Quantum Field Theory from Harmonic Oscillator to QCD.

Lecture Notes

Ian Balitsky
Department of Physics, Old Dominion University,
Norfolk, VA 23529
and
Theory Group of Jefferson Lab
12000 Jefferson Ave, Newport News, VA 23606

Contents

---

i

Contents

1 Notations

Mostly, I use the notations from the book by Peskin and Schoeder.
However, the totally antisymmetric tensor $\epsilon^{\mu\nu\lambda\rho}$ differs in sign from Peskin. I define $\epsilon^{0123} = -1$, same as Bjorken & Drell textbook.

In addition:

Fourier transform - $(2\pi)^{-1}$ goes with $\int dk$

$$f(x) = \int \frac{d^n k}{(2\pi)^n} e^{-ikx} f(k) \equiv \int \bar{d}^n k e^{-ikx} f(k) : \qquad f(k) = \int d^n x e^{ikx} f(x) \tag{1.1}$$

I use the space-saving notation $\int \bar{d}^n k \equiv \int \frac{d^n k}{(2\pi)^n}$.

Dirac *delta*-function:

Definition: $\delta(x-y) = 0$ if $x \neq y$ and $\int dx\ \delta(x) = 1$. Property:

$$\int dy\ \delta(x-y) f(y) = f(x) \tag{1.2}$$

In multi-dimensional space

$$\delta^{(n)}(x-y) \stackrel{\text{def}}{\equiv} \delta(x_1-y_1)\delta(x_2-y_2)...\delta(x_n-y_n) \tag{1.3}$$

Sometimes I will omit the upper label (n) for brevity (if there is no confusion about the dimension of the δ-function, for example $\delta(\vec{x}-\vec{y}) \equiv \delta^{(3)}(\vec{x}-\vec{y})$).

Properties:

$$\delta(F(x)-F(y)) = \frac{1}{|F'(x)|}\delta(x-y) \tag{1.4}$$

θ-function:

$$\begin{aligned} \theta(x) &= 1 \quad x \geq 0 \\ \theta(x) &= 0 \quad x < 0 \end{aligned} \tag{1.5}$$

Derivative:

$$\frac{d}{dx}\theta(x) = \delta(x) \tag{1.6}$$

Spatial components of 4-vectors and 3-vectors

I denote the components of three-dimensional vector $\vec{a} = (a_x, a_y, a_z)$ by $\vec{a}_i$: $\vec{a}_1 \equiv a_x$, $\vec{a}_2 \equiv a_y$, $\vec{a}_3 \equiv a_z$. This (unusual) notation is introduced in order to avoid confusion with covariant components in four-dimensional notations. The contravariant components of 4-vector $a = (a^0, \vec{a})$ are defined as $a^1 \equiv a_x, a^2 \equiv a_y, a^3 \equiv a_z$ so the covariant components of this 4-vector will be $a_1 = -a_x, a_2 = -a_y, a_3 = -a_z$ which differ in sign from $\vec{a}_i$.

Part I

2 Harmonic oscillator as a trivial (0+1) field theory

2.1 Harmonic oscillator in classical mechanics

The Lagrangian of a harmonic oscillator is

$$L(\phi, \dot{\phi}) \;=\; \frac{\dot{\phi}^2}{2} - \frac{\omega^2\phi^2}{2} \tag{2.1}$$

where $\phi(t)$ is a coordinate. We use the notation $\phi(t)$ rather than common notation $x(t)$ to ease future transition to field theory where the field is denoted by $\phi(\vec{x}, t)$.

2.1.1 Classical equations of motion from least action principle.

Action:

$$S(\phi) \;=\; \int_{t_1}^{t_2} dt\; L(\phi(t), \dot{\phi}(t)) \tag{2.2}$$

$S(\phi)$ is a functional of $\phi(t)$. (Mathematically, $S: L_2 \to R$)

Q: How to find minimum of a functional?

A: Same way as for a function: consider small deviation of the argument from the would-be minimum and check if the linear term of deviation of the function vanishes.

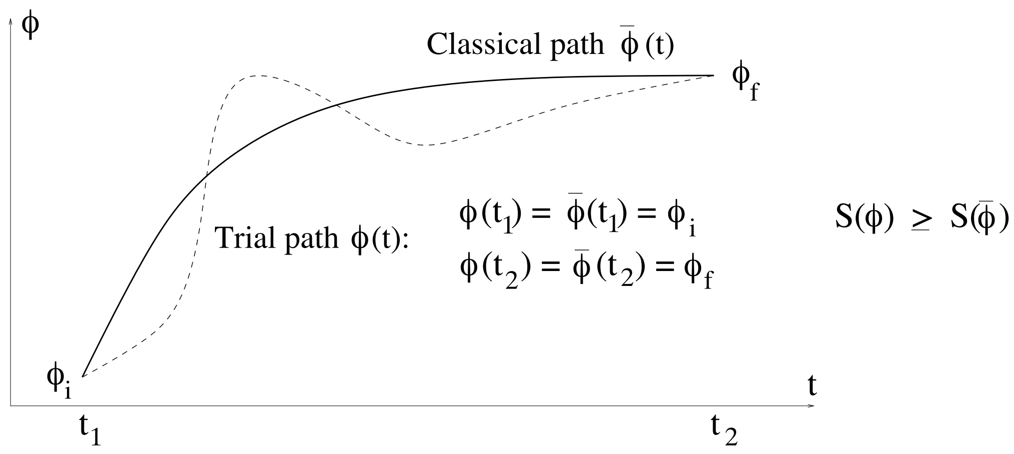


Figure 1. Least action principle: given the initial and final points, the classical path $\bar{\phi}(t)$ is a path of minimal action

For a function $F(x)$: let x_* be a minimum of $F(x)$, then

$$
\begin{aligned}
F(x_* + \Delta x) &= F(x_*) + \Delta x F'(x_*) + \frac{1}{2}(\Delta x)^2 F''(x_*) + ... \\
\Rightarrow \; F'(x_*) &= 0 \quad (\text{and } F''(x) \; < \; 0)
\end{aligned} \tag{2.3}
$$

Now we repeat the same steps for the functional $S(\phi(t))$. Suppose $S(\phi(t))$ is minimal for $\phi(t) = \bar{\phi}(t)$, then

$$S\big(\bar{\phi}(t) + \delta\phi(t)\big) \;\geq S\big(\bar{\phi}(t)\big) \tag{2.4}$$

for arbitrary $\delta\phi(t)$ with boundary conditions

$$\delta\phi(t_1) \;=\; \delta\phi(t_2) \;=\; 0 \tag{2.5}$$

(recall that $\phi(t_1) = \bar{\phi}(t_1) = \phi_i$ and $\phi(t_2) = \bar{\phi}(t_2) = \phi_f$ for any trial path $\phi(t)$).

Expanding the action (2.4) in powers of small $\delta\phi(t)$ we get

$$
\begin{aligned}
S\big(\bar{\phi}(t) + \delta\phi(t)\big) \;&=\; S\big(\bar{\phi}(t)\big) + \int_{t_1}^{t_2} dt\; \delta\phi(t)\, \frac{\delta S}{\delta\phi}(t)\Big|_{\phi(t)=\bar{\phi}(t)} \\
&+ \int_{t_1}^{t_2} dt dt'\; \delta\phi(t)\delta\phi(t')\, \frac{\delta^2 S}{\delta\phi^2}(t,t')\Big|_{\phi(t)=\bar{\phi}(t)} + O(\delta\phi^3)
\end{aligned} \tag{2.6}
$$

The function $\frac{\delta S}{\delta\phi}(t)$ defined by the above equation is called a first variational derivative of the functional $S(\phi)$ (and $\frac{\delta^2 S}{\delta\phi^2}(t,t')$ is called a second variational derivative of the action).

Similarly to the case of a function $F(x_*)$, in order for $S\big(\bar{\phi}(t)\big)$ be a minimum, the linear term in Eq. (2.6) should vanish and the quadratic term should be positive. Since $\delta\phi(t)$ is arbitrary we get

$$\int_{t_1}^{t_2} dt\; \delta\phi(t)\, \frac{\delta S}{\delta\phi}(t)\Big|_{\phi(t)=\bar{\phi}(t)} \;=\; 0 \;\;\Rightarrow\;\; \frac{\delta S}{\delta\phi}(t)\Big|_{\phi(t)=\bar{\phi}(t)} \;=\; 0 \tag{2.7}$$

(the second requirement means that $\frac{\delta^2 S}{\delta\phi^2}(t,t')\Big|_{\phi(t)=\bar\phi(t)}$ should be a positive-definite operator).

Let us find $\frac{\delta S}{\delta\phi}(t)$ for the harmonic oscillator (2.1).

$$S\big(\bar\phi(t)+\delta\phi(t)\big) \;=\; \int_{t_1}^{t_2} dt\; L(\phi+\delta\phi,\dot\phi+\delta\dot\phi) \tag{2.8}$$
$$=\; \int_{t_1}^{t_2} dt\,\Big\{L(\phi,\dot\phi) \;+\; \frac{\partial L(\phi,\dot\phi)}{\partial\phi}(t)\Big|_{\phi(t)=\bar\phi(t)}\delta\phi(t) + \;\frac{\partial L(\phi,\dot\phi)}{\partial\dot\phi}(t)\Big|_{\phi(t)=\bar\phi(t)}\delta\dot\phi(t) \;+\; O(\delta\phi^2)\Big\}$$

Integrating by parts the term proportional to $\delta\dot\phi(t)$ we get

$$\int_{t_1}^{t_2} dt\;\frac{\partial L(\phi,\dot\phi)}{\partial\dot\phi}(t)\Big|_{\phi(t)=\bar\phi(t)}\delta\dot\phi(t) \;=\; \int_{t_1}^{t_2} dt\;\frac{\partial L(\phi,\dot\phi)}{\partial\dot\phi}(t)\Big|_{\phi(t)=\bar\phi(t)}\frac{d}{dt}\delta\phi(t) \tag{2.9}$$
$$=\; \frac{\partial L(\phi,\dot\phi)}{\partial\dot\phi}(t)\delta\phi(t)\Big|_{t=t_1}^{t=t_2} - \;\int_{t_1}^{t_2} dt\;\delta\phi(t)\frac{d}{dt}\frac{\partial L(\phi,\dot\phi)}{\partial\dot\phi}(t)\Big|_{\phi(t)=\bar\phi(t)} \;=\; -\int_{t_1}^{t_2} dt\;\delta\phi(t)\frac{d}{dt}\frac{\partial L(\phi,\dot\phi)}{\partial\dot\phi}(t)\Big|_{\phi(t)=\bar\phi(t)}$$

where the end-point terms vanish since $\delta\phi(t_1)=\delta\phi(t_2)=0$. We get

$$S\big(\bar\phi(t)+\delta\phi(t)\big) \;=\; \int_{t_1}^{t_2} dt\; L(\phi,\dot\phi)+\int_{t_1}^{t_2} dt\;\delta\phi(t)\Big\{\frac{\partial L(\phi,\dot\phi)}{\partial\phi}(t)\Big|_{\phi(t)=\bar\phi(t)} - \frac{d}{dt}\frac{\partial L(\phi,\dot\phi)}{\partial\dot\phi}(t)\Big|_{\phi(t)=\bar\phi(t)}\Big\} \tag{2.10}$$

Since $\delta\phi(t)$ is arbitrary we get **Euler-Lagrange equation**

$$\Big(\frac{\partial L(\phi,\dot\phi)}{\partial\phi} - \frac{d}{dt}\frac{\partial L(\phi,\dot\phi)}{\partial\dot\phi}\Big)(t)\Big|_{\phi(t)=\bar\phi(t)} \;=\; 0 \tag{2.11}$$

For the harmonic oscillator

$$L(\phi,\dot\phi) \;=\; \frac{\dot\phi^2}{2} - \frac{\omega^2\phi^2}{2} \;\Rightarrow\; \frac{\partial L(\phi,\dot\phi)}{\partial\dot\phi} = \dot\phi, \qquad \frac{\partial L(\phi,\dot\phi)}{\partial\phi} \;=\; -\,\omega^2\phi \tag{2.12}$$

and the Euler-Lagrange equation takes the form

$$\frac{d}{dt}\dot\phi(t) \;=\; -\,\omega^2\phi(t) \quad\Leftrightarrow\quad \ddot\phi(t) \;=\; -\,\omega^2\phi(t) \tag{2.13}$$

This is the familiar second-order differential equation for harmonic oscillator with solutions $e^{\pm i\omega t}$ (or $\cos\omega t$ and $\sin\omega t$).

2.1.2 Classical Hamiltonian for the harmonic oscillator

The canonical momentum is defined as

$$\pi(t) \;\equiv\; \frac{\partial L}{\partial\dot\phi}(t) \tag{2.14}$$

For the harmonic oscillator

$$\pi(t) \equiv \frac{\partial}{\partial\dot{\phi}}\Big(\frac{\dot{\phi}^2}{2} - \frac{\omega^2\phi^2}{2}\Big) = \dot{\phi} \tag{2.15}$$

In general, classical Hamiltonian is defined as

$$H = \pi\dot{\phi} - L(\phi, \dot{\phi}) \tag{2.16}$$

where we must express $\dot{\phi}$ in terms of π using Eq. (2.14).

For the harmonic oscillator we get

$$H = \pi\dot{\phi} - \Big(\frac{\dot{\phi}^2}{2} - \frac{\omega^2\phi^2}{2}\Big)\Big|_{\dot{\phi}=\pi} = \frac{\pi^2}{2} + \frac{\omega^2\phi^2}{2} \tag{2.17}$$

2.2 Harmonic oscillator in quantum mechanics

2.2.1 Quantization in the Schrödinger picture

Quantization recipe: We promote ϕ and π to operators $\hat{\phi}$ and $\hat{\pi}$ satisfying canonical commutation relations

$$[\phi, \pi] = i\hbar, \tag{2.18}$$

define QM Hamiltonian

$$\hat{H} = \frac{\hat{\pi}^2}{2} + \frac{\omega^2\hat{\phi}^2}{2} \tag{2.19}$$

and solve Schrödinger equation

$$i\hbar\frac{d}{dt}|\Psi_t\rangle = \hat{H}|\Psi_t\rangle \tag{2.20}$$

where $|\Psi_t\rangle$ is the QM vector of state. Hereafter we set $\hbar = 1$ as common in QFT.

Usually we write down the Schrödinger equation (2.20) is so-called coordinate representation where $|\phi\rangle$ are eigenstates of the coordinate operator $\hat{\phi}$

$$\hat{\phi}|\phi\rangle = \phi|\phi\rangle \tag{2.21}$$

In this representation the state vector is given by the Schrödinger wave function

$$\langle\phi|\Psi_t\rangle = \Psi(t, \phi) \tag{2.22}$$

where the wave function $\Psi(t, \phi)$ is the amplitude to discover harmonic oscillator at the position ϕ at the time t. The operators ϕ and π in the coordinate representation have the form

$$\langle\phi|\hat{\phi}|\Phi_t\rangle = \hat{\phi}\Psi(t, \phi) = \phi(t)\Psi(t, \phi), \qquad \hat{\pi}\Psi(t, \phi) = -i\frac{\partial}{\partial\phi}\Psi(t, \phi) \tag{2.23}$$

where the first equation is the trivial consequence of the definition (2.20) and the second follows from the commutation relation $[\hat{\phi}, \hat{\pi}] = i$:

$$\begin{aligned} [\hat{\phi}, \hat{\pi}]\Psi(t, \phi) &= \hat{\phi}\hat{\pi}\Psi(t, \phi) - \hat{\pi}\hat{\phi}\Psi(t, \phi) = \hat{\phi}\Big(-i\frac{\partial}{\partial\phi}\Big)\Psi(t, \phi) + i\frac{\partial}{\partial\phi}\Big(\phi\Psi(t, \phi)\Big) \\ &= -i\phi\frac{\partial}{\partial\phi}\Psi(t, \phi) + i\frac{\partial}{\partial\phi}\Big(\phi\Psi(t, \phi)\Big) = i\Psi(t, \phi) \end{aligned} \tag{2.24}$$

With operators $\hat\phi$ and $\hat\pi$ in the form (2.22) we get the familiar for of the Schrödinger equation

$$i\frac{\partial}{\partial t}\Psi(t,\phi) \;=\; \Big(\frac{\hat\pi^2}{2}+\frac{\omega^2\hat\phi^2}{2}\Big)\Psi(t,\phi) \;=\; -\frac{1}{2}\frac{\partial^2\Phi(t,\phi)}{\partial\phi^2}+\frac{\omega^2}{2}\phi^2(t)\Phi(t,\phi) \tag{2.25}$$

For the stationary states

$$\Psi(t,\phi) \;=\; e^{-iEt}\Psi(\phi) \tag{2.26}$$

so the stationary Schrödinger equation turns to

$$\hat H\Psi(\phi) \;=\; E\Psi(\phi) \tag{2.27}$$

or, in the explicit form,

$$-\frac{1}{2}\frac{\partial^2\Phi(\phi)}{\partial\phi^2}+\frac{\omega^2}{2}\phi^2\Phi(\phi) \;=\; E\Psi(\phi) \tag{2.28}$$

Such scheme of quantization (when vector of state depends on time and the operators $\hat\phi$ and $\hat\pi$ do not) is called the Schrödinger picture. It is very convenient in quantum mechanics, but, as we shall see below, extremely inconvenient in quantum field theory.

2.2.2 Quantization in the Heisenberg picture

In Heisenberg picture vector of state $|\Psi\rangle$ does not depend on time but operators $\hat\phi$ and $\hat\pi$ do.

We take Schrödinger vector of state at $t=0$ and define

$$\begin{aligned} &|\Psi\rangle_{\rm Heis} \;\equiv\; |\Psi_t\rangle_{\rm Schro}\big|_{t=0} \\ &\hat\phi(t) \;\equiv\; e^{i\hat Ht}\hat\phi e^{-i\hat Ht}, \qquad \hat\pi(t) \;\equiv\; e^{i\hat Ht}\hat\pi e^{-i\hat Ht}, \end{aligned} \tag{2.29}$$

In this picture instead of Schrödinger equation for vector of state (2.20) we have two Heisenberg equations for operators $\hat\phi(t)$ and $\hat\pi(t)$

$$\frac{d}{dt}\hat\phi(t) \;=\; i[\hat H,\hat\phi(t)], \qquad \frac{d}{dt}\hat\pi(t) \;=\; i[\hat H,\hat\pi(t)] \tag{2.30}$$

It is instructive to see that the commutation relation between $\hat\phi$ and $\hat\pi$ does not depend on time

$$\begin{aligned} [\hat\phi(t),\hat\pi(t)] \;&=\; \hat\phi(t)\hat\pi(t)-\hat\pi(t)\hat\phi(t) \;=\; e^{i\hat Ht}\hat\phi e^{-i\hat Ht}e^{i\hat Ht}\hat\pi e^{-i\hat Ht}-e^{i\hat Ht}\hat\pi e^{-i\hat Ht}e^{i\hat Ht}\hat\phi e^{-i\hat Ht} \\ &=\; e^{i\hat Ht}\hat\phi\hat\pi e^{-i\hat Ht}-e^{i\hat Ht}\hat\pi\hat\phi e^{-i\hat Ht} \;=\; e^{i\hat Ht}[\hat\phi,\hat\pi]e^{-i\hat Ht} \;=\; e^{i\hat Ht}ie^{-i\hat Ht} \;=\; i \end{aligned} \tag{2.31}$$

As we shall see below, the Heisenberg quantization picture easily generalizes to quantum field theory.

3 Classical theory of a scalar Klein-Gordon field

For simplicity, we start the discussion of the field theory using a simplest scalar field $\phi(x)$ as an example. We denote 4-dimensional vectors by Latin letters $x \equiv (ct, \vec{x})$ and, in what follows, we set $c = 1$ as usual in QFT.

The Lagrangian of the scalar Klein-Gordon field has the form

$$L(t) \ = \ \int d^3x \ \mathcal{L}(t, \vec{x}) \tag{3.1}$$

where $\mathcal{L}(\vec{x}, t)$ is the **Largangian density**

$$\mathcal{L}(\vec{x}, t) \ = \ \mathcal{L}\big(\phi(t, \vec{x})\partial_\mu\phi(t, \vec{x})\big) \ = \ \frac{1}{2}\partial_\mu\phi\partial^\mu\phi(x) - \frac{m^2}{2}\phi^2(x) \tag{3.2}$$

The corresponding action takes the form

$$S(\phi) \ = \ \int dt \ L(t) \ = \ \int dt d^3x \mathcal{L}(t, \vec{x}) \ = \ \int d^4x \mathcal{L}(x) \ = \ \int d^4x \mathcal{L}\big(\phi(x), \partial_\mu\phi(x)\big) \tag{3.3}$$

Canonical coordinates and canonical momenta are

$$\phi(t, x) \qquad \text{and} \qquad \pi(t, x) \ \equiv \ \frac{\partial\mathcal{L}}{\partial\dot{\phi}}(t, x) \ = \ \dot{\phi}(t, x) \tag{3.4}$$

so the classical Hamiltonian for the KG field takes the form

$$\begin{aligned} H \ &= \int d^3x \pi(t, x)\dot{\phi}(t, x) - L(t) \ = \ \int d^3x \big[\pi(t, x)\dot{\phi}(t, x) - \frac{\dot{\phi}^2(t, x)}{2} + \frac{|\nabla\phi(t, x)|^2}{2} + \frac{m^2}{2}\phi^2(t, x)\big] \\ &= \ \int d^3x \big[\frac{\pi^2(t, x)}{2} + \frac{\big|\nabla\phi(t, x)\big|^2}{2} + \frac{m^2}{2}\phi^2(t, x)\big] \end{aligned} \tag{3.5}$$

3.1 Least action principle for Klein-Gordon field

Given the initial and final field configurations

$$\phi(t_1, \vec{x}) \ = \ \phi_i(\vec{x}), \qquad \phi(t_2, \vec{x}) \ = \ \phi_f(\vec{x}) \tag{3.6}$$

the classical field $\bar{\phi}(x)$ varies in such a way that the action $S(\bar{\phi})$ is minimal of all possible field configurations. In other words, if we take the trial field configuration $\phi(t, \vec{x})$ with the same initial and final conditions $\phi(t_1, \vec{x}) = \phi_i(\vec{x})$ and $\phi(t_2, \vec{x}) = \phi_f(\vec{x})$ and compare its action to that of classical field $\bar{\phi}(x)$ we will get

$$S(\phi) \ \geq \ S(\bar{\phi}) \tag{3.7}$$

similarly to equation in Fig. 1 for classical mechanics.

3.1.1 Classical field equations from least action principle

Similarly to the case of harmonic oscillator (see Fig. 1) we consider trial field configuration

$$\phi(x) \;=\; \bar\phi(x) + \delta\phi(x) \tag{3.8}$$

where $\delta\phi(x)$ is a small deviation from the classical field $\bar\phi(x)$, and expand in powers of $\delta\phi$:

$$S(\phi) \;=\; S(\bar\phi) + \int_{t_1}^{t_2} dt \int d^3x\ \delta\phi(x)\ \frac{\delta S(\phi)}{\delta\phi}\Big|_{\phi(x)=\bar\phi(x)} + O(\delta\phi^2) \tag{3.9}$$

If $S(\phi) \geq S(\bar\phi)$ the linear term in the r.h.s. of Eq. (3.9) must vanish (and the quadratic term must be positive, but this is a separate issue which we will not discuss now) so we obtain

$$\int_{t_1}^{t_2} dt \int d^3x\ \delta\phi(x)\ \frac{\delta S(\phi)}{\delta\phi}\Big|_{\phi(x)=\bar\phi(x)} \;=\; 0 \tag{3.10}$$

Since the above equation holds true at any $\delta\phi(x)$ we get

$$\frac{\delta S(\phi)}{\delta\phi}\Big|_{\phi(x)=\bar\phi(x)} \;=\; 0 \qquad\quad - \text{ classical field equation} \tag{3.11}$$

Let us find now the explicit form of $\frac{\delta S(\phi)}{\delta\phi}$ for the action (3.3).

$$\begin{aligned} S(\phi+\delta\phi) - S(\phi) \;&=\; \int_{t_1}^{t_2} dt \int d^3x\ \left[\mathcal{L}(\phi+\delta\phi, \partial_\alpha\phi + \partial_\alpha\delta\phi) - \mathcal{L}(\phi, \partial_\alpha\phi)\right] \\ &=\; \int_{t_1}^{t_2} dt \int d^3x\ \left[\delta\phi(x)\frac{\partial\mathcal{L}(\phi,\partial_\alpha\phi)}{\partial\phi} + \partial_\mu\delta\phi(x)\frac{\partial\mathcal{L}(\phi,\partial_\alpha\phi)}{\partial\partial_\mu\phi} + O(\delta\phi^2)\right] \end{aligned} \tag{3.12}$$

The second term in the r.h.s. may be integrated by parts similarly to Eq. (2.9):

$$\begin{aligned} &\int_{t_1}^{t_2} dt \int d^3x\ \partial_\mu\delta\phi(x)\frac{\partial\mathcal{L}(\phi,\partial_\alpha\phi)}{\partial\partial_\mu\phi} \\ &= \int_{t_1}^{t_2} dt \int d^3x\ \left[\frac{d}{dt}\delta\phi(x)\frac{\partial\mathcal{L}(\phi,\partial_\alpha\phi)}{\partial\dot\phi} + \frac{\partial}{\partial x^i}\delta\phi(x)\frac{\partial\mathcal{L}(\phi,\partial_\alpha\phi)}{\partial\partial_i\phi} + O(\delta\phi^2)\right] \\ &= \int d^3x\delta\phi(t,\vec x)\ \frac{\partial\mathcal{L}(\phi,\partial_\alpha\phi)}{\partial\dot\phi}\Big|_{t_1}^{t_1} - \int_{t_1}^{t_2} dt \int d^3x\ \left[\frac{d}{dt}\delta\phi(x)\frac{\partial\mathcal{L}(\phi,\partial_\alpha\phi)}{\partial\dot\phi} + \delta\phi(x)\frac{\partial}{\partial x^i}\frac{\partial\mathcal{L}(\phi,\partial_\alpha\phi)}{\partial\partial_i\phi} + O(\delta\phi^2)\right] \end{aligned} \tag{3.13}$$

Since $\delta\phi(t_1,\vec x) = \delta\phi(t_2,\vec x) = 0$ the first (boundary) term in the r.h.s of the above equation vanishes and we obtain

$$\begin{aligned} &S(\bar\phi+\delta\phi) - S(\bar\phi) \;= \\ &=\; \int_{t_1}^{t_2} dt \int d^3x\ \delta\phi(x)\Big[\frac{\partial\mathcal{L}(\phi,\partial_\alpha\phi)}{\partial\phi} - \frac{\partial}{\partial x^\mu}\frac{\partial\mathcal{L}(\phi,\partial_\alpha\phi)}{\partial\partial_\mu\phi}\Big] \;+\; O(\delta\phi^2) \end{aligned} \tag{3.14}$$

By definition, the expression in square brackets is $\frac{\delta S}{\delta\phi}$ so we get

$$\frac{\delta S(\phi)}{\delta\phi} \;=\; \frac{\partial\mathcal{L}(\phi,\partial_\alpha\phi)}{\partial\phi} - \frac{\partial}{\partial x^\mu}\frac{\partial\mathcal{L}(\phi,\partial_\alpha\phi)}{\partial\partial_\mu\phi} \tag{3.15}$$

and the classical field equation (3.11) takes the familiar Euler-Lagrange form

$$\frac{\partial\mathcal{L}(\phi,\partial_\alpha\phi)}{\partial\phi}\Big|_{\phi=\bar\phi} ~=~ \frac{\partial}{\partial x^\mu}\Big(\frac{\partial\mathcal{L}(\phi,\partial_\alpha\phi)}{\partial\partial_\mu\phi}\Big|_{\phi=\bar\phi}\Big) \tag{3.16}$$

For the Klein-Gordon field $\mathcal{L} ~=~ \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{m^2}{2}\phi^2$ so

$$\frac{\partial\mathcal{L}(\phi,\partial_\alpha\phi)}{\partial\phi} ~=~ -m^2\phi, \qquad \frac{\partial\mathcal{L}(\phi,\partial_\alpha\phi)}{\partial\partial_\mu\phi} ~=~ \partial^\mu\phi \tag{3.17}$$

and the equation (3.15) takes the form

$$\frac{\partial}{\partial x^\mu}\partial^\mu\phi ~=~ \partial_\mu\partial^\mu\phi ~=~ -m^2\phi \tag{3.18}$$

or, in short,

$$(\partial^2+m^2)\phi(x) ~=~ 0 \tag{3.19}$$

This is the Klein-Gordon equation studied in the AQM course. (Right now m^2 is just a parameter in this equation but it will be a mass of the scalar particle after quantization of the Klein-Gordon field.)

Part II

4 Quantization of the Klein-Gordon field

For simplicity consider (1+1) dimensional Klein-Gordon field $\phi(t,x)$. Lagrangian for this field is

$$L(t) ~=~ \int dx~\mathcal{L}(t,x) \tag{4.1}$$
$$\mathcal{L}(t,x) ~=~ \frac{\dot\phi^2}{2} - \frac{\phi'^2}{2} - \frac{m^2\phi^2}{2}$$

where $\phi' \equiv \frac{\partial\phi(t,x)}{\partial x}$ and $\dot\phi' \equiv \frac{\partial\phi(t,x)}{\partial t}$ as usual).

Similarly to the case of (3+1) dimensions one can get the Euler-Lagrange equation in the form of Klein-Gordon equation

$$\ddot\phi(t,x) - \phi''(t,x) ~=~ m^2\phi(t,x) \tag{4.2}$$

4.1 Lattice model for the KG field

Lattice model: a harmonic oscillator in each point of the grid and similar interaction between two adjacent oscillators:

$$L(t) ~=~ \sum_{n=-\infty}^{\infty}\Big[\frac{a}{2}\dot\phi_n^2(t) - \frac{a}{2}m^2\phi_n^2(t) - \frac{1}{2a}[\phi_{n+1}(t)-\phi_n(t)]^2\Big] \tag{4.3}$$

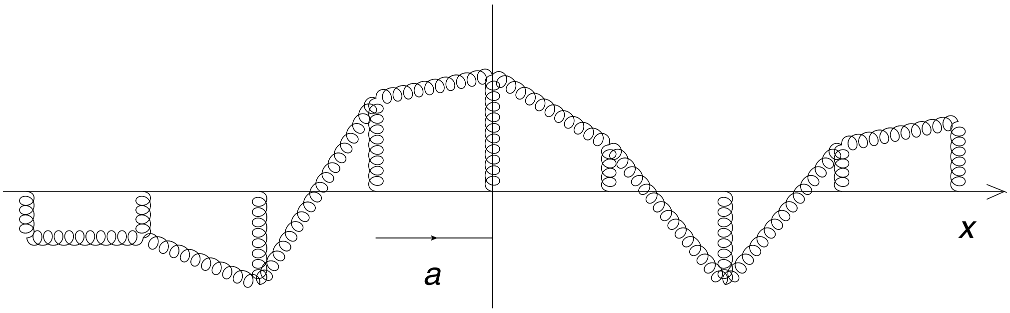


Figure 2. Lattice model: a harmonic oscillator in each point on the grid and similar interaction between two adjacent oscillators. The lattice spacing a is the same for all oscillators.

