eq. (9.4.14) depends on the acceleration, and for large $\varrho$ it varies like $1/\varrho$. It is related to the electromagnetic waves radiated by the particle. It is given by

$$\mathbf{E}_{\text{rad}}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2 \varrho} \frac{\hat{\boldsymbol{\varrho}}\times(\mathbf{u}\times\mathbf{a})}{(\hat{\boldsymbol{\varrho}}\cdot\mathbf{u})^3} = \frac{q}{4\pi\epsilon_0 c^2 \varrho} \frac{\hat{\boldsymbol{\varrho}}\times(\mathbf{u}\times\mathbf{a})}{(1-\hat{\boldsymbol{\varrho}}\cdot\mathbf{v}/c)^3}\ . \tag{9.4.18}$$

The radiation magnetic field,

$$\mathbf{B}_{\text{rad}}(\mathbf{r},t) = \frac{\hat{\boldsymbol{\varrho}}}{c}\times\mathbf{E}_{\text{rad}}(\mathbf{r},t) \tag{9.4.19}$$

also decreases as $1/\varrho$ for large $\varrho$.

The Poynting vector is

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \frac{1}{\mu_0 c}\mathbf{E}\times(\hat{\boldsymbol{\varrho}}\times\mathbf{E}) = \frac{1}{\mu_0 c}[E^2\hat{\boldsymbol{\varrho}} - (\hat{\boldsymbol{\varrho}}\cdot\mathbf{E})\mathbf{E}] \tag{9.4.20}$$

Some of the energy is radiation; another part is just a field energy carried along by the particle as it moves. To calculate the power radiated by the particle at time $t_*$, we draw a large sphere with radius $\varrho = R$, wait for $t - t_* = \frac{R}{c}$, and integrate Poynting vector over the surface. Since the velocity field is $\sim 1/R^2$ the corresponding $P_{\text{rad}}$ is $\sim R^2\frac{1}{R^4} = \frac{1}{R^2}$ so it does not contribute to the radiated power at large $R$. The power due to the acceleration field ($\sim 1/R$) is finite: $P_{\text{rad}} \sim R^2\frac{1}{R^2} = 1$. We get

$$\begin{aligned} \mathbf{E}_{\text{rad}}(\mathbf{r},t) &= \frac{q}{4\pi\epsilon_0 c^2\varrho}\frac{\hat{\boldsymbol{\varrho}}\times(\mathbf{u}\times\mathbf{a})}{(\hat{\boldsymbol{\varrho}}\cdot\mathbf{u})^3} \\ &\to\ \boldsymbol{\varrho}\cdot\mathbf{E}_{\text{rad}}(\mathbf{r},t) = 0 \ \Rightarrow\ \mathbf{S}_{\text{rad}} = \frac{\hat{\boldsymbol{\varrho}}}{\mu_0 c}E^2_{\text{rad}}. \end{aligned} \tag{9.4.21}$$

For simplicity, consider the charge which is instantaneously at rest at $t = t_*$. Since $\mathbf{v}(t_*) = 0$, $\mathbf{u}(t_*) = \hat{\boldsymbol{\varrho}}$ so the Eq. (9.4.21) reduces to

$$\mathbf{S}_{\rm rad} = \frac{\hat{\boldsymbol{\varrho}}}{\mu_0 c}\left(\frac{\mu_0 q}{4\pi R}\right)^2 [a^2 - (\hat{\boldsymbol{\varrho}}\cdot\mathbf{a})^2] = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\frac{\sin^2\theta}{R^2}\hat{\boldsymbol{\varrho}} \tag{9.4.22}$$

The total power is given by the following Larmor formula

$$P_{\rm rad} = \oint_S \mathbf{S}_{\rm rad}\cdot\mathbf{dS} = \frac{\mu_0 q^2 a^2}{16\pi^2 c R^2}\int \frac{\sin^2\theta}{R^2} R^2 \sin\theta d\theta d\Phi \;\; = \frac{\mu_0 q^2 a^2}{6\pi c} \tag{9.4.23}$$

which we have already obtained using the electric dipole radiation, see Eq. (9.3.10).
We have derived the Larmor formula under the assumption that $v = 0$ but one can demonstrate that it holds true as long as $v \ll c$. In the general case of arbitrary velocity, the radiation is given by the Lienard formula

$$P_{\rm rad} = \frac{\mu_0 q^2\gamma^6}{6\pi c}\left(a^2 - \frac{(\mathbf{v}\cdot\mathbf{a})^2}{c^2}\right) \tag{9.4.24}$$

where $\gamma \equiv 1/\sqrt{1-\frac{v^2}{c^2}}$.

