842 midterm (20 points) Due Thursday Nov 16 at the lecture.

For the first two problems, consider a theory of self-interacting neutral π-mesons with the four-particle interaction described by a vertex $\frac{i\lambda}{4!}$ in the set of Feynman rules in the coordinate space.

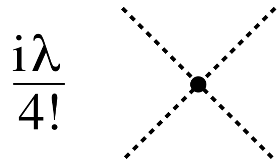


FIG. 1. Four-meson interaction vertex. Dotted lines denote neutral π-mesons.

Problem 1 (2 points).

Write down the momentum integral (and symmetry coefficient) for the reduced two-point Green function in the λ^2 order in perturbation theory.

Solution

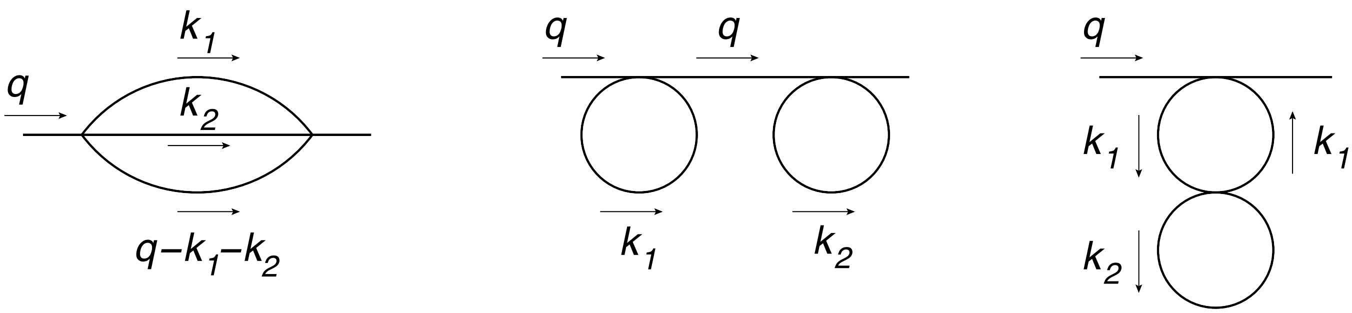


FIG. 2. Diagrams for 2-point Green function in λ^2 order.

First diagram

$$\frac{1}{6}\int \frac{d^4k_1}{(2\pi)^4 i}\frac{d^4k_2}{(2\pi)^4 i}\frac{1}{m^2-k_1^2-i\epsilon}\frac{1}{m^2-k_2^2-i\epsilon}\frac{1}{m^2(q-k_1-k_2)^2-i\epsilon}$$

Second diagram

$$\frac{1}{4}\frac{1}{m^2-q^2-i\epsilon}\int \frac{d^4k_1}{(2\pi)^4 i}\frac{d^4k_2}{(2\pi)^4 i}\frac{1}{m^2-k_1^2-i\epsilon}\frac{1}{m^2-k_2^2-i\epsilon}$$

Third diagram

$$\frac{1}{4}\int \frac{d^4k_1}{(2\pi)^4 i}\frac{d^4k_2}{(2\pi)^4 i}\frac{1}{[m^2-k_1^2-i\epsilon]^2}\frac{1}{m^2-k_2^2-i\epsilon}$$

Problem 2 (3 points).

Find the differential and total cross section of the elastic $\pi\pi$ scattering in this model in the lowest order in perturbation theory. “Elastic scattering” means that the final particles are the same as initial ones (with different momenta).

Solution

From the above expression in coordinate space one easily finds that the vertex in a set of Feynman rules for reduced Green functions is λ. The reduced amputated 4-pion Green function in the leading order is also λ so the differential cross section reads

$$\frac{d\sigma}{d\Omega} = \frac{\lambda^2}{64\pi^2 s}$$

and the total cross section is

$$\sigma_{\rm tot} = \frac{1}{2}\int d\Omega \, \frac{d\sigma}{d\Omega} = \frac{\lambda^2}{32\pi s}$$

where $\frac{1}{2}$ is due to identical particles in the final state.

