HW2 solution

Problem 1.

From Eq. (9.166)-(9.168) we get

$$\hat{J}_x|+1\rangle = \hat{J}_x|-1\rangle = \frac{\hbar}{\sqrt{2}}|0\rangle, \quad \hat{J}_x|0\rangle = \frac{\hbar}{\sqrt{2}}(|+1\rangle + |-1\rangle) \tag{1}$$

$$\hat{J}_y|+1\rangle = i\frac{\hbar}{\sqrt{2}}|0\rangle, \quad \hat{J}_y|-1\rangle = -i\frac{\hbar}{\sqrt{2}}|0\rangle, \quad \hat{J}_y|0\rangle = i\frac{\hbar}{\sqrt{2}}(|+1\rangle - |-1\rangle)$$

and therefore for $|\psi\rangle = \alpha|+1\rangle + \beta|0\rangle + \gamma|-1\rangle$

$$\begin{aligned}
\hat{J}_x|\psi\rangle &= (\alpha+\gamma)\frac{\hbar}{\sqrt{2}}|0\rangle + \beta\frac{\hbar}{\sqrt{2}}(|+1\rangle + |-1\rangle) \\
\hat{J}_y|\psi\rangle &= i(\alpha-\gamma)\frac{\hbar}{\sqrt{2}}|0\rangle - i\beta\frac{\hbar}{\sqrt{2}}(|+1\rangle - |-1\rangle) \\
\hat{J}_z|\psi\rangle &= \alpha\hbar|+1\rangle - \gamma\hbar|-1\rangle
\end{aligned} \tag{2}$$

so the answer to (a) is

$$\begin{aligned}
\langle\psi|\vec{\hat{J}}|\psi\rangle &= \vec{e}_x\langle\psi|\hat{J}_x|\psi\rangle + \vec{e}_y\langle\psi|\hat{J}_y|\psi\rangle + \vec{e}_z\langle\psi|\hat{J}_z|\psi\rangle \\
&= \sqrt{2}\Re[\vec{e}_x(\alpha+\gamma)\beta^* + i\vec{e}_y(\alpha-\gamma)\beta^*] + \vec{e}_z(|\alpha|^2 - |\gamma|^2)
\end{aligned} \tag{3}$$

and to (b)

$$\begin{aligned}
\langle\psi|\hat{J}_x^2|\psi\rangle &= \hbar^2(|\beta|^2 + \frac{1}{2}|\alpha+\gamma|^2) \\
\langle\psi|\hat{J}_y^2|\psi\rangle &= \hbar^2(|\beta|^2 + \frac{1}{2}|\alpha-\gamma|^2) \\
\langle\psi|\hat{J}_z^2|\psi\rangle &= \hbar^2(|\alpha|^2 + |\gamma|^2)
\end{aligned} \tag{4}$$

Check:

$$\langle\psi|\hat{J}^2|\psi\rangle = \hbar^2(2\beta^2 + \frac{1}{2}|\alpha+\gamma|^2 + \frac{1}{2}|\alpha-\gamma|^2 + |\alpha|^2 + |\gamma|^2) = 2\hbar^2$$

Problem 2.

(a): $|1,+1\rangle, \quad |1,0\rangle, \quad |1,-1\rangle, \quad |0,0\rangle$

(b1):

From Eq. (2) it is clear that the eigenstate of $\hat{J}_x$ (with eigenvalue $\hbar$) is

$$|1,+1_x\rangle \;=\; \frac{1}{2}|1,+1\rangle + \frac{1}{\sqrt{2}}|1,0\rangle + \frac{1}{2}|1,-1\rangle$$

Obviously, it is also an eigenstate of $\hat{J}^2$ with eigenvalue $2\hbar^2$. The probability to measure $\hat{J}^2$ to be $2\hbar^2$ and $\hat{J}_x$ to be $\hbar$ is

$$|\langle 1,+1_x|\psi\rangle|^2 \;=\; |\frac{\alpha+\gamma}{2} + \frac{\beta}{\sqrt{2}}|^2 \tag{5}$$