The classical Euler-Lagrange equations for the set of oscillators are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi} \Leftrightarrow \ddot{\phi}_n = -m^2\phi_n + \frac{\phi_{n+1} + \phi_{n-1} - 2\phi_n}{a^2} \tag{4.4}$$

The normal modes are

$$\phi_p(x_n, t) = \alpha_p e^{ianp - i\omega_p t} + \alpha_p^* e^{-ianp + i\omega_p t} \tag{4.5}$$

where p is a real number, α_p is an arbitrary complex number, and

$$\omega_p^2 = m^2 + \frac{4}{a^2}\sin^2\frac{ap}{2} \tag{4.6}$$

so the general solution has the form

$$\phi_n(t) = \int_{-\pi/a}^{\pi/a} \not{d}p \; (\alpha_p e^{ianp - i\omega_p t} + \alpha_p^* e^{-ianp + i\omega_p t}) \tag{4.7}$$

Let us now label each point on the grid not by its number, but by the position of the corresponding point on the lattice $x_n = na$

$$L(t) = \frac{1}{2}\sum_{n=-\infty}^{\infty} a\Big[\dot{\phi}^2(t, x_n) - m^2\phi^2(t, x_n) - \Big(\frac{\phi(t, x_{n+1}) - \phi(t, x_n)}{a}\Big)^2\Big] \tag{4.8}$$

In the "continuum limit" $a \to 0$ the above sum turns into integral

$$L(t) = \frac{1}{2}\int_{-\infty}^{\infty} dx \; [\dot{\phi}^2(t, x) - \phi'^2(t, x) - m^2\phi(t, x)^2] \tag{4.9}$$

which is KG lagrangian (3.2) (for 1+1 dimensions).

Thus, the classical Lagrangian for our lattice model turn to the Lagrangian of KG field in the continuum limit $a \to 0$. It is therefore natural to assume that quantization of our model will give a theory of quantum KG field in the limit $a \to 0$.

4.2 Quantization of the set of harmonic osciillators.

First we need to identify canonical momenta for the lattice Lagrangian (4.3).

$$\pi_n(t) \;=\; \frac{\partial L}{\partial \dot{\phi}_n}(t) \;=\; a\dot{\phi}_n(t) \tag{4.10}$$

Let us now label π_n by the point on the lattice $x_n = na$ and define

$$\pi(t,x_n)) \;\equiv\; \frac{1}{a}\pi_n(t) \;=\; \dot{\phi}_n(t) \;=\; \dot{\phi}(t,x_n) \tag{4.11}$$

(recall that $\pi(t,x) \;=\; \dot{\phi}(t,x)$ in the continuum limit, see Eq. (3.4).

Hamiltonian of our set of harmonic oscillators reads

$$\begin{aligned} H(t) \;&=\; \sum_{n=-\infty}^{\infty}\big[\pi_n(t)\dot{\phi}_n(t) - L(t)\big]\Big|_{\dot{\phi}_n=\pi_n} \\ &=\; \sum_{n=-\infty}^{\infty}\big[\frac{1}{2a}\pi_n^2(t) + \frac{1}{2a}[\phi_{n+1}(t)-\phi_n(t)]^2 + \frac{a}{2}m^2\phi_n^2(t)\big] \end{aligned} \tag{4.12}$$

In the notations $\phi(t,x_n)$ and $\pi(t,x_n)$ (see Eq. (4.10) it takes the form

$$H(t) \;=\; \frac{a}{2}\sum_{n=-\infty}^{\infty}\big[\pi^2(t,x_n) + \Big(\frac{\phi(t,x_{n+1})-\phi(t,x_n)}{a}\Big)^2 + m^2\phi(t,x_n)^2\big] \tag{4.13}$$

which reduces to the classical KG lagrangian in the continuum limit $a\to 0$

$$\int dx\big[\frac{\pi^2(t,x)}{2} + \frac{\big(\nabla\phi(t,x)\big)^2}{2} + \frac{m^2}{2}\phi^2(t,x)\big] \tag{4.14}$$

(cf. Eq. (3.5) for three space dimensions).

As usual, to quantize a set of harmonic oscillators we promote ϕ_n and π_n to operators $\hat{\phi}_n$ and $\hat{\pi}_n$ satisfying the canonical commutation relations

$$[\phi_n,\pi_n] \;=\; i, \qquad [\phi_m,\pi_n]\Big|_{m\neq n} \;=\; [\phi_m,\phi_n] \;=\; [\pi_m,\pi_n] \;=\; 0 \tag{4.15}$$

and solve the corresponding Schrödinger equaton

$$i\frac{d}{dt}|\Psi\rangle \;=\; \hat{H}|\Psi\rangle \tag{4.16}$$

where

$$\hat{H} \;=\; \frac{a}{2}\sum_{n=-\infty}^{\infty}\big[\hat{\pi}^2(x_n) + \Big(\frac{\hat{\phi}(x_{n+1})-\hat{\phi}(x_n)}{a}\Big)^2 + m^2\hat{\phi}(x_n)^2\big] \tag{4.17}$$

and $|\Phi\rangle$ is a vector of state for this system. In coordinate representation $|\Psi\rangle$ is described by a wave function depending on all coordinates:

$$\langle ...\phi_{-n}...\phi_{-1},\phi_0,\phi_1...\phi_n...|\Phi_t\rangle \;=\; \Phi(t,...\phi_{-n}...\phi_{-1},\phi_0,\phi_1...\phi_n...) \;=\; \Psi(t,\{\phi_n\}) \tag{4.18}$$

As usual, $|\Psi(t,\{\phi_n\})|^2$ is the probability to find the zeroth oscillator at the position ϕ_0, the first oscillator at the position ϕ_1, the (-1) oscillator at the position ϕ_{-1} etc. at time t.

The action of canonical operators $\hat{\phi}_n$ and $\hat{\pi}_n$ on the wave function in the coordinate representation is given by

$$\hat{\phi}_n\Psi(t,\{\phi_k\}) \;=\; \phi_n(t)\Psi(t,\{\phi_k\}), \qquad \hat{\pi}_n\Psi(t,\{\phi_k\}) \;=\; -i\frac{\partial}{\partial\phi_n}\Psi(t,\{\phi_k\}), \tag{4.19}$$

It is easy to check the canonical commutation relations:

$$\begin{aligned}
[\hat{\phi}_m,\hat{\pi}_n]\Psi(t,\{\phi_n\}) &= -i\phi_m\frac{\partial}{\partial\phi_n}\Psi(t,\{\phi_n\})+i\frac{\partial}{\partial\phi_n}\phi_m\Psi(t,\{\phi_k\}) \;=\; i\delta_{mn}\Psi(t,\{\phi_k\})\\
[\hat{\phi}_m,\hat{\phi}_n]\Psi(t,\{\phi_n\}) &= (\phi_m\phi_n - m\leftrightarrow n)\Psi(t,\{\phi_n\}) \;=\; 0\\
[\hat{\pi}_m,\hat{\pi}_n]\Psi(t,\{\phi_n\}) &= -i\big(\frac{\partial}{\partial\phi_m}\frac{\partial}{\partial\phi_n} - (m\leftrightarrow n)\big)\Psi(t,\{\phi_n\}) \;=\; 0
\end{aligned} \tag{4.20}$$

4.2.1 Normal modes for the set of oscillators

To find a solution to Schrödinger equation (4.16) we need to rewrite Hamiltonian in terms of normal modes (4.5). A general solution to classical equations for the set of oscillators is given by (4.7), the corresponding set of canonical momenta (4.7) have the form

$$\pi(x_n,t) \;=\; -i\int_{-\pi/a}^{\pi/a} đp\;\omega_p(\alpha_p e^{ianp-i\omega_p t} - \alpha_p^* e^{-ianp+i\omega_p t}) \tag{4.21}$$

and therefore the classical Hamiltonian (4.13) rewritten in terms of normal modes looks like

$$H(t) \;=\; \int_{-\pi/a}^{\pi/a} đp\;\omega_p\alpha_p^*\alpha_p \tag{4.22}$$

where we have used the formula

$$\sum_{n=-\infty}^{\infty} e^{ian(p-p')} \;=\; 2\pi\delta(a(p-p')) \;=\; \frac{2\pi}{a}\delta(p-p') \tag{4.23}$$

The r.h.s. of Eq. (4.22) does not depend on time which reflects the conservation of energy.

For the operators $\hat{\phi}(x_n)$ and $\hat{\pi}(x_n)$ the expansion in normal modes reads

$$\begin{aligned}
\hat{\phi}(x_n) &= \int_{-\pi/a}^{\pi/a}\frac{đp}{\sqrt{2\omega_p}}\,(\hat{\alpha}_p e^{ianp} + \hat{\alpha}_p^\dagger e^{-ianp})\\
\hat{\pi}(x_n) &= -i\int_{-\pi/a}^{\pi/a}\frac{đp}{\sqrt{2\omega_p}}\;\omega_p(\hat{\alpha}_p e^{ianp} - \hat{\alpha}_p^\dagger e^{-ianp})
\end{aligned} \tag{4.24}$$

where the factor $\sqrt{2\omega_p}$ is for convenience and ω_p is given by Eq. (4.6). The inverse formulas are

$$\begin{aligned}
\hat{\alpha}_p &= a\sum_{n=-\infty}^{\infty}\frac{\omega_p\hat{\phi}(x_n)+i\hat{\pi}(x_n)}{\sqrt{2\omega_p}}e^{-ianp}\\
\hat{\alpha}_p^\dagger &= a\sum_{n=-\infty}^{\infty}\frac{\omega_p\hat{\phi}(x_n)-i\hat{\pi}(x_n)}{\sqrt{2\omega_p}}e^{ianp}
\end{aligned} \tag{4.25}$$

It is easy to see that the operators $\hat{\alpha}_p$ satisfy the canonical commutation relations

$$[a_p, a^\dagger_{p'}] \;=\; 2\pi\delta(p-p'), \qquad [a_p, a_{p'}] \;=\; [a^\dagger_p, a^\dagger_{p'}] \;=\; 0 \tag{4.26}$$

One can check self-consistency

$$\begin{aligned}
[\hat{\phi}(x_m), \hat{\pi}(x_n)] \;&=\; -i\int_{-\pi/a}^{\pi/a} \frac{{\mathchar'26\mkern-12mu d} p}{\sqrt{2\omega_p}} \frac{{\mathchar'26\mkern-12mu d} p'}{\sqrt{2\omega_{p'}}}\,\omega_{p'}[\hat{\alpha}_p e^{iamp} + \hat{\alpha}^\dagger_p e^{-iamp}, \hat{\alpha}_{p'} e^{ianp'} - \hat{\alpha}^\dagger_{p'} e^{-ianp'}] \\
=\; i\int_{-\pi/a}^{\pi/a} {\mathchar'26\mkern-12mu d} p\, \cos a(m-n)p \;&=\; \frac{i}{a}\delta_{mn} \;\Leftrightarrow\; [\hat{\phi}_m, \hat{\pi}_n] \;=\; i\delta_{mn}
\end{aligned} \tag{4.27}$$

Similarly, one can check that $[\hat{\phi}_m, \hat{\phi}_n] \;=\; [\hat{\pi}_m, \hat{\pi}_n] \;=\; 0$.

4.3 Quantum KG field as a continuum limit of the set of harmonic osciillators.

In the continuum limit $a \to 0$ the lattice Hamiltonian (4.9) reduces to

$$\begin{aligned}
\hat{H} \;&=\; \frac{a}{2}\sum_{n=-\infty}^{\infty}\Big[\hat{\pi}_n^2(x) + \Big(\frac{\hat{\phi}(x_{n+1}) - \hat{\phi}(x_n)}{a}\Big)^2 + m^2\hat{\phi}(x_n)^2\Big] \\
&\overset{a\to 0}{=}\; \frac{1}{2}\int dx\big[\hat{\pi}^2(x) + \hat{\phi}'^2(x) + m^2\hat{\phi}^2(x)\big]
\end{aligned} \tag{4.28}$$

(recall that $\hat{\pi}(x_n) \;=\; \frac{1}{a}\pi_n$), see Eq. (4.7).

In the continuum limit the wave function (4.14) depends on the continuum set of coordinates $\phi(x)$ (each of $\phi(x)$ is an independent coordinate) so $\Psi(t, \phi(x))$ is actually a wave functional.

Let us find the explicit form of the canonical operators $\hat{\phi}$ and $\hat{\pi}$ in this "coordinate representation" of wave functional $\Psi(t, \phi(x))$.

For the operator $\hat{\phi}$ the limit $a \to 0$ is trivial:

$$\hat{\phi}(x_n)\Psi(t, \{\phi(x_k)\}) \;=\; \phi(x_n)\Psi(t, \{\phi(x_k)\}) \;\overset{a\to 0}{\to}\; \hat{\phi}(x)\Psi(t, \phi(z)) \;=\; \phi(x)\Psi(t, \phi(z)) \tag{4.29}$$

For the operator $\hat{\pi}$ the limit $a \to 0$ is more subtle.

Let us prove that

$$\lim_{a\to 0} \hat{\pi}(x_n)\Psi(t, \{\phi(x_k)\}) \;=\; \frac{\delta\Psi(t, \phi(z))}{\delta\phi(x)} \tag{4.30}$$

where the r.h.s. is a variational derivative defined in a usual way

$$\Psi(t, \phi(z) + h(x)) - \Psi(t, \phi(z)) \;=\; \int dx\; h(x)\frac{\delta\Psi(t, \phi(z))}{\delta\phi(x)} \;+\; O(h^2) \tag{4.31}$$

as a linear part of the deviation of the value of functional Ψ due to the small deviation of the argument of the functional $\Psi(t, \phi(z))$, same as in the Eq. (2.6) (here $h(x) \equiv \delta\phi(x)$) in Eq. (2.6)).

Proof: consider

$$\Psi(t, \{\phi(x_k) + h(x_k)\}) - \Psi(t, \phi(x_k)) \;=\; \sum_{n=-\infty}^{\infty} h(x_n)\frac{\partial\Psi(t, \{\phi(x_k)\})}{\partial\phi(x_n)} \;+\; O(h^2) \tag{4.32}$$

The canonical momentum operator is defined as

$$\hat{\pi}(x_n)\Psi(t,\{\phi(x_k)\}) \;=\; \frac{1}{a}\hat{\pi}_n\Psi(t,\{\phi(x_k)\}) \;=\; -\frac{i}{a}\frac{\partial\Psi(t,\{\phi(x_k)\})}{\partial\phi(x_n)} \tag{4.33}$$

so

$$\Psi(t,\{\phi(x_k)+h(x_k)\})-\Psi(t,\{\phi(x_k)\}) \;=\; i\sum_{n=-\infty}^{\infty} ah(x_n)\hat{\pi}(x_n)\Psi(t,\{\phi(x_k)\})$$

$$\overset{a\to 0}{\to}\;\; \Psi(t,\phi(z)+h(z))-\Psi(t,\phi(z)) \;=\; i\int dx\; h(x)\hat{\pi}(x)\Psi(t,\phi(z)) \tag{4.34}$$

Comparing the above formula to Eq. (4.43) we see that

$$\hat{\pi}(x)\Psi(t,\phi(z)) \;=\; -i\frac{\delta\Psi(t,\phi(z))}{\delta\phi(x)}, \qquad\qquad \text{Q.E.D.} \tag{4.35}$$

Let us now demonstrate that the canonical commutation relation (CCR)

$$[\hat{\phi}(x_m),\hat{\pi}(x_n)] \;=\; \frac{i}{a}\delta_{mn} \tag{4.36}$$

turns to

$$[\hat{\phi}(x),\hat{\pi}(y)] \;=\; i\delta(x-y) \tag{4.37}$$

in the limit $a\to 0$. Indeed, from Eq. (4.36) we see that if $x\neq y$ we get $[\hat{\phi}(x),\hat{\pi}(y)]=0$. In addition,

$$\sum_{m=-\infty}^{\infty} a[\hat{\phi}(x_m),\hat{\pi}(x_n)] \;=\; i \qquad \overset{a\to 0}{\to} \qquad \int dx'\;[\phi(x'),\hat{\pi}(x)] \;=\; i \tag{4.38}$$

which means that $[\hat{\phi}(x),\hat{\pi}(y)]=\delta(x-y)$ by definition of δ-function (1.2).

Check of self-consistency of our formulas: CCR (4.37) in the coordinate representation of vector of state

$$[\hat{\phi}(x),\hat{\pi}(y)]\Psi(t,\{\phi(z)\}) \;=\; -i\phi(x)\frac{\delta}{\delta\phi(y)}\Psi(t,\{\phi(z)\})+i\frac{\delta}{\delta\phi(y)}\Big(\phi(x)\Psi(t,\{\phi(z)\})\Big)$$

$$=\; i\Psi(t,\{\phi(z)\})\Big(\frac{\delta}{\delta\phi(y)}\phi(x)\Big) \;=\; i\delta(x-y)\Psi(t,\{\phi(z)\}) \tag{4.39}$$

To find $\frac{\delta}{\delta\phi(y)}\phi(x)$ let us note that for the functional $F(\phi)=\int dx' f(x-x')\phi(x')$ the definition of variational derivative (2.6) gives

$$\frac{\delta}{\delta\phi(y)}F(\phi) \;=\; \frac{\delta}{\delta\phi(y)}\int dx'\;\phi(x')f(x'-x) \;=\; f(x-y) \tag{4.40}$$

Next, because by definition (1.2) of δ-function

$$\phi(x) \;=\; \int dx'\;\phi(x')\delta(x'-x) \tag{4.41}$$

we can use Eq. (4.28) for the functional $F(\phi)=\phi(x)=\int dx'\;\phi(x')\delta(x'-x)$ and get

$$\frac{\delta}{\delta\phi(y)}\phi(x) \;=\; \delta(x-y) \tag{4.42}$$

so the commutation relation (4.41) turns to CCR

$$[\hat{\phi}(x), \hat{\pi}(y)]\Psi(t, \{\phi(z)\}) \;=\; i\delta(x-y)\Psi(t, \{\phi(z)\}) \tag{4.43}$$

Now we are in a position to write down the Schrödinger equation for the wave functional $\Phi(t, \phi(z))$

$$\begin{aligned} i\frac{d}{dt}\Psi(t, \{\phi(z)\}) \;&=\; \hat{H}\Psi(t, \{\phi(z)\}) \;=\; \frac{1}{2}\int dx\big[\hat{\pi}^2(x) + \hat{\phi'}^2(x) + m^2\hat{\phi}^2(x)\big]\Psi(t, \{\phi(z)\}) \;= \\ =\; &\frac{1}{2}\int dx\big[\big(\frac{\delta}{\delta\phi(x)}\big)^2 + \phi'^2(x) + m^2\phi^2(x)\big]\Psi(t, \{\phi(z)\}) \end{aligned} \tag{4.44}$$

For the stationary states $\Psi(t, \{\phi(z)\}) \;=\; e^{-iEt}\Psi_E(\{\phi(z)\})$ so the Schrödinger equation (4.44) takes the form

$$\frac{1}{2}\int dx\big[\frac{\delta}{\delta\phi(x)}\frac{\delta}{\delta\phi(x)} + \phi'^2(x) + m^2\phi^2(x)\big]\Psi_E(t, \{\phi(z)\}) \;=\; E\Psi_E(t, \{\phi(z)\}) \tag{4.45}$$

4.3.1 3-dimensional KG field

At this point it is convenient to remember that our Universe has three spatial dimensions so the KG field ϕ is a function of $x = (t, \vec{x})$. To quantize the 3d KG field $\phi(t, \vec{x})$ we can repeat the steps discussed in the previous Section: consider 3d lattice of harmonic oscillators with nearest-neighbor interactions, quantize this grid of oscillators and take the limit of lattice spacing $a \to 0$. The wave functional $\Psi(t, \phi(x)$ satisfies the Schrödinger equation

$$\begin{aligned} i\frac{d}{dt}\Psi(t, \phi(x) \;&=\; \hat{H}\Psi(t, \phi(x) \\ \hat{H} \;&=\; \frac{1}{2}\int d^3x[\hat{\pi}^2(t, \vec{x}) + |\nabla\phi(t, \vec{x})|^2 + m^2\phi^2(t, \vec{x})] \end{aligned} \tag{4.46}$$

where operators of canonical coordinate and canonical momentum

$$\begin{aligned} \hat{\phi}(\vec{x})\Psi(t, \phi) \;&=\; \phi(\vec{x})\Psi(t, \phi) \\ \hat{\pi}(\vec{x}) \;&=\; -i\frac{\delta}{\delta\phi(\vec{x})}\Psi(t, \phi) \end{aligned} \tag{4.47}$$

Check of the canonical commutation relation :

$$[\hat{\phi}(\vec{x}), \hat{\pi}(\vec{y})] \;=\; i\frac{\delta}{\delta\phi(\vec{y})}\phi(\vec{x}) \;=\; i\delta^{(3)}(\vec{x}-\vec{y}) \tag{4.48}$$

The Schrödinger equation for stationary states

$$\Psi(t, \phi) \;=\; e^{-iEt}\Psi_E(\phi)$$

takes the form

$$\hat{H}\Psi_E(\phi) \;=\; E\Psi_E(\phi) \;\Leftrightarrow\; \frac{1}{2}\int d^3x\,\Big[-\Big(\frac{\delta}{\delta\phi(\vec{x})}\Big)^2 + |\nabla\phi(\vec{x})|^2 + m^2\phi^2(\vec{x})\Big]\Psi_E(\phi) \;=\; E\Psi_E(\phi) \tag{4.49}$$

Part III

4.3.2 Trivial solution of QFT Schrödinger equation: vacuum state

For the harmonic oscillator

$$\langle\phi|0\rangle \;\sim\; e^{-\omega\frac{\phi^2}{2}} \tag{4.50}$$

(Educated) guess: for the Klein-Gordon field the vacuum wave function(al) should be similar

$$\langle\phi(\vec{x})|0\rangle \;\sim\; e^{-\frac{1}{2}\int d^3x\;(\omega?)\;\phi^2(\vec{x})} \tag{4.51}$$

Q: What is the analog of $\omega\phi^2$?
A: $\phi(\vec{x})\sqrt{m^2+\nabla^2}\phi(\vec{x})$

Motivation:

For the oscillator: the classical path is $\phi(t) \;=\; e^{\pm i\omega t}$.
For the KG field: the classical solution is [1]

$$\phi(x) \;\sim\; \int d\!\!{}^-{}^3p\; e^{i\vec{p}\cdot\vec{x}\pm i\omega_p t}, \qquad \omega_p \;=\; \sqrt{m^2+\vec{p}^2)} \tag{4.52}$$

The Eq. (4.52) looks like a superposition of oscillators with frequencies depending on $|\vec{p}|$ $\Rightarrow$

Our guess for the wave functional of vacuum state for the KG field is a product of wave functions for these oscillators:

$$\Pi_x e^{-\frac{1}{2}\phi(\vec{x})\omega_p\phi(\vec{x})} \;\sim\; e^{-\frac{1}{2}\int d^3x\phi(\vec{x})\omega_p\phi(\vec{x})} \;=\; e^{-\frac{1}{2}\int d^3x\phi(\vec{x})\sqrt{m^2-\nabla^2}\phi(\vec{x})} \tag{4.53}$$

because

$$\sqrt{m^2-\nabla^2}\phi(\vec{x}) \;=\; \sqrt{m^2-\nabla^2}\int d^3p\; e^{i\vec{p}\cdot\vec{x}}\phi(\vec{p}) \;=\; \int d\!\!{}^-{}^3p\;\sqrt{m^2+\vec{p}^2}e^{i\vec{p}\cdot\vec{x}}\phi(\vec{p}) \tag{4.54}$$

Thus, our guess for the KG wave functional for the vacuum state $|\Omega\rangle$ is

$$\langle\phi(\vec{x})|\Omega\rangle \;=\; \Psi_{\rm vac}\big(\phi(\vec{x})\big) \;=\; N^{-1}e^{-\frac{1}{2}\int d^3x\phi(\vec{x})\hat{W}\phi(\vec{x})} \tag{4.55}$$

where W is a differential operator (4.54) defined as

$$W\phi(x) \equiv \sqrt{m^2-\nabla^2}\phi(\vec{x}) \;=\; \int d\!\!{}^-{}^3p\;\sqrt{m^2+\vec{p}^2}e^{i\vec{p}\cdot\vec{x}}\phi(\vec{p}) \tag{4.56}$$

and N^{-1} is some normalization factor. In the momentum representation it is even more simple

$$\langle\phi(\vec{p})|\Omega\rangle \;=\; \Psi_{\rm vac}\big(\phi(\vec{p})\big) \;=\; N^{-1}e^{-\frac{1}{2}\int d\!\!{}^-{}^3p\;\omega_p\phi(\vec{p})\phi(\vec{p})} \tag{4.57}$$

Let us now prove our guess (4.55). To this end we need to check that the Eq. (4.55) satisfies Schrödinger equation (4.49)

$$\hat{H}\Psi_{\rm vac}(\phi) \;\equiv\; \frac{1}{2}\int d^3x\;\Big[-\Big(\frac{\delta}{\delta\phi(\vec{x})}\Big)^2+|\nabla\phi(\vec{x})|^2+m^2\phi^2(\vec{x})\Big]\Psi_{\rm vac}(\phi) \;=\; E_{\rm vac}\Psi_{\rm vac}(\phi) \tag{4.58}$$

[1] Note that ω_p from Eq. (4.6) turns to $\sqrt{m^2+\vec{p}^2}$ in the continuum limit $a\to 0$. This is relation between energy and momentum of a Klein-Gordon scalar particle as we shall see below.

Second two terms are evident so we need to find $\left(\frac{\delta}{\delta\phi(\vec{x})}\right)^2\Psi_{\rm vac}(\phi)$. A first step is to find

$$\frac{\delta}{\delta\phi(\vec{x})}\Psi_{\rm vac}(\phi) \;=\; ? \tag{4.59}$$

According to general definition of variational derivative (4.31)

$$\begin{aligned}
\Psi_{\rm vac}(\phi+\delta\phi)-\Psi_{\rm vac}(\phi) \;&=\; \exp\Big[-\frac{1}{2}\int d^3x\;(\phi+\delta\phi)\hat{W}(\phi+\delta\phi)\Big]-\exp\big[-\frac{1}{2}\int d^3x\;\phi\hat{W}\phi\big]\\
=\;& \exp\Big[-\frac{1}{2}\int d^3x\;\Big(\phi\hat{W}\phi+\delta\phi\hat{W}\phi+\phi\hat{W}\delta\phi+\delta\phi\hat{W}\delta\phi\Big)\Big]-\exp\big[-\frac{1}{2}\int d^3x\;\phi\hat{W}\phi\big]\\
\simeq\;& \exp\big[-\frac{1}{2}\int d^3x\;\phi\hat{W}\phi\big]\Big\{1-\int d^3x\;\delta\phi(\vec{x})\hat{W}\phi(\vec{x})+O(\delta\phi^2)\Big\}-\exp\big[-\frac{1}{2}\int d^3x\;\phi\hat{W}\phi\big]\\
=\;& -\int d^3x\;\delta\phi(\vec{x})\hat{W}\phi(\vec{x})\Psi_{\rm vac}(\phi)+O(\delta\phi^2)
\end{aligned}\tag{4.60}$$

so

$$\frac{\delta}{\delta\phi(\vec{x})}\Psi_{\rm vac}(\phi) \;=\; -\hat{W}\phi(\vec{x})\Psi_{\rm vac}(\phi) \tag{4.61}$$

Now

$$\begin{aligned}
\Big(\frac{\delta}{\delta\phi(\vec{x})}\Big)^2\Psi_{\rm vac}(\phi) \;&=\; -\frac{\delta}{\delta\phi(\vec{x})}\Big[\hat{W}\phi(\vec{x})\Psi_{\rm vac}(\phi)\Big]\\
&=\; -\Big[\frac{\delta}{\delta\phi(\vec{x})}\hat{W}\phi(\vec{x})\Big]\Psi_{\rm vac}(\phi)-\big[\hat{W}\phi(\vec{x})\big]\frac{\delta}{\delta\phi(\vec{x})}\Psi_{\rm vac}(\phi)
\end{aligned}\tag{4.62}$$

Let us find first $\frac{\delta}{\delta\phi(\vec{x})}\frac{\delta}{\delta\phi(\vec{y})}\Psi_{\rm vac}(\phi)$ at $x\neq y$

$$\frac{\delta}{\delta\phi(\vec{x})}\frac{\delta}{\delta\phi(\vec{y})}\Psi_{\rm vac}(\phi) \;=\; -\Big[\frac{\delta}{\delta\phi(\vec{x})}\hat{W}\phi(\vec{y})\Big]\Psi_{\rm vac}(\phi)+\big[\hat{W}\phi(\vec{x})\big]\big[\hat{W}\phi(\vec{y})\big]\Psi_{\rm vac}(\phi) \tag{4.63}$$

and take the limit $x\to y$ afterwards.

