Solution

The projection operators $\hat{P}_0$ and $\hat{P}_1$ can be rewritten as

$$\hat{P}_1 \;=\; \frac{3}{4}-\vec{\hat{s}}_1\cdot\vec{\hat{s}}_2 \;=\; \frac{1}{2}\hat{s}^2, \quad \hat{P}_0 \;=\; 1-\frac{1}{2}\hat{s}^2 \quad \Rightarrow \quad V \;=\; [v_c(r)-\frac{3}{4}v_\sigma(r)]\hat{P}_0+[v_c(r)+\frac{1}{4}v_\sigma(r)]\hat{P}_1$$

Since $\hat{s}^2|1,m\rangle \;=\; 2|1,m\rangle$ and $\hat{s}^2|0,m=0\rangle \;=\; 0$ it is clear that

$$\hat{P}_1|1,m\rangle \;=\; |1,m\rangle, \qquad \hat{P}_1|0,m=0\rangle \;=\; 0$$

and

$$\hat{P}_0|0,m=0\rangle \;=\; |0,m=0\rangle, \qquad \hat{P}_0|1,m\rangle \;=\; 0.$$

The Hamiltonian of "relative particle" is

$$\hat{H} \;=\; \frac{\hat{p}^2}{2\mu}-E+(v_c-\frac{3}{4}v_\sigma)\hat{P}_0+(v_c+\frac{1}{4}v_\sigma)\hat{P}_1 \;=\; \frac{p^2}{2\mu}-E+[v_c(r)-\frac{3}{4}v_\sigma(r)](1-\frac{\hat{s}^2}{2})+[v_c(r)+\frac{1}{4}v_\sigma(r)]\frac{\hat{s}^2}{2}$$

Schrödinger equation:

$$(\hat{H}-E)|\psi_E\rangle \;=\; \lambda_{00}\Big[\frac{\hat{p}^2}{2\mu}-E+[v_c(r)-\frac{3}{4}v_\sigma(r)](1-\frac{\hat{s}^2}{2})+[v_c(r)+\frac{1}{4}v_\sigma(r)]\frac{\hat{s}^2}{2}\Big]|\psi_E^{(0)}\rangle|s=0,m=0\rangle$$
$$+\sum_{m=-1}^{m=1}\lambda_{1m}\Big[\frac{\hat{p}^2}{2\mu}-E+[v_c(r)-\frac{3}{4}v_\sigma(r)](1-\frac{\hat{s}^2}{2})+[v_c(r)+\frac{1}{4}v_\sigma(r)]\frac{\hat{s}^2}{2}\Big]|\psi_E^{(1)}\rangle|s=1,m\rangle \;=\; 0$$

Since $\hat{P}_0|s=1,m\rangle \;=\; \hat{P}_1|s=0,m=0\rangle \;=\; 0$ we get

$$\begin{aligned}(\hat{H}-E)|\psi_E\rangle \;&=\; \lambda_{00}\Big[\frac{\hat{p}^2}{2\mu}-E+[v_c(r)-\frac{3}{4}v_\sigma(r)](1-\frac{\hat{s}^2}{2})\Big]|\psi_E^{(0)}\rangle|s=0,m=0\rangle\\
&\quad+\sum_{m=-1}^{m=1}\lambda_{1m}\Big[\frac{\hat{p}^2}{2\mu}+[v_c(r)+\frac{1}{4}v_\sigma(r)]\frac{\hat{s}^2}{2}\Big]|\psi_E^{(1)}\rangle|s=1,m\rangle\\
&=\; \lambda_{00}\Big[\frac{\hat{p}^2}{2\mu}-E+[v_c(r)-\frac{3}{4}v_\sigma(r)]\Big]|\psi_E^{(0)}\rangle|s=0,m=0\rangle\\
&\quad+\sum_{m=-1}^{m=1}\lambda_{1m}\Big[\frac{\hat{p}^2}{2\mu}+[v_c(r)+\frac{1}{4}v_\sigma(r)]\Big]|\psi_E^{(1)}\rangle|s=1,m\rangle \;=\; 0\end{aligned}$$

Multiplying from the left by the bra states $\langle s=1,m|$ and $\langle s=0,m=0|$ we see that the equations for $\psi_E^{(0)}(r)$ and $\psi_E^{(1)}(r)$ are

$$\Big[-\frac{\hbar^2}{2\mu}\nabla^2+[v_c(r)-\frac{3}{4}v_\sigma(r)]\Big]\psi_E^{(0)}(r) \;=\; E\psi_E^{(0)}(r)$$

and

$$\Big[-\frac{\hbar^2}{2\mu}\nabla^2+[v_c(r)+\frac{1}{4}v_\sigma(r)]\Big]\psi_E^{(1)}(r) \;=\; E\psi_E^{(1)}(r)$$