### 9.4.4 Electromagnetic fields due to a point charge moving with constant velocity.

#### Potentials

For a point charge moving with constant velocity $\mathbf{v}$ the trajectory is $\mathbf{w} = t\mathbf{v}$, so that $\boldsymbol{\varrho}(t) \equiv \mathbf{r} - \mathbf{w}(t)$ is given by $\boldsymbol{\varrho}(t) = \mathbf{r} - t\mathbf{v}$. Hence, $\boldsymbol{\varrho}(t_r) = \mathbf{r} - t_r\mathbf{v}$. Recalling that we have also $\varrho(t_r) = c(t - t_r)$, we see that, in this case, the difference

$$\boldsymbol{\varrho}(t_r) - \frac{\mathbf{v}}{c}\,\varrho(t_r) = [\mathbf{r} - t_r\mathbf{v}] - (t - t_r)\,\mathbf{v} = \mathbf{r} - t\mathbf{v} = \boldsymbol{\varrho}(t)$$

is the distance $\boldsymbol{\varrho}(t)$ from the charge to the point of observation at precisely the moment $t$ of observation. The retarded time may be also calculated explicitly:

$$c(t - t_r) = |\mathbf{r} - t_r\mathbf{v}| \quad\Rightarrow\quad c^2(t^2 - 2tt_r + t_r^2) = r^2 - 2t_r\mathbf{v}\cdot\mathbf{r} + v^2 t_r^2$$

Thus, we have a quadratic equation for $t_r$:

$$(c^2 - v^2)t_r^2 - 2t_r(c^2 - \mathbf{v}\cdot\mathbf{r}) + c^2t^2 - r^2 = 0\ .$$

Its solution (with $t_r \le t$) is given by

$$\Rightarrow\;\; t_r = \frac{c^2 t - \mathbf{v}\cdot\mathbf{r} - \sqrt{(c^2 t - \mathbf{v}\cdot\mathbf{r})^2 - (c^2 - v^2)(c^2t^2 - r^2)}}{c^2 - v^2}\ .$$

This gives

$$c^2t - \mathbf{v}\cdot\mathbf{r} - (c^2 - v^2)t_r = \sqrt{(c^2t - \mathbf{v}\cdot\mathbf{r})^2 - (c^2 - v^2)(c^2t^2 - r^2)}\ .$$

The Liénard-Wiechert potentials (9.4.13) then take the form

$$\begin{aligned}\Phi(\mathbf{r},t) &= \frac{qc}{4\pi\epsilon_0}\,\frac{1}{c|\mathbf{r} - t_r\mathbf{v}| - \mathbf{v}\cdot(\mathbf{r} - t_r\mathbf{v})} \\ &= \frac{qc}{4\pi\epsilon_0[c^2t - (c^2 - v^2)t_r - \mathbf{v}\cdot\mathbf{r}]} = \frac{qc}{4\pi\epsilon_0}\Big[(c^2t - \mathbf{v}\cdot\mathbf{r})^2 - (c^2 - v^2)(c^2t^2 - r^2)\Big]^{-1/2}\end{aligned} \tag{9.4.25}$$

and

$$\mathbf{A}(\mathbf{r},t) = \frac{\mathbf{v}}{c^2}\Phi(\mathbf{r},t). \tag{9.4.26}$$

Let us demonstrate that $\Phi(\mathbf{r},t)$ can be rewritten as

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 R}\Big(1 - \frac{v^2}{c^2}\sin^2\theta\Big)^{-1/2}\ , \tag{9.4.27}$$

where $\mathbf{R} = \mathbf{r} - t\mathbf{v}$ and $\theta$ is the angle between $\mathbf{R}$ and $\mathbf{v}$. As we have seen, for a constant velocity $R$ (denoted as $\varrho$ in general case) is the distance to the position of the moving charge at the time of measurement of the fields.
We have

$$\begin{aligned}&(c^2t - \mathbf{v}\cdot\mathbf{r})^2 - (c^2 - v^2)(c^2t^2 - r^2) = [c^2t - \mathbf{v}\cdot(\mathbf{R} + t\mathbf{v})]^2 - (c^2 - v^2)[c^2t^2 - (\mathbf{R} + t\mathbf{v})^2] \\ &= [(c^2 - v^2)t - \mathbf{v}\cdot\mathbf{R}]^2 - (c^2 - v^2)[(c^2 - v^2)t^2 - 2t\mathbf{v}\cdot\mathbf{R} - R^2] = (c^2 - v^2)R^2 + (\mathbf{v}\cdot\mathbf{R})^2 \\ &= c^2R^2 - v^2R^2\sin^2\theta\ ,\end{aligned} \tag{9.4.28}$$

and therefore

$$\sqrt{(c^2t - \mathbf{v}\cdot\mathbf{r})^2 - (c^2 - v^2)(c^2t^2 - r^2)} = Rc\sqrt{1 - \frac{v^2}{c^2}\sin^2\theta}\ . \tag{9.4.29}$$