To get the first term let us find $\frac{\delta}{\delta\phi(\vec{x})}\hat{W}\phi(\vec{y})$

$$\hat{W}\phi(\vec{y}) \;=\; \int d^3p\,\omega_p\phi(\vec{p})e^{i\vec{p}\cdot\vec{y}} \;=\; \int d^3p\,\omega_p e^{i\vec{p}\cdot\vec{y}}\int d^3z\,e^{-i\vec{p}\cdot\vec{z}}\phi(\vec{z}) \;=\; \int d^3z\,\phi(\vec{z})\int d^3p\,\omega_p e^{i\vec{p}\cdot(\vec{y}-\vec{z})} \tag{4.64}$$

and therefore

$$\begin{aligned}
\hat{W}\big(\phi(\vec{y})+\delta\phi(\vec{y})\big) \;&=\; \int d^3z\;[\phi(\vec{z})+\delta\phi(\vec{z})]\int d^3p\;\omega_p e^{i\vec{p}\cdot(\vec{y}-\vec{z})}\\
\Rightarrow\;\frac{\delta}{\delta\phi(\vec{x})}\hat{W}\big(\phi(\vec{y})\big) \;&=\; \int d^3p\;\omega_p e^{i\vec{p}\cdot(\vec{y}-\vec{x})}
\end{aligned}\tag{4.65}$$

Finally

$$-\frac{\delta}{\delta\phi(\vec{x})}\frac{\delta}{\delta\phi(\vec{y})}\Psi_{\rm vac}(\phi) \;=\; \Big\{\int d^3p\;\omega_p e^{i\vec{p}\cdot(\vec{y}-\vec{x})}-[\hat{W}\phi(\vec{x})][\hat{W}\phi(\vec{y})]\Big\}\Psi_{\rm vac}(\phi) \tag{4.66}$$

Now let us take the limit $\vec{y}\to\vec{x}$

$$-\frac{\delta}{\delta\phi(\vec{x})}\frac{\delta}{\delta\phi(\vec{x})}\Psi_{\rm vac}(\phi) \;=\; \Big\{\int d^3p\;\omega_p-[\hat{W}\phi(\vec{x})][\hat{W}\phi(\vec{x})]\Big\}\Psi_{\rm vac}(\phi) \tag{4.67}$$

and integrate over $\vec{x}$. We get

$$-\int d^3x \frac{\delta}{\delta\phi(\vec{x})}\frac{\delta}{\delta\phi(\vec{x})}\Psi_{\rm vac}(\phi) = \Big\{\int d^3x \int d^3p\ \omega_p - \int d^3x [\hat{W}\phi(\vec{x})][\hat{W}\phi(\vec{x})]\Big\}\Psi_{\rm vac}(\phi)$$
$$= \int d^3x \int d^3p\ \omega_p - \int d^3x\ [m^2\phi^2(\vec{x}) + |\vec{\nabla}\phi(\vec{x})|^2] \tag{4.68}$$

where we used

$$\int d^3x [\hat{W}\phi(\vec{x})][\hat{W}\phi(\vec{x})]\Big\} = \int d^3x \int \text{đ}^3p\ \omega_p e^{i\vec{p}\cdot\vec{x}}\phi(\vec{p}) \int \text{đ}^3p'\ \omega_{p'} e^{i\vec{p}\cdot\vec{x}}\phi(\vec{p}') \tag{4.69}$$
$$= \int \text{đ}^3p \int \text{đ}^3p'\ \omega_p\omega_{p'}\phi(\vec{p})\phi(\vec{p}')(2\pi)^3\delta(\vec{p}+\vec{p}') = \int \text{đ}^3p\ \omega_p^2\phi(\vec{p})\phi(-\vec{p})$$
$$= \int \text{đ}^3p\ (m^2+\vec{p}^2)\phi(\vec{p})\phi(-\vec{p}) = \int \text{đ}^3p\ (m^2+\vec{p}^2)\int d^3y\ e^{-i\vec{p}\cdot\vec{y}}\phi(\vec{y}) \int d^3z\ e^{i\vec{p}\cdot\vec{z}}\phi(\vec{z})$$
$$= m^2\phi^2(\vec{x}) + \int \text{đ}^3p \int d^3y\ \frac{\partial}{\partial y_i} e^{-i\vec{p}\cdot\vec{y}}\phi(\vec{y}) \int d^3z\ \frac{\partial}{\partial z_i} e^{i\vec{p}\cdot\vec{z}}\phi(\vec{z}) = m^2\phi^2(\vec{x}) + \partial_i\phi(\vec{x})\partial_i\phi(\vec{x})$$

Thus, we get Eq. (4.58)

$$\hat{H}\Psi_{\rm vac}(\phi) \equiv \frac{1}{2}\int d^3x\ \Big[-\Big(\frac{\delta}{\delta\phi(\vec{x})}\Big)^2 + |\nabla\phi(\vec{x})|^2 + m^2\phi^2(\vec{x})\Big]\Psi_{\rm vac}(\phi) = E_{\rm vac}\Psi_{\rm vac}(\phi) \tag{4.70}$$

with eigenvalue being the vacuum energy

$$E_{\rm vac} = \int d^3x \int \text{đ}^3p\ \frac{\omega_p}{2} = V\int d^3p\ \frac{\omega_p}{2} \tag{4.71}$$

where V is the volume of the 3-dimensional space. The vacuum energy density

$$\mathcal{E}_{\rm vac} = \int \text{đ}^3p\ \frac{\omega_p}{2} = V\int d^3p\ \frac{\omega_p}{2} \tag{4.72}$$

is a sum of energies of oscillators with different momenta and each oscillator brings $\frac{\omega_p}{2}$. This sum (strictly speaking, integral) diverges. It is a general feature of all QFTs except the supersymmetric ones where the vacuum energy vanishes.

4.3.3 Perturbation series for QFT Schrödinger equation

The perturbative series for a quantum theory is constructed like that: suppose we have a Hamiltonian of the form

$$\hat{H} = \hat{H}_0 + \lambda\hat{H}_I \qquad \text{with } \lambda \ll 1 \tag{4.73}$$

The task is to find the spectrum of $\hat{H}$ and the probabilities of transitions between different states as a series in small parameter λ (in QFT, these probabilities are expressed in terms of cross sections of particle scattering).

For example, let us consider the interaction Hamiltonian of the form

$$\lambda\hat{H}_I = \lambda\int d^3x\ \hat{\phi}^4(\vec{x}) \tag{4.74}$$

The usual procedure to get a perturbative series would be to solve the Schrödinger equation by iterations, namely write down

$$\Psi(\phi) = \Psi_0(\phi) + \lambda\Psi_1(\phi) + \lambda^2\Psi_2(\phi) + ... \tag{4.75}$$
$$(\hat{H}_0 + \lambda\hat{H}_I)(\Psi_0(\phi) + \lambda\Psi_1(\phi) + \lambda^2\Psi_2(\phi) + ...) = (E_0 + \lambda E_1 + \lambda^2 E_2 + ...)(\Psi_0(\phi) + \lambda\Psi_1(\phi) + \lambda^2\Psi_2(\phi) + ...)$$

and solve:

$$\begin{aligned} \hat{H}_0\Psi_0(\phi) &= E_0\Psi_0(\phi), \\ (\hat{H}_0 - E_0)\Psi_1(\phi) &= E_1\Psi_0(\phi) - \hat{H}_I\Psi_0(\phi) \quad \Rightarrow \quad E_1 = \frac{\langle\Psi_0|\hat{H}_I|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle} \\ \Psi_1(\phi) &= \frac{1}{\hat{H}_0 - E_0}(E_1\Psi_0 - \hat{H}_I\Psi_0) \end{aligned} \tag{4.76}$$

The first equation for E_1 for wave functional reads

$$E_1 = \frac{\int D\phi(\vec{x})\Psi_0(\phi(\vec{x}))\hat{H}_I\Psi_0(\phi(\vec{x}))}{\int D\phi(\vec{x})\Psi_0(\phi(\vec{x})\Psi_0(\phi(\vec{x})} \tag{4.77}$$

for wave functionals $\Psi_0(\phi(\vec{x})$. If somebody will calculate this ratio (with functional integrals insted of usual integrals) he will definitely be lost of the second step of inverting of a variational derivative operator. Probably, this procedure can be implemented on the lattice, but in this case people just calculate functional integrals for the full Hamiltonian (4.73). Bottom line: standard construction of QM perturbative series does not easily generalize to QFT. Fortunately, there is another way - the QM ladder operator formalism can be easily generalized to QFT.

5 Ladder formalism

5.1 Ladder operator formalism for harmonic oscillator

Reminder: we define

$$\left.\begin{aligned} \hat{a} &= \tfrac{\omega\hat{\phi}+i\hat{\pi}}{\sqrt{2\omega}} \\ \hat{a}^\dagger &= \tfrac{\omega\hat{\phi}-i\hat{\pi}}{\sqrt{2\omega}} \end{aligned}\right\} \quad \text{ladder operators} \tag{5.1}$$

Commutation relation for these operators reads

$$[\hat{a}, \hat{a}^\dagger] = \frac{1}{2\omega}(-i\omega[\hat{\phi}, \hat{\pi}] + i\omega[\hat{\pi}, \hat{\phi}]) = 1 \tag{5.2}$$

Hamiltonian

$$\hat{H} = \frac{\hat{\pi}^2}{2} + \frac{\omega^2}{2}\hat{\phi}^2 = \omega\hat{a}^\dagger\hat{a} + \frac{\omega}{2} \tag{5.3}$$

Property:

$$\hat{a}|0\rangle = \frac{\omega\hat{\phi} + i\hat{\pi}}{\sqrt{2\omega}}e^{-\frac{\omega}{2}\phi^2} = 0 \tag{5.4}$$

so $\hat{a}$ is called an "annihilation operator". This equation

$$\hat{a}|0\rangle = 0 \tag{5.5}$$

may serve as a definition of the vacuum state $|0\rangle$.

Excited states:

$$|n\rangle \;=\; c_n(\hat{a}^\dagger)^n|0\rangle, \qquad E_n \;=\; \omega(n+\frac{1}{2}) \tag{5.6}$$

where $c_n = \frac{1}{\sqrt{n!}}$ - normalization factor. Let us check that Eq. (5.6) is an eigenstate of the Hamiltonian (5.3). First, note that

$$\begin{aligned} [\hat{H},\hat{a}^\dagger] \;&=\; \omega\hat{a}^\dagger \\ \Rightarrow\quad [\hat{H},(\hat{a}^\dagger)^n] \;&=\; [\hat{H},\hat{a}^\dagger](\hat{a}^\dagger)^{n-1} + \hat{a}^\dagger[\hat{H},\hat{a}^\dagger](\hat{a}^\dagger)^{n-1} + ... + (\hat{a}^\dagger)^{n-1}[\hat{H},\hat{a}^\dagger] \;=\; n\omega(\hat{a}^\dagger)^n \end{aligned} \tag{5.7}$$

and therefore

$$\hat{H}|n\rangle \;=\; c_n\hat{H}(\hat{a}^\dagger)^n|0\rangle \;=\; c_n[\hat{H},(\hat{a}^\dagger)^n]|0\rangle + c_n(\hat{a}^\dagger)^n\hat{H}|0\rangle \;=\; c_n(n\omega+\frac{1}{2})(\hat{a}^\dagger)^n|0\rangle \;=\; (n\omega+\frac{1}{2})|n\rangle \tag{5.8}$$

so $|n\rangle$ is an eigenstate with energy $E_n = \omega(n+\frac{1}{2})$. (The normalization factor $c_n = \frac{1}{\sqrt{n!}}$ is derived in QM courses but we will not need its explicit form in what follows).

Thus, the state $|0\rangle$ is the vacuum state, the state $\hat{a}^\dagger|0\rangle$ is the first excited state, $\frac{1}{\sqrt{2}}(\hat{a}^\dagger)^2|0\rangle$ is the second excited state etc. For this reason the operator $\hat{a}^\dagger$ is called a "creation operator".

Similarly to Eq. (5.7) one can show that $[\hat{H},\hat{a}] \;=\; -\omega\hat{a}$. Note also that

$$\hat{a}^\dagger|n\rangle \;=\; \sqrt{n+1}|n+1\rangle, \qquad \hat{a}|n\rangle \;=\; \sqrt{n}|n-1\rangle \tag{5.9}$$

which justifies the name "ladder operators" (you climb up and down a ladder $|n\rangle$).

So, why these operators are more convenient that $\hat{\phi}$ and $\hat{\pi}$? Let us consider an example: shift of vacuum state energy for anharmonic oscillator with $\hat{H}_I \;=\; \lambda\hat{\phi}^4$. Conventional calculation reads

$$\langle 0|\hat{H}_I|0\rangle \;=\; \sqrt{\frac{\omega}{\pi}}\int d\phi\; e^{-\frac{\omega}{2}\phi^2}\lambda\phi^4 e^{-\frac{\omega}{2}\phi^2} \;=\; \lambda\sqrt{\frac{\omega}{\pi}}\int d\phi\;\phi^4 e^{-\omega\phi^2} \;=\; \frac{3\lambda}{4\omega^2} \tag{5.10}$$

On the other hand, in terms of ladder operators we get (recall that $a|0\rangle \;=\; \langle 0|a^\dagger \;=\; 0$)

$$\langle 0|\hat{H}_I|0\rangle \;=\; \frac{\lambda}{4\omega^2}\langle 0|(\hat{a}+\hat{a}^\dagger)^4|0\rangle \;=\; \frac{\lambda}{4\omega^2}\langle 0|(\hat{a}+\cancel{\hat{a}^\dagger})(\hat{a}+\hat{a}^\dagger)^2(\cancel{\hat{a}}+\hat{a}^\dagger)|0\rangle \tag{5.11}$$

$$\begin{aligned} &=\; \frac{\lambda}{4\omega^2}\langle 0|(\hat{a}^2+[\hat{a},\hat{a}^\dagger]+\cancel{\hat{a}^\dagger}\hat{a})((\hat{a}^\dagger)^2+[\hat{a},\hat{a}^\dagger]+\hat{a}^\dagger\cancel{\hat{a}})|0\rangle \;=\; \frac{\lambda}{4\omega^2}\langle 0|\hat{a}^2(\hat{a}^\dagger)^2+1|0\rangle \\ &=\; \frac{\lambda}{4\omega^2}\langle 0|\hat{a}([\hat{a},\hat{a}^\dagger]+\hat{a}^\dagger\hat{a})\hat{a}^\dagger+1|0\rangle \;=\; \frac{\lambda}{4\omega^2}\langle 0|\hat{a}\hat{a}^\dagger+\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger+1|0\rangle \\ &=\; \frac{\lambda}{4\omega^2}\langle 0|[\hat{a},\hat{a}^\dagger]+[\hat{a},\hat{a}^\dagger][\hat{a},\hat{a}^\dagger]+1 \;=\; \frac{3\lambda}{4\omega^2} \end{aligned} \tag{5.12}$$

We see that the integration over ϕ is replaced by commuting various ladder operators. For harmonic oscillator, it is about equally difficult (or equally easy). However, in QFT we have an infinite (and worse, continuous) set of coordinates $\phi(x)$ so we will have an infinite and continuous number of integrations if we try to generalize Eq. (5.10) to QFT. On the contrary, as we shall see in below, the ladder operator formalism easily generalizes to QFT.

5.2 Ladder operator formalism in quantum field theory

To generalize ladder operator formalism to QFT we consider the lattice model for Klein-Gordon field and take the limit $a \to 0$. The expansion (4.24) in normal modes turns to

$$
\begin{aligned}
\hat{\phi}(x) &= \int \frac{đp}{\sqrt{2\omega_p}} (\hat{a}_{\vec{p}} e^{i\vec{p}\vec{x}} + \hat{a}^\dagger_{\vec{p}} e^{-i\vec{p}\vec{x}}) = \int \frac{đp}{\sqrt{2\omega_p}} (\hat{a}_{\vec{p}} + \hat{a}^\dagger_{-\vec{p}}) e^{i\vec{p}\vec{x}} \\
\hat{\pi}(x) &= -i \int \frac{đp}{\sqrt{2\omega_p}} \omega_{\vec{p}} (\hat{a}_p e^{i\vec{p}\vec{x}} - \hat{a}^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}}) = -i \int \frac{đp}{\sqrt{2\omega_p}} \omega_{\vec{p}} (\hat{a}_p - \hat{a}^\dagger_{-\vec{p}}) e^{i\vec{p}\cdot\vec{x}}
\end{aligned}
\tag{5.13}
$$

where we relabeled the normal modes as $\hat{a}_p$ and $\hat{a}^\dagger_p$. The inverse formulas can be obtained by taking the limit $a \to 0$, $n \to \infty$ in Eq. (4.25)

$$
\begin{aligned}
\hat{a}_{\vec{p}} &= \int d^3x \frac{1}{\sqrt{2\omega_p}} [\hat{\phi}(\vec{x}) + i\hat{\pi}(\vec{x})] e^{-i\vec{p}\cdot\vec{x}} \\
\hat{a}^\dagger_{\vec{p}} &= \int d^3x \frac{1}{\sqrt{2\omega_p}} [\hat{\phi}(\vec{x}) - i\hat{\pi}(\vec{x})] e^{i\vec{p}\cdot\vec{x}}
\end{aligned}
\tag{5.14}
$$

From Eq. (4.26) we see that $[\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{p}'}] = (2\pi)^3 \delta(\vec{p} - \vec{p}')$.
Self-consistency check:

$$
\begin{aligned}
[\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{p}'}] &= \int d^3x d^3y \ e^{i\vec{p}\cdot\vec{x} - i\vec{p}'\cdot\vec{y}} \frac{1}{2\sqrt{\omega_p \omega_{p'}}} (-i\omega_p [\hat{\phi}(\vec{x}), \hat{\pi}(\vec{y})] + i\omega_{p'} [\hat{\pi}(\vec{x}), \hat{\phi}(\vec{y})]) \\
&= \int d^3x d^3y \ e^{i\vec{p}\cdot\vec{x} - i\vec{p}'\cdot\vec{y}} \frac{\omega_p + \omega_{p'}}{2\sqrt{\omega_p \omega_{p'}}} \delta(\vec{x} - \vec{y}) = \frac{\omega_p + \omega_{p'}}{2\sqrt{\omega_p \omega_{p'}}} \int d^3x \ e^{i(\vec{p} - \vec{p}')\cdot\vec{x}} = (2\pi)^3 \delta(\vec{p} - \vec{p}')
\end{aligned}
\tag{5.15}
$$

Similarly, one can demonstrate that $[\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{p}'}] = [\hat{a}^\dagger_{\vec{p}}, \hat{a}^\dagger_{\vec{p}'}] = 0$.

Lets summarize CCR in ladder formalism

$$
\begin{aligned}
[\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{p}'}] &= (2\pi)^3 \delta(\vec{p} - \vec{p}') \\
[\hat{a}_{\vec{p}}, \hat{a}_{\vec{p}'}] &= [\hat{a}^\dagger_{\vec{p}}, \hat{a}^\dagger_{\vec{p}'}] = 0
\end{aligned}
\tag{5.16}
$$

Hamiltonian in terms of ladder operators

$$
\begin{aligned}
\hat{H} &= \frac{1}{2} \int d^3x \ [\hat{\pi}^2(\vec{x}) + |\nabla\hat{\phi}(\vec{x})|^2 + m^2 \hat{\phi}^2(x)] \\
&= \frac{1}{2} \int d^3x đ^3p đ^3p' \ e^{i(\vec{p}+\vec{p}')\cdot\vec{x}} \Big[-\frac{1}{2}\sqrt{\omega_p \omega_{p'}} (\hat{a}_{\vec{p}} - \hat{a}^\dagger_{-\vec{p}})(\hat{a}_{\vec{p}'} - \hat{a}^\dagger_{-\vec{p}'}) + \frac{-\vec{p}\cdot\vec{p}' + m^2}{\sqrt{\omega_p \omega_{p'}}} (\hat{a}_{\vec{p}} + \hat{a}^\dagger_{-\vec{p}})(\hat{a}_{\vec{p}'} + \hat{a}^\dagger_{-\vec{p}'}) \Big] \\
&= \frac{1}{2} \int đ^3p \ \Big[-\frac{\omega_p}{2} (\hat{a}_{\vec{p}} - \hat{a}^\dagger_{-\vec{p}})(\hat{a}_{\vec{p}'} - \hat{a}^\dagger_{-\vec{p}'}) + \frac{\omega_p}{2} (\hat{a}_{\vec{p}} + \hat{a}^\dagger_{-\vec{p}})(\hat{a}_{\vec{p}'} + \hat{a}^\dagger_{-\vec{p}'}) \\
&= \int đ^3p \ \frac{\omega_p}{2} (\hat{a}_{\vec{p}} \hat{a}^\dagger_{\vec{p}} + \hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{p}}) = \int đ^3p \ \frac{\omega_p}{2} (2\hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{p}} + [\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{p}}]) = \int đ^3p \ \{\omega_p \hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{p}} + \frac{\omega_p}{2} (2\pi)^3 \delta(0)\} \\
&= \int đ^3p \ \omega_p \hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{p}} \ + \ V \int đ^3p \ \frac{\omega_p}{2}
\end{aligned}
\tag{5.17}
$$

where $(2\pi)^3 \delta(0) = \int d^3x = V$ where V is the total volume of 3-dim space. This (last) term in Eq. (5.17) is the infinite total vacuum energy (4.71) which does not affect any transition amplitudes (cross sections). In what follows we omit this term so for our purposes the KG Hamiltonian in terms of ladder operators has the form

$$\hat{H} \ = \ \int đ^3p\ \omega_p \hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}} \tag{5.18}$$

Commutators of $\hat{H}$ with ladder operators

$$\begin{aligned}
[\hat{H},\hat{a}^\dagger_{\vec{p}}] \ &= \ \int đ^3k\ \omega_k[\hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{k}},\hat{a}^\dagger_{\vec{p}}] \ = \ \int đ^3k\ \omega_k\hat{a}^\dagger_{\vec{k}}[\hat{a}_{\vec{k}},\hat{a}^\dagger_{\vec{p}}] \ = \ \int đ^3k\ \omega_k\hat{a}^\dagger_{\vec{k}}(2\pi)^3\delta(\vec{p}-\vec{k}) \ = \ \omega_p\hat{a}^\dagger_{\vec{p}} \\
[\hat{H},\hat{a}_{\vec{p}}] \ &= \ \int đ^3k\ \omega_k[\hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{k}},\hat{a}_{\vec{p}}] \ = \ \int đ^3k\ \omega_k[\hat{a}^\dagger_{\vec{k}},\hat{a}_{\vec{p}}]\hat{a}_{\vec{k}} \ = \ -\int đ^3k\ \omega_k\hat{a}_{\vec{k}}(2\pi)^3\delta(\vec{p}-\vec{k}) \ = \ -\omega_p\hat{a}_{\vec{p}}
\end{aligned} \tag{5.19}$$

Similarly to the case of harmonic oscillator, $\hat{a}_p$ is an "annihilation operator" in a sense that

$$\hat{a}_{\vec{p}}|0\rangle \ = \ 0 \tag{5.20}$$

Proof: from Eqs. (4.55) and (4.35) we see that

$$\begin{aligned}
&\langle\{\phi(\vec{z})\}|\hat{a}_{\vec{p}}|0\rangle \ = \ \langle\{\phi(\vec{z})\}|\int d^3x\frac{e^{-i\vec{p}\cdot\vec{x}}}{\sqrt{2\omega_p}}\ [\omega_p\hat{\phi}(\vec{x})+i\hat{\pi}(\vec{x})]|0\rangle \ = \ \int d^3x\frac{e^{-i\vec{p}\cdot\vec{x}}}{\sqrt{2\omega_p}}\langle\{\phi(\vec{z})\}|\omega_p\hat{\phi}(\vec{x})+i\hat{\pi}(\vec{x})|0\rangle \\
&= \ \int d^3x\frac{e^{-i\vec{p}\cdot\vec{x}}}{\sqrt{2\omega_p}}\Big(\omega_p\phi(\vec{x})+\frac{\delta}{\delta\phi(\vec{x})}\Big)\langle\{\phi(\vec{z})\}|0\rangle \ = \ \int d^3x\frac{e^{-i\vec{p}\cdot\vec{x}}}{\sqrt{2\omega_p}}\Big(\omega_p\phi(\vec{x})+\frac{\delta}{\delta\phi(\vec{x})}\Big)N^{-1}e^{-\frac{1}{2}\int d^3z\phi(\vec{z})\hat{W}\phi(\vec{z})} \\
&= \ N^{-1}e^{-\frac{1}{2}\int d^3z\phi(\vec{z})\hat{W}\phi(\vec{z})}\int d^3x\frac{e^{-i\vec{p}\cdot\vec{x}}}{\sqrt{2\omega_p}}[\omega_p\phi(\vec{x})-\hat{W}\phi(\vec{x})] \ = \ 0
\end{aligned} \tag{5.21}$$

because

$$\int d^3x\frac{e^{-i\vec{p}\cdot\vec{x}}}{\sqrt{2\omega_p}}\hat{W}\phi(\vec{x}) \ = \ \int d^3x\frac{e^{-i\vec{p}\cdot\vec{x}}}{\sqrt{2\omega_p}}\int đ^3k\ \omega_k e^{i\vec{k}\cdot\vec{x}}\phi(\vec{k}) \ = \ \sqrt{\frac{\omega_p}{2}}\phi(p) \ = \ \int d^3x\frac{e^{-i\vec{p}\cdot\vec{x}}}{\sqrt{2\omega_p}}\omega_p\phi(\vec{x}) \tag{5.22}$$

Please note that $\hat{W}$ is a usual differential operator (4.64) rather than the quantum operator like $\hat{\phi}$ and $\hat{\pi}$.

We have defined vacuum state as a solution of Schrödinger equation (4.58). Alternatively, one can define a vacuum state as a state annihilated by operators $\hat{a}_{\vec{p}}$ (see Eq. (5.20)). It turns out that for the purpose of calculation of cross sections of particle scattering one does not need to know the explicit form of the vacuum state $\Psi_0(\phi) = \langle\{\phi(\vec{x}\}|0\rangle$ - the rule (5.20) is sufficient. (For that reason, the explicit form of wave functionals is rarely discussed in QFT textbooks).

Thus, vacuum state is a state annihilated by $\hat{a}_{\vec{p}}$ (for any $\vec{p}$). The excited states are

$$|p\rangle \ = \ \hat{a}^\dagger_{\vec{p}}\,|0\rangle \tag{5.23}$$

Let us prove that $\hat{a}^\dagger_{\vec{p}}\,|0\rangle$ is a one-particle state - the eigenstate of the KG Hamiltonian (5.18) with momentum $\vec{p}$ and energy $E_p = \omega_p = \sqrt{m^2+\vec{p}^2}$.

It is easy to start with the energy of the state (5.23). From Eq. (5.19) we see that

$$\hat{H}|p\rangle = \hat{H}\hat{a}^\dagger_{\vec{p}}|0\rangle = [\hat{H},\hat{a}^\dagger_{\vec{p}}]|0\rangle + \hat{a}^\dagger_{\vec{p}}\hat{H}|0\rangle = [\hat{H},\hat{a}^\dagger_{\vec{p}}]|0\rangle = \omega_p\hat{a}^\dagger_{\vec{p}}|0\rangle = \omega_p|p\rangle \tag{5.24}$$

so (5.23) is an eigenstate of Hamiltonian with energy $E_p = \omega_p = \sqrt{m^2+p^2}$. But what about the momentum of the state $|\vec{p}\rangle$? For now, $\vec{p}$ in the definition (5.23) is just a label and we need to demonstrate that it has a meaning of the momentum. To this end we need to construct momentum operator for quantized KG field.

Part IV

5.3 Momentum operator in a quantum field theory

5.3.1 Reminder: momentum in a classical field theory

A momentum of the classical system (particles or fields) describes the response of the system with respect to translations.

Suppose we make an infinitesimal translation $x_\mu \to x_\mu + \epsilon_\mu$. [2] The change in the Lagrangian (density) is $\mathcal{L}(x+\epsilon) = \mathcal{L}(x) + \epsilon^\mu \frac{d\mathcal{L}}{dx^\mu}$ where

$$\frac{d}{dx_\mu}\mathcal{L}(\phi,\partial_\alpha\phi) = \frac{\partial\mathcal{L}}{\partial\phi}\frac{\partial\phi}{\partial x_\mu} + \frac{\partial\mathcal{L}}{\partial\partial_\nu\phi}\frac{\partial\partial_\nu\phi}{\partial x_\mu} \tag{5.25}$$

From Euler-Lagrange equations $\frac{\partial\mathcal{L}}{\partial\phi} = \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}$ we get

$$\begin{aligned}\frac{d\mathcal{L}}{dx_\mu} &= \frac{\partial\phi}{\partial x_\mu}\frac{\partial}{\partial x^\nu}\frac{\partial\mathcal{L}}{\partial\partial_\nu\phi} + \frac{\partial^2\phi}{\partial x_\mu\partial x^\nu}\frac{\partial\mathcal{L}}{\partial\partial_\nu\phi} = \frac{\partial}{\partial x^\nu}\Big(\frac{\partial\phi}{\partial x_\mu}\frac{\partial\mathcal{L}}{\partial\partial_\nu\phi}\Big)\\ \Rightarrow \frac{\partial}{\partial x^\nu}\Big(\partial^\mu\phi\frac{\partial\mathcal{L}}{\partial\partial_\nu\phi} - g^{\mu\nu}\mathcal{L}\Big) &= 0 \quad\Leftrightarrow\quad \partial_\nu T^{\mu\nu} = 0\end{aligned} \tag{5.26}$$

where

$$T_{\mu\nu} = \partial_\mu\phi\frac{\partial\mathcal{L}}{\partial\partial^\nu\phi} - g_{\mu\nu}\mathcal{L} = \partial_\mu\phi\partial_\nu\phi - \frac{g_{\mu\nu}}{2}(\partial^\alpha\phi\partial_\alpha\phi - m^2\phi^2) \tag{5.27}$$

is a stress-energy tensor of the scalar field $\phi(x)$.

Conservation of energy

$$\begin{aligned}&\int_{t_2}^{t_1}\int d^3x\;\partial^\nu T_{\nu 0} = \int_{t_2}^{t_1}\int d^3x\;(\partial^0 T_{00} + \partial^i T_{i0}) = \int_{t_2}^{t_1}\int d^3x\;\partial^0 T_{00} = 0 \qquad (5.28)\\ &\Rightarrow \int d^3x\; T_{00}(t_1,\vec{x}) = \int d^3x\; T_{00}(t_2,\vec{x})\\ &\Rightarrow \int d^3x\; T_{00}(t,\vec{x}) = \int d^3x\;\Big(\frac{1}{2}(\partial_0\phi(t,\vec{x}))^2 + \frac{1}{2}|\nabla\phi(t,\vec{x})|^2 + \frac{m^2}{2}\phi^2(t,\vec{x})\Big) = \text{const}\end{aligned}$$

[2] Throughout these notes the Greek letters will denote components of 4-vectors a_μ while Latin indices will mean components of 3-dim vectors $\vec{a}_i$. (I try to avoid notation a_i since it can mean both the covariant component of 4-vector a and usual component of 3-vector $\vec{a}$ which differ in sign)

Since $\partial_0\phi(t,\vec{x}) \;=\; \pi(t,\vec{x})$

$$\int d^3x\; T_{00}(t,\vec{x}) \;=\; \int d^3x\;\Big(\frac{1}{2}\pi^2(t,\vec{x}) + \frac{1}{2}|\nabla\phi(t,\vec{x})|^2 + \frac{m^2}{2}\phi^2(t,\vec{x})\Big) \;=\; \text{const} \tag{5.29}$$

The expression in the r.h.s. is the classical Hamiltonian (3.5). Thus, we reobtained the conservation of energy for the Klein-Gordon field.