Problem 3 (2 points).

Write down the momenta integrals (and symmetry coefficients, if applicable) for reduced amputated Green functions for the following diagrams

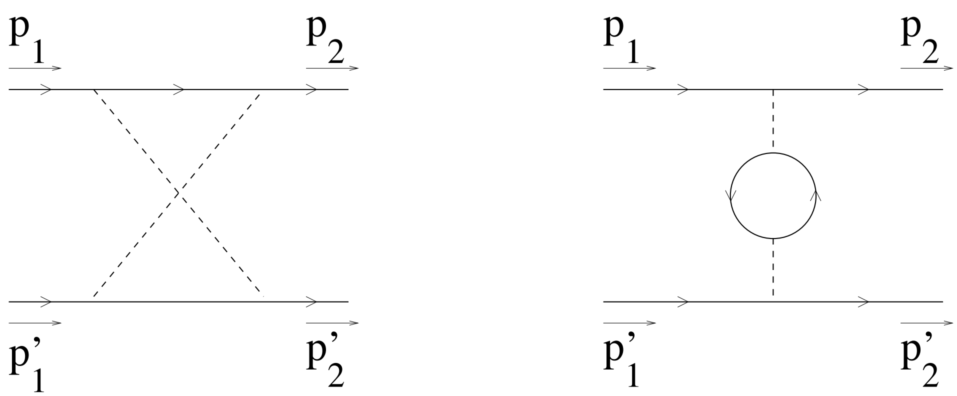


FIG. 3. Dashed lines denote photons and solid lines charged π-mesons. (No 4-γ vertex in the first diagram!)

describing scattering of two π^+ mesons. (Do not try to calculate the integrals, they are divergent.)

Solution

Diagram 1

$$\int \frac{d^4k}{16\pi^4 i} \frac{(2p_1+k, 2p_2'+k)(p_1+p_2+k, p_1'+p_2'+k)}{[m^2-(p_1+k)^2-i\epsilon][m^2-(p_1+k)^2-i\epsilon](k^2+i\epsilon)[(p_1-p_2+k)^2+i\epsilon]}$$

Diagram 2

$$\int \frac{d^4k}{16\pi^4 i} \frac{(p_1+p_2, 2k-p_1+p_2)(2k-p_1+p_2, p_1'+p_2')}{[m^2-k^2-i\epsilon][m^2-(p_1-p_2-k)^2-i\epsilon][(p_1-p_2)^2+i\epsilon]^2}$$

(Here $(a,b) \equiv (a\cdot b)$). There are no symmetrical (sub)diagrams here $\Rightarrow$ no symmetry coefficients.

Problem 4 (2 points).

As you probably have heard, the Higgs boson (spin-0 scalar meson with mass m_H) can decay in two photons. At moderate energies, this interaction can be modeled by a local vertex, see Fig. 3a where the wavy line denote Higgs boson. Suppose this is the only vertex of interaction of Higgs bosons with photons.

Can this vertex for the set of reduced Green functions in the momentum space have the form
(a) $gg^{\mu\nu}$ (b) $gk_1^\mu k_2^\nu$ (c) $gk_2^\mu k_1^\nu$ (d) $g(g^{\mu\nu}k_1\cdot k_2 - k_1^\mu k_2^\nu)$ (e) $g(g^{\mu\nu}k_1\cdot k_2 - k_2^\mu k_1^\nu)$,