For the vector potential, we have

$$\mathbf{A}(\mathbf{r},t) = \frac{\mathbf{v}}{c^2}\Phi(\mathbf{r},t) = \frac{q\mu_0}{4\pi R}\,\mathbf{v}\,\Big(1 - \frac{v^2}{c^2}\sin^2\theta\Big)^{-1/2}. \tag{9.4.30}$$

**Fields**

Since the potentials in this case are given by explicit functions of $t$ and $\mathbf{r}$, the calculation of

$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

is straightforward. We need

$$-\frac{1}{c^2}\frac{\partial}{\partial t}\Big[(c^2t-\mathbf{v}\cdot\mathbf{r})^2-(c^2-v^2)(c^2t^2-r^2)\Big]^{-1/2}=\frac{c^2t-\mathbf{v}\cdot\mathbf{r}-(c^2-v^2)t}{[(c^2t-\mathbf{v}\cdot\mathbf{r})^2-(c^2-v^2)(c^2t^2-r^2)]^{3/2}}$$
$$=\frac{-\mathbf{v}\cdot\mathbf{r}+v^2t}{[(c^2t-\mathbf{v}\cdot\mathbf{r})^2-(c^2-v^2)(c^2t^2-r^2)]^{3/2}}$$

and

$$-\nabla\Big[(c^2t-\mathbf{v}\cdot\mathbf{r})^2-(c^2-v^2)(c^2t^2-r^2)\Big]^{-1/2}=\frac{-(c^2t-\mathbf{v}\cdot\mathbf{r})\mathbf{v}+(c^2-v^2)\mathbf{r}}{[(c^2t-\mathbf{v}\cdot\mathbf{r})^2-(c^2-v^2)(c^2t^2-r^2)]^{3/2}}$$

Then

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}=\frac{qc}{4\pi\epsilon_0}\frac{-(c^2t-\mathbf{v}\cdot\mathbf{r})\mathbf{v}+(c^2-v^2)\mathbf{r}+\mathbf{v}(-\mathbf{v}\cdot\mathbf{r}+v^2t)}{[(c^2t-\mathbf{v}\cdot\mathbf{r})^2-(c^2-v^2)(c^2t^2-r^2)]^{3/2}}$$
$$=\frac{qc}{4\pi\epsilon_0}\frac{(c^2-v^2)(\mathbf{r}-t\mathbf{v})}{[(c^2t-\mathbf{v}\cdot\mathbf{r})^2-(c^2-v^2)(c^2t^2-r^2)]^{3/2}}=\frac{qc(c^2-v^2)}{4\pi\epsilon_0}\ \frac{\mathbf{R}}{(R^2c^2-R^2v^2\sin^2\theta)^{3/2}}$$

or

$$\mathbf{E}=\frac{q\mathbf{R}}{4\pi\epsilon_0R^3}\ \frac{1-v^2/c^2}{\Big(1-v^2\sin^2\theta/c^2\Big)^{3/2}}\ . \tag{9.4.31}$$

For magnetic field, we get

$$\mathbf{B}=-\frac{q}{4\pi\epsilon_0c}\frac{(c^2-v^2)(\mathbf{r}\times\mathbf{v})}{[(c^2t-\mathbf{v}\cdot\mathbf{r})^2-(c^2-v^2)(c^2t^2-r^2)]^{3/2}}=\frac{q(1-v^2/c^2)}{4\pi\epsilon_0R^3c^2}\ \frac{\mathbf{v}\times\mathbf{R}}{(1-v^2\sin^2\theta/c^2)^{3/2}}$$
$$=\frac{1}{c^2}\mathbf{v}\times\mathbf{E}(\mathbf{r},t)\ . \tag{9.4.32}$$

It can be demonstrated that the fields (9.4.31) , (9.4.32) are Lorentz transforms of the usual Coulomb field of a point charge ($\mathbf{E}(\mathbf{r},t)=q\mathbf{R}/4\pi\epsilon_0R^3$, $\mathbf{B}=0$).