Conservation of momentum

$$\int_{t_2}^{t_1}\int d^3x\;\partial^\nu T_{\nu i} \;=\; \int_{t_2}^{t_1}\int d^3x\;(\partial^0 T_{0i} + \partial^k T_{ki}) \;=\; \int_{t_2}^{t_1}\int d^3x\;\partial^0 T_{0i} \;=\; 0 \tag{5.30}$$

$$\Rightarrow \int d^3x\; T_{0i}(t_1,\vec{x}) \;=\; \int d^3x\; T_{0i}(t_2,\vec{x})$$

$$\Rightarrow \int d^3x\; T_{0i}(t,\vec{x}) \;=\; \int d^3x\;\partial_0\phi(t,\vec{x}))\partial_i\phi(t,\vec{x}) \;=\; \int d^3x\;\pi(t,\vec{x})\partial_i\phi(t,\vec{x}) \;=\; \text{const}$$

The expression in the r.h.s.

$$P_i \;\equiv\; \int d^3x\; T_{0i}(t,\vec{x}) \;=\; \int d^3x\;\pi(t,\vec{x})\partial_i\phi(t,\vec{x}) \tag{5.31}$$

is the (conserved) classical momentum of the KG field $\phi(t,\vec{x})$.

5.3.2 Quantum momentum operator

Now we construct the corresponding quantum operator. As usual, we take $t = 0$ and promote $\phi(0,\vec{x})$ and $\pi(0,\vec{x})$ to operators $\hat{\phi}(\vec{x})$ and $\hat{\pi}(\vec{x})$. We get

$$\hat{P}^i \;=\; \int d^3x\;\hat{\pi}(\vec{x})\partial^i\hat{\phi}(\vec{x}) \tag{5.32}$$

This is the quantum operator of momentum. In terms of ladder operators

$$\begin{aligned}
\hat{P}^i \;&=\; \int d^3x đ^3p\;(-i)\sqrt{\frac{\omega_p}{2}}e^{i\vec{p}\cdot\vec{x}}(\hat{a}_{\vec{p}} - \hat{a}^\dagger_{-\vec{p}})\int đ^3p'\;\frac{-ip'^i}{\sqrt{2\omega_{\vec{p}'}}}e^{i\vec{p}'\cdot\vec{x}}(\hat{a}_{\vec{p}'} + \hat{a}^\dagger_{-\vec{p}'}) \qquad (5.33)\\
&\overset{\vec{p}'=-\vec{p}}{=}\; \frac{1}{2}\int đ^3p\; p^i(\hat{a}_{\vec{p}} - \hat{a}^\dagger_{-\vec{p}})(\hat{a}_{-\vec{p}} + \hat{a}^\dagger_{\vec{p}}) \;=\; \frac{1}{2}\int đ^3p\; p^i[\hat{a}_p\hat{a}_{-\vec{p}} - \hat{a}^\dagger_{-\vec{p}}\hat{a}_{-\vec{p}} + \hat{a}_{\vec{p}}\hat{a}^\dagger_{\vec{p}} - \hat{a}^\dagger_{-\vec{p}}\hat{a}^\dagger_{\vec{p}}]\\
&\overset{\vec{p}\leftrightarrow-\vec{p}}{=}\; \frac{1}{2}\int đ^3p\; p^i[\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}} + \hat{a}_{\vec{p}}\hat{a}^\dagger_{\vec{p}}] \;=\; \int đ^3p\; p^i(\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}} + \frac{1}{2}(2\pi)^3\delta(\vec{0})) \;= \int đ^3p\; p^i\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}} + \; V\int đ^3p\; p^i
\end{aligned}$$

The integral $\int đ^3p\; p_i$ is formally divergent but it should be put to zero since there is no preferred direction of "vacuum momentum" due to rotational invariance:

$$\hat{P}_i|0\rangle \;=\; \int đ^3p\; p_i|0\rangle \;=\; \text{should be } 0 \tag{5.34}$$

Finally, the quantum operator of momentum for the KG field has the form

$$\hat{P}_i \;=\; \int đ^3p\; p_i\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}} \tag{5.35}$$

(cf. Eq. (5.18) for the Hamiltonian: $\hat{P}_0 \;=\; \hat{H} \;=\; \int đ^3p\;\omega_p\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}}$).

Commutators:

$$
\begin{aligned}
[\hat{P}^i, \hat{a}^\dagger_{\vec{p}}] \;&=\; \int \text{đ}^3p'\; p'^i [\hat{a}^\dagger_{\vec{p}'}\hat{a}_{\vec{p}'}, \hat{a}^\dagger_{\vec{p}}] \qquad (5.36)\\
&=\; \int \text{đ}^3p'\; p'^i \hat{a}^\dagger_{\vec{p}'}[\hat{a}_{\vec{p}'}, \hat{a}^\dagger_{\vec{p}}] \;=\; \int \text{đ}^3p' p'^i \hat{a}^\dagger_{\vec{p}'}(2\pi)^3\delta(\vec{p}-\vec{p}') \;=\; p^i \hat{a}^\dagger_{\vec{p}}\\
[\hat{P}^i, \hat{a}_{\vec{p}}] \;&=\; \int \text{đ}^3p'\; p'^i [\hat{a}^\dagger_{\vec{p}'}\hat{a}_{\vec{p}'}, \hat{a}_{\vec{p}}]\\
&=\; \int \text{đ}^3p'\; p'^i [\hat{a}^\dagger_{\vec{p}'}, \hat{a}_{\vec{p}}]\hat{a}_{\vec{p}'} \;=\; -\int \text{đ}^3p' p'^i \hat{a}_{\vec{p}'}(2\pi)^3\delta(\vec{p}-\vec{p}') \;=\; -\,p^i \hat{a}_{\vec{p}}
\end{aligned}
$$

$\Rightarrow$

$$
\begin{aligned}
[\hat{P}^i, \hat{\phi}(\vec{x})] \;&=\; \int \frac{\text{đ}p}{\sqrt{2\omega_p}}\left([\hat{P}^i, \hat{a}_{\vec{p}}]e^{i\vec{p}\vec{x}} + [\hat{P}^i, \hat{a}^\dagger_{\vec{p}}]e^{-i\vec{p}\vec{x}}\right) \;=\; \int \frac{\text{đ}p}{\sqrt{2\omega_p}}(-\vec{p}_i\hat{a}_{\vec{p}}e^{i\vec{p}\vec{x}} + \vec{p}_i\hat{a}^\dagger_{\vec{p}}e^{-i\vec{p}\vec{x}})\\
=\; i\frac{\partial}{\partial\vec{x}_i}\int \frac{\text{đ}p}{\sqrt{2\omega_p}}&(\hat{a}_{\vec{p}}e^{i\vec{p}\vec{x}} + \hat{a}^\dagger_{\vec{p}}e^{-i\vec{p}\vec{x}}) \;=\; i\frac{\partial}{\partial x^i}\hat{\phi}(\vec{x}) \;=\; i\partial_i\hat{\phi}(\vec{x}) \;=\; -\,i\partial^i\hat{\phi}(\vec{x}) \qquad (5.37)
\end{aligned}
$$

Let us check that the momentum operator commutes with the Hamiltonian

$$
\begin{aligned}
[\hat{H}, \hat{P}_i] \;&=\; \int \text{đ}^3p\text{đ}^3q\; \omega_p q_i [\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{q}}\hat{a}_{\vec{q}}] \qquad (5.38)\\
&=\; \int \text{đ}^3p\text{đ}^3q\; \omega_p q_i (\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{q}}[\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{q}}] + \hat{a}_{\vec{p}}\hat{a}^\dagger_{\vec{q}}[\hat{a}^\dagger_{\vec{p}}, \hat{a}_{\vec{q}}])\\
&=\; \int \text{đ}^3p\text{đ}^3q\; \omega_p q_i (2\pi)^3\delta(\vec{p}-\vec{q})(\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{q}} - \hat{a}_{\vec{p}}\hat{a}^\dagger_{\vec{q}}) \;=\; \int \text{đ}^3p\text{đ}^3q\; \omega_p q_i [\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{p}}] \;=\; 0
\end{aligned}
$$

Now we are in a position to check that the state $\hat{a}^\dagger_{\vec{p}}$ has a definite momentum $\vec{p}$

$$\hat{P}_i\hat{a}^\dagger_{\vec{p}}|0\rangle \;=\; \hat{a}^\dagger_{\vec{p}}\hat{P}_i|0\rangle + [\hat{P}_i, \hat{a}^\dagger_{\vec{p}}]|0\rangle \;=\; p_i\hat{a}^\dagger_{\vec{p}}|0\rangle \qquad (5.39)$$

so $\hat{a}^\dagger_{\vec{p}}|0\rangle$ is an eigenstate of the momentum operator $\hat{P}_i$ with eigenvalue $p_i \;\Rightarrow\; |\vec{p}\rangle \;\equiv\; \hat{a}^\dagger_{\vec{p}}|0\rangle$ is a state with momentum $\vec{p}$.

Thus, the state $|\vec{p}\rangle$ is an eigenstate of both Hamiltonian and the momentum operator

$$\hat{H}|\vec{p}\rangle \;=\; \omega_p|\vec{p}\rangle \;=\; E_p|\vec{p}\rangle, \qquad\qquad \hat{P}_i|\vec{p}\rangle \;=\; p_i|\vec{p}\rangle \qquad (5.40)$$

and the relation between energy and momentum of the state is a characteristic of a relativistic particle with mass m

$\Rightarrow$

$|\vec{p}\rangle$ is a state of a (scalar) particle with momentum $\vec{p}$ and energy $E_p = \omega_p = \sqrt{m^2+\vec{p}^2}$.

Next, from commutators (5.19) and (5.36) we see that

$$
\begin{aligned}
\hat{P}_i\hat{a}^\dagger_{\vec{p}}\hat{a}^\dagger_{\vec{q}}|0\rangle \;&=\; [\hat{P}_i, \hat{a}^\dagger_{\vec{p}}]\hat{a}^\dagger_{\vec{q}}|0\rangle + \hat{a}^\dagger_{\vec{p}}[\hat{P}_i, \hat{a}^\dagger_{\vec{q}}]|0\rangle + \hat{a}^\dagger_{\vec{p}}\hat{a}^\dagger_{\vec{q}}\cancel{\hat{P}_i}|0\rangle \;=\; (p_i+q_i)\hat{a}^\dagger_{\vec{p}}\hat{a}^\dagger_{\vec{q}}|0\rangle\\
\hat{H}\hat{a}^\dagger_{\vec{p}}\hat{a}^\dagger_{\vec{q}}|0\rangle \;&=\; [\hat{H}, \hat{a}^\dagger_{\vec{p}}]\hat{a}^\dagger_{\vec{q}}|0\rangle + \hat{a}^\dagger_{\vec{p}}[\hat{H}, \hat{a}^\dagger_{\vec{q}}]|0\rangle + \hat{a}^\dagger_{\vec{p}}\hat{a}^\dagger_{\vec{q}}\cancel{\hat{H}}|0\rangle \;=\; (E_p+E_q)\hat{a}^\dagger_{\vec{p}}\hat{a}^\dagger_{\vec{q}}|0\rangle \qquad (5.41)
\end{aligned}
$$

(here $E_p \equiv \omega_p = \sqrt{m^2+\vec{p}^2}$)

$\Rightarrow$ the state $\hat{a}^\dagger_{\vec{p}}\hat{a}^\dagger_{\vec{q}}|0\rangle$ is a two-particle state which we will denote

$$|\vec{p},\vec{q}\rangle \;\equiv\; \hat{a}^\dagger_{\vec{p}}\hat{a}^\dagger_{\vec{q}}|0\rangle \qquad (5.42)$$

Note that $\hat{a}^\dagger_{\vec{p}}\hat{a}^\dagger_{\vec{q}}|0\rangle \;=\; \hat{a}^\dagger_{\vec{q}}\hat{a}^\dagger_{\vec{p}}|0\rangle \Rightarrow$ Bose-Einstein statistics.

Similarly one can define a n-particle state

$$|\vec{p}_1,\vec{p}_2,...\vec{p}_n\rangle \;\equiv\; \hat{a}^\dagger_{\vec{p}_1}\hat{a}^\dagger_{\vec{p}_2}....\hat{a}^\dagger_{\vec{p}_n}|0\rangle \tag{5.43}$$

This is a state of n particles with momenta $\vec{p}_1,\vec{p}_2,...\vec{p}_n$.

6 Heisenberg picture in QFT

Reminder: in quantum mechanics

Schrödinger picture:

$$\begin{array}{lcll} \Psi(t) & - & \text{vector of state} & \text{depends on time} \\ \hat{\phi} & - & \text{canonical coordinate} & \text{does not depend on time} \\ \hat{\pi} & - & \text{canonical momentum} & \text{does not depend on time} \end{array} \tag{6.1}$$

Dynamics is governed by Schrödinger equation

$$i\frac{d\Psi(t)}{dt} \;=\; \hat{H}\Psi(t) \tag{6.2}$$

Heisenberg picture

$$\begin{array}{lcccl} \Psi \;=\; \Psi_{\rm Schro}(t)|_{t=0} & - & \text{vector of state} & & \text{does not depend on time} \\ \hat{\phi}(t) \;=\; e^{i\hat{H}t}\hat{\phi}e^{-i\hat{H}t} & - & \text{canonical coordinate} & & \text{depends on time} \\ \hat{\phi}(t) \;=\; e^{i\hat{H}t}\hat{\phi}e^{-i\hat{H}t} & - & \text{canonical coordinate} & & \text{depends on time} \end{array} \tag{6.3}$$

Dynamics is governed by Heisenberg equations

$$\frac{d\hat{\phi}(t)}{dt} \;=\; i[\hat{H},\hat{\phi}(t)], \qquad\qquad \frac{d\hat{\pi}(t)}{dt} \;=\; i[\hat{H},\hat{\pi}(t)] \tag{6.4}$$

Schrödinger picture in QFT:

$$\begin{array}{lcll} \Psi(t,\{\phi(\vec{x})\}) & - & \text{vector of state} & \text{depends on time} \\ \hat{\phi}(\vec{x}) & - & \text{canonical coordinate} & \text{does not depend on time} \\ \hat{\pi}(\vec{x}) & - & \text{canonical momentum} & \text{does not depend on time} \end{array} \tag{6.5}$$

Dynamics: Schrödinger equation (4.44)

$$i\frac{d}{dt}\Psi(t,\{\phi(\vec{z})\}) \;=\; \frac{1}{2}\int d^3x\big[\big(\frac{\delta}{\delta\phi(\vec{x})}\big)^2 + |\nabla\phi(\vec{x})|^2(x) + m^2\phi^2(\vec{x})\big]\Psi(t,\{\phi(\vec{z})\}) \tag{6.6}$$

Transition to Heisenberg picture in QFT: same as in QM

$$\begin{array}{rcccc} \Psi(\{\phi(\vec{x})\}) & = & \Psi_{\rm Schro}(t,\{\phi(\vec{x})\})|_{t=0} & - & \text{vector of state} & \text{does not depend on time} \\ \hat{\phi}(t,\vec{x}) & = & e^{i\hat{H}t}\hat{\phi}(\vec{x})e^{-i\hat{H}t} & - & \text{canonical coordinate} & \text{depends on time} \\ \hat{\phi}(t) & = & e^{i\hat{H}t}\hat{\pi}(\vec{x})e^{-i\hat{H}t} & - & \text{canonical coordinate} & \text{depends on time} \end{array} \tag{6.7}$$

(for example, vacuum state is $\Psi_{\rm vac}(\{\phi(\vec{x})\}) \;=\; N^{-1}e^{-\frac{1}{2}\int d^3x\phi(\vec{x})\sqrt{m^2-\nabla^2}\phi(\vec{x})}$)

Dynamics is governed by Heisenberg equations

$$\frac{d\hat{\phi}(t,\vec{x})}{dt} \;=\; i[\hat{H},\hat{\phi}(t,\vec{x})], \qquad\qquad \frac{d\hat{\pi}(t,\vec{x})}{dt} \;=\; i[\hat{H},\hat{\pi}(t,\vec{x})]) \tag{6.8}$$

NB: instead of the variational derivatives in Eq. (6.6) we have ordinary derivatives in Heisenberg equations (7.13).

In terms of ladder operators

$$\begin{aligned} \hat{\phi}(x) &= \int \frac{đp}{\sqrt{2E_p}} \, (\hat{a}_{\vec{p}} e^{-ipx} + \hat{a}^\dagger_{\vec{p}} e^{ipx}) \\ \hat{\pi}(x) &= -i\int \frac{đp}{\sqrt{2E_p}} \, E_p(\hat{a}_{\vec{p}} e^{-ipx} - \hat{a}^\dagger_{\vec{p}} e^{ipx}) \end{aligned} \tag{6.9}$$

Proof of Eq. (6.9)

$$\begin{aligned} e^{i\hat{H}t}\hat{a}_{\vec{p}}e^{-i\hat{H}t} &= \sum_{n=0}^{\infty} i^n \frac{t^n}{n!}[\hat{H},[\hat{H},....[\hat{H},\hat{a}_{\vec{p}}]]] = \sum_{n=0}^{\infty} i^n \frac{t^n}{n!}(-E_p)^n \hat{a}_{\vec{p}} = \hat{a}_{\vec{p}} e^{-iE_pt} \\ e^{i\hat{H}t}\hat{a}^\dagger_{\vec{p}}e^{-i\hat{H}t} &= \sum_{n=0}^{\infty} i^n \frac{t^n}{n!}[\hat{H},[\hat{H},....[\hat{H},\hat{a}^\dagger_{\vec{p}}]]] = \sum_{n=0}^{\infty} i^n \frac{t^n}{n!}(E_p)^n \hat{a}_{\vec{p}} = \hat{a}^\dagger_{\vec{p}} e^{iE_pt} \end{aligned} \tag{6.10}$$

(recall that $[\hat{H},\hat{a}_{\vec{p}}] = -E_p\hat{a}_{\vec{p}}$, $[\hat{H},\hat{a}^\dagger_{\vec{p}}] = E_p\hat{a}^\dagger_{\vec{p}}$ where $E_p \equiv \omega_p = \sqrt{m^2+\vec{p}^2}$).

From Eq. (5.13) we see that

$$\begin{aligned} e^{i\hat{H}t}\hat{\phi}(\vec{x})e^{-i\hat{H}t} &= \int \frac{đp}{\sqrt{2E_p}} \, (e^{i\hat{H}t}\hat{a}_{\vec{p}}e^{-i\hat{H}t}e^{i\vec{p}\vec{x}} + e^{i\hat{H}t}\hat{a}^\dagger_{\vec{p}}e^{-i\hat{H}t}e^{-i\vec{p}\vec{x}}) = \int \frac{đp}{\sqrt{2E_p}} \, \hat{a}_{\vec{p}}e^{-iE_pt}e^{i\vec{p}\vec{x}} + \hat{a}^\dagger_{\vec{p}}e^{iE_pt}e^{-i\vec{p}\vec{x}}) \\ e^{i\hat{H}t}\hat{\pi}(\vec{x})e^{-i\hat{H}t} &= \int \frac{đp}{i} \, \sqrt{\frac{E_p}{2}}(e^{i\hat{H}t}\hat{a}_{\vec{p}}e^{-i\hat{H}t}e^{i\vec{p}\vec{x}} - e^{i\hat{H}t}\hat{a}^\dagger_{\vec{p}}e^{-i\hat{H}t}e^{-i\vec{p}\cdot\vec{x}}) = \int \frac{đp}{i} \, \sqrt{\frac{E_p}{2}}(\hat{a}_{\vec{p}}e^{-iE_pt}e^{i\vec{p}\vec{x}} - \hat{a}^\dagger_{\vec{p}}e^{iE_pt}e^{-i\vec{p}\cdot\vec{x}}) \end{aligned} \tag{6.11}$$

which coinsides with Eq. (6.9).

Heisenberg equations:

$$\begin{aligned} \frac{\partial}{\partial t}\hat{\phi}(t,\vec{x}) &= \frac{\partial}{\partial t}\int \frac{đp}{\sqrt{2E_p}} \, (\hat{a}_{\vec{p}}e^{-iE_pt}e^{i\vec{p}\vec{x}} + \hat{a}^\dagger_{\vec{p}}e^{iE_pt}e^{-i\vec{p}\vec{x}}) = -i\int đp\sqrt{\frac{E_p}{2}} \, (\hat{a}_{\vec{p}}e^{-iE_pt}e^{i\vec{p}\vec{x}} - \hat{a}^\dagger_{\vec{p}}e^{iE_pt}e^{-i\vec{p}\vec{x}}) = \hat{\pi}(t,\vec{x}) \\ \frac{\partial}{\partial t}\hat{\pi}(t,\vec{x}) &= \frac{\partial}{\partial t}\int \frac{đp}{i} \, \sqrt{\frac{E_p}{2}}(\hat{a}_{\vec{p}}e^{-iE_pt}e^{i\vec{p}\vec{x}} - \hat{a}^\dagger_{\vec{p}}e^{iE_pt}e^{-i\vec{p}\cdot\vec{x}}) = -\int \frac{đp}{\sqrt{2E_p}}E_p^2 \, (\hat{a}_{\vec{p}}e^{-iE_pt}e^{i\vec{p}\vec{x}} + \hat{a}^\dagger_{\vec{p}}e^{iE_pt}e^{-i\vec{p}\vec{x}}) \\ &= -\int \frac{đp}{\sqrt{2E_p}}(m^2+\vec{p}^2) \, (\hat{a}_{\vec{p}}e^{-iE_pt}e^{i\vec{p}\vec{x}} + \hat{a}^\dagger_{\vec{p}}e^{iE_pt}e^{-i\vec{p}\vec{x}}) = -\int \frac{đp}{\sqrt{2E_p}}(m^2-\nabla^2) \, (\hat{a}_{\vec{p}}e^{-iE_pt}e^{i\vec{p}\vec{x}} + \hat{a}^\dagger_{\vec{p}}e^{iE_pt}e^{-i\vec{p}\vec{x}}) \\ &= (-m^2+\nabla^2)\int \frac{đp}{\sqrt{2E_p}} \, (\hat{a}_{\vec{p}}e^{-iE_pt}e^{i\vec{p}\vec{x}} + \hat{a}^\dagger_{\vec{p}}e^{iE_pt}e^{-i\vec{p}\vec{x}}) = -(m^2-\nabla^2)\hat{\phi}(t,\vec{x}) \end{aligned} \tag{6.12}$$

Combining these two equations we get the Klein-Gordon equation for the operator $\hat{\phi}(t,\vec{x})$

$$\frac{\partial^2}{\partial t^2}\hat{\phi}(t,\vec{x}) = \frac{\partial}{\partial t}\hat{\pi}(t,\vec{x}) = (-m^2+\nabla^2)\hat{\phi}(t,\vec{x}) \quad \Rightarrow \quad (\partial^2+m^2)\hat{\phi}(x) = 0 \tag{6.13}$$

which has the same form as the KG equation for the classical field $\phi(x)$.

Part V

6.1 Momentum operator and shifts of coordinates

We define

$$\hat{P}^\mu = (\hat{H}, \hat{\vec{P}}) \quad - \quad \text{operator of } 4-\text{momentum} \tag{6.14}$$

In terms of ladder operators it has the form ($p_0 = E_p = \sqrt{m^2 + \vec{p}^2}$)

$$\hat{P}^\mu \;=\; \int đ^3 p\; p^\mu \hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{p}} \tag{6.15}$$

where we combined Eq. (5.18) and (5.35).

Let us prove that

$$\hat{\phi}(x+a) \;=\; e^{i\hat{P}a}\hat{\phi}(x)e^{-i\hat{P}a} \tag{6.16}$$

As a first step, we check that

$$e^{-i\hat{\vec{P}}\cdot\vec{a}}\hat{\phi}(x_0,\vec{x})e^{i\hat{\vec{P}}\cdot\vec{a}} \;=\; \hat{\phi}(x_0,\vec{x}+\vec{a}) \tag{6.17}$$

The r.h.s. can be expanded in commutators

$$\begin{aligned} e^{-i\hat{\vec{P}}\cdot\vec{a}}\hat{\phi}(x_0,\vec{x})e^{i\hat{\vec{P}}\cdot\vec{a}} \;&=\; \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}[\hat{\vec{P}}\cdot\vec{a},[\hat{\vec{P}}\cdot\vec{a},....[\hat{\vec{P}}\cdot\vec{a},\hat{\phi}(x_0,\vec{x})]]] \\ =\; \sum_{n=0}^{\infty}\frac{1}{n!}(\vec{a}_i\frac{\partial}{\partial\vec{x}_i})^n\hat{\phi}(x_0,\vec{x}) \;&=\; \sum_{n=0}^{\infty}\frac{1}{n!}(\vec{a}\cdot\vec{\nabla})^n\hat{\phi}(x_0,\vec{x}) \;=\; \hat{\phi}(x_0,\vec{x}+\vec{a}) \end{aligned} \tag{6.18}$$

where we used

$$\begin{aligned} [\hat{\vec{P}}_i,\hat{\phi}(x)] \;&=\; [\hat{P}^i,\hat{\phi}(x)] \;=\; [\hat{P}^i, e^{i\hat{H}x_0}\hat{\phi}(\vec{x})e^{-i\hat{H}x_0}] \;=\; e^{i\hat{H}x_0}[\hat{P}^i,\hat{\phi}(\vec{x})]e^{-i\hat{H}x_0} \;=\; -ie^{i\hat{H}x_0}\partial^i\hat{\phi}(\vec{x})e^{-i\hat{H}x_0} \\ =\; -i\partial^i e^{i\hat{H}x_0}\hat{\phi}(\vec{x})e^{-i\hat{H}x_0} \;&=\; -i\partial^i\hat{\phi}(x) \;=\; -i\frac{\partial}{\partial x_i}\hat{\phi}(x) \;=\; i\frac{\partial}{\partial x^i}\hat{\phi}(x) \;=\; i\frac{\partial}{\partial\vec{x}_i}\hat{\phi}(x) \end{aligned} \tag{6.19}$$

see Eq. (5.37) and Eq. (5.38).

Next,

$$e^{i\hat{H}a_0}\hat{\phi}(x)e^{-i\hat{H}a_0} \;=\; \sum_{n=0}^{\infty}\frac{(ia_0)^n}{n!}[\hat{H},[\hat{H},....[\hat{H},\hat{\phi}(x)]]] \;=\; \sum_{n=0}^{\infty}\frac{(a_0)^n}{n!}\partial_0^n\hat{\phi}(x) \;=\; \hat{\phi}(x_0+a,\vec{x}) \tag{6.20}$$

because

$$[\hat{H},\hat{\phi}(x)] \;=\; [\hat{H}, e^{i\hat{H}x_0}\hat{\phi}(\vec{x})e^{-i\hat{H}x_0}] \;=\; e^{i\hat{H}x_0}[\hat{H},\hat{\phi}(\vec{x})]e^{-i\hat{H}x_0} \;=\; -ie^{i\hat{H}x_0}\partial_0\hat{\phi}(\vec{x})e^{-i\hat{H}x_0} \;=\; -i\partial_0\hat{\phi}(x) \tag{6.21}$$

where we used Heisenberg equation (7.13).

