HW2 solution

Problem 1.

Wave function in spherical coordinates has the form

$$\psi(\vec{r}) \;=\; rf(r)(\sin\theta\cos\phi + \sin\theta\sin\phi + 3\cos\theta) \tag{1}$$

and therefore

$$\begin{aligned}
L^2\psi(\vec{r}) \;&=\; -\hbar^2[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}]\psi(\vec{r}) \\
=\;& -\hbar^2 rf(r)[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})](\sin\theta\cos\phi + \sin\theta\sin\phi + 3\cos\theta) \\
=\;& -\hbar^2\frac{rf(r)}{\sin\theta}[-6\sin\theta\cos\theta + (\cos^2\theta - \sin^2\theta)(\cos\phi + \sin\phi)] \\
& + \hbar^2\frac{rf(r)}{\sin\theta}(\cos\phi + \sin\phi) \\
=\;& 2\hbar^2 rf(r)[\sin\theta\cos\phi + \sin\theta\sin\phi + 3\cos\theta] \;=\; 2\hbar^2\phi(\vec{r})
\end{aligned}$$

so the function (1) is an eigenfunction of L^2 operator with $l = 1$

In terms of spherical harmonics

$$x \;=\; \sqrt{\frac{2\pi}{3}}r(Y_{1,-1} - Y_{1,+1}), \quad y \;=\; i\sqrt{\frac{2\pi}{3}}r(Y_{1,-1} + Y_{1,+1}), \quad z \;=\; \sqrt{\frac{2\pi}{3}}rY_{1,0}$$

and therefore

$$\begin{aligned}
\psi(\vec{r}) \;&=\; \sqrt{\frac{2\pi}{3}}rf(r)[3\sqrt{2}Y_{1,0} + (i+1)Y_{1,-1} + (i-1)Y_{1,+1}] \\
\Rightarrow\; |\psi\rangle \;&\sim\; \frac{1}{\sqrt{22}}(3\sqrt{2}|1,0\rangle \;+\; (i-1)|1,1\rangle \;+\; (i+1)|1,-1\rangle)
\end{aligned}$$

The probabilities are

$$
\begin{aligned}
P(m=0) &= |\langle 1,0|\psi\rangle|^2 = \frac{9}{11} \\
P(m=1) &= |\langle 1,1|\psi\rangle|^2 = \frac{1}{11} \\
P(m=-1) &= |\langle 1,-1|\psi\rangle|^2 = \frac{1}{11}
\end{aligned}
\qquad (2)
$$

Problem 2.

For $|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Leftrightarrow \sigma_x|\psi\rangle = \frac{\hbar}{2}|\psi\rangle
$$

$\Rightarrow$

$$
\langle s_z = \frac{\hbar}{2}|s_x = \frac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}} \Rightarrow |\langle s_z = \frac{\hbar}{2}|s_x = \frac{\hbar}{2}\rangle| = \frac{1}{2}
$$

The result of the first measurement is $\frac{1}{2}$, of the second one $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, and the result of the third measurement is obviously 0.