## 9.5 Magnetic Dipole Radiation

Let us now return to our previous approach in which we assumed that the source currents in

$$\mathbf{A}(\mathbf{x},t)=\frac{\mu_0}{4\pi}\int d^3x'\,\frac{\mathbf{J}(\mathbf{x}',t_r)}{|\mathbf{x}-\mathbf{x}'|}\ , \tag{9.5.1}$$

have an oscillating dependence on time, i.e., take

$$\mathbf{J}(\mathbf{x}',t_r)=\mathbf{J}(\mathbf{x}')e^{-i\omega t_r}=\mathbf{J}(\mathbf{x}')e^{-i\omega(t-|\mathbf{x}-\mathbf{x}'|/c)}=\mathbf{J}(\mathbf{x}')e^{-i\omega t}e^{ik|\mathbf{x}-\mathbf{x}'|}\ .$$

We see that $\mathbf{A}(\mathbf{x},t)$ has now $e^{-i\omega t}$ dependence on time, $\mathbf{A}(\mathbf{x},t) = \mathbf{A}(\mathbf{x})e^{-i\omega t}$, with $\mathbf{A}(\mathbf{x})$ given by

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\exp[ik\,|\mathbf{x}-\mathbf{x}'|] \; . \tag{9.5.2}$$

Using $|\mathbf{x}-\mathbf{x}'| = r-(\hat{\mathbf{n}}\cdot\mathbf{x}')+\ldots$ we obtain that the next, linear in $\mathbf{n}\cdot\mathbf{x}'$, term in the multipole expansion of

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x' \, \mathbf{J}(\mathbf{x}')\Big(1+\frac{\hat{n}\cdot\mathbf{x}'}{r}+...\Big)[1-ik\,\mathbf{n}\cdot\mathbf{x}'+\ldots] \tag{9.5.3}$$

is

$$\mathbf{A}^{\text{next}}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\frac{1}{r}-ik\right)\int d^3x' \, \mathbf{J}(\mathbf{x}')(\mathbf{n}\cdot\mathbf{x}') \; , \tag{9.5.4}$$

where the $\mathcal{O}(1/r^2)$ term is kept to ensure the expansion is valid at all distances. To exhibit the form of this potential, we express the integrand as pieces symmetric and anti-symmetric in $\mathbf{J}$ and $\mathbf{x}'$, by writing

$$\begin{aligned}(\mathbf{n}\cdot\mathbf{x}')\mathbf{J} =& \frac{1}{2}[(\mathbf{n}\cdot\mathbf{x}')\mathbf{J}+(\mathbf{n}\cdot\mathbf{J})\mathbf{x}'] + \frac{1}{2}[(\mathbf{n}\cdot\mathbf{x}')\mathbf{J}-(\mathbf{n}\cdot\mathbf{J})\mathbf{x}'] \\ =& \frac{1}{2}[(\mathbf{n}\cdot\mathbf{x}')\mathbf{J}+(\mathbf{n}\cdot\mathbf{J})\mathbf{x}'] - \frac{1}{2}\mathbf{n}\times(\mathbf{x}'\times\mathbf{J})\end{aligned} \tag{9.5.5}$$

We now introduce the magnetization density

$$\mathbf{M} = \frac{1}{2}\mathbf{x}'\times\mathbf{J}. \tag{9.5.6}$$

Then the second term gives rise to a vector potential

$$\mathbf{A}^{\text{mag.dip.}}(\mathbf{x}) = \frac{ik\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\mathbf{n}\times\mathbf{m} = \nabla\times\left(\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\mathbf{m}\right), \tag{9.5.7}$$

where $\mathbf{m}$ is the **magnetic dipole moment**.

Let us find now electric and magnetic fields of the magnetic dipole radiation. Taking the curl of Eq. (9.5.7), we find

$$\begin{aligned}\mathbf{H}^{\text{mag.dip.}} =& \frac{1}{\mu_0}\nabla\times\mathbf{A}^{\text{mag.dip.}} = \frac{ik}{4\pi}\nabla\times(\mathbf{n}\times\mathbf{m})\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right) \\ =& \; \frac{1}{4\pi}\left\{k^2(\mathbf{n}\times\mathbf{m})\times\mathbf{n}\frac{e^{ikr}}{r}+[3\mathbf{n}(\mathbf{n}\cdot\mathbf{m})-\mathbf{m}]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\}.\end{aligned} \tag{9.5.8}$$