Now we are in a position to prove Eq. (6.16). Combining Eqs. (6.18) and (6.21) we get

$$\begin{aligned} e^{i\hat{P}a}\hat{\phi}(x)e^{-i\hat{P}a} \;&=\; e^{i\hat{H}a_0 - i\hat{\vec{P}}\cdot\vec{a}}\hat{\phi}(x)e^{-i\hat{H}a_0 + i\hat{\vec{P}}\cdot\vec{a}} \;=\; e^{i\hat{H}a_0}e^{-i\hat{\vec{P}}\cdot\vec{a}}\hat{\phi}(x)e^{i\hat{\vec{P}}\cdot\vec{a}}e^{-i\hat{H}a_0} \\ =\; e^{i\hat{H}a_0}\hat{\phi}(x_0,\vec{x}+\vec{a})e^{-i\hat{H}a_0} \;&=\; \hat{\phi}(x_0+a_0,\vec{x}+\vec{a}), \qquad \text{Q.E.D.} \end{aligned} \tag{6.22}$$

Another proof of Eq. (6.16): in terms of ladder operators

$$e^{i\hat{P}a}\hat{a}_{\vec{p}}e^{-i\hat{P}a} = e^{i\hat{H}a_0}e^{-i\vec{\hat{P}}\cdot a}\hat{a}_{\vec{p}}e^{i\vec{\hat{P}}\cdot\vec{a}}e^{-i\hat{H}a_0} = e^{i\hat{H}a_0}\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}[\vec{\hat{P}}\cdot a,[\vec{\hat{P}}\cdot a,...[\vec{\hat{P}}\cdot a,\hat{a}_{\vec{p}}]]]e^{-i\hat{H}a_0}$$

$$= e^{i\hat{H}a_0}\sum_{n=0}^{\infty}\frac{(i\vec{p}\cdot\vec{a})^n}{n!}\hat{a}_{\vec{p}}e^{-i\hat{H}a_0} = e^{i\vec{p}\cdot\vec{a}}e^{i\hat{H}a_0}\hat{a}_{\vec{p}}e^{-i\hat{H}a_0} = e^{i\vec{p}\cdot\vec{a}}\sum_{n=0}^{\infty}\frac{(ia_0)^n}{n!}[\hat{H},[\hat{H},...[\hat{H},\hat{a}_{\vec{p}}]]]$$

$$= e^{i\vec{p}\cdot\vec{a}}\sum_{n=0}^{\infty}\frac{(-ia_0E_p)^n}{n!}\hat{a}_{\vec{p}} = e^{-iE_pa_0+i\vec{p}\cdot\vec{a}}\hat{a}_{\vec{p}} = e^{-ipa}\hat{a}_{\vec{p}} \tag{6.23}$$

Similarly,

$$e^{i\hat{P}a}\hat{a}^\dagger_{\vec{p}}e^{-i\hat{P}a} = e^{i\hat{H}a_0}e^{-i\vec{\hat{P}}\cdot a}\hat{a}^\dagger_{\vec{p}}e^{i\vec{\hat{P}}\cdot\vec{a}}e^{-i\hat{H}a_0} = e^{i\hat{H}a_0}\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}[\vec{\hat{P}}\cdot a,[\vec{\hat{P}}\cdot a,...[\vec{\hat{P}}\cdot a,\hat{a}^\dagger_{\vec{p}}]]]e^{-i\hat{H}a_0}$$

$$= e^{i\hat{H}a_0}\sum_{n=0}^{\infty}\frac{(-i\vec{p}\cdot\vec{a})^n}{n!}\hat{a}^\dagger_{\vec{p}}e^{-i\hat{H}a_0} = e^{-i\vec{p}\cdot\vec{a}}e^{i\hat{H}a_0}\hat{a}^\dagger_{\vec{p}}e^{-i\hat{H}a_0} = e^{-i\vec{p}\cdot\vec{a}}\sum_{n=0}^{\infty}\frac{(ia_0)^n}{n!}[\hat{H},[\hat{H},...[\hat{H},\hat{a}^\dagger_{\vec{p}}]]]$$

$$= e^{i\vec{p}\cdot\vec{a}}\sum_{n=0}^{\infty}\frac{(ia_0E_p)^n}{n!}\hat{a}^\dagger_{\vec{p}} = e^{iE_pa_0-i\vec{p}\cdot\vec{a}}\hat{a}^\dagger_{\vec{p}} = e^{ipa}\hat{a}^\dagger_{\vec{p}} \tag{6.24}$$

and therefore

$$e^{i\hat{P}a}\hat{\phi}(x)e^{-i\hat{P}a} = \int\frac{d\!\!\bar{}\,p}{\sqrt{2E_p}}\,(e^{i\hat{P}a}\hat{a}_{\vec{p}}e^{-i\hat{P}a}e^{-ipx} + e^{i\hat{P}a}\hat{a}^\dagger_{\vec{p}}e^{-i\hat{P}a}e^{ipx}) \tag{6.25}$$

$$= \int\frac{d\!\!\bar{}\,p}{\sqrt{2E_p}}\,(\hat{a}_{\vec{p}}e^{-ipa}e^{-ipx} + \hat{a}^\dagger_{\vec{p}}e^{ipa}e^{ipx}) = \int\frac{d\!\!\bar{}\,p}{\sqrt{2E_p}}\,(\hat{a}_{\vec{p}}e^{-ip(x+a)} + \hat{a}^\dagger_{\vec{p}}e^{ip(x+a)}) = \hat{\phi}(x+a), \qquad \text{Q.E.D.}$$

For shifts in $\hat{\pi}(x)$ we have similar formula

$$e^{i\hat{P}a}\hat{\pi}(x)e^{-i\hat{P}a} = \hat{\pi}(x+a) \tag{6.26}$$

Indeed, since $\hat{\pi}(x_0,\vec{x}) = \partial_0\hat{\phi}(x_0,\vec{x})$ we obtain

$$e^{i\hat{P}a}\hat{\pi}(x)e^{-i\hat{P}a} = e^{i\hat{P}a}\frac{\partial}{\partial x_0}\hat{\phi}(x_0,\vec{x})e^{i\hat{P}a} = \frac{\partial}{\partial x_0}e^{i\hat{P}a}\hat{\phi}(x_0,\vec{x})e^{i\hat{P}a} = \frac{\partial}{\partial x_0}\hat{\phi}(x+a) = \hat{\pi}(x+a) \tag{6.27}$$

6.1.1 Quantum operator $\hat{P}$ and differential operator P (generator of shifts)

Shifts in 4-dim space-time are generated by the differential operator

$$P^\mu\hat{\phi}(x) \equiv i\partial^\mu\hat{\phi}(x) = i\frac{\partial}{\partial x_\mu}\hat{\phi}(x), \qquad \hat{\phi}(x+a) = e^{-iPa}\hat{\phi}(x) \tag{6.28}$$

where the last equation is easily checked by Taylor expansion. The same formula will be evidently true for quantum operator $\hat{\phi}(x)$

$$P^\mu\hat{\phi}(x) \equiv i\partial^\mu\hat{\phi}(x) = i\frac{\partial}{\partial x_\mu}\hat{\phi}(x), \qquad \hat{\phi}(x+a) = e^{-iPa}\hat{\phi}(x) \tag{6.29}$$

so the relation between the action of quantum momentum operator $\hat{P}$ and differential operator P is

$$[\hat{P}^\mu,\hat{\phi}(x)] = -i\partial_\mu\hat{\phi}(x) = -P^\mu\hat{\phi}(x) \Rightarrow e^{i\hat{P}a}\hat{\phi}(x)e^{-i\hat{P}a} = \hat{\phi}(x+a) = e^{-iPa}\hat{\phi}(x) \tag{6.30}$$

To avoid confusion, hereafter we mark by hat only quantum operators.

6.2 Equal-time commutators

By definition

$$[\hat{\phi}(\vec{x}), \hat{\pi}(\vec{y})] \;=\; i\delta(\vec{x}-\vec{y}) \tag{6.31}$$

Let us compute

$$\begin{aligned}
[\hat{\phi}(t,\vec{x}), \hat{\pi}(t,\vec{y})] \;&=\; [e^{i\hat{H}t}\hat{\phi}(\vec{x})e^{-i\hat{H}t}, e^{i\hat{H}t}\hat{\pi}(\vec{y})e^{-i\hat{H}t}] \;=\; e^{i\hat{H}t}[\hat{\phi}(\vec{x}), \hat{\pi}(\vec{y})]e^{-i\hat{H}t} \;=\; i\delta(\vec{x}-\vec{y}) \\
[\hat{\phi}(t,\vec{x}), \hat{\phi}(t,\vec{y})] \;&=\; [e^{i\hat{H}t}\hat{\phi}(\vec{x})e^{-i\hat{H}t}, e^{i\hat{H}t}\hat{\phi}(\vec{y})e^{-i\hat{H}t}] \;=\; e^{i\hat{H}t}[\hat{\phi}(\vec{x}), \hat{\phi}(\vec{y})]e^{-i\hat{H}t} \;=\; 0 \\
[\hat{\pi}(t,\vec{x}), \hat{\pi}(t,\vec{y})] \;&=\; [e^{i\hat{H}t}\hat{\pi}(\vec{x})e^{-i\hat{H}t}, e^{i\hat{H}t}\hat{\pi}(\vec{y})e^{-i\hat{H}t}] \;=\; e^{i\hat{H}t}[\hat{\pi}(\vec{x}), \hat{\pi}(\vec{y})]e^{-i\hat{H}t} \;=\; 0
\end{aligned} \tag{6.32}$$

$\Rightarrow$ Equal-time commutation relations:

$$[\hat{\phi}(t,\vec{x}), \hat{\pi}(t,\vec{y})] \;=\; i\delta(\vec{x}-\vec{y}), \qquad [\hat{\phi}(t,\vec{x}), \hat{\phi}(t,\vec{y})] = [\hat{\pi}(t,\vec{x}), \hat{\pi}(t,\vec{y})] \;=\; 0 \tag{6.33}$$

6.2.1 Normalization of one-particle state

$$\begin{aligned}
|p\rangle \;&=\; \sqrt{2E_p}a^\dagger_{\vec{p}}|0\rangle \qquad && \text{Peskin} \\
|p\rangle \;&=\; a^\dagger_{\vec{p}}|0\rangle && \text{Bjorken \& Drell}
\end{aligned} \tag{6.34}$$

The factor $\sqrt{2E_p}$ makes $\langle p|p'\rangle \;=\; 2E_p 02\pi^3\delta(\vec{p}-\vec{p}')$ relativistic invariant, see Peskin's textbook.

6.3 Propagators

6.3.1 Wightman propagator

We start with the definition of Wightman propagator

$$D(x,y) \;\equiv\; \langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle \tag{6.35}$$

Using formula (6.9)

$$\begin{aligned}
D(x,y) \;&=\; \int \frac{\bar{d}^3p}{\sqrt{2E_p}}\frac{\bar{d}^3p'}{\sqrt{2E_{p'}}}\; \langle 0|(\hat{a}_{\vec{p}}e^{-ipx} + \cancel{\hat{a}^\dagger_{\vec{p}}}e^{ipx})(\cancel{\hat{a}_{\vec{p}'}}e^{-ip'y} + \hat{a}^\dagger_{\vec{p}'}e^{ip'y})|0\rangle \\
&=\; \int \frac{\bar{d}^3p}{2E_p}\; e^{-ip(x-y)}\Big|_{p_0=E_p} \;=\; \int \frac{\bar{d}^3p}{2E_p}\; e^{-iE_p(x_0-y_0)+i\vec{p}(\vec{x}-\vec{y})}
\end{aligned} \tag{6.36}$$

Large-distance behavior of $D(x-y)$.

First, consider the time-like intervals $(x-y)^2 > 0$. At time-like interval we can find a frame such that $x-y = (t,\vec{0})$. In this frame

$$\begin{aligned}
D(x,y) \;&=\; \int \frac{\bar{d}^3p}{2E_p}\; e^{-iE_p(x_0-y_0)} \;=\; \frac{1}{4\pi^2}\int_0^\infty dp\; \frac{p^2}{\sqrt{p^2+m^2}}e^{-it\sqrt{p^2+m^2}} \\
&=\; \frac{1}{4\pi^2}\int_m^\infty dE\; \sqrt{E^2-m^2}e^{-iEt} \;\stackrel{E=\mathcal{E}+m}{=}\; \frac{e^{-imt}}{4\pi^2}\int_0^\infty d\mathcal{E}\; \sqrt{2m\mathcal{E}+\mathcal{E}^2}e^{-i\mathcal{E}t} \\
&\stackrel{t\to\infty}{=}\; \frac{e^{-imt}}{4\pi^2}\int_0^\infty d\mathcal{E}\; \sqrt{2m\mathcal{E}+\cancel{\mathcal{E}^2}}e^{-i\mathcal{E}t} \;\stackrel{t\to\infty}{=}\; \frac{\sqrt{m}}{2(2\pi i t)^{3/2}}e^{-imt}
\end{aligned} \tag{6.37}$$

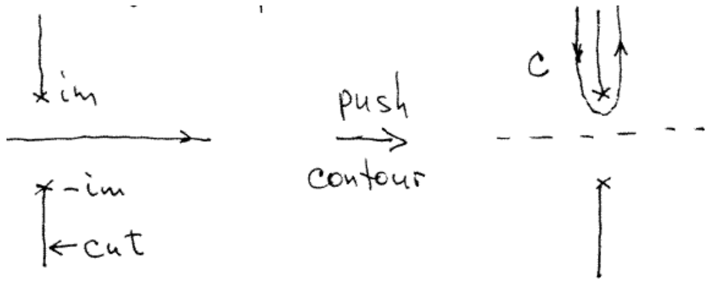


Figure 3. Contour

At space-like intervals we can find a frame where $x - y = (0, \vec{r})$. In this frame

$$
\begin{aligned}
D(x,y) \;&=\; \int \frac{\bar{d}^3p}{2E_p}\, e^{i\vec{p}(\vec{x}-\vec{y})} \;=\; \frac{1}{8\pi^2}\int_0^\infty dp\, \frac{p^2}{\sqrt{p^2+m^2}} \int_{-\pi}^{\pi} d\theta\; \sin\theta e^{ipr\cos\theta} \\
&= \frac{1}{8\pi^2}\int_0^\infty dp\, \frac{p^2}{\sqrt{p^2+m^2}} \int_{-1}^{1} du\; e^{ipru} \;=\; \frac{1}{8\pi^2}\int_0^\infty dp\, \frac{p^2}{\sqrt{p^2+m^2}} \frac{e^{ipr}-e^{-ipr}}{ipr} \\
&= \frac{-i}{8\pi^2 r}\int_{-\infty}^{\infty} dp\, \frac{p}{\sqrt{p^2+m^2}} \;=\; \frac{-i}{8\pi^2 r}\int_C dp\, \frac{p}{\sqrt{p^2+m^2}} e^{ipr} \\
&= \;-\frac{1}{4\pi^2 r}\int_m^\infty d\rho\, \frac{\rho e^{-\rho r}}{\sqrt{\rho^2-m^2}} \;\overset{\rho=\lambda+m}{=}\; -\frac{i}{8\pi^2 r}\int_0^\infty d\lambda\, \frac{(m+\lambda)e^{-\lambda r}}{\sqrt{2\lambda m+\lambda^2}} \\
&\overset{\lambda\sim 1/r}{=} \;-\frac{i}{8\pi^2 r}\int_0^\infty d\lambda\, \frac{(m+\cancel{\lambda})e^{-\lambda r}}{\sqrt{2\lambda m+\cancel{\lambda}^2}} \;\overset{r\to\infty}{=}\; \frac{\sqrt{m}}{2(2\pi r)^{3/2}} e^{-mr} \qquad (6.38)
\end{aligned}
$$

6.3.2 Causality

Causality: no signal should go faster than the speed of light.
In other words: measurement performed at the point x should not affect measurement performed at the point y if $(x-y)^2 < 0$.

"Elementary measurement" in QFT is $[\hat{\phi}(x), \hat{\phi}(y)] \Rightarrow$ causality requires that $[\hat{\phi}(x), \hat{\phi}(y)] \;=\; 0$ for $(x-y)^2 < 0$.

Check:

$$
\begin{aligned}
[\hat\phi(x),\hat\phi(y)] &= \int \frac{đ^3p}{\sqrt{2E_p}}\frac{đ^3p'}{\sqrt{2E_{p'}}}\;[\hat a_{\vec p}e^{-ipx}+\hat a^\dagger_{\vec p}e^{ipx},\hat a_{\vec p'}e^{-ip'y}+\hat a^\dagger_{\vec p'}e^{ip'y}] \\
&= \int \frac{đ^3p}{\sqrt{2E_p}}\frac{đ^3p'}{\sqrt{2E_{p'}}}\;\big([\hat a_{\vec p},\hat a^\dagger_{\vec p'}]e^{-ipx+ip'y}+[\hat a^\dagger_{\vec p},\hat a_{\vec p'}]e^{ipx-ip'y}\big) \\
&= \int \frac{đ^3p}{2E_p}\;\big[e^{-ip(x-y)}-e^{ip(x-y)}\big] \;=\; D(x-y)-D(y-x)
\end{aligned}
\tag{6.39}
$$

Now, if $(x-y)^2<0$ there exists a frame where $x_0=y_0$ so $x-y\;=\;(0,\vec r)$ and

$$
D(x-y)\;=\;\int\frac{đ^3p}{2E_p}\,e^{-i\vec p\cdot\vec r},\quad D(y-x)\;=\;\int\frac{đ^3p}{2E_p}\,e^{i\vec p\cdot\vec r}\overset{\vec p\leftrightarrow-\vec p}{=}\int\frac{đ^3p}{2E_p}\,e^{i\vec p\cdot\vec r}\;=\;D(x-y)
\tag{6.40}
$$

which proves $[\hat\phi(x),\hat\phi(y)]\;=\;0$ at $(x-y)^2<0$.

6.3.3 Feynman propagator

In AQM course you've studied Feynman propagators. In terms of operators the Feynman propagator reads

$$
D_F(x-y)\;\equiv\;\theta(x_0-y_0)D(x-y)+\theta(y_0-x_0)D(y-x)\;=\;\langle 0|T\{\hat\phi(x)\hat\phi(y)\}|0\rangle
\tag{6.41}
$$

where

$$
T\{\hat\phi(x)\hat\phi(y)\}\;\equiv\;\theta(x_0-y_0)\hat\phi(x)\hat\phi(y)+\theta(y_0-x_0)\hat\phi(y)\hat\phi(x)
\tag{6.42}
$$

is called a"T-product" of operators.
NB: since $[\hat\phi(x),\hat\phi(y)]\;=\;0$ the T-product (6.42) is relativistic invariant $\to$ the Feynman propagator is relativistic invariant.

Proof: if $(x-y)^2>0$, $\theta(x_0-y_0)$ singles out upper cone, $\theta(x_0-y_0)$ singles out lower cone, and for $(x-y)^2<0$ the order of operators does not matter anyway.

Explicit form of Feynman propagator:

$$
D_F(x-y)\;=\;\lim_{\epsilon\to0}\int\frac{đ^4p}{i}\,\frac{1}{m^2-p^2-i\epsilon}
\tag{6.43}
$$

Proof: perform the integration over p_0

$$
\begin{aligned}
\lim_{\epsilon\to0}\int\frac{đ^4p}{i}\,\frac{e^{-ip(x-y)}}{m^2-p^2-i\epsilon} &= \int đ^3p\;e^{i\vec p(\vec x-\vec y)}\lim_{\epsilon\to0}\int\frac{dp_0}{2\pi i}\,\frac{e^{-ip_0(x-y)_0}}{m^2+\vec p^2-p_0^2-i\epsilon} \\
&= \int đ^3p\;e^{i\vec p(\vec x-\vec y)}\lim_{\epsilon\to0}\int\frac{dp_0}{2\pi i}\,\frac{-e^{-ip_0(x-y)_0}}{(p_0-E_p+i\epsilon)(p_0+E_p-i\epsilon)}
\end{aligned}
\tag{6.44}
$$

If $x_0>y_0$ we can close the contour of integration over p_0 in the lower half-plane and get a residue at $p_0\;=\;E_p-i\epsilon$, see Fig. 4

$$
\lim_{\epsilon\to0}\int\frac{dp_0}{2\pi i}\,\frac{-e^{-ip_0(x-y)_0}}{(p_0-E_p+i\epsilon)(p_0+E_p-i\epsilon)}\;=\;\frac{1}{2E_p}e^{-iE_p(x_0-y_0)}
\tag{6.45}
$$

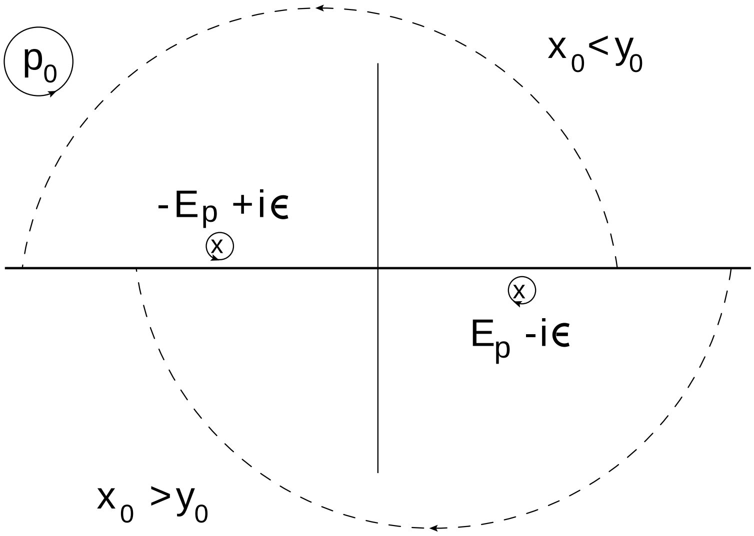


Figure 4. Contour for Feynman propagator

and therefore

$$\lim_{\epsilon\to 0}\int \frac{d\!\!{}^-\!{}^4 p}{i}\,\frac{e^{-ip(x-y)}}{m^2-p^2-i\epsilon} \;=\; \int \frac{d\!\!{}^-\!{}^3 p}{2E_p}\, e^{-ip_0(x-y)_0+i\vec{p}(\vec{x}-\vec{y})} \;=\; D(x-y) \tag{6.46}$$

Similarly, if $x_0 < y_0$ we can close the contour of integration over p_0 in the upper half-plane and get a residue at $p_0 \;=\; -E_p + i\epsilon$ so

$$\lim_{\epsilon\to 0}\int \frac{dp_0}{2\pi i}\,\frac{-e^{-ip_0(x-y)_0}}{(p_0-E_p+i\epsilon)(p_0+E_p-i\epsilon)} \;=\; \frac{1}{2E_p}e^{iE_p(x_0-y_0)} \tag{6.47}$$

and therefore

$$\lim_{\epsilon\to 0}\int \frac{d\!\!{}^-\!{}^4 p}{i}\,\frac{e^{-ip(x-y)}}{m^2-p^2-i\epsilon} \;=\; \int \frac{d\!\!{}^-\!{}^3 p}{2E_p}\, e^{ip_0(x-y)_0+i\vec{p}(\vec{x}-\vec{y})} \overset{\vec{p}\leftrightarrow-\vec{p}}{=} D(y-x) \tag{6.48}$$

so we get $D_F(x-y) \;=\; \theta(x_0-y_0)D(x-y)+\theta(y_0-x_0)D(y-x)$ as in the definition (6.41).

Mathematically, the Feynman propagator (6.43) is a Green function of the KG equation:

$$\begin{aligned}\lim_{\epsilon\to 0}(\partial_x^2+m^2)\int \frac{d\!\!{}^-\!{}^4 p}{i}\,\frac{e^{-ip(x-y)}}{m^2-p^2-i\epsilon} \;&=\; \lim_{\epsilon\to 0}\int \frac{d\!\!{}^-\!{}^4 p}{i}\,(m^2-p^2)\frac{e^{-ip(x-y)}}{m^2-p^2-i\epsilon}\\ =\; \int \frac{d\!\!{}^-\!{}^4 p}{i}\,e^{-ip(x-y)} \;&=\; -i\delta^{(4)}(x-y)\end{aligned} \tag{6.49}$$

6.3.4 Retarded, advanced and Feynman Green functions

A green function of the KG operator is a function satisfying the equation

$$(\partial_x^2+m^2)D_G(x-y) \;=\; -i\delta^{(4)}(x-y) \tag{6.50}$$

This equation can be solved by Fourier transformation and the answer is

$$D_G(x-y) \;=\; \int \frac{d\!\!{}^-\!{}^4 p}{i}\,\frac{1}{m^2-p^2} \tag{6.51}$$

However, this answer is ill-defined since there is a singularity on the path of integration over p_0 which need to be circumvent one way or another. There are 3 possible ways to go around the singularity. They correspond to retarded, advanced and Feynman Green functions.

Retarded propagator (retarded Green function)

$$D_R(x-y) \;\equiv\; \theta(x_0-y_0)\langle[\hat{\phi}(x),\hat{\phi}(y)]\rangle \tag{6.52}$$

let us demonstrate that

$$D_R(x-y) \;=\; \lim_{\epsilon\to 0}\int\frac{đ^4p}{i}\;\frac{e^{-ip(x-y)}}{m^2-p^2-i\epsilon p_0} \tag{6.53}$$

Indeed,

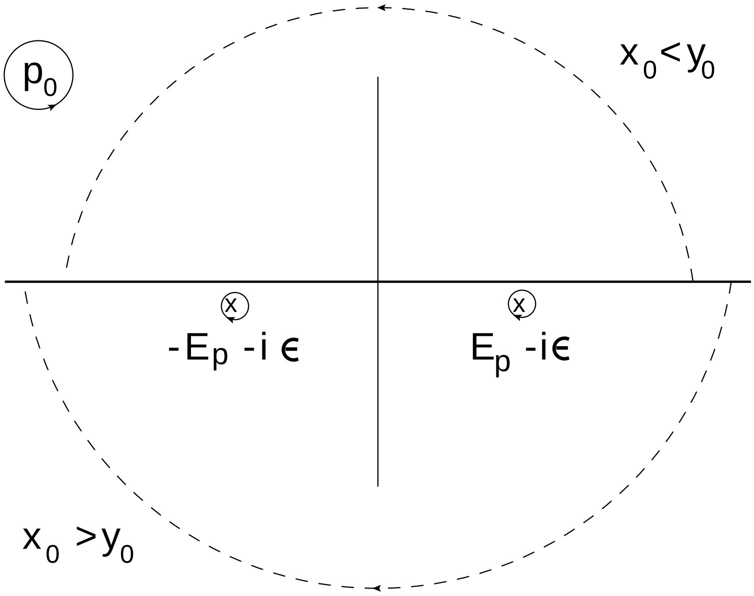


Figure 5. Contour for retarded Green function

$$\begin{aligned}\lim_{\epsilon\to 0}\int\frac{đ^4p}{i}\;\frac{e^{-ip(x-y)}}{m^2-p^2-i\epsilon p_0} \;&=\; \lim_{\epsilon\to 0}\int đ^3p\; e^{i\vec{p}(\vec{x}-\vec{y})}\lim_{\epsilon\to 0}\int\frac{dp_0}{2\pi i}\;\frac{e^{-ip_0(x-y)_0}}{m^2+\vec{p}^2-p_0^2-i\epsilon p_0}\\ =\; \int đ^3p\; e^{i\vec{p}(\vec{x}-\vec{y})}\lim_{\epsilon\to 0}\int\frac{dp_0}{2\pi i}\;&\frac{-e^{-ip_0(x-y)_0}}{(p_0-E_p+i\epsilon)(p_0+E_p+i\epsilon)}\end{aligned} \tag{6.54}$$

Now both poles in the integral over p_0 lie in the lower half-plane (see Fig. 5) so at $x_0<y_0$ one can close the contour in the upper half-plane and get 0 while at $x_0>y_0$ one gets a sum of two residues:

$$\lim_{\epsilon\to 0}\int\frac{dp_0}{2\pi i}\;\frac{-e^{-ip_0(x-y)_0}}{(p_0-E_p+i\epsilon)(p_0+E_p+i\epsilon)} \;=\; \theta(x_0-y_0)\frac{1}{2E_p}\left[e^{-iE_p(x_0-y_0)}-e^{iE_p(x_0-y_0)}\right] \tag{6.55}$$

Thus,

$$\lim_{\epsilon\to 0}\int\frac{d\!\!{}^-{}^4p}{i}\,\frac{e^{-ip(x-y)}}{m^2-p^2-i\epsilon p_0} \;=\; \theta(x_0-y_0)\int\frac{d\!\!{}^-{}^3p}{2E_p}\,e^{i\vec{p}(\vec{x}-\vec{y})}\big[e^{-iE_p(x_0-y_0)}-e^{iE_p(x_0-y_0)}\big]$$
$$=\; \theta(x_0-y_0)[D(x-y)-D(y-x)] \;=\; \theta(x_0-y_0)\langle[\hat{\phi}(x),\hat{\phi}(y)]\rangle, \qquad \text{Q.E.D.} \tag{6.56}$$

Similarly one can prove that the advanced Green function

$$D_A(x-y) \;\equiv\; \theta(y_0-x_0)\langle[\hat{\phi}(y),\hat{\phi}(x)]\rangle \tag{6.57}$$

can be represented as

$$D_A(x-y) \;=\; \lim_{\epsilon\to 0}\int\frac{d\!\!{}^-{}^4p}{i}\,\frac{e^{-ip(x-y)}}{m^2-p^2+i\epsilon p_0} \tag{6.58}$$

(Actually, the easiest way to prove the above equation is to make a change of variables $p\leftrightarrow -p$ in the integral (6.53) which corresponds to $x\leftrightarrow y$).

Part VI

7 Self-interacting KG field

In classical physics

$$\mathcal{L} \;=\; \frac{1}{2}\partial^\mu\phi\partial_\mu\phi-\frac{m^2}{2}\phi^2-\frac{\lambda}{4!}\phi^4 \tag{7.1}$$

Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial\phi} \;=\; -m^2\phi-\frac{\lambda}{3!}\phi^3, \qquad \frac{\partial\mathcal{L}}{\partial\partial^\mu\phi} \;=\; \partial_\mu\phi, \qquad \frac{\partial\mathcal{L}}{\partial\phi} \;=\; \partial^\mu\frac{\partial\mathcal{L}}{\partial\partial^\mu\phi}$$
$$\Rightarrow\; -m^2\phi-\frac{\lambda}{3!}\phi^3 \;=\; \partial^2\phi(x) \tag{7.2}$$

$\Rightarrow$ the equation of motion is non-linear

$$(\partial^2+m^2)\phi(x) \;=\; -\frac{\lambda}{3!}\phi^3(x) \tag{7.3}$$

In classical physics, we try to solve the non-linear equation (7.3). In QFT, the exact solutions were found only for some simple $1+1$-dimensional models. Instead

- Perturbation theory at small $\lambda \ll 1$.
- Semiclassical methods (analog of WKB method in QM).
- Calculations of functional integrals by lattice simulations.

In this course we will discuss only the perturbation theory (in KG model, then in Yukawa theory, in QED and finally in QCD).

7.1 Perturbation theory for self-interacting KG scalar field in QFT

The Lagrangian density:

$$\mathcal{L} \;=\; \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{7.4}$$

The canonical momentum:

$$\pi \;=\; \frac{\partial\mathcal{L}}{\partial\partial^0\phi} \;=\; \partial^0\phi \tag{7.5}$$

The Hamiltonian density is given by

$$\mathcal{H} \;=\; \pi\partial^0\phi - \mathcal{L} \;=\; \frac{\pi^2}{2} + \frac{1}{2}|\vec{\nabla}\phi|^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{7.6}$$

7.1.1 Quantization (at $t = 0$)

As usually, we promote ϕ and π to operators

$$\phi(t,\vec{x}) \;\to\; \hat{\phi}(\vec{x}), \quad \pi(t,\vec{x}) \;\to\; \hat{\pi}(\vec{x}), \tag{7.7}$$

satisfying the canonical commutation relations (CCRs)

$$[\hat{\phi}(\vec{x}),\hat{\pi}(\vec{y})] \;=\; i\delta(\vec{x}-\vec{y}), \quad [\hat{\phi}(\vec{x}),\hat{\phi}(\vec{y})] \;=\; [\hat{\pi}(\vec{x}),\hat{\pi}(\vec{y})] \;=\; 0 \tag{7.8}$$

The quantum Hamiltonian is

$$\hat{H} \;=\; \int d^3x\Big[\frac{1}{2}\hat{\pi}^2(\vec{x}) + \frac{1}{2}|\vec{\nabla}\hat{\phi}(\vec{x})|^2 + \frac{m^2}{2}\hat{\phi}^2(\vec{x}) + \frac{\lambda}{4!}\hat{\phi}^4(\vec{x})\Big] \tag{7.9}$$

We define vacuum state $|\Omega\rangle$ as an eigenstate of $\hat{H}$ with the lowest energy (we suppose that it is non-degenerate).

$$\hat{H}|\Omega\rangle \;=\; E_{\rm vac}|\Omega\rangle \qquad - \qquad \text{stationary Schrodinger equation} \tag{7.10}$$

In the explicit form it reads

$$\int d^3x\,\Big[-\frac{1}{2}\Big(\frac{\delta}{\delta\phi(\vec{x})}\Big)^2 + \frac{1}{2}|\nabla\phi(\vec{x})|^2 + \frac{m^2}{2}\phi^2(\vec{x}) + \frac{\lambda}{4!}\phi^4(\vec{x})\Big]\Psi_{\rm vac}(\phi) \;=\; E_{\rm vac}\Psi_{\rm vac}(\phi) \tag{7.11}$$

(here $\langle\{\phi(\vec{x})\}|\Omega\rangle \;=\; \Psi_{\rm vac}(\phi(\vec{x})$ as usual).

Heisenberg picture

We start with $\Psi(\{\phi(\vec{x})\}) \;=\; \Psi_{\rm Schro}(t = 0, \{\phi(\vec{x})\})$ and time-dependent canonical operators

$$\begin{aligned}\hat{\phi}(x) &\equiv e^{i\hat{H}t}\hat{\phi}(\vec{x})e^{-i\hat{H}t}\\ \hat{\pi}(x) &\equiv e^{i\hat{H}t}\hat{\pi}(\vec{x})e^{-i\hat{H}t}\end{aligned} \tag{7.12}$$

where $x = (t,\vec{x})$ and $\hat{H}$ is given by Eq. (7.9). Note that formulas (7.12) look like Eqs. (6.7), only $\hat{H}$ now means the Hamiltonian (7.9) with the interaction term.

