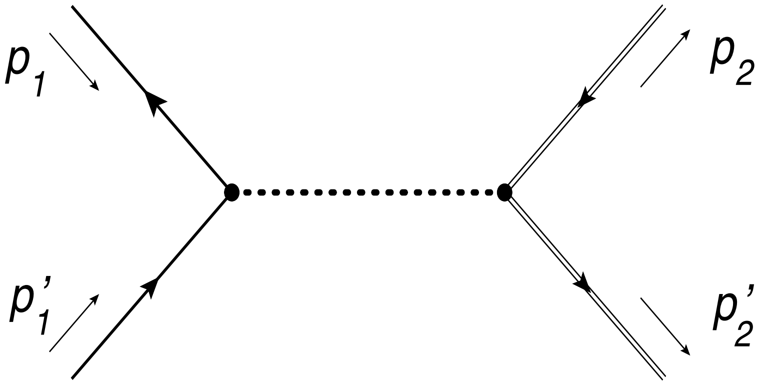
Solution to HW 7

The differential cross section of the unpolarized $e^+e^- \to \mu^+\mu^-$ annihilation is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{c.m.}} = \frac{1}{64\pi^2 s}\frac{|\vec{p}_2|}{|\vec{p}_1|}\frac{1}{4}\sum_{\text{polarizations}}|T|^2 \quad (0.1)$$

where the transition matrix is determined by the diagram



$$T^{\lambda_1,\lambda_1'\lambda_2\lambda_2'} = \frac{e^2}{s}[\bar{U}^{\lambda_2}(p_2)\gamma^\mu V^{\lambda_2'}(p_2')][\bar{v}^{\lambda_1'}(p_1')\gamma^\mu u^{\lambda_1}(p_1)] \quad (0.2)$$

We get

$$\begin{aligned}
&\frac{1}{4e^4}\sum_{\lambda_1,\lambda_1',\lambda_2,\lambda_2'}|T^{\lambda_1,\lambda_1'\lambda_2\lambda_2'}|^2\\
&= \frac{1}{4s^2}[\bar{U}^{\lambda_2}(p_2)\gamma_\mu V^{\lambda_2'}(p_2')][\bar{v}^{\lambda_1'}(p_1')\gamma^\mu u^{\lambda_1}(p_1)][\bar{V}^{\lambda_2'}(p_2')\gamma^\nu U^{\lambda_2}(p_2)][\bar{u}^{\lambda_1}(p_1)\gamma_\nu v^{\lambda_1'}(p_1')]\\
&= \frac{1}{4s^2}\mathrm{Tr}\{(\not{p}_2+\mathrm{M})\gamma^\mu(\not{p}_2'-\mathrm{M})\gamma^\nu\}\mathrm{Tr}\{(\not{p}_1+\mathrm{m})\gamma_\nu(\not{p}_1'-\mathrm{m})\gamma_\mu\}\\
&= \frac{4}{s^2}\left[p_2^\mu p_2'^\nu + p_2^\nu p_2'^\mu - (p_2\cdot p_2'+M^2)g^{\mu\nu}\right]\left[p_{1\mu}p'_{1\nu}+p_{1\nu}p'_{1\mu}-(p_1\cdot p_1'+m^2)g_{\mu\nu}\right]\\
&= \frac{8}{s^2}\left[(p_1\cdot p_2)(p_1'\cdot p_2')+(p_1'\cdot p_2)(p_1\cdot p_2')+M^2(p_1\cdot p_1')+m^2(p_2\cdot p_2')+2m^2M^2\right] \quad (0.3)
\end{aligned}$$

In the c.m. frame $E_1 = E_2 = \frac{1}{2}\sqrt{s}$ and

$$p_1\cdot p_2 = p_1'\cdot p_2' = E_1E_2 - |\vec{p}_1||\vec{p}_2|\cos\theta, \quad p_1\cdot p_2' = p_1'\cdot p_2 = E_1E_2 + |\vec{p}_1||\vec{p}_2|\cos\theta$$
$$p_1\cdot p_1' = \frac{s}{2}-m^2, \quad p_2\cdot p_2' = \frac{s}{2}-M^2, \quad |\vec{p}_1| = \frac{1}{2}\sqrt{s-4m^2}, \quad |\vec{p}_2| = \frac{1}{2}\sqrt{s-4M^2} \quad (0.4)$$

so we get

$$\begin{aligned}
&\frac{1}{4e^4}\sum_{\lambda_1,\lambda_1',\lambda_2,\lambda_2'}|T^{\lambda_1,\lambda_1'\lambda_2\lambda_2'}|^2\\
&= \frac{4}{s^2}\left[4E_1^2E_2^2 + 4\vec{p}_1^2\vec{p}_2^2\cos^2\theta + s(M^2+m^2)\right] \quad (0.5)
\end{aligned}$$

and therefore

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{c.m.}} = \frac{1}{4}\sum_{\text{polarizations}}\frac{|T|^2}{64\pi^2 s}\frac{|\vec{p}_2|}{|\vec{p}_1|}$$

$$= \frac{e^4}{16\pi^2 s^3}\frac{|\vec{p}_2|}{|\vec{p}_1|}\left[4E_1^2E_2^2 + 4\vec{p}_1^{\,2}\vec{p}_2^{\,2}\cos^2\theta + s(M^2+m^2)\right] \tag{0.6}$$

The total cross section is given by

$$\sigma_{\text{tot}} = \int d\Omega\left(\frac{d\sigma}{d\Omega}\right) = \frac{e^4}{4\pi s^3}\frac{|\vec{p}_2|}{|\vec{p}_1|}\left[4E_1^2E_2^2 + \frac{4}{3}\vec{p}_1^{\,2}\vec{p}_2^{\,2} + s(M^2+m^2)\right]$$

$$= \frac{e^4}{12\pi s}\left(1 + 2\frac{M^2+m^2}{s} + \frac{4M^2m^2}{s^2}\right)\sqrt{\frac{s-4M^2}{s-4m^2}} \tag{0.7}$$