This outcome may be easily understood if we recall that in the electric dipole case we had

$$\mathbf{H}^{\text{el.dip.}} = \frac{ck^2}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{e^{ikr}}{r}\Big(1-\frac{1}{ikr}\Big) \tag{9.5.9}$$

for the magnetic field, with the electric field obtained from it by taking the curl

$$\mathbf{E}^{\text{el.dip.}} = \frac{i}{ck\epsilon_0}\nabla\times\mathbf{H}^{\text{el.dip.}} = \frac{ik}{4\pi\epsilon_0}\nabla\times(\mathbf{n}\times\mathbf{p})\frac{e^{ikr}}{r}\Big(1-\frac{1}{ikr}\Big)\ , \tag{9.5.10}$$

and the result (see Eq. (9.2.13)) was

$$\mathbf{E}^{\text{el.dip.}} = \frac{1}{4\pi\epsilon_0}\Big[k^2(\mathbf{n}\times\mathbf{p})\times\mathbf{n}\frac{e^{ikr}}{r} + (3(\mathbf{n}\cdot\mathbf{p})\mathbf{n}-\mathbf{p})\Big(\frac{1}{r^3}-\frac{ik}{r^2}\Big)e^{ikr}\Big]\ . \tag{9.5.11}$$

Thus, the field $\mathbf{H}^{\text{mag.dip.}}$ due to the *magnetic dipole* is of the same form as the field $\mathbf{E}^{\text{el.dip.}}$ due to the *electric* dipole.

$$H^{\text{mag.dip.}} = \frac{\epsilon_0 m}{p}E^{\text{el.dip.}} \quad \text{or} \quad H^{\text{mag.dip.}} = \frac{m}{p}D^{\text{el.dip.}}\ . \tag{9.5.12}$$

Similarly we have

$$\begin{aligned}\mathbf{E}^{\text{mag.dip.}} =& \frac{i}{ck\epsilon_0}\nabla\times\mathbf{H}^{\text{mag.dip.}} = \frac{i}{ck\epsilon_0\mu_0}\nabla\times(\nabla\times\mathbf{A}^{\text{mag.dip.}})\\ =& \frac{ic}{k}\left[\nabla(\nabla\cdot\mathbf{A}^{\text{mag.dip.}})-\nabla^2\mathbf{A}^{\text{mag.dip.}}\right]\ .\end{aligned} \tag{9.5.13}$$

Noticing that

$$\mathbf{A}^{\text{mag.dip.}}(\mathbf{x}) = \nabla\times\left(\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\mathbf{m}\right)\ , \tag{9.5.14}$$

we conclude that $\nabla\cdot\mathbf{A}^{\text{mag.dip.}} = 0$, i.e. $\mathbf{A}^{\text{mag.dip.}}$ satisfies the Coulomb gauge condition. Hence, it also satisfies the Helmholtz wave equation $\nabla^2\mathbf{A}^{\text{mag.dip.}} = -k^2\mathbf{A}^{\text{mag.dip.}}$, which may also be checked directly:

$$\begin{aligned}\nabla^2\mathbf{A}^{\text{mag.dip.}}(\mathbf{x}) =& \nabla^2\nabla\times\left(\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\mathbf{m}\right) = \nabla\times\nabla^2\left(\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\mathbf{m}\right)\\ =& \nabla\times(-k^2)\left(\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\mathbf{m}\right) = -k^2\mathbf{A}^{\text{mag.dip.}}\end{aligned} \tag{9.5.15}$$

(recall that $\nabla^2 f(r) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf(r))$. Using it, we obtain

$$\mathbf{E}^{\text{mag.dip.}} = ick\mathbf{A}^{\text{mag.dip.}} = -\frac{\mu_0}{4\pi}ck^2(\mathbf{n}\times\mathbf{m})\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right), \tag{9.5.16}$$

or

$$\mathbf{E}^{\text{mag.dip.}} = -\frac{Z_0}{4\pi}k^2(\mathbf{n}\times\mathbf{m})\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right), \tag{9.5.17}$$

so that the *electric field* due to a *magnetic dipole* is of the same form as the *magnetic field* due to an *electric dipole*:

$$E^{\text{mag.dip.}} = -\frac{\mu_0 m}{p}H^{\text{el.dip.}} \quad \text{or} \quad E^{\text{mag.dip.}} = -\frac{m}{p}B^{\text{el.dip.}}\ . \tag{9.5.18}$$

Since the radiated power is proportional to $\mathbf{n} \cdot (\mathbf{E} \times \mathbf{H})$,

$$P_{\rm rad}^{\rm mag.dip.} = \frac{m^2}{p^2c^2} P_{\rm rad}^{\rm el.dip.} = \frac{\mu_0 m^2 \omega^4}{12\pi c^3} \ . \tag{9.5.19}$$