The Heisenberg equations look similarly to Eq. (6.3) (with $\hat{H}$ given by (7.9))

$$\frac{d\hat{\phi}(t,\vec{x})}{dt} = i[\hat{H},\hat{\phi}(t,\vec{x})], \qquad \frac{d\hat{\pi}(t,\vec{x})}{dt} = i[\hat{H},\hat{\pi}(t,\vec{x})]) \tag{7.13}$$

Let us prove equal-time commutators

$$\begin{aligned}
[\hat{\phi}(t,\vec{x}),\hat{\pi}(t,\vec{y})] &= [e^{i\hat{H}t}\hat{\phi}(\vec{x})e^{-i\hat{H}t}, e^{i\hat{H}t}\hat{\pi}(\vec{y})e^{-i\hat{H}t}] = e^{i\hat{H}t}[\hat{\phi}(\vec{x}),\hat{\pi}(\vec{y})]e^{-i\hat{H}t} = i\delta(\vec{x}-\vec{y})\\
[\hat{\phi}(t,\vec{x}),\hat{\phi}(t,\vec{y})] &= [e^{i\hat{H}t}\hat{\phi}(\vec{x})e^{-i\hat{H}t}, e^{i\hat{H}t}\hat{\phi}(\vec{y})e^{-i\hat{H}t}] = e^{i\hat{H}t}[\hat{\phi}(\vec{x}),\hat{\phi}(\vec{y})]e^{-i\hat{H}t} = 0\\
[\hat{\pi}(t,\vec{x}),\hat{\pi}(t,\vec{y})] &= [e^{i\hat{H}t}\hat{\pi}(\vec{x})e^{-i\hat{H}t}, e^{i\hat{H}t}\hat{\pi}(\vec{y})e^{-i\hat{H}t}] = e^{i\hat{H}t}[\hat{\pi}(\vec{x}),\hat{\pi}(\vec{y})]e^{-i\hat{H}t} = 0 \qquad (7.14)
\end{aligned}$$

which is identical to Eq. (6.32) albeit with different $\hat{H}$. From this equation it is easy to see that

$$\begin{aligned}
\hat{H}(t) &\equiv \int d^3x\big[\frac{1}{2}\hat{\pi}^2(t,\vec{x}) + \frac{1}{2}|\vec{\nabla}\hat{\phi}(t,\vec{x})|^2 + \frac{m^2}{2}\hat{\phi}^2(t,\vec{x}) + \frac{\lambda}{4!}\hat{\phi}^4(t,\vec{x})\big] \qquad (7.15)\\
&= \int d^3x\big[\frac{1}{2}e^{i\hat{H}t}\hat{\pi}^2(\vec{x})e^{-i\hat{H}t} + \frac{1}{2}e^{i\hat{H}t}|\vec{\nabla}\hat{\phi}(\vec{x})|^2e^{-i\hat{H}t} + \frac{m^2}{2}e^{i\hat{H}t}\hat{\phi}^2(\vec{x})e^{-i\hat{H}t} + \frac{\lambda}{4!}e^{i\hat{H}t}\hat{\phi}^4(\vec{x})e^{-i\hat{H}t}\big]\\
&= e^{i\hat{H}t}\int d^3x\big[\frac{1}{2}\hat{\pi}^2(\vec{x}) + \frac{1}{2}|\vec{\nabla}\hat{\phi}(\vec{x})|^2 + \frac{m^2}{2}\hat{\phi}^2(\vec{x}) + \frac{\lambda}{4!}\hat{\phi}^4(\vec{x})\big]e^{-i\hat{H}t} = e^{i\hat{H}t}\hat{H}e^{-i\hat{H}t} = \hat{H}
\end{aligned}$$

so $\hat{H}(t)$ does not actually depend on t.

Let us prove now that the quantum operator $\hat{\phi}(x)$ satisfies the same KG equation (7.3) as the classical field

$$(\partial^2+m^2)\hat{\phi}(x) = -\frac{\lambda}{3!}\hat{\phi}^3(x) \tag{7.16}$$

Proof: first, consider $\frac{\partial}{\partial t}\hat{\phi}(t,\vec{x})$. Due to Eqs. (7.13) and (7.15) we get

$$\begin{aligned}
\frac{\partial}{\partial t}\hat{\phi}(t,\vec{x}) &= i[\hat{H},\hat{\phi}(t,\vec{x})] \qquad (7.17)\\
&= i\int d^3z\big[\frac{1}{2}\hat{\pi}^2(t,\vec{z}) + \frac{1}{2}|\vec{\nabla}\hat{\phi}(t,\vec{z})|^2 + \frac{m^2}{2}\hat{\phi}^2(t,\vec{z}) + \frac{\lambda}{4!}\hat{\phi}^4(t,\vec{z}),\hat{\phi}(t,\vec{x})\big]\\
&= i\int d^3z\big[\frac{1}{2}\hat{\pi}^2(t,\vec{z}),\hat{\phi}(t,\vec{x})\big] = \int d^3z\ \hat{\pi}(t,\vec{z})\delta(\vec{x}-\vec{z}) = \hat{\pi}(t,\vec{x})
\end{aligned}$$

Next,

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}\hat{\phi}(t,\vec{x}) &= \frac{\partial}{\partial t}\hat{\pi}(t,\vec{x}) = i[\hat{H},\hat{\pi}(t,\vec{x})] \qquad (7.18)\\
&= i\int d^3z\big[\frac{1}{2}\hat{\pi}^2(t,\vec{z}) + \frac{1}{2}|\vec{\nabla}\hat{\phi}(t,\vec{z})|^2 + \frac{m^2}{2}\hat{\phi}^2(t,\vec{z}) + \frac{\lambda}{4!}\hat{\phi}^4(t,\vec{z}),\hat{\pi}(t,\vec{x})\big]\\
&= i\int d^3z\big[\frac{1}{2}|\vec{\nabla}\hat{\phi}(t,\vec{z})|^2 + \frac{m^2}{2}\hat{\phi}^2(t,\vec{z}) + \frac{\lambda}{4!}\hat{\phi}^4(t,\vec{z}),\hat{\pi}(t,\vec{x})\big]\\
&= \nabla^2\hat{\phi}(t,\vec{x}) - m^2\hat{\phi}(t,\vec{x}) - \frac{\lambda}{3!}\hat{\phi}^3(t,\vec{x}) \quad\Rightarrow\quad (\partial^2+m^2)\hat{\phi}(x) = -\frac{\lambda}{3!}\hat{\phi}^3(x)
\end{aligned}$$

where we used formulas

$$
\begin{aligned}
\int d^3z\big[\frac{m^2}{2}\hat{\phi}^2(t,\vec{z}),\hat{\pi}(t,\vec{x})\big] &= m^2\int d^3z\ \hat{\phi}^2(t,\vec{z})\big[\hat{\phi}(t,\vec{z}),\hat{\pi}(t,\vec{x})\big] = im^2\hat{\phi}(t,\vec{x})\\
\int d^3z\big[\frac{\lambda}{4!}\hat{\phi}^4(t,\vec{z}),\hat{\pi}(t,\vec{x})\big] &= \frac{\lambda}{3!}\int d^3z\ \hat{\phi}^3(t,\vec{z})\big[\hat{\phi}(t,\vec{z}),\hat{\pi}(t,\vec{x})\big] = i\frac{\lambda}{3!}\hat{\phi}^3(t,\vec{x})\\
\int d^3z\big[\frac{1}{2}|\vec{\nabla}\hat{\phi}(t,\vec{z})|^2,\hat{\pi}(t,\vec{x})\big] &= \int d^3z\ \vec{\partial}_i\hat{\phi}(t,\vec{z})[\vec{\partial}_i\hat{\phi}(t,\vec{z}),\hat{\pi}(t,\vec{x})] = i\int d^3z\ \vec{\partial}_i\hat{\phi}(t,\vec{z})\frac{\partial}{\partial\vec{z}_i}\delta(\vec{x}-\vec{z})\\
= -i\int d^3z\ \nabla^2\hat{\phi}(t,\vec{z})\delta(\vec{x}-\vec{z}) &= -i\nabla^2\hat{\phi}(t,\vec{x}) \qquad (7.19)
\end{aligned}
$$

7.2 Green functions

Definition of the n-point Feynman Green function

$$
\begin{aligned}
&\langle\Omega|T\{\hat{\phi}(x_1)\hat{\phi}(x_2)...\hat{\phi}(x_n)\}|\Omega\rangle \qquad (7.20)\\
&= \theta(x_{10}>x_{20}>...x_{n0})\langle\Omega|\hat{\phi}(x_1)\hat{\phi}(x_2)...\hat{\phi}(x_n)|\Omega\rangle + \text{permutations}
\end{aligned}
$$

Examples:

1. Two-point Geeen function

$$
\begin{aligned}
&\langle\Omega|T\{\hat{\phi}(x_1)\hat{\phi}(x_2)\}|\Omega\rangle\\
&= \theta(x_{10}-x_{20})\langle\Omega|\hat{\phi}(x_1)\hat{\phi}(x_2)|\Omega\rangle + \theta(x_{20}-x_{10}>)\langle\Omega|\hat{\phi}(x_2)\hat{\phi}(x_1)|\Omega\rangle \qquad (7.21)
\end{aligned}
$$

2. 3-point Green function

$$
\begin{aligned}
&\langle\Omega|T\{\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\}|\Omega\rangle \qquad (7.22)\\
&= \theta(x_{10}-x_{20})\theta(x_{20}-x_{30})\langle\Omega|\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)|\Omega\rangle + \theta(x_{20}-x_{10})\theta(x_{10}-x_{30})\langle\Omega|\hat{\phi}(x_2)\hat{\phi}(x_1)\hat{\phi}(x_3)|\Omega\rangle\\
&+ \theta(x_{20}-x_{30})\theta(x_{30}-x_{10})\langle\Omega|\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_1)|\Omega\rangle + \theta(x_{10}-x_{30})\theta(x_{30}-x_{20})\langle\Omega|\hat{\phi}(x_1)\hat{\phi}(x_3)\hat{\phi}(x_2)|\Omega\rangle\\
&+ \theta(x_{30}-x_{10})\theta(x_{10}-x_{20})\langle\Omega|\hat{\phi}(x_3)\hat{\phi}(x_1)\hat{\phi}(x_2)|\Omega\rangle + \theta(x_{30}-x_{20})\theta(x_{20}-x_{10})\langle\Omega|\hat{\phi}(x_3)\hat{\phi}(x_2)\hat{\phi}(x_1)|\Omega\rangle
\end{aligned}
$$

Later we will prove that the Green function (7.20) can be represented by a sum of Feynman diagrams with n tails, but at first we discuss the relation between Green functions and scattering amplitudes.

8 LSZ reduction formula

8.1 In- and out- states

A typical setup for scattering:

Let us define the operator $\hat{\phi}_{\rm in}$ by

$$
\hat{\phi}_{\rm in}(x) \equiv \hat{\phi}(x) + i\frac{\lambda}{3!}\int d^4z\ G^0_R(x-z)\ \hat{\phi}^3(z) \qquad (8.1)
$$

where $G^0_R(x-y)$ is the retarded Green function (6.53)

$$
G^0_R(x-y) \equiv \int\frac{d\!\!{}^-{}^4p}{i}\frac{e^{-ip(x-y)}}{m^2-p^2-i\epsilon p_0}
$$

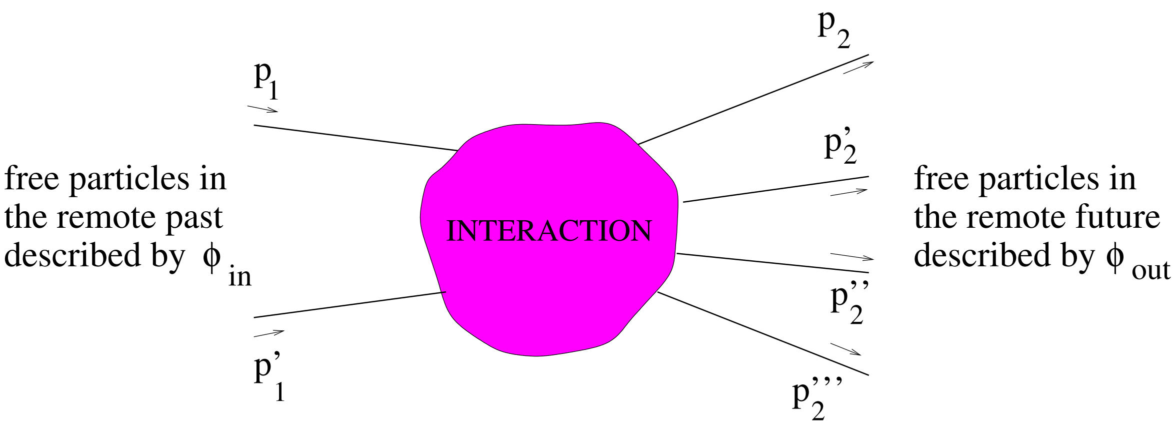


Figure 6. Scattering

satisfying the equation

$$(\partial^2 + m^2)\; G_R^0(x-y) \;=\; -\,i\,\delta^{(4)}(x-y), \qquad G_R^0(x-y) \;=\; 0 \;\;(\text{ if } \;\mathrm{x}_0 < \mathrm{y}_0\;) \tag{8.2}$$

Using this equation one obtains

$$\begin{aligned}\left(\partial^2 + m^2\right)\;\hat\phi_{\rm in}(x) \;&=\; (\partial^2+m^2)\hat\phi(x) + i\frac{\lambda}{3!}\int d^4z\;\;(\partial_x^2+m^2)G_R^0(x-z)\;\hat\phi^3(z)\\ &=\; -\,\frac{\lambda}{3!}\hat\phi^3(x) \;+\; \frac{\lambda}{3!}\hat\phi^3(x) \;=\; 0\end{aligned} \tag{8.3}$$

so the field $\phi_{\rm in}$ describes free particles at $t \to -\infty$.

Let us look now at the relation between at $\hat\phi_{\rm in}(t,\vec{x})$ and $\hat\phi(t,\vec{x})$ as $t \to -\infty$

$$\begin{aligned}\hat\phi_{\rm in}(x) \;&\equiv\; \hat\phi(x) + i\int d^4z\;\; G_R^0(x-z)\;\hat\phi^3(z)\\ &=\; \hat\phi(x) + i\int_{-\infty}^{t} dz_0 \int d^3z\;\; G_R^0(x-z)\;\hat\phi^3(z) \quad\Rightarrow\quad \hat\phi(x) \stackrel{t\to-\infty}{\rightarrow} \hat\phi_{\rm in}(x)\end{aligned} \tag{8.4}$$

so the field $\hat\phi(x)$ approaches the free field $\hat\phi_{\rm in}(x)$ in the remote past.

Similarly, we define

$$\hat\phi_{\rm out}(x) \;\;\equiv\;\; \hat\phi(x) \;+\; i\frac{\lambda}{3!}\int d^4z\;\; G_A^0(x-z)\;\hat\phi^3(z) \tag{8.5}$$

where $G_R^0(x-y)$ is the advanced Green function (6.58):

$$G_A^0(x-y) \;\equiv\; \int \frac{d\!\!{}^-{}^4p}{i}\,\frac{e^{-ip(x-y)}}{m^2-p^2+i\epsilon p_0}$$

satisfying the equation

$$\left(\partial^2 + m^2\right)\;G_A^0(x-y) \;=\; -\,i\,\delta^{(4)}(x-y), \qquad G_A^0(x-y) \;=\; 0 \;\;(\text{ if }\;\mathrm{x}_0 > \mathrm{y}_0\;) \tag{8.6}$$

Using this equation one obtains

$$\left(\partial^2+m^2\right)\ \hat{\phi}_{\rm out}(x) \;=\; (\partial^2+m^2)\hat{\phi}(x)+i\frac{\lambda}{3!}\int d^4z\ \ (\partial_x^2+m^2)G_A^0(x-z)\ \hat{\phi}^3(z)$$
$$=\; -\frac{\lambda}{3!}\hat{\phi}^3(x)\ +\ \frac{\lambda}{3!}\hat{\phi}^3(x)\ =\ 0 \tag{8.7}$$

Hence, the wave function $\phi_{\rm out}$ describes free particles at $t\to\infty$. Similarly to Eq. (8.4) one obtains

$$\hat{\phi}_{\rm out}(x) \;\equiv\; \hat{\phi}(x)+i\int d^4z\ \ G_A^0(x-z)\ \hat{\phi}^3(z)$$
$$=\; \hat{\phi}(x)+i\int_t^\infty dz_0\int d^3z\ \ G_A^A(x-z)\ \hat{\phi}^3(z) \quad\Rightarrow\quad \hat{\phi}(x)\overset{t\to\infty}{\to}\hat{\phi}_{\rm out}(x) \tag{8.8}$$

so the field $\hat{\phi}(x)$ approaches the free field $\hat{\phi}_{\rm out}(x)$ in the remote future. [3]
In terms of ladder operators:

$$\hat{\phi}_{\rm in} = \int\frac{đ^3p}{\sqrt{2E_p}}\ \left[\hat{a}_{\rm in}(p)\ e^{-ipx}+\hat{a}^\dagger_{\rm in}(p)\ e^{ipx}\right]\Bigg|_{p_0=E_p}$$

with commutation relation $[\hat{a}_{\rm in}(p),\ \hat{a}^\dagger_{\rm in}(p')]\ =\ (2\pi)^3\ \delta(\vec{p}-\vec{p}\,')$,
and $\hat{a}_{\rm in}(p)\ |0_{\rm in}\rangle\ =\ 0$, where $|0_{\rm in}\rangle\ \equiv$ ground state of $\hat{H}^0_{\rm in}$.

Similarly,

$$\hat{\phi}_{\rm out} = \int\frac{đ^3p}{\sqrt{2E_p}}\ \left[\hat{a}_{\rm out}(p)\ e^{-ipx}+\hat{a}^\dagger_{\rm out}(p)\ e^{ipx}\right]\Bigg|_{p_0=E_p}$$

with commutation relation $[\hat{a}_{\rm out}(p),\ \hat{a}^\dagger_{\rm out}(p')]\ =\ (2\pi)^3\ \delta(\vec{p}-\vec{p}\,')$,
and $\hat{a}_{\rm out}\ |0_{\rm out}\rangle\ =\ 0$, where $|0_{\rm out}\rangle\ \equiv$ ground state of $\hat{H}^0_{\rm out}$.

Main Hypothesis: $|0_{\rm in}\rangle\ =\ |0_{\rm out}\rangle\ =\ |\Omega\rangle$
"In" and "out" states:

$$|p_1,...p_n\rangle_{\rm in}\ \equiv\ \Pi\sqrt{2E_{p_k}}\ a^\dagger_{p_k}\ |0_{\rm in}\rangle$$
$$|p_1,...p_n\rangle_{\rm out}\ \equiv\ \Pi\sqrt{2E_{p_k}}\ a^\dagger_{p_k}\ |0_{\rm out}\rangle \tag{8.9}$$

The amplitude of the $m\to n$ transition is given by the matrix element of S-matrix:

$$S\big(p_1,p_1',...p_1^{(m)}\to p_2,p_2',...p_2^{(n)}\big)\ =\ {}_{\rm out}\langle p_2,p_2',...p_2^{(n)}\ |\ p_1,p_1',...,p_1^{(m)}\rangle_{\rm in} \tag{8.10}$$

8.2 LSZ reduction formula (for 2→2 scattering)

$$S(p_1,p_1'\ \to\ p_2,p_2')\ =\ i^4\lim_{p_i^2\to m^2}(m^2-p_1^2)\ (m^2-p_2^2)\ (m^2-p'^2_1)\ (m^2-p'^2_2) \tag{8.11}$$
$$\times\int dx\ dx'\ dy\ dy'\ e^{-ip_1x_1-ip_1'x'+ip_2y+ip_2'y'}\ \langle\Omega|T\{\hat{\phi}(x)\hat{\phi}(x')\hat{\phi}(y)\hat{\phi}(y')\}|\Omega\rangle$$

[3]The rigorous statement is $\hat{\phi}(x)\overset{t\to-\infty}{\to}Z^{\frac12}\hat{\phi}_{\rm in}(x)$ and $\hat{\phi}(x)\overset{t\to\infty}{\to}Z^{\frac12}\hat{\phi}_{\rm out}(x)$ where Z is a number (to be discussed with the theory of renormalization)

Proof of the LSZ theorem:
For any free KG field

$$\begin{aligned}\sqrt{2E_p}\;\hat{a}(p) &= i\int d^3x\;\; e^{-i\vec{p}\vec{x}+iE_pt}\;\overset{\leftrightarrow}{\partial_0}\;\hat{\phi}(x)\\ \sqrt{2E_p}\;\hat{a}^\dagger(p) &= -\,i\int d^3x\;\; e^{i\vec{p}\vec{x}-iE_pt}\;\overset{\leftrightarrow}{\partial_0}\;\hat{\phi}(x)\end{aligned}\tag{8.12}$$

and therefore

$$\begin{aligned}&{}_{\rm out}\langle p_2,p_2'\mid p_1,p_1'\rangle_{\rm in} = {}_{\rm out}\langle p_2,p_2'\mid\;\hat{a}^\dagger_{\rm in}(p_1)\;|p_1'\rangle_{\rm in}\;\;\sqrt{2E_1} =\\ &= {}_{\rm out}\langle p_2,p_2'\mid\;\hat{a}^\dagger_{\rm out}(p_1)+\Big(\hat{a}^\dagger_{\rm in}(p_1)-\hat{a}^\dagger_{\rm out}(p_1)\Big)\;|p_1'\rangle_{\rm in}\;\;\sqrt{2E_1} =\\ &= {}_{\rm out}\langle p_2,p_2'\mid\;\int d^3x\;e^{i\vec{p}_1\vec{x}-iE_1t}\;i\;\overset{\leftrightarrow}{\partial_0}\;\Big(\hat{\phi}_{\rm out}(x)-\hat{\phi}_{\rm in}(x)\Big)\;|p_1'\rangle_{\rm in}\end{aligned}\tag{8.13}$$

The l.h.s. does not depend on t so

$$\begin{aligned}&{}_{\rm out}\langle p_2,p_2'\mid\;\int d^3x\;e^{i\vec{p}_1\vec{x}-iE_1t}\;i\;\overset{\leftrightarrow}{\partial_0}\;\hat{\phi}_{\rm out}(x)\;|p_1'\rangle_{\rm in}\;=\;\text{take } t\to\infty\;=\\ &= {}_{\rm out}\langle p_2,p_2'\mid\;\int d^3x\;e^{i\vec{p}_1\vec{x}-iE_1t}\;i\;\overset{\leftrightarrow}{\partial_0}\;\hat{\phi}(x)\;|p_1'\rangle_{\rm in}\;\Big|_{t=\infty}\end{aligned}\tag{8.14}$$

Similarly,

$$\begin{aligned}&{}_{\rm out}\langle p_2,p_2'\mid\;\int d^3x\;e^{i\vec{p}_1\vec{x}-iE_1t}\;i\;\overset{\leftrightarrow}{\partial_0}\;\hat{\phi}_{\rm in}(x)\;|p_1'\rangle_{\rm in}\;=\;\text{take } t\to-\infty\;=\\ &= {}_{\rm out}\langle p_2,p_2'\mid\;\int d^3x\;e^{i\vec{p}_1\vec{x}-iE_1t}\;i\;\overset{\leftrightarrow}{\partial_0}\;\hat{\phi}(x)\;|p_1'\rangle_{\rm in}\;\Big|_{t=-\infty}\end{aligned}\tag{8.15}$$

Using the formula

$$\int d^3x\;\; g_1(t,\vec{x})\;\overset{\leftrightarrow}{\partial_0}\;g_2(t,\vec{x})\Big|_{t=-\infty}^{t=\infty} \;=\; \int d^4x\;\;\left[g_1(x)\;\frac{\partial^2}{\partial t^2}\;g_2(x)-g_2(x)\;\frac{\partial^2}{\partial t^2}\;g_1(x)\right]\tag{8.16}$$

for $\;g_1(x)\equiv e^{i\vec{p}_1\vec{x}-iE_1t},\;g_2\equiv\hat{\phi}(x)\;$ we get:

$$\begin{aligned}{}_{\rm out}\langle p_2,p_2'\mid p_1,p_1'\rangle_{\rm in} &= {}_{\rm out}\langle p_2,p_2'\mid i\int d^4x\;e^{-ip_1x}\;\;\big(E_1^2+\partial_0^2\big)\;\;\hat{\phi}(x)\;|p_1'\rangle_{\rm in} = \qquad (8.17)\\ &= \lim_{p_1^2\to m^2}\big(m^2-p_1^2\big)\;\;i\int d^4x\;e^{-ip_1x}{}_{\rm out}\langle p_2,p_2'\mid\hat{\phi}(x)|p_1'\rangle_{\rm in}\end{aligned}$$

Next

$$\begin{aligned}&{}_{\rm out}\langle p_2,p_2'\mid\hat{\phi}(x)\mid p_1'\rangle_{\rm in} \;=\; {}_{\rm out}\langle p_2'\mid a_{\rm out}(p_2)\hat{\phi}(x)-\hat{\phi}(x)a_{\rm in}(p_2)\mid p_1'\rangle_{\rm in}\;\sqrt{2E_2} =\\ &= i\int d^3y\;e^{-i\vec{p}_2\vec{y}+iE_2t}{}_{\rm out}\langle p_2'|\;\frac{\overset{\leftrightarrow}{\partial}}{\partial t}\;\hat{\phi}_{\rm out}(t,\vec{y})\;\hat{\phi}(x)|p_1'\rangle_{\rm in}-i\int d^3y\;e^{-i\vec{p}_2\vec{y}+iE_2t}{}_{\rm out}\langle p_2'|\hat{\phi}(x)\;\frac{\overset{\leftrightarrow}{\partial}}{\partial t}\;\hat{\phi}_{\rm in}(t,\vec{y})|p_1'\rangle_{\rm in}\\ &= i\int d^3y\;e^{-i\vec{p}_2\vec{y}+iE_2t}{}_{\rm out}\langle p_2'|\;\frac{\overset{\leftrightarrow}{\partial}}{\partial t}\;\hat{\phi}(t,\vec{y})\hat{\phi}(x)|p_1'\rangle_{\rm in}\Big|_{t=\infty}-i\int d^3y\;e^{-i\vec{p}_2\vec{y}+iE_2t}{}_{\rm out}\langle p_2'|\hat{\phi}(x)\;\frac{\overset{\leftrightarrow}{\partial}}{\partial t}\;\hat{\phi}(t,\vec{y})|p_1'\rangle_{\rm in}\Big|_{t=-\infty}\\ &= i\int d^3y\;e^{-i\vec{p}_2\vec{y}+iE_2t}{}_{\rm out}\langle p_2'|T\{\hat{\phi}(x)\;\frac{\overset{\leftrightarrow}{\partial}}{\partial t}\;\hat{\phi}(t,\vec{y})\}|p_1'\rangle_{\rm in}\Big|_{t=-\infty}^{t=\infty}\end{aligned}\tag{8.18}$$

Using again formula (8.16) for $g_1(y) = e^{-i\vec{p}_2\vec{y}+iE_2t}$ and $g_2(y) = T\{\hat{\phi}(x)\hat{\phi}(t,\vec{y})\}$ we get:

$$\begin{aligned}
{}_{\text{out}}\langle p_2, p_2'|\hat{\phi}(x)\ |p_1'\rangle_{\text{in}} &= i\int d^4y\ e^{ip_2y}\ \left(E_2^2 + \partial_0^2\right)\ \langle p_2'|T\{\hat{\phi}(y)\hat{\phi}(x)\}|p_1'\rangle_{\text{in}} \qquad (8.19)\\
&= \lim_{p_2^2\to m^2}(m^2 - p_2^2)\ i\int d^4y\ e^{ip_2y}\ {}_{\text{out}}\langle p_2'|T\{\hat{\phi}(y)\hat{\phi}(x)\}|p_1'\rangle_{\text{in}}
\end{aligned}$$

and therefore

$$\begin{aligned}
&{}_{\text{out}}\langle p_2, p_2'\ \mid\ p_1, p_1'\rangle_{\text{in}}\ = \qquad (8.20)\\
&= \lim_{p_1^2,p_2^2\to m^2}(m^2 - p_1^2)(m^2 - p_2^2)\ i^2\int d^4x\ d^4y\ e^{-ip_1x+ip_2y}\ {}_{\text{out}}\langle p_2'\mid T\{\hat{\phi}(x)\hat{\phi}(y)\}\ |p_1'\rangle_{\text{in}}
\end{aligned}$$

Repeating this trick two more times, we get

$$\begin{aligned}
S(p_1, p_1' \to p_2, p_2') &= i^4\lim_{p_i^2\to m^2}(m^2 - p_1^2)\ (m^2 - p_2^2)\ (m^2 - p_1'^2)\ (m^2 - p_2'^2) \qquad (8.21)\\
&\times\int dx dx' dy dy'\ e^{-ip_1x_1 - ip_1'x' + ip_2y + ip_2'y'}\ \langle\Omega|T\{\hat{\phi}(x)\hat{\phi}(x')\hat{\phi}(y)\hat{\phi}(y')\}|\Omega\rangle
\end{aligned}$$

which is the LSZ formula (8.11.)

8.3 LSZ formula in General Case:

$$\begin{aligned}
S\big(p_1, p_1', ...p_1^{(m)} \to p_2, p_2', ...p_2^{(n)}\big) &= {}_{\text{out}}\langle p_2, p_2', ...p_2^{(n)}\mid p_1, p_1', ..., p_1^{(m)}\rangle_{\text{in}} = \qquad (8.22)\\
&= i^{m+n}\lim_{p_i^2\to m^2}\Pi(m^2 - p_i^2)\ \int \Pi dx_1^{(i)}\ \Pi dx_2^{(j)}\ \ e^{-i\sum p_1^{(i)}x_1^{(i)} + i\sum p_2^{(j)}x_2^{(j)}}\\
&\quad\langle\Omega|T\{\hat{\phi}(x_1)\ ...\ \hat{\phi}(x_1^{(m)})\ \ \hat{\phi}(x_2)\ ...\ \hat{\phi}(x_2^{(n)})\}|\Omega\rangle
\end{aligned}$$

Part VII

9 Perturbation theory for self-interacting KG scalar field in QFT

Reminder: we define vacuum state $|\Omega\rangle$ as an eigenstate of $\hat{H}$ with the lowest energy

$$\hat{H}|\Omega\rangle\ =\ E_{\text{vac}}|\Omega\rangle \qquad -\qquad \text{stationary Schrodinger equation} \qquad (9.1)$$

In the explicit form it reads

$$\int d^3x\ \Big[-\frac{1}{2}\Big(\frac{\delta}{\delta\phi(\vec{x})}\Big)^2 + \frac{1}{2}|\nabla\phi(\vec{x})|^2 + \frac{m^2}{2}\phi^2(\vec{x}) + \frac{\lambda}{4!}\phi^4(\vec{x})\Big]\Psi_{\text{vac}}(\phi)\ =\ E_{\text{vac}}\Psi_{\text{vac}}(\phi) \qquad (9.2)$$

(here $\langle\{\phi(\vec{x})\}|\Omega\rangle\ =\ \Psi_{\text{vac}}(\phi(\vec{x})$ as usual).

