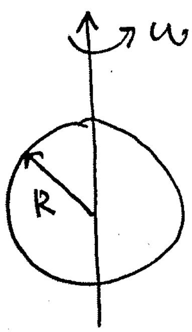
HW 3 solution



Fields:

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}\theta(r-R) = \frac{e}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}\theta(r-R)$$

$$\vec{B}_{inside} = \frac{2}{3}\mu_0 \sigma R \vec{\omega}$$

$B_{outside} \simeq$ mag. field of a pure dipole

From Griffiths: $$\vec{A}(r,\theta,\varphi) = \frac{\mu_0 R^4}{3}\omega\sigma \frac{\sin\theta}{r^2}\hat{\varphi} =$$

$$= \frac{\mu_0}{4\pi}\underbrace{\left(\frac{4\pi}{3}R^4\omega\sigma\right)}_{m}\frac{\sin\theta}{r^2}\hat{\varphi}$$

$$\Rightarrow \vec{B}(r,\theta,\varphi) = \frac{\mu_0}{4\pi}\left(\frac{4\pi}{3}R^4\omega\sigma\right)\frac{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}}{r^3} \qquad \sigma = \frac{e}{4\pi R^2}$$

$$\Rightarrow \vec{B} = \frac{\mu_0 R^4}{3}\omega\sigma\frac{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}}{r^3}\theta(r-R) + \frac{2}{3}\mu_0\sigma R\omega\,\hat{e}_3\,\theta(R-r)$$

Energy:

$$u = \frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2 = \frac{e^2\theta(r-R)}{32\pi^2\epsilon_0 r^4}\left\{1 + \mu_0\epsilon_0\frac{\omega^2 R^4}{9r^2}(1+3\cos^2\theta)\right\}$$

$$+ \frac{\mu_0}{72\pi^2}\frac{e^2\omega^2}{R^2}\theta(R-r)$$

$$\int d^3x\, u =$$

$$= \frac{4}{3}\pi R^3\frac{\mu_0}{72\pi^2}\frac{e^2\omega^2}{R^2} + 2\pi\int_R^\infty r^2 dr\int_0^\pi d\theta\,\frac{e^2\sin\theta}{32\pi^2\epsilon_0}\left[\frac{1}{r^4} + \frac{\mu_0\epsilon_0\omega^2 R^4}{9r^6}(1+3\cos^2\theta)\right.$$

$$= \frac{\mu_0}{54\pi}e^2\omega^2 R + \frac{e^2}{8\pi\epsilon_0}\left(\frac{1}{R} + \frac{2}{27}\mu_0\epsilon_0\omega^2 R\right)$$

$$= \frac{e^2}{8\pi\epsilon_0 R}\left(1 + \left[\frac{8}{54} + \frac{2}{27}\right]\mu_0\epsilon_0\omega^2 R^2\right) = W_{e.m.} = \frac{e^2}{8\pi\epsilon_0 R}\left(1 + \frac{2}{9}\frac{\omega^2 R^2}{c^2}\right)$$

Angular momentum

$$\vec{\ell} = \epsilon_0\, \vec{r} \times (\vec{E} \times \vec{B})$$

$$\vec{E} \times \vec{B} = \frac{e^2 \mu_0 R^2 \omega}{48\pi^2 \epsilon_0 r^5}\, \hat{r} \times (\cancel{2\cos\theta\, \hat{r}} + \sin\theta\, \hat{\theta})\, \theta(r-R)$$

$$\epsilon_0 \vec{r} \times (\vec{E} \times \vec{B}) = \frac{e^2 \mu_0 R^2 \omega}{48\pi^2 r^4}\, \hat{r} \times (\hat{r} \times \hat{\theta}) \sin\theta = -\frac{e^2 \mu_0 R^2 \omega}{48\pi^2} \frac{\sin\theta}{r^4}\, \hat{\theta}$$

$$\int d^3x\, \vec{\ell} = -\frac{\mu_0 \omega e^2 R^2}{48\pi^2} \int_R^\infty r^2 dr \int_0^\pi d\theta \sin^2\theta \int_0^{2\pi} d\varphi\, \frac{\hat{\theta}}{r^4}$$

By symmetry $\int d\varphi\, \hat{\theta} \uparrow\uparrow \hat{e}_3 \Rightarrow \quad \hat{\theta}_z = -\sin\theta$

$$\Rightarrow \int d^3x\, \ell_3 = +\frac{\mu_0 \omega e^2 R^2}{48\pi^2} \frac{2\pi}{R} \int_0^\pi d\theta \sin^3\theta = \frac{\mu_0 \omega R e^2}{18\pi} \Rightarrow$$

$$\Rightarrow \vec{L} = \frac{\mu_0 \omega R}{18\pi} e^2\, \hat{z}$$

Classical model?

$$m_e c^2 = \frac{e^2}{8\pi\epsilon_0 R}$$

$$\frac{\mu_0 \omega R}{18\pi} e^2 = \frac{\hbar}{2} \Rightarrow \mu_0 \omega R = \frac{9\pi\hbar}{e^2} \Rightarrow \mu_0 \epsilon_0 \omega R = \frac{9\pi\hbar\epsilon_0}{e^2}$$

$$\Rightarrow \frac{\omega R}{c} = \frac{v}{c} = \frac{9\pi\hbar c\epsilon_0}{e^2} = \frac{9}{4} \frac{1}{e^2/4\pi\hbar c\epsilon_0} = \frac{9}{4} \cdot 137 \gg 1$$

$\frac{e^2}{4\pi\hbar c\epsilon_0}$ = "fine structure constant" $\simeq \frac{1}{137}$