In order to get an estimate of the relative strength of the electric and magnetic dipole radiation, consider a physical dipole $p = qd$ made from two charges $q$ and $-q$ separated by distance $d$ which rotate with angular velocity $\omega$ around the center of the dipole. The magnetic moment of this system can be approximated by an oscillating current $I = \frac{q}{T} = \frac{q\omega}{2\pi}$ so we get an oscillating magnetic moment $m = qd^2\omega/8$. The ratio of powers for this example is

$$\frac{P_{\rm mag}}{P_{\rm el}} = \frac{\omega^2 d^2}{64c^2} \sim \frac{v^2}{c^2} \tag{9.5.20}$$

where $v$ is the linear velocity of the rotating charges. We see that for charges moving with non-relativistic velocities the electric dipole radiation is the most important part while the magnetic dipole radiation is of the size of the relativistic corrections.

As an example of magnetic dipole radiation, consider the circular loop of radius $b$ with current

$$I(t) = I\cos\omega t = {\rm Re}\, I e^{-i\omega t}$$

in the $XY$ plane. The magnetic dipole moment of this loop oscillates in time as

$$m(t) = m\cos\omega t = {\rm Re}\, \pi b^2 I e^{-i\omega t} \ .$$

Let us calculate the magnetic vector potential due to this setup. W.l.o.g. we can assume that the point $\mathbf{x}$ lies in the $XZ$ plane. The general formula for the magnetic vector potential has the form

$$\mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \oint dl' \frac{e^{-i\omega t_{r'}}}{|\mathbf{x} - \mathbf{x}'|} I \hat{e}_{\varphi'}$$

Expanding $t_{r'} \simeq t - \frac{r}{c} + \frac{\mathbf{n}\cdot\mathbf{X}'}{c}$ and $\frac{1}{|\mathbf{X}-\mathbf{X}'|} \simeq \frac{1}{r}\Big(1 + \frac{\hat{n}\cdot\mathbf{X}'}{r}\Big)$ we get

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0 b I}{4\pi} \frac{e^{ikr}}{r} \int_0^{2\pi} d\varphi' (-\hat{e}_1 \sin\varphi' + \hat{e}_2 \cos\varphi') \Big(1 + \frac{b}{r} \sin\theta \cos\varphi'\Big) e^{-ikb\sin\theta\cos\varphi'}$$

Since $kb = 2\pi\frac{b}{\lambda} \ll 1$ we can expand the exponential in the r.h.s. of this equation and get

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0 b I}{4\pi} \frac{e^{ikr}}{r} \int_0^{2\pi} d\varphi' (-\hat{e}_1 \sin\varphi' + \hat{e}_2 \cos\varphi') \Big(1 + \frac{b}{r} \sin\theta \cos\varphi' - ikb \sin\theta\cos\varphi'\Big)$$

Performing integration over $\varphi'$ we obtain

$$\mathbf{A}(\mathbf{x}) = -\frac{ik\mu_0 I b^2}{4r} \hat{e}_2 \Big(1 - \frac{1}{ikr}\Big) e^{ikr} \sin\theta$$

For our setup $\hat{e}_2 = \hat{e}_\varphi$ so the final result for the vector potential takes the form

$$\mathbf{A}(\mathbf{x}) = -\frac{ik\mu_0\hat{m}}{4\pi r}\hat{e}_\varphi\Big(1 - \frac{1}{ikr}\Big)e^{ikr}\sin\theta$$

which coincides with Eq. (9.5.7).

## 9.6 Electric Quadrupole Radiation

The first term of the decomposition in Eq. (9.5.5), obtained from the **symmetric part**, is related to the **quadrupole moment**.

$$\mathbf{A}^{\text{el. quadr.}}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\frac{1}{r} - ik\right)\int d^3x'\,\frac{1}{2}\Big[(\mathbf{n}\cdot\mathbf{x}')\mathbf{J} + (\mathbf{n}\cdot\mathbf{J})\mathbf{x}'\Big]\ , \tag{9.6.1}$$

where the $\mathcal{O}(1/r^2)$ term is kept to ensure the expansion is valid at all distances.
Recall that for the lowest term we had

$$\int d^3x'\,\mathbf{J} = -i\omega\int d^3x'\,\mathbf{x}'\rho(\mathbf{x}')\ .$$

Let us show that now one can use

$$\frac{1}{2}\int d^3x'\ \{(\mathbf{n}\cdot\mathbf{x}')\mathbf{J} + (\mathbf{n}\cdot\mathbf{J})\mathbf{x}'\} = -\frac{i\omega}{2}\int d^3x'\,\rho\,\mathbf{x}'(\mathbf{n}\cdot\mathbf{x}'), \tag{9.6.2}$$

and write

$$\mathbf{A}^{\text{quad.mom.}}(\mathbf{x}) = -\frac{\mu_0 ck^2}{8\pi}\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)\int d^3x'\rho(\mathbf{x}')\mathbf{x}'(\mathbf{n}\cdot\mathbf{x}')\ . \tag{9.6.3}$$