We split the Hamiltonian in two parts:

$$\begin{aligned} H(t) &= H_0(t) + H_{\text{int}}(t) \\ H_0(t) &= \int d^3x\, \mathcal{H}_0 = \int d^3x \big[\frac{1}{2}\pi^2(t,\vec{x}) + \frac{1}{2}|\vec{\nabla}\phi(t,\vec{x})|^2 + \frac{m^2}{2}\phi^2(t,\vec{x})\big] \\ H_{\text{int}}(t) &= \int d^3x\, \mathcal{H}_{\text{int}} = \int d^3x\, \frac{\lambda}{4!}\phi^4(t,\vec{x}) \end{aligned} \tag{9.3}$$

Our goal is to develop the perturbation theory at small $\lambda \ll 1$. In QM that would be stationary perturbation theory given by Eq. (9.4):

$$\begin{aligned} \Psi(\phi) &= \Psi_0(\phi) + \lambda\Psi_1(\phi) + \lambda^2\Psi_2(\phi) + ... \\ (\hat{H}_0 + \lambda\hat{H}_I)(\Psi_0(\phi) + \lambda\Psi_1(\phi) + ...) &= (E_0 + \lambda E_1 + ...)(\Psi_0(\phi) + \lambda\Psi_1(\phi) + ...) \end{aligned} \tag{9.4}$$

As we discussed above, in QFT it is extremely inconvenient (if only possible) to solve the Schrödinger equation (9.2) by iterations.
Reminder: two reasons why

- At each intermediate step we have "functional" integral over infinite number of canonical coordinates $\phi(x)$
- Schrödinger equation is not relativistic invariant, the invariance should be restored for the final results for scattering amplitudes

$\Rightarrow$ In QFT, instead of solution of Schrödinger equation, we use Heisenberg picture and the formalism of Green functions.

Now we can try to solve the operator equation (7.16) perturbatively

$$\begin{aligned} \hat{\phi}(x) &= \hat{\phi}_0(x) + \lambda\hat{\phi}_1(x) + \lambda^2\hat{\phi}_2(x) + ... \\ (\partial^2 + m^2)\hat{\phi}_0(x) &= 0 \\ (\partial^2 + m^2)\hat{\phi}_1(x) &= -\frac{1}{3!}\hat{\phi}_0^3(x) \\ (\partial^2 + m^2)\hat{\phi}_2(x) &= -\frac{1}{2}\hat{\phi}_0^2(x)\hat{\phi}_1(x) \\ &... \end{aligned}$$

but technically it turns out to be more convenient to develop a perturbation theory for Green functions - vacuum expectation values (VEVs) of the field operators rather than for the operators themselves.

Technical trick - "Interaction picture" (somewhere in between Scrödinger and Heisenberg pictures)

9.1 The interaction picture

Define

$$\hat{\phi}_I(t,\vec{x}) \equiv e^{i\hat{H}_0 t}\hat{\phi}(\vec{x})\, e^{-i\hat{H}_0 t} \tag{9.5}$$

Expanding $\hat{\phi}(\vec{x})$ in ladder operators (5.14) and using Eqs. (6.10) one obtains

$$\hat{\phi}_I(t,\vec{x}) \;=\; \int \frac{d\!\!{}^-{}^3p}{\sqrt{2E_p}}\big(a_{\vec{p}}e^{-iE_pt+i\vec{p}\vec{x}} + a^\dagger_{\vec{p}}e^{iE_pt-i\vec{p}\vec{x}}\big) \tag{9.6}$$

which is identical to Eq. (6.9) for a free KG theory (recall that the Hamiltonian which we denoted by $\hat{H}$ ther is now $\hat{H}_0$). Thus, the interaction picture is a Heisenberg picture at $\lambda = 0$.

The relation between $\hat{\phi}(x)$ and $\hat{\phi}_I(x)$

$$\begin{aligned}\hat{\phi}(t,\vec{x}) \;&=\; e^{i\hat{H}t}\hat{\phi}(\vec{x})e^{-i\hat{H}t} \;=\; e^{i\hat{H}t}e^{-i\hat{H}_0t}\big(e^{i\hat{H}_0t}\hat{\phi}(\vec{x})e^{-i\hat{H}_0t}\big)e^{i\hat{H}_0t}e^{-i\hat{H}t}\\ &=\; \hat{U}^\dagger(t)\hat{\phi}_I(t)\hat{U}(t)\end{aligned} \tag{9.7}$$

where

$$\hat{U}(t) \;\equiv\; e^{i\hat{H}_0t}e^{-i\hat{H}t} \qquad \text{and} \qquad \hat{U}^\dagger(t) \;\equiv\; e^{-i\hat{H}_0t}e^{i\hat{H}t} \tag{9.8}$$

the operator $\hat{U}(t)$ in the above equation is defined in terms of operators $\hat{\phi}(\vec{x})$ and $\hat{\pi}(\vec{x})$ since both $\hat{H}_0$ and $\hat{H}$ are written in terms of these operators (see Eq. (9.3). Let us write it down in terms of the operator $\hat{\phi}_I(x)$ instead.

To do this, we need to prove the following theorem:

$$e^{i\hat{A}t}e^{-i(\hat{A}+\hat{B})t} \;=\; \mathrm{Texp}\Big\{-i\int_0^t dt'\hat{B}(t')\Big\} \tag{9.9}$$

where $\hat{B}(t) \;\equiv\; e^{i\hat{A}t}\hat{B}e^{-i\hat{A}t}$ and T-exponent is defined as follows

$$\begin{aligned}\mathrm{Texp}\Big\{-i\int_0^t dt'\hat{B}(t')\Big\} \;&\equiv\; 1 - i\int_0^t dt'\hat{B}(t') + i^2\int_0^t dt'\int_0^{t'} dt''\hat{B}(t')\hat{B}(t'')\\ &- i^3\int_0^t dt'\int_0^{t'} dt''\int_0^{t''} dt'''\hat{B}(t')\hat{B}(t'')\hat{B}(t''') \;+\; ...\end{aligned} \tag{9.10}$$

Similarly to the definition of T-product in Eq. (7.22) the operators in Texp are arranged according to their times.

Proof of Eq. (9.9)
Let us differentiate both sides of Eq. (9.9) with respect to time t. We get

$$\begin{aligned}\frac{d}{dt}(\text{l.h.s.}) \;&=\; e^{i\hat{A}t}(i\hat{A} - i\hat{A} - i\hat{B})e^{-i(\hat{A}+\hat{B})t} \;=\; -ie^{i\hat{A}t}\hat{B}e^{-i(\hat{A}+\hat{B})t}\\ &=\; -ie^{i\hat{A}t}\hat{B}e^{-i\hat{A}t}e^{i\hat{A}t}e^{-i(\hat{A}+\hat{B})t} \;=\; -i\hat{B}(t)\times(\text{l.h.s.})\end{aligned} \tag{9.11}$$

$$\frac{d}{dt}(\text{r.h.s.}) \;=\; -i\hat{B}(t) + i^2\int_0^t dt''\hat{B}(t)\hat{B}(t'') - i^3\int_0^t dt''\int_0^{t''} dt'''\hat{B}(t)\hat{B}(t'')\hat{B}(t''') + ... \tag{9.12}$$

$$= \;-i\hat{B}(t)\Big[1 - i\int_0^t dt''\hat{B}(t'') + i^2\int_0^t dt''\int_0^{t''} dt'''\hat{B}(t'')\hat{B}(t''') \;+\; ...\Big] \;=\; -i\hat{B}(t)\times(\text{r.h.s.})$$

In addition,

$$\text{l.h.s.}|_{t=0} \;=\; \text{r.h.s.}|_{t=0} \;=\; 1 \tag{9.13}$$

Thus, the differential equations and the initial conditions for the l.h.s and the r.h.s. of Eq. (9.9) are identical $\Rightarrow$ the l.h.s. of Eq. (9.9) is equal to the r.h.s. of Eq. (9.9), Q.E.D.

In our case $\hat{A} = \hat{H}_0$ and $\hat{B} = \hat{H}_I$ so

$$\begin{aligned} U(t) &\equiv e^{i\hat{H}_0 t}e^{-i\hat{H}_0 t - i\hat{H}_I t} = \mathrm{Texp}\Big\{-i\int_0^t dt' \hat{H}_I(t')\Big\}, \\ \hat{H}_I(t) &= e^{i\hat{H}_0 t}\hat{H}_{\rm int}e^{-i\hat{H}_0 t} \end{aligned} \tag{9.14}$$

Explicit form of $\hat{H}_I(t)$

$$\begin{aligned} \hat{H}_I(t) &= e^{i\hat{H}_0 t}\hat{H}_{\rm int}e^{-i\hat{H}_0 t} = e^{i\hat{H}_0 t}\int d^3x\big[\frac{\lambda}{4!}\hat{\phi}^4(\vec{x})\big]e^{-i\hat{H}_0 t} \\ &= \frac{\lambda}{4!}\int d^3x\ \big(e^{i\hat{H}_0 t}\hat{\phi}(\vec{x})e^{-i\hat{H}_0 t}\big)\big(e^{i\hat{H}_0 t}\hat{\phi}(\vec{x})e^{-i\hat{H}_0 t}\big)\big(e^{i\hat{H}_0 t}\hat{\phi}(\vec{x})e^{-i\hat{H}_0 t}\big)\big(e^{i\hat{H}_0 t}\hat{\phi}(\vec{x})e^{-i\hat{H}_0 t}\big) \\ &= \int d^3x\ \frac{\lambda}{4!}\big(\hat{\phi}_I(\vec{x},t)\big)^4 \end{aligned} \tag{9.15}$$

so

$$\hat{U}(t) = \mathrm{Texp}\Big\{-i\int_0^t dt'\int d^3x\ \frac{\lambda}{4!}\hat{\phi}_I^4(t',\vec{x})\Big\}, \tag{9.16}$$

It is convenient to define

$$\hat{U}(t,t') \equiv \hat{U}(t)\hat{U}^\dagger(t') \tag{9.17}$$

Let us prove that

$$\hat{U}(t_1,t_2) = \mathrm{Texp}\Big\{-i\int_{t_2}^{t_1} dt\int d^3x\ \frac{\lambda}{4!}\hat{\phi}_I^4(t,\vec{x})\Big\}, \tag{9.18}$$

Proof: similarly to Eqs. (9.11) and (9.12) we compare the time derivatives of l.h.s. and r.h.s. of Eq. (9.18)

$$\frac{d}{dt}(\text{l.h.s.}) = \big(\frac{d}{dt}\hat{U}(t_1)\big)\hat{U}^\dagger(t_2) = -i\hat{H}_I(t_1)\hat{U}(t_1)\hat{U}^\dagger(t_2) = -i\hat{H}_I(t_1)\times(\text{l.h.s.}) \tag{9.19}$$

$$\begin{aligned} \frac{d}{dt_1}(\text{r.h.s.}) &= \frac{d}{dt_1}\Big[1 - i\int_{t_2}^{t_1}dt'\hat{H}_I(t') \\ &+ i^2\int_{t_2}^{t_1}dt'\int_{t_2}^{t'}dt''\hat{H}_I(t')\hat{H}_I(t'') - i^3\int_{t_2}^{t_1}dt'\int_{t_2}^{t'}dt''\int_{t_2}^{t''}dt'''\hat{H}_I(t')\hat{H}_I(t'')\hat{H}_I(t''') + ...\Big] \\ &= -i\hat{H}_I(t_1) + i^2\int_{t_2}^{t_1}dt''\hat{H}_I(t_1)\hat{H}_I(t'') - i^3\int_{t_2}^{t_1}dt''\int_{t_2}^{t''}dt'''\hat{H}_I(t_1)\hat{H}_I(t'')\hat{H}_I(t''') + ... \\ &= -i\hat{H}_I(t_1)\Big[1 - i\int_{t_2}^{t_1}dt''\hat{H}_I(t'') + i^2\int_{t_2}^{t_1}dt''\int_{t_2}^{t''}dt'''\hat{H}_I(t'')\hat{H}_I(t''') + ...\Big] = -i\hat{H}_I(t_1)\times(\text{r.h.s.}) \end{aligned} \tag{9.20}$$

In addition,

$$\text{l.h.s. of Eq. (9.18)} = \text{r.h.s. of Eq. (9.18)} = 1 \text{ at } t_1 = t_2$$

so the l.h.s. of Eq. (9.18) = r.h.s. of Eq. (9.18), Q.E.D.

Group property

$$\hat{U}(t_1,t_2)\hat{U}(t_2,t_3) \;=\; \hat{U}(t_1)\hat{U}^\dagger(t_2)\hat{U}(t_2)\hat{U}^\dagger(t_3) \;=\; \hat{U}(t_1)\hat{U}^\dagger(t_3) \;=\; \hat{U}(t_1,t_3) \tag{9.21}$$

Now we can rewrite the two-point Wightman function

$$\langle\Omega|\hat{\phi}(x)\hat{\phi}(y)|\Omega\rangle \;=\; \langle\Omega|\hat{U}^\dagger(x_0)\hat{\phi}_I(\vec{x},x_0)\hat{U}(x_0)\hat{U}^\dagger(y_0)\hat{\phi}_I(\vec{y},y_0)\hat{U}(y_0)|\Omega\rangle \tag{9.22}$$

as

$$\langle\Omega|\hat{\phi}(x)\hat{\phi}(y)|\Omega\rangle \;=\; \langle\Omega|\hat{U}(0,x_0)\hat{\phi}_I(x)\hat{U}(x_0,y_0)\hat{\phi}_I(y)\hat{U}(y_0,0)|\Omega\rangle \tag{9.23}$$

All operators in the r.h.s. can be expressed in terms of ladder $\hat{a}_{\vec{p}}$ and $\hat{a}^\dagger_{\vec{p}}$ [4], so if we had a rule $\hat{a}_{\vec{p}}|\Omega\rangle \;=\; 0$ we could reduce the r.h.s. of Eq. (9.23) to commutators of various $\hat{a}_{\vec{p}}$'s and $\hat{a}^\dagger_{\vec{p}}$'s. Unfortunately, we do <u>not</u> know the action of the operator $\hat{a}_{\vec{p}}$ on the vacuum Ω. (We know that $\hat{a}^{\rm in}_{\vec{p}}|\Omega\rangle \;=\; \hat{a}^{\rm out}_{\vec{p}}|\Omega\rangle \;=\; 0$ but these in- and out- operators are completely different objects).

Way around this difficulty: define "perturbative vacuum" $|0\rangle$ as lowest eigenstate of the free Hamiltonian $\hat{H}_0$. The explicit form is of course Eq. (4.55) but we will need only the property (5.20)

$$\hat{a}_{\vec{p}}|0\rangle \;=\; 0 \tag{9.24}$$

which, as explained in Sect. 5, can serve as a definition of perturbative vacuum $|0\rangle$. Note that if in Eq. (9.23) we had $\langle 0|....|0\rangle$ instead of $\langle\Omega|....|\Omega\rangle$ we could easily calculate the r.h.s. of that equation.

Now comes the central idea: if we take perturbative vacuum $|0\rangle$ and wait long enough, we get true vacuum $|\Omega\rangle$. Indeed, let us consider the evolution $e^{-i\hat{H}T}|0\rangle$ and insert full set of eigenstates $|n\rangle$ of full Hamiltonian $\hat{H}$

$$\begin{aligned} e^{-i\hat{H}T}|0\rangle \;&=\; \sum_{\{n\}} e^{-i\hat{H}T}|n\rangle\langle n|0\rangle \;=\; \sum_{\{n\}} e^{-iE_nT}|n\rangle\langle n|0\rangle \\ &=\; e^{-iE_0T}\Big[|\Omega\rangle\langle\Omega|0\rangle \;+\; \sum_{\{n\neq\Omega\}} e^{-i(E_n-E_0)T}|n\rangle\langle n|0\rangle\Big] \end{aligned} \tag{9.25}$$

Now, if we take $T=\tau(1-i\epsilon)$ and first take the limit $\tau\to\infty$ (and then $\epsilon\to 0$), only the first term in the r.h.s. of Eq. (9.25) survives:

$$\lim_{\epsilon\to 0}\lim_{\tau\to\infty} e^{-i\hat{H}\tau(1-i\epsilon)}|0\rangle \;=\; e^{-iE_0(\tau(1-i\epsilon))}|\Omega\rangle\langle\Omega|0\rangle \tag{9.26}$$

and therefore

$$|\Omega\rangle \;=\; \lim_{\epsilon\to 0}\lim_{\tau\to\infty}\frac{1}{e^{-iE_0\tau(1-i\epsilon)}\langle\Omega|0\rangle}e^{-i\hat{H}\tau(1-i\epsilon)}|0\rangle \tag{9.27}$$

Now, since $e^{-i\hat{H}_0t}|0\rangle \;=\; 1$ for any t (our convention is $\hat{H}_0|0\rangle \;=\; 0$) we can formally insert $e^{-i\hat{H}_0\tau(1-i\epsilon)}|0\rangle$ in the above equation and get

$$|\Omega\rangle \;=\; \lim_{\epsilon\to 0}\lim_{\tau\to\infty}\frac{1}{e^{-iE_0\tau(1-i\epsilon)}\langle\Omega|0\rangle}e^{-i\hat{H}\tau(1-i\epsilon)}e^{-i\hat{H}_0\tau(1-i\epsilon)}|0\rangle \tag{9.28}$$

---

[4] Recall Eq. (9.6): $\hat{\phi}_I(t,\vec{x}) \;=\; \int\frac{d\!\!^-{}^3p}{\sqrt{2E_p}}\big(a_{\vec{p}}e^{-iE_pt+i\vec{p}\vec{x}}+a^\dagger_{\vec{p}}e^{iE_pt-i\vec{p}\vec{x}}\big)$.

Since $e^{-i\hat{H}(\tau(1-i\epsilon))}e^{-i\hat{H}_0(\tau(1-i\epsilon))} = \hat{U}^\dagger(T) = \hat{U}(0,-T)$ one can write down

$$|\Omega\rangle = \lim_{\epsilon\to 0}\lim_{\tau\to\infty}\frac{1}{e^{-iE_0T}\langle\Omega|0\rangle}\hat{U}(0,-T)|0\rangle\Big|_{T=\tau(1-i\epsilon)} \tag{9.29}$$

Similarly,

$$\langle\Omega| = \lim_{\epsilon\to 0}\lim_{\tau\to\infty}\frac{1}{e^{-iE_0\tau(1-i\epsilon)}\langle 0|\Omega\rangle}\langle 0|e^{-i\hat{H}\tau(1-i\epsilon)} \tag{9.30}$$

$$= \lim_{\epsilon\to 0}\lim_{\tau\to\infty}\frac{1}{e^{-iE_0T}\langle 0|\Omega\rangle}\langle 0|e^{iH_0T}e^{-i\hat{H}T}\Big|_{T=\tau(1-i\epsilon)} = \lim_{\epsilon\to 0}\lim_{\tau\to\infty}\frac{1}{e^{-iE_0T}\langle 0|\Omega\rangle}\langle 0|\hat{U}(T,0)\Big|_{T=\tau(1-i\epsilon)}$$

Now we can substitute these expressions in the r.h.s. of Eq. (9.23) and get

$$\langle\Omega|\hat{\phi}(x)\hat{\phi}(y)|\Omega\rangle = \langle\Omega|\hat{U}(0,x_0)\hat{\phi}_I(x)\hat{U}(x_0,y_0)\hat{\phi}_I(y)\hat{U}(y_0,0)|\Omega\rangle \tag{9.31}$$

$$= \lim_{\epsilon\to 0}\lim_{\tau\to\infty}\frac{1}{e^{-2iE_0T}\langle 0|\Omega\rangle\langle\Omega|0\rangle}\langle 0|\hat{U}(T,0)\hat{U}(0,x_0)\hat{\phi}_I(x)\hat{U}(x_0,y_0)\hat{\phi}_I(y)\hat{U}(y_0,0)\}U(0,-T)|0\rangle\Big|_{T=\tau(1-i\epsilon)}$$

$$= \lim_{\epsilon\to 0}\lim_{\tau\to\infty}\frac{1}{e^{-2iE_0T}\langle 0|\Omega\rangle\langle\Omega|0\rangle}\langle 0|\hat{U}(T,x_0)\hat{\phi}_I(x)\hat{U}(x_0,y_0)\hat{\phi}_I(y)\hat{U}(y_0,-T)|0\rangle\Big|_{T=\tau(1-i\epsilon)}$$

where we used the group property (9.21).

Let us consider now Feynman Green function which is a v.e.v. of the T-product of field operators.

Suppose $x_0 > y_0$, then

$$\langle\Omega|\mathrm{T}\{\hat{\phi}(x)\hat{\phi}(y)\}|\Omega\rangle = \langle\Omega|\hat{\phi}(x)\hat{\phi}(y)|\Omega\rangle \tag{9.32}$$

$$= \lim_{\epsilon\to 0}\lim_{\tau\to\infty}\frac{1}{e^{-2iE_0T}\langle 0|\Omega\rangle\langle\Omega|0\rangle}\langle 0|\hat{U}(T,x_0)\hat{\phi}_I(x)\hat{U}(x_0,y_0)\hat{\phi}_I(y)\hat{U}(y_0,-T)|0\rangle\Big|_{T=\tau(1-i\epsilon)}$$

Since all $\hat{\phi}_I$ operators in the evolution operator $U(t,t')$ are ordered according to their times (see Eq. (9.18) we can rewrite Eq. (9.33) as

$$\langle\Omega|\mathrm{T}\{\hat{\phi}(x)\hat{\phi}(y)\}|\Omega\rangle \tag{9.33}$$

$$\stackrel{x_0\geq y_0}{=} \lim_{\epsilon\to 0}\lim_{\tau\to\infty}\frac{1}{e^{-2iE_0T}\langle 0|\Omega\rangle\langle\Omega|0\rangle}\langle 0|\mathrm{T}\{\hat{U}(T,x_0)\hat{\phi}_I(x)\hat{U}(x_0,y_0)\hat{\phi}_I(y)\hat{U}(y_0,-T)\}|0\rangle\Big|_{T=\tau(1-i\epsilon)}$$

$$\stackrel{x_0\geq y_0}{=} \lim_{\epsilon\to 0}\lim_{\tau\to\infty}\frac{1}{e^{-2iE_0T}\langle 0|\Omega\rangle\langle\Omega|0\rangle}\langle 0|\mathrm{T}\{\hat{U}(T,-T)\hat{\phi}_I(x)\hat{\phi}_I(y)\}|0\rangle\Big|_{T=\tau(1-i\epsilon)} \tag{9.34}$$

Similarly, at $y_0 > x_0$

$$\langle\Omega|\mathrm{T}\{\hat{\phi}(x)\hat{\phi}(y)\}|\Omega\rangle \tag{9.35}$$

$$\stackrel{y_0\geq x_0}{=} \lim_{\epsilon\to 0}\lim_{\tau\to\infty}\frac{1}{e^{-2iE_0T}\langle 0|\Omega\rangle\langle\Omega|0\rangle}\langle 0|\mathrm{T}\{\hat{U}(T,y_0)\hat{\phi}_I(y)\hat{U}(y_0,x_0)\hat{\phi}_I(x)\hat{U}(y_0,-T)\}|0\rangle\Big|_{T=\tau(1-i\epsilon)}$$

$$\stackrel{y_0\geq x_0}{=} \lim_{\epsilon\to 0}\lim_{\tau\to\infty}\frac{1}{e^{-2iE_0T}\langle 0|\Omega\rangle\langle\Omega|0\rangle}\langle 0|\mathrm{T}\{\hat{U}(T,-T)\hat{\phi}_I(y)\hat{\phi}_I(x)\}|0\rangle\Big|_{T=\tau(1-i\epsilon)} \tag{9.36}$$

so the general formula for Feynman Green function can be written as

$$\langle\Omega|\mathrm{T}\{\hat{\phi}(x)\hat{\phi}(y)\}|\Omega\rangle \tag{9.37}$$

$$= \lim_{\epsilon\to 0}\lim_{\tau\to\infty}\frac{1}{e^{-2iE_0T}\langle 0|\Omega\rangle\langle\Omega|0\rangle}\langle 0|\mathrm{T}\{\hat{U}(T,-T)\hat{\phi}_I(x)\hat{\phi}_I(y)\}|0\rangle\Big|_{T=\tau(1-i\epsilon)}$$

Let us now consider the denominator. It can be represented as

$$\lim_{\epsilon\to 0}\lim_{\tau\to\infty} e^{-2iE_0T}\langle 0|\Omega\rangle\langle\Omega|0\rangle = \lim_{\epsilon\to 0}\lim_{\tau\to\infty} \langle 0|\mathrm{T}\{\hat{U}(T,-T)|0\}\rangle\Big|_{T=\tau(1-i\epsilon)} \tag{9.38}$$

Indeed,

$$\langle 0|\mathrm{T}\{\hat{U}(T,-T)|0\}\rangle\Big|_{T=\tau(1-i\epsilon)} = \langle 0|\hat{U}(T,-T)|0\rangle\Big|_{T=\tau(1-i\epsilon)} = \langle 0|e^{-2i\hat{H}(\tau(1-i\epsilon))}|0\rangle \tag{9.39}$$
$$\Rightarrow \lim_{\epsilon\to 0}\lim_{\tau\to\infty} \langle 0|\mathrm{T}\{\hat{U}(T,-T)|0\}\rangle\Big|_{T=\tau(1-i\epsilon)} = \langle 0|\Omega\rangle e^{-2iE_0T}\langle\Omega|0\rangle$$

Now we are in a position to assemble the final result for the 2-point Green function

$$\langle\Omega|\mathrm{T}\{\hat{\phi}(x)\hat{\phi}(y)\}|\Omega\rangle = \lim_{\epsilon\to 0}\lim_{\tau\to\infty} \frac{\langle 0|\mathrm{T}\{\hat{U}(T,-T)\hat{\phi}_I(x)\hat{\phi}_I(y)\}|0\rangle}{\langle 0|\mathrm{T}\{\hat{U}(T,-T)\}|0\rangle}\Bigg|_{T=\tau(1-i\epsilon)} \tag{9.40}$$

If we recall Eq. (9.18) $\hat{U}(t,t') = \mathrm{T}\exp\big\{-i\int_{t'}^{t}dt''\frac{\lambda}{4!}\hat{\phi}_I^4(t'',\vec{x})\big\}$ we can rewrite this equation as

$$\langle\Omega|\mathrm{T}\{\hat{\phi}(x)\hat{\phi}(y)\}|\Omega\rangle = \lim_{\epsilon\to 0}\lim_{\tau\to\infty} \frac{\langle 0|\mathrm{T}\big\{e^{-i\int_{-T}^{T}dt\int d^3x\ \frac{\lambda}{4!}\hat{\phi}_I^4(t,\vec{x})}\hat{\phi}_I(x)\hat{\phi}_I(y)\big\}|0\rangle}{\langle 0|\mathrm{T}\big\{e^{-i\int_{-T}^{T}dt\int d^3x\ \frac{\lambda}{4!}\hat{\phi}_I^4(t,\vec{x})}\big\}||0\rangle}\Bigg|_{T=\tau(1-i\epsilon)} \tag{9.41}$$

Now we can use the rule $\hat{a}_p|0\rangle = 0$ and reduce the r.h.s. of Eq. (9.41) to commutators.

9.2 Wick's theorem

First, we separate the operator $\hat{\phi}_I$ into positive-frequency $\hat{\phi}_I^+$ and negative-frequency $\hat{\phi}_I^-$ parts:

$$\hat{\phi}_I(x) = \hat{\phi}_I^+(x) + \hat{\phi}_I^-(x), \tag{9.42}$$
$$\hat{\phi}_I^+(x) \equiv \int\frac{d\!\!\bar{}^3p}{\sqrt{2E_p}}\hat{a}_{\vec{p}}e^{-ipx}, \qquad \hat{\phi}_I^-(x) \equiv \int\frac{d\!\!\bar{}^3p}{\sqrt{2E_p}}\hat{a}_{\vec{p}}^{\dagger}e^{ipx}$$

Next, we define normal product of operators. (You may encounter two different notations in the textbooks: $N(ABC...D)$ and $:ABC...D:$).

Definition of $:\hat{\phi}_I(x_1)\hat{\phi}_I(x_2)....\hat{\phi}_I(x_n):$
We write down each $\hat{\phi}_I(x_i)$ as $\hat{\phi}_I^+(x_i)+\hat{\phi}_I^-(x_i)$, open all parentheses and put all $\hat{\phi}_I^-$ operators in each term to the left of all $\hat{\phi}_I^+$ operators:

$$:\hat{\phi}_I(x): \equiv \hat{\phi}_I(x) \tag{9.43}$$

$$:\hat{\phi}_I(x)\hat{\phi}_I(y): = :(\hat{\phi}_I^+(x)+\hat{\phi}_I^-(x))(\hat{\phi}_I^+(y)+\hat{\phi}_I^-(y)):$$
$$\equiv \hat{\phi}_I^+(x)\hat{\phi}_I^+(y)+\hat{\phi}_I^-(x)\hat{\phi}_I^+(y)+\hat{\phi}_I^-(y)\hat{\phi}_I^+(x)+\hat{\phi}_I^-(x)\hat{\phi}_I^-(y) \tag{9.44}$$

$$:\hat{\phi}_I(x)\hat{\phi}_I(y)\hat{\phi}_I(z): = :(\hat{\phi}_I^+(x)+\hat{\phi}_I^-(x))(\hat{\phi}_I^+(y)+\hat{\phi}_I^-(y)(\hat{\phi}_I^+(z)+\hat{\phi}_I^-(z)): \equiv$$
$$\equiv \hat{\phi}_I^+(x)\hat{\phi}_I^+(y)\hat{\phi}_I^+(z)+\hat{\phi}_I^-(x)\hat{\phi}_I^+(y)\hat{\phi}_I^+(z)+\hat{\phi}_I^-(y)\hat{\phi}_I^+(z)\hat{\phi}_I^+(x)+\hat{\phi}_I^-(z)\hat{\phi}_I^+(x)\hat{\phi}_I^+(y)$$
$$+\hat{\phi}_I^-(x)\hat{\phi}_I^-(y)\hat{\phi}_I^+(z)+\hat{\phi}_I^-(y)\hat{\phi}_I^-(z)\hat{\phi}_I^+(x)+\hat{\phi}_I^-(z)\hat{\phi}_I^-(x)\hat{\phi}_I^+(y)+\hat{\phi}_I^-(x)\hat{\phi}_I^-(y)\hat{\phi}_I^-(z)$$

and so on. Note that since $[\hat\phi_I^-(x_i),\hat\phi_I^-(x_j)]=0$ and $[\hat\phi_I^+(x_i),\hat\phi_I^+(x_j)]=0$ the relative order of operators inside the (-) or $(+)$ blocks does not matter.