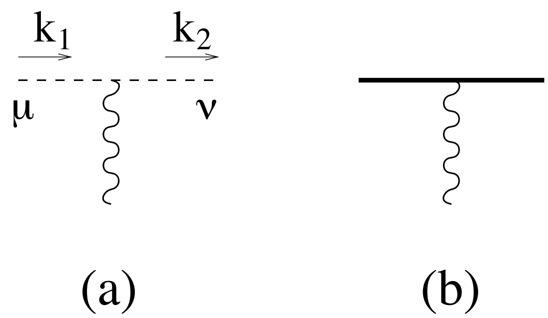


FIG. 4. Higgs vertices: $\gamma\gamma H$ (a) and MMH (b). Higgs is denoted by wavy line, photon by dashed line, and netral M-meson by bold solid line.

where g is some constant (do not confuse it with metric tensor $g_{\mu\nu}$!). Circle the correct answer. (Here $a\cdot b \equiv a^\alpha b_\alpha$).

Solution

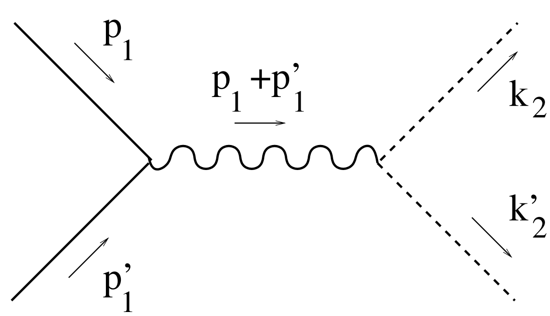
By Ward identity $k_1^\mu \Gamma_{\mu\nu} \;=\; 0$ for photon with arbitrary $k_1^2 \Rightarrow$ only choice (e) is correct.

Problem 5 (6 points).

Suppose, in addition to the $H\gamma\gamma$ vertex shown in Fig. 2a, we have also the "MMH" vertex describing the emission of Higgs boson by the M-boson (see Fig 3b) and let the vertex be λ ($\Leftrightarrow \frac{i\lambda}{2}$ in the coordinate space), same as in our πM model. Using the $H\gamma\gamma$ vertex which you circled in Problem 2, find the total cross section of the $MM \to \gamma\gamma$ transition in the lowest order in g and λ in the c.m. frame. Also, recall that the notion of *total* cross section includes summation over the polarizations of the final photons).

Solution

A single Feynman diagram in the lowest order in perturbation theory is



In the c.m. frame, the total cross section is given by (see f-la (721) from Appendix A):

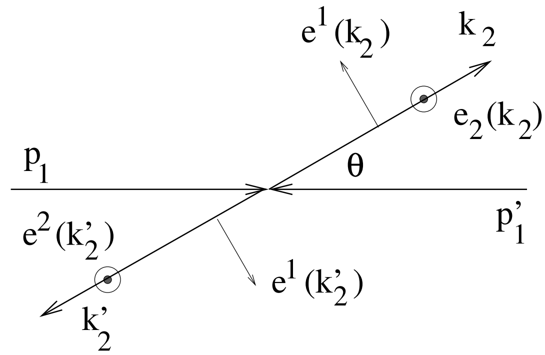
$$\sigma_{\rm tot} \;=\; \frac{1}{2}\int d\Omega\, \frac{1}{64\pi^2 s}\frac{\sqrt{s}}{\sqrt{s-4M^2}}\left[|T^{++}|^2 + |T^{--}|^2 + |T^{+-}|^2| + T^{-+}|^2\right]$$

where the factor $\frac{1}{2}$ comes from the fact that particles in the final state are identical.

The transition matrix is

$$T^{\lambda_2,\lambda_2'} = e_\mu^{\lambda_2 *} T^{\mu\nu} e_\nu^{\lambda_2' *} = \frac{\lambda}{m_H^2 - s}\left[(k_2 \cdot k_2')(e^{\lambda_2 *} \cdot e^{\lambda_2' *}) - (k_1 \cdot k_2)(e^{\lambda_2 *} \cdot k_2')(e^{\lambda_2' *} \cdot k_2)\right]$$

Let us set up polarization vectors as shown below.