Indeed, using $\mathbf{J} = (\mathbf{J}\cdot\nabla')\mathbf{x}'$, we have

$$\begin{aligned}
\int d^3x'\ \{(\mathbf{n}\cdot\mathbf{x}')\mathbf{J} + \mathbf{x}'(\mathbf{J}\cdot\mathbf{n})\} &= \int d^3x'\ \{(\mathbf{n}\cdot\mathbf{x}')(\mathbf{J}\cdot\nabla')\mathbf{x}' + \mathbf{x}'(\mathbf{J}\cdot\nabla')(\mathbf{x}'\cdot\mathbf{n})\}\Big|_{\text{by parts}} \\
&= -\int d^3x'\,\{\mathbf{x}'(\nabla'\cdot\mathbf{J})(\mathbf{n}\cdot\mathbf{x}') + (\mathbf{x}'\cdot\mathbf{n})(\nabla'\cdot\mathbf{J})\mathbf{x}'\} \\
&= -\int d^3x'\,\{\mathbf{x}'(\mathbf{n}\cdot\mathbf{x}')(\nabla'\cdot\mathbf{J}) + \mathbf{x}'(\mathbf{n}\cdot\mathbf{J}) + (\mathbf{x}'\cdot\mathbf{n})\mathbf{x}'(\nabla'\cdot\mathbf{J}) + (\mathbf{x}'\cdot\mathbf{n})\mathbf{J}\} \\
&= -2\int d^3x'\mathbf{x}'(\mathbf{n}\cdot\mathbf{x}')(\nabla'\cdot\mathbf{J}) - \int d^3x'\,\{\mathbf{x}'(\mathbf{n}\cdot\mathbf{J}) + (\mathbf{x}'\cdot\mathbf{n})\mathbf{J}\}\ .
\end{aligned} \tag{9.6.4}$$

The second integral here coincides with (minus) original expression. Hence,

$$\int d^3x'\ \{(\mathbf{n}\cdot\mathbf{x}')\mathbf{J} + \mathbf{x}'(\mathbf{J}\cdot\mathbf{n})\} = -\int d^3x'\mathbf{x}'(\mathbf{n}\cdot\mathbf{x}')(\nabla'\cdot\mathbf{J})\ . \tag{9.6.5}$$

Using the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \ , \tag{9.6.6}$$

which with our assumed time dependence becomes

$$-i\omega\rho + \nabla \cdot \mathbf{J} = 0 \ , \tag{9.6.7}$$

we get the desired relation

$$\frac{1}{2}\int d^3x' \ \{(\mathbf{n}\cdot\mathbf{x}')\mathbf{J} + \mathbf{x}'(\mathbf{J}\cdot\mathbf{n})\} = -\frac{i\omega}{2}\int d^3x' \, \mathbf{x}'(\mathbf{n}\cdot\mathbf{x}')\rho(\mathbf{x}') \ . \tag{9.6.8}$$

In components, the integral can be written as

$$\int d^3x' \, \rho(\mathbf{x}') \, x'_\alpha \sum_\beta n_\beta x'_\beta = \sum_\beta n_\beta \int d^3x' \, \rho(\mathbf{x}') \, x'_\alpha x'_\beta \ . \tag{9.6.9}$$

If we now recall our expression for the quadrupole moment

$$Q_{\alpha\beta} = \int d^3x' \rho(\mathbf{x}')(3x'_\alpha x'_\beta - r'^2\delta_{\alpha\beta}) \ , \tag{9.6.10}$$

we see that

$$\sum_\beta n_\beta \int d^3x' \, \rho(\mathbf{x}') \, x'_\alpha x'_\beta = \frac{1}{3}\sum_\beta n_\beta Q_{\alpha\beta} + \frac{n_\alpha}{3}\int d^3x' \, r'^2 \, \rho(\mathbf{x}') \tag{9.6.11}$$

or

$$\int d^3x' \, \mathbf{x}'(\mathbf{n}\cdot\mathbf{x}')\rho(\mathbf{x}') = \frac{1}{3}\mathbf{Q}(\mathbf{n}) + \frac{\mathbf{n}}{3}\int d^3x' \, r'^2 \, \rho(\mathbf{x}') \ \equiv \frac{1}{3}\mathbf{Q}(\mathbf{n}) + q\,\frac{\mathbf{n}}{3}\,\langle r'^2\rangle \tag{9.6.12}$$