(b2):

From Eq. (2)

$$\langle\psi|\hat{J}_z|\psi\rangle \;=\; \hbar(|\alpha|^2 - |\gamma|^2)$$

If one measures $f(J_z)$ one gets $f(\hbar)$ with probability $\langle 1,+1|\psi\rangle|^2 = |\alpha|^2$, $f(-\hbar)$ with probability $\langle 1,-1|\psi\rangle|^2 = |\gamma|^2$, and $f(0)$ with probability $\langle 1,0|\psi\rangle|^2 + \langle 0,0|\psi\rangle|^2 = |\beta|^2 + |\delta|^2$. Check: the probabilities sum up to $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$ as required.

(b3.1):

Mean value of J^2 is $\langle\psi|\hat{J}^2|\psi\rangle \;=\; 2\hbar^2(|\alpha|^2 + |\beta^2| + |\gamma|^2)$. If one measures $f(J^2)$ one gets $f(2\hbar^2)$ with probability $|\alpha|^2 + |\beta^2| + |\gamma|^2$

and $f(0)$ with probability$|\delta|^2$.

(b3.2):

From Eq. (3) we see that the mean value of J_x is

$\langle\psi|\hat{J}_x|\psi\rangle \;=\; \sqrt{2}\Re(\alpha+\gamma)\beta^*$.

For the second part of the problem one needs to know the eigenstates of $\hat{J}_x$. From Eq. (2) we get

$$
\begin{aligned}
|1,+1_x\rangle &= \frac{1}{2}|1,+1\rangle + \frac{1}{\sqrt{2}}|1,0\rangle + \frac{1}{2}|1,-1\rangle \\
|1,0_x\rangle &= \frac{1}{\sqrt{2}}|1,+1\rangle - \frac{1}{\sqrt{2}}|1,-1\rangle \\
|1,-1_x\rangle &= \frac{1}{2}|1,+1\rangle - \frac{1}{\sqrt{2}}|1,0\rangle + \frac{1}{2}|1,-1\rangle
\end{aligned} \qquad (6)
$$

So, if one measures $f(J_x)$ one gets $f(\hbar)$ with probability $|\langle 1,+1_x|\psi\rangle|^2 \;=\; |\frac{\alpha+\gamma}{2}+\frac{\beta}{\sqrt{2}}|^2$, $f(-\hbar)$ with probability $|\langle 1,-1_x|\psi\rangle|^2 \;=\; |\frac{\alpha+\gamma}{2}-\frac{\beta}{\sqrt{2}}|^2$, and $f(0)$ with probability $|\langle 1,0_x|\psi\rangle|^2+|\langle 0,0|\psi\rangle|^2 \;=\; \frac{|\alpha-\gamma|^2}{2}+|\delta|^2$.

Check: $|\frac{\alpha+\gamma}{2}+\frac{\beta}{\sqrt{2}}|^2+|\frac{\alpha+\gamma}{2}-\frac{\beta}{\sqrt{2}}|^2+\frac{|\alpha-\gamma|^2}{2}+|\delta|^2 \;=\; 1$.

(b4):

If one measures J_z^2 possible results are $\hbar^2$ with probability $|\langle 1,+1|\psi\rangle|^2+|\langle 1,-1|\psi\rangle|^2 \;=\; |\alpha|^2+|\gamma|^2$ and 0 with probability $|\langle 1,0|\psi\rangle|^2+|\langle 0,0|\psi\rangle|^2 \;=\; |\beta|^2+|\delta|^2$. Again, sum of the probabilities is 1. Also, if we weigh probabilities with values, we get mean of J_x^2, see Eq. (4):

$$\hbar^2(|\alpha|^2+|\gamma|^2) \;=\; \langle\psi|\hat{J}_z^2|\psi\rangle$$