For this setup, $(e^{\lambda_2' *} \cdot k_2) = (e^{\lambda_2 *} \cdot k_2') = 0$ for all polarizations so

$$T^{\lambda_2,\lambda_2'} = \frac{\lambda^2}{m_H^2 - s}(k_2 \cdot k_2')(e^{\lambda_2 *} \cdot e^{\lambda_2' *}) = \frac{\lambda^2 s}{2(s - m_H^2)}(\vec{e}^{\lambda_2 *} \cdot \vec{e}^{\lambda_2' *})$$

which gives

$$T^{22} = -T^{11} = \frac{\lambda^2 s}{2(s - m_H^2)}, \quad T^{12} = T^{21} = 0 \quad \Rightarrow \quad T^{++} = T^{--} = -\frac{\lambda^2 s}{2(s - m_H^2)}, \quad T^{-+} = T^{+-} = 0$$

and therefore

$$\sigma_{\text{tot}} = \frac{4\pi}{2}\frac{1}{64\pi^2 s}\frac{\sqrt{s}}{\sqrt{s - 4M^2}}\frac{\lambda^4 s^2}{2(s - m_H^2)^2} = \frac{\sqrt{s}}{\sqrt{s - 4M^2}}\frac{\lambda^4 s}{64\pi(s - m_H^2)^2}$$

Problem 6 (5 points).

Pretending that electron e^- (and positron e^+) are scalar mesons with spin 0, calculate the differential and total cross section of the $e^+e^- \Rightarrow \pi^+\pi^-$ transition in the lowest order in perturbation theory in the c.m. frame. (Electrons do not interact with π-mesons except by exchange of photons).

Solution

According to general formula

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{1}{64\pi^2 s}\sqrt{\frac{(s - m_a^2 - m_b^2)^2 - 4m_a^2 m_b^2}{(s - M_A^2 - M_B^2)^2 - 4M_A^2 M_B^2}}|T|^2 = \frac{1}{64\pi^2 s}\sqrt{\frac{s - 4m^2}{s - 4m_e^2}}|T|^2 \tag{1}$$

The lowest-order Feynman diagram is shown below

so the relevant term in the T-matrix is

$$\frac{(p_1 - p_1') \cdot (p_2 - p_2')}{(p_1 + p_1')^2} = \frac{u - t}{s} \tag{2}$$

The differential cross section in the c.m. frame takes the form

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{e^4}{64\pi^2 s}\sqrt{\frac{s - 4m^2}{s - 4m_e^2}}\frac{(u - t)^2}{s^2} \tag{3}$$

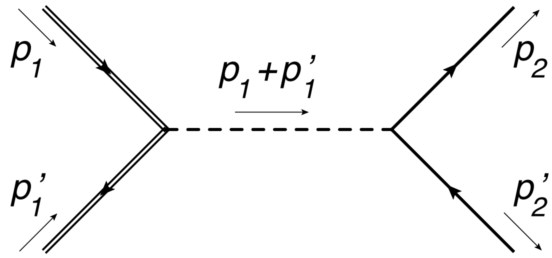


FIG. 5. Scalar charged “electron” is denoted by double line, photon by dashed line, charged π-meson by solid line.

In the c.m. frame

$$(p_1 - p'_1) \cdot (p_2 - p'_2) = -4\vec{p}_1 \cdot \vec{p}_2 = -\sqrt{s-4m^2}\sqrt{s-4m_e^2}\cos\theta \tag{4}$$

and the total cross section

$$\sigma^{\rm tot} = \frac{e^4}{64\pi^2 s}\sqrt{\frac{s-4m^2}{s-4m_e^2}}\int d\Omega \frac{(s-4m^2)(s-4m_e^2)}{s^2}\cos^2\theta = \frac{e^4}{48\pi s^3}(s-4m^2)^{\frac{3}{2}}\sqrt{s-4m_e^2} \tag{5}$$