where $\mathbf{Q}(\mathbf{n})$ is defined by

$$Q_\alpha = \sum_\beta Q_{\alpha\beta} n_\beta. \tag{9.6.13}$$

The next step is to use the expression for $\mathbf{A}$

$$\mathbf{A}^{\text{quad.mom.}}(\mathbf{x}) = -\frac{\mu_0 ck^2}{8\pi}\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)\underbrace{\int d^3x'\rho(\mathbf{x}')\mathbf{x}'(\mathbf{n}\cdot\mathbf{x}')}_{\mathbf{Q}(\mathbf{n})/3 + q\,\mathbf{n}\,\langle r'^2\rangle/3} \ . \tag{9.6.14}$$

to find the fields in the $r \gg \lambda$ limit,

$$\begin{aligned} \mathbf{H} &= ik\mathbf{n}\times\mathbf{A}/\mu_0 \\ \mathbf{E} &= ikZ_0(\mathbf{n}\times\mathbf{A})\times\mathbf{n}/\mu_0. \end{aligned} \tag{9.6.15}$$

We find that fields can be written as

$$\mathbf{H} = -\frac{ick^3}{24\pi}\frac{e^{ikr}}{r}\mathbf{n}\times\mathbf{Q}(\mathbf{n}) \ , \quad \mathbf{E} = -Z_0\frac{ick^3}{24\pi}\frac{e^{ikr}}{r}[\mathbf{n}\times\mathbf{Q}(\mathbf{n})]\times\mathbf{n} \tag{9.6.16}$$

and the power dissipation is

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{1152\pi^2}k^6|[\mathbf{n}\times\mathbf{Q}(\mathbf{n})]\times\mathbf{n}|^2 \tag{9.6.17}$$

(where $1152 = 2(24)^2$).
A simple model of a quadrupole moment is given by

$$\begin{aligned} Q_{33} &= Q_0 \\ Q_{11} = Q_{22} &= -\frac{1}{2}Q_0, \end{aligned} \tag{9.6.18}$$

which is clearly traceless. Then

$$\mathbf{Q}(\mathbf{n}) = -\frac{1}{2}\,Q_0 n_1\hat{\mathbf{e}}_1 - \frac{1}{2}\,Q_0 n_2\hat{\mathbf{e}}_2 + Q_0 n_3\hat{\mathbf{e}}_3 = -\frac{1}{2}\,Q_0\mathbf{n} + \frac{3}{2}\,Q_0\,n_3\hat{\mathbf{e}}_3 \ , \tag{9.6.19}$$

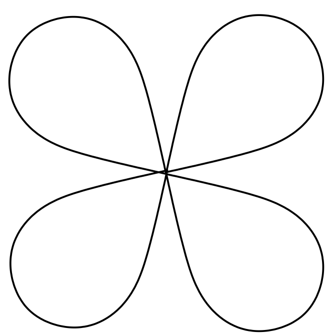
and

$$\mathbf{n}\times\mathbf{Q}(\mathbf{n}) = \frac{3}{2}\,Q_0\,n_3(\mathbf{n}\times\hat{\mathbf{e}}_3) = \frac{3}{2}\,Q_0\,\cos\theta(\mathbf{n}\times\hat{\mathbf{e}}_3) = -\frac{3}{2}\,Q_0\,\cos\theta\,\sin\theta\,\hat{\mathbf{e}}_\varphi \ . \tag{9.6.20}$$

Since $\hat{\mathbf{e}}_\varphi$ is a unit vector orthogonal to a unit vector $\mathbf{n}$, the product $\hat{\mathbf{e}}_\varphi\times\mathbf{n}$ is also a unit vector, i.e., $|\mathbf{e}_\varphi\times\mathbf{n}| = 1$, and the angular power distribution is given by

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^6}{512\pi^2}Q_0^2\sin^2\theta\cos^2\theta \ . \tag{9.6.21}$$

Thus, for quadrupole radiation, we have a four-lobe pattern of power distribution



Using

$$\int_{-1}^{1}\cos^2\theta\,\sin^2\theta\,d(\cos\theta) = \int_{-1}^{1}x^2(1-x^2)\,dx = \frac{4}{15} \ , \tag{9.6.22}$$

we find that the total power radiated is

$$P = \frac{c^2 Z_0 k^6 Q_0^2}{960\pi}. \tag{9.6.23}$$

The complete description requires the full **multipole expansion** which is beyond what we are going to do in this course.