Property (evident)

$$\langle 0|:\hat\phi_I(x_1)\hat\phi_I(x_2)....\hat\phi_I(x_n):|0\rangle \;=\; 0 \tag{9.45}$$

Wick's theorem is a relation between T-product and N-product of operators. Let us find this relation for two operators.

$$\begin{aligned}
\mathrm{T}\{\hat\phi_I(x)\hat\phi_I(y)\} \;&=\; \theta(x_0-y_0)(\hat\phi_I^+(x)\hat\phi_I^+(y)+\hat\phi_I^-(x)\hat\phi_I^+(y)+\hat\phi_I^+(x)\hat\phi_I^-(y)+\hat\phi_I^-(x)\hat\phi_I^-(y))\\
&+\theta(y_0-x_0)(\hat\phi_I^+(y)\hat\phi_I^+(x)+\hat\phi_I^-(y)\hat\phi_I^+(x)+\hat\phi_I^+(y)\hat\phi_I^-(x)+\hat\phi_I^-(y)\hat\phi_I^-(x))\\
&=\;\theta(x_0-y_0)(:\hat\phi_I(x)\hat\phi_I(y):+[\hat\phi_I^+(x),\hat\phi_I^-(y)])+\theta(y_0-x_0)(:\hat\phi_I(x)\hat\phi_I(y):+[\hat\phi_I^+(y),\hat\phi_I^-(x)])\\
&=\;:\hat\phi_I(x)\hat\phi_I(y):+\theta(x_0-y_0)[\hat\phi_I^+(x),\hat\phi_I^-(y)]+\theta(y_0-x_0)[\hat\phi_I^+(y),\hat\phi_I^-(x)]
\end{aligned} \tag{9.46}$$

Let us take now vacuum expectation value (v.e.v.):

$$\begin{aligned}
\langle 0|\mathrm{T}\{\hat\phi_I(x)\hat\phi_I(y)\}|0\rangle \;&=\; \langle 0|:\hat\phi_I(x)\hat\phi_I(y):|0\rangle+\theta(x_0-y_0)[\hat\phi_I^+(x),\hat\phi_I^-(y)]+\theta(y_0-x_0)[\hat\phi_I^+(y),\hat\phi_I^-(x)]\\
&=\;\theta(x_0-y_0)[\hat\phi_I^+(x),\hat\phi_I^-(y)]+\theta(y_0-x_0)[\hat\phi_I^+(y),\hat\phi_I^-(x)]
\end{aligned} \tag{9.47}$$

where we used Eq. (9.45). On the other hand, we know that $\langle 0|\mathrm{T}\{\hat\phi_I(x)\hat\phi_I(y)\}|0\rangle \;=\; D_F(x-y)$ (see Eq. (6.41)) so

$$\mathrm{T}\{\hat\phi_I(x)\hat\phi_I(y)\} \;=\; :\hat\phi_I(x)\hat\phi_I(y):+D_F(x-y) \tag{9.48}$$

A convenient notation:

$$\hat\phi_I\overbrace{(x)\hat\phi}_I(y) \;=\; D_F(x-y) \qquad \text{“contraction''} \tag{9.49}$$

Thus,

$$\mathrm{T}\{\hat\phi_I(x)\hat\phi_I(y)\} \;=\; :\hat\phi_I(x)\hat\phi_I(y): \;+\; \hat\phi_I\overbrace{(x)\hat\phi}_I(y) \tag{9.50}$$

Next

$$\begin{aligned}
&\mathrm{T}\{\hat\phi_I(x)\hat\phi_I(y)\hat\phi_I(z)\} \;-\; :\hat\phi_I(x)\hat\phi_I(y)\hat\phi_I(z): \;=\; \theta(x_0>y_0,z_0)\hat\phi_I(x)\big(:\hat\phi_I(y)\hat\phi_I(z):+\hat\phi_I\overbrace{(y)\hat\phi}_I(z)\big)\\
&+\theta(y_0>x_0,z_0)\hat\phi_I(y)\big(:\hat\phi_I(x)\hat\phi_I(z):+\hat\phi_I\overbrace{(x)\hat\phi}_I(z)\big) \;+\; \theta(z_0>x_0,y_0)\hat\phi_I(z)\big(:\hat\phi_I(x)\hat\phi_I(y):+\hat\phi_I\overbrace{(x)\hat\phi}_I(y)\big)\\
&-\;:\hat\phi_I(x)\hat\phi_I(y)\hat\phi_I(z):\big(\theta(x_0>y_0,z_0)+\theta(y_0>x_0,z_0)+\theta(z_0>x_0,y_0)\big)\\
&=\;\theta(x_0>y_0,z_0)\big\{[\hat\phi_I^+(x),:\hat\phi_I(y)\hat\phi_I(z):]+\hat\phi_I(x)\hat\phi_I\overbrace{(y)\hat\phi}_I(z)\big\} \;+\; (x\leftrightarrow y) \;+\; (x\leftrightarrow z)\\
&=\;\theta(x_0>y_0,z_0)\big\{[\hat\phi_I^+(x),\hat\phi_I^-(y)\hat\phi_I^+(z)+\hat\phi_I^-(z)\hat\phi_I^+(y)+\hat\phi_I^-(y)\hat\phi_I^-(z)]+\hat\phi_I(x)\hat\phi_I\overbrace{(y)\hat\phi}_I(z)\big\} \;+\; (x\leftrightarrow y) \;+\; (x\leftrightarrow z)
\end{aligned}$$

$$
\begin{aligned}
&= \theta(x_0>y_0,z_0)\{[\hat\phi_I^+(x),\hat\phi_I^-(y)]\hat\phi_I(z)+[\hat\phi_I^+(x),\hat\phi_I^-(z)]\hat\phi_I(y)+\hat\phi_I(x)\overbrace{\hat\phi_I(y)\hat\phi_I}(z)\}\\
&+\theta(y_0>x_0,z_0)\{[\hat\phi_I^+(y),\hat\phi_I^-(x)]\hat\phi_I(z)+[\hat\phi_I^+(y),\hat\phi_I^-(z)]\hat\phi_I(x)+\hat\phi_I(y)\overbrace{\hat\phi_I(x)\hat\phi_I}(z)\}\\
&+\theta(z_0>x_0,y_0)\{[\hat\phi_I^+(z),\hat\phi_I^-(y)]\hat\phi_I(x)+[\hat\phi_I^+(z),\hat\phi_I^-(x)]\hat\phi_I(y)+\hat\phi_I(z)\overbrace{\hat\phi_I(x)\hat\phi_I}(y)\}\\
&= \hat\phi_I(z)\Big\{\theta(x_0>y_0>z_0)[\hat\phi_I^+(x),\hat\phi_I^-(y)]+\theta(x_0>z_0>y_0)[\hat\phi_I^+(x),\hat\phi_I^-(y)]+\theta(y_0>z_0>x_0)[\hat\phi_I^+(y),\hat\phi_I^-(x)]\\
&+\theta(y_0>x_0>z_0)[\hat\phi_I^+(y),\hat\phi_I^-(x)]+\theta(x_0>y_0,z_0)\overbrace{\hat\phi_I(x)\hat\phi_I}(y)\Big\}\ +\ (z\leftrightarrow x)\ +\ (z\leftrightarrow y)\\
&= \hat\phi_I(z)\Big\{\theta(x_0>y_0,z_0)\overbrace{\hat\phi_I(x)\hat\phi_I}(y)+\theta(x_0,y_0>z_0)\big(\theta(x_0-y_0)[\hat\phi_I^+(x),\hat\phi_I^-(y)]+\theta(y_0-x_0)[\hat\phi_I^+(y),\hat\phi_I^-(x)]\big)\\
&+\big(1-\theta(x_0,y_0>z_0)-\theta(z_0>x_0,y_0)\big)\big(\theta(x_0-y_0)[\hat\phi_I^+(x),\hat\phi_I^-(y)]+\theta(y_0-x_0)[\hat\phi_I^+(y),\hat\phi_I^-(x)]\big)\Big\}\ +\ (z\leftrightarrow x,y)\\
&= \hat\phi_I(z)\Big\{\theta(x_0>y_0,z_0)\overbrace{\hat\phi_I(x)\hat\phi_I}(y)+\theta(x_0,y_0>z_0)\overbrace{\hat\phi_I(x)\hat\phi_I}(y)\\
&\qquad+\big(1-\theta(x_0,y_0>z_0)-\theta(z_0>x_0,y_0)\big)\overbrace{\hat\phi_I(x)\hat\phi_I}(y)\Big\}\ +\ (z\leftrightarrow x)\ +\ (z\leftrightarrow y)\\
&= \hat\phi_I(z)\overbrace{\hat\phi_I(x)\hat\phi_I}(y)+\hat\phi_I(x)\overbrace{\hat\phi_I(y)\hat\phi_I}(z)+\hat\phi_I(y)\overbrace{\hat\phi_I(x)\hat\phi_I}(z)
\end{aligned}
\tag{9.51}
$$

where I used formulas

$$
\begin{aligned}
\theta(x_0>z_0>y_0) &= \theta(x_0-y_0)\big[1-\theta(x_0,y_0>z_0)-\theta(z_0>x_0,y_0)\big],\\
\theta(y_0>z_0>x_0) &= \theta(y_0-x_0)\big[1-\theta(x_0,y_0>z_0)-\theta(z_0>x_0,y_0)\big]
\end{aligned}
\tag{9.52}
$$

Thus, we get

$$
\mathrm{T}\{\hat\phi_I(x)\hat\phi_I(y)\hat\phi_I(z)\} = \ :\hat\phi_I(x)\hat\phi_I(y)\hat\phi_I(z):+\hat\phi_I(z)\overbrace{\hat\phi_I(x)\hat\phi_I}(y)+\hat\phi_I(x)\overbrace{\hat\phi_I(y)\hat\phi_I}(z)+\hat\phi_I(y)\overbrace{\hat\phi_I(x)\hat\phi_I}(z)
\tag{9.53}
$$

In a similar way one can prove that

$$
\begin{aligned}
\mathrm{T}\{\hat\phi_1\hat\phi_2\hat\phi_3\hat\phi_4\} &= \ :\hat\phi_1\hat\phi_2\hat\phi_3\hat\phi_4:\\
&+\ :\overbrace{\hat\phi_1\hat\phi_2}\hat\phi_3\hat\phi_4+\overbrace{\hat\phi_1\hat\phi_2\hat\phi_3}\hat\phi_4+\overbrace{\hat\phi_1\hat\phi_2\hat\phi_3\hat\phi_4}+\hat\phi_1\overbrace{\hat\phi_2\hat\phi_3}\hat\phi_4+\hat\phi_1\overbrace{\hat\phi_2\hat\phi_3\hat\phi_4}+\hat\phi_1\hat\phi_2\overbrace{\hat\phi_3\hat\phi_4}:\\
&+\overbrace{\hat\phi_1\hat\phi_2}\overbrace{\hat\phi_3\hat\phi_4}+\overbrace{\hat\phi_1\hat\phi_3}\overbrace{\hat\phi_2\hat\phi_4}+\overbrace{\hat\phi_1\hat\phi_4}\overbrace{\hat\phi_2\hat\phi_3}
\end{aligned}
\tag{9.54}
$$

where $\hat\phi_i\equiv\hat\phi_I(x_i)$ and $:\overbrace{\hat\phi_i\hat\phi_j}\hat\phi_k\hat\phi_l:\equiv\overbrace{\hat\phi_i\hat\phi_j}\ :\hat\phi_k\hat\phi_l:$.

Wick's theorem in general case

$$
\mathrm{T}\{\hat\phi_I(x_1)\hat\phi_I(x_2)...\hat\phi_I(x_n)\} = \ :\hat\phi_I(x_1)\hat\phi_I(x_2)...\hat\phi_I(x_n):\ +\ \text{all possible contractions}
\tag{9.55}
$$

is proved by induction.

Taking v.e.v. we get a version of Wick's theorem convenient for conversion of Green functions into a set of Feynman diagrams:

$$
\begin{aligned}
\langle 0|\mathrm{T}\{\hat\phi_I(x_1)\hat\phi_I(x_2)...\hat\phi_I(x_n)\}|0\rangle &= \overbrace{\hat\phi_I(x_1)\hat\phi_I}(x_2)\overbrace{\hat\phi_I(x_3)\hat\phi_I}(x_4)...\overbrace{\hat\phi_I(x_{n-1})\hat\phi_I}(x_n) \qquad (9.56)\\
&+\overbrace{\hat\phi_I(x_1)\hat\phi_I}(x_3)\overbrace{\hat\phi_I(x_2)\hat\phi_I}(x_4)...\overbrace{\hat\phi_I(x_{n-1})\hat\phi_I}(x_n)\ +\ ...(\text{all possible contractions of all operators})
\end{aligned}
$$

Some examples:

$$
\begin{aligned}
\langle 0|\mathrm{T}\{\hat\phi_I(x_1)\hat\phi_I(x_2)\}|0\rangle &= \overbrace{\hat\phi_I(x_1)\hat\phi_I(x_2)} = D_F(x_1-x_2) \qquad (9.57)\\
\langle 0|\mathrm{T}\{\hat\phi_I(x_1)\hat\phi_I(x_2)\hat\phi_I(x_3)\}|0\rangle &= 0\\
\langle 0|\mathrm{T}\{\hat\phi_I(x_1)\hat\phi_I(x_2)\hat\phi_I(x_3)\hat\phi_I(x_4)\}|0\rangle &\\
&= \overbrace{\hat\phi_I(x_1)\hat\phi_I(x_2)}\overbrace{\hat\phi_I(x_3)\hat\phi_I(x_4)} + \overbrace{\hat\phi_I(x_1)\hat\phi_I(x_3)}\overbrace{\hat\phi_I(x_2)\hat\phi_I(x_4)} + \overbrace{\hat\phi_I(x_1)\hat\phi_I(x_4)}\overbrace{\hat\phi_I(x_2)\hat\phi_I(x_3)}\\
&= D_F(x_1-x_2)D_F(x_3-x_4) + D_F(x_1-x_3)D_F(x_2-x_4) + D_F(x_1-x_4)D_F(x_2-x_3)
\end{aligned}
$$

Feynman diagrams: a line for each $\overbrace{\hat\phi_I(x)\hat\phi_I(y)} = D_F(x-y)$

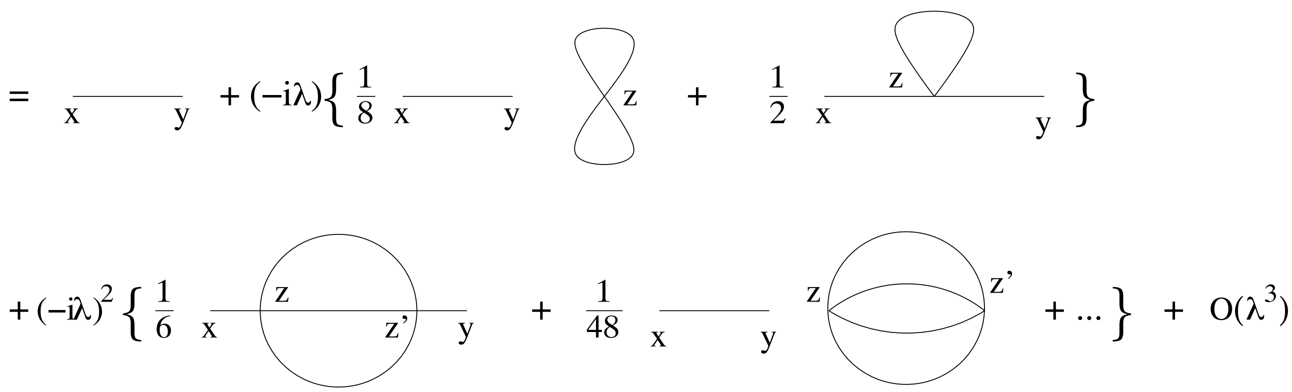
Let us apply Wick's theorem to the calculation of two-point Green function (9.41)

$$
\langle\Omega|\mathrm{T}\{\hat\phi(x)\hat\phi(y)\}|\Omega\rangle = \frac{\langle 0|\mathrm{T}\{e^{-i\int d^4z\ \frac{\lambda}{4!}\hat\phi_I^4(z)}\hat\phi_I(x)\hat\phi_I(y)\}|0\rangle}{\langle 0|\mathrm{T}\{e^{-i\int d^4z\ \frac{\lambda}{4!}\hat\phi_I^4(z)}\}||0\rangle} \qquad (9.58)
$$

We will discuss the limit $T=\tau(1-i\epsilon)\to\infty$ later and for now I just replaced $\int_{-T}^{T}dt$ by $\int dt$.

First, let us expand the numerator in Eq. (9.58) in powers of λ

$$
\begin{aligned}
&\langle 0|\mathrm{T}\{e^{-i\int d^4z\ \frac{\lambda}{4!}\hat\phi_I^4(z)}\hat\phi_I(x)\hat\phi_I(y)\}|0\rangle\\
&= \langle 0|\mathrm{T}\Big\{\hat\phi_I(x)\hat\phi_I(y)\Big(1-i\frac{\lambda}{4!}\int d^4z\ \hat\phi_I^4(z) - \frac{1}{2}\big(\frac{\lambda}{4!}\big)^2\int d^4z\ \hat\phi_I^4(z)\int d^4z'\ \hat\phi_I^4(z')+\dots\Big)\Big\}|0\rangle\\
&= \langle 0|\mathrm{T}\{\hat\phi_I(x)\hat\phi_I(y)\}|0\rangle - i\frac{\lambda}{4!}\int d^4z\langle 0|\mathrm{T}\{\hat\phi_I(x)\hat\phi_I(y)\hat\phi_I^4(z)\}|0\rangle - \frac{1}{2}\big(\frac{\lambda}{4!}\big)^2\int d^4z d^4z'\langle 0|\mathrm{T}\{\hat\phi_I(x)\hat\phi_I(y)\hat\phi_I^4(z)\hat\phi_I^4(z')\}|0\rangle\\
&= \langle 0|\mathrm{T}\{\hat\phi_I(x)\hat\phi_I(y)\}|0\rangle - i\frac{\lambda}{4!}\int d^4z\langle 0|\mathrm{T}\{\hat\phi_I(x)\hat\phi_I(y)\hat\phi_I(z)\hat\phi_I(z)\hat\phi_I(z)\hat\phi_I(z)\}|0\rangle\\
&- \frac{1}{2}\big(\frac{\lambda}{4!}\big)^2\int d^4z d^4z'\langle 0|\mathrm{T}\{\hat\phi_I(x)\hat\phi_I(y)\hat\phi_I(z)\hat\phi_I(z)\hat\phi_I(z)\hat\phi_I(z)\hat\phi_I(z')\hat\phi_I(z')\hat\phi_I(z')\hat\phi_I(z')\}|0\rangle + O(\lambda^3)\\
&\overset{\text{Wick's theorem}}{=} \overbrace{\hat\phi_I(x)\hat\phi_I(y)} - i\frac{\lambda}{4!}\int d^4z\big[\overbrace{\hat\phi_I(x)\hat\phi_I(y)}\times 3\overbrace{\hat\phi_I(z)\hat\phi_I(z)}\overbrace{\hat\phi_I(z)\hat\phi_I(z)} + 4\overbrace{\hat\phi_I(x)\hat\phi_I(z)}\times 3\overbrace{\hat\phi_I(z)\hat\phi_I(y)}\overbrace{\hat\phi_I(z)\hat\phi_I(z)}\big]\\
&- \frac{\lambda^2}{2(4!)^2}\int d^4z d^4z'\big[2\times 4\overbrace{\hat\phi_I(x)\hat\phi_I(z)}\times 4\overbrace{\hat\phi_I(z')\hat\phi_I(y)}\times 6\big(\overbrace{\hat\phi_I(z)\hat\phi_I(z')}\big)^3+\dots\big]\\
&= \overbrace{\hat\phi_I(x)\hat\phi_I(y)} - i\lambda\int d^4z\big[\frac{1}{8}\overbrace{\hat\phi_I(x)\hat\phi_I(y)}\overbrace{\hat\phi_I(z)\hat\phi_I(z)}\overbrace{\hat\phi_I(z)\hat\phi_I(z)} + \frac{1}{2}\overbrace{\hat\phi_I(x)\hat\phi_I(z)}\overbrace{\hat\phi_I(z)\hat\phi_I(y)}\overbrace{\hat\phi_I(z)\hat\phi_I(z)}\big]\\
&- \frac{\lambda^2}{2(4!)^2}\int d^4z d^4z'\big[\frac{1}{6}\overbrace{\hat\phi_I(x)\hat\phi_I(z)}\overbrace{\hat\phi_I(z')\hat\phi_I(y)}\big(\overbrace{\hat\phi_I(z)\hat\phi_I(z')}\big)^3+\dots\big] \qquad (9.59)
\end{aligned}
$$



Thus,

$$\langle 0|\mathrm{T}\{e^{-i\int d^4z\ \frac{\lambda}{4!}\hat\phi_I^4(z)}\hat\phi_I(x)\hat\phi_I(y)\}|0\rangle = \text{ sum of all possible diagrams with two external legs} \tag{9.60}$$

Part VIII

9.3 Vacuum bubbles

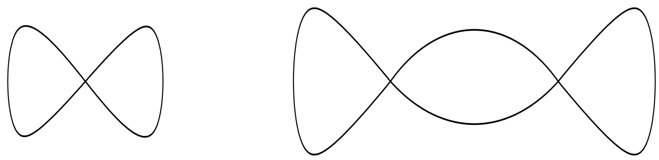
Vacuum bubble is a Feynman diagram (or sub-diagram) without external legs. For example, in the second term in the r.h.s. of Eq. (9.59)

$$-i\lambda\int d^4z\Big[\frac{1}{8}\widehat{\hat\phi_I(x)\hat\phi_I(y)}\widehat{\hat\phi_I(z)\hat\phi_I(z)}\widehat{\hat\phi_I(z)\hat\phi_I(z)} = \widehat{\hat\phi_I(x)\hat\phi_I(y)}\times\Big[-i\frac{\lambda}{8}\int d^4z\widehat{\hat\phi_I(z)\hat\phi_I(z)}\widehat{\hat\phi_I(z)\hat\phi_I(z)}\Big] \tag{9.61}$$

the expression in $\Big[...\Big]$ is a vacuum bubble:

$$\begin{aligned} -i\frac{\lambda}{8}\int d^4z\widehat{\hat\phi_I(z)\hat\phi_I(z)}\widehat{\hat\phi_I(z)\hat\phi_I(z)} &= -i\frac{\lambda}{8}\int d^4z\ \big(D_F(z,z)\big)^2 = -i\frac{\lambda}{8}\int d^4z\ \big(D_F(0)\big)^2 \\ &= -i\frac{\lambda}{8}\mathcal{V}\big(D_F(0)\big)^2 \end{aligned} \tag{9.62}$$

where $\mathcal{V}\ \equiv\int d^4z$ is a 4-volume of the space-time. Two disconnected bubbles will give $\mathcal{V}^2$, for example



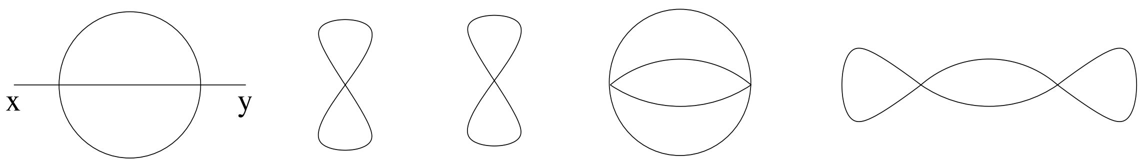
$$=$$

$$\begin{aligned} &-i\frac{\lambda}{8}\int d^4x\widehat{\hat\phi_I(x)\hat\phi_I(x)}\widehat{\hat\phi_I(x)\hat\phi_I(x)}\frac{(-i\lambda)^2}{16}\int d^4z\widehat{\hat\phi_I(z)\hat\phi_I(z)}\widehat{\hat\phi_I(z)\hat\phi_I(z')}\widehat{\hat\phi_I(z)\hat\phi_I(z')}\widehat{\hat\phi_I(z')\hat\phi_I(z')} \\ &= -i\frac{\lambda}{8}\mathcal{V}\big(D_F(0)\big)^2\frac{(-i\lambda)^2}{16}\big(D_F(0)\big)^2\int d^4zd^4z'D_F^2(z-z') \\ &= -i\frac{\lambda}{8}\mathcal{V}\big(D_F(0)\big)^2\frac{(-i\lambda)^2}{16}\mathcal{V}\big(D_F(0)\big)^2\int d^4zD_F^2(z) = \frac{(-i\lambda)^3}{128}\mathcal{V}^2\big(D_F(0)\big)^4\Big[\int d^4zD_F^2(z)\Big] \end{aligned} \tag{9.63}$$

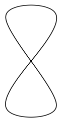
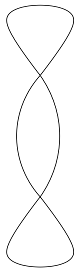
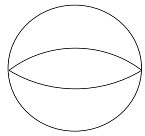
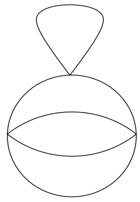
Similarly one can show that three disconnected bubbles give the contribution $\sim\mathcal{V}^3$, and so on.

9.3.1 Exponentiation of vacuum bubbles

A typical diagram:



V_k - value of k-th vacuum bubble, n_k is a number of such vacuum bubbles.

V_1 =  V_2 =  V_3 =  V_4 = 

Sum of all diagrams is ($\{n_i\} \equiv n_1, n_2 ... n_k$)

$$\sum_{\text{connected parts}} \sum_{\text{all } \{n_i\}} (\text{Value of connected part}) \left(\prod \frac{1}{n_k!} V^{n_k}\right) \tag{9.64}$$

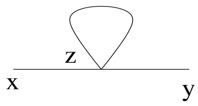
$$= \sum_{\text{connected parts}} \big(\text{Value of connected part}\big) \Big(\sum_{n_1} \frac{1}{n_1!} V_1^{n_1}\Big) \Big(\sum_{n_2} \frac{1}{n_2!} V_1^{n_2}\Big) ... \Big(\sum_{n_k} \frac{1}{n_k!} V_1^{n_k}\Big) ...$$

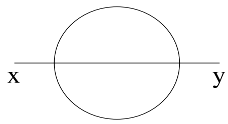
$$= \sum_{\text{connected parts}} \big(\text{Value of connected part}\big) \prod_{k=0}^{\infty} e^{V_k} = \sum_{\text{connected parts}} \big(\text{Value of connected part}\big) \times e^{\sum_{k=0}^{\infty} V_k}$$

Thus, the numerator in Eq. (9.58) is

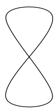
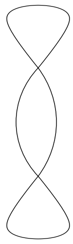
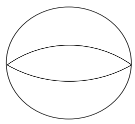
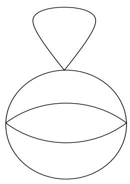
$$\langle 0 | \mathrm{T} \big\{ e^{-i \int d^4 z \, \frac{\lambda}{4!} \hat{\phi}_I^4(z)} \hat{\phi}_I(x) \hat{\phi}_I(y) \big\} | 0 \rangle \ = \tag{9.65}$$

$=$ $\Big\{$ 

$+$ 

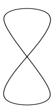
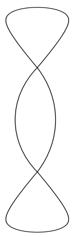
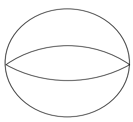
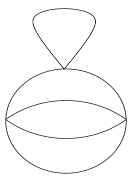
$+$ 

$+ ... \Big\} \times$

$\times$ exp $\Big\{$  $+$  $+$  $+$  $+ ... \Big\}$

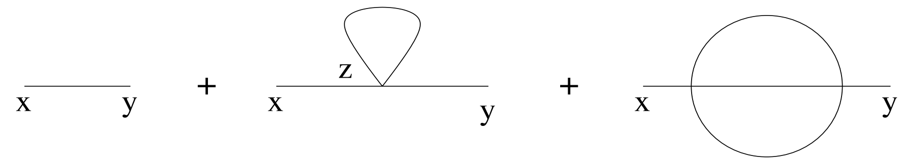
Let us now consider the denominator in the Eq. (9.58). Repeating the same steps, we get

$$\langle 0 | \mathrm{T} \big\{ e^{-i \int d^4 z \, \frac{\lambda}{4!} \hat{\phi}_I^4(z)} \big\} | 0 \rangle = \ \exp \big\{ \text{sum of all vacuum bubbles} \big\} \tag{9.66}$$

$=$ exp $\Big\{$  $+$  $+$  $+$  $+ ... \Big\}$

so the sums of vacuum bubbles in the numerator and in the denominator cancel and we obtain

$$\langle\Omega|\mathrm{T}\{\hat\phi(x)\hat\phi(y)\}|\Omega\rangle = \frac{\langle 0|\mathrm{T}\big\{e^{-i\int d^4z\ \frac{\lambda}{4!}\hat\phi_I^4(z)}\hat\phi_I(x)\hat\phi_I(y)\big\}|0\rangle}{\langle 0|\mathrm{T}\big\{e^{-i\int d^4z\ \frac{\lambda}{4!}\hat\phi_I^4(z)}\big\}||0\rangle}$$

$=$ 

$+\ \ldots$

$$\Rightarrow\ \ \langle\Omega|\mathrm{T}\{\hat\phi(x)\hat\phi(y)\}|\Omega\rangle\ =\ \text{sum of all } \underline{\text{connected}} \text{ diagrams with two external legs} \tag{9.67}$$

NB: Sum of vacuum bubbles is actually a shift of the ground state energy

$$\langle 0|\mathrm{T}\big\{e^{-i\int d^4z\ \frac{\lambda}{4!}\hat\phi_I^4(z)}\big\}|0\rangle = \lim_{T\to\infty}\langle 0|U(T,-T)|0\rangle \tag{9.68}$$
$$= \lim_{T\to\infty}\langle 0|e^{i\hat H_0T}e^{-i\hat HT}e^{-i\hat HT}e^{i\hat H_0T}|0\rangle = \lim_{T\to\infty}\langle 0|e^{-2i\hat HT}|0\rangle = \lim_{\tau\to\infty}\langle 0|e^{-2i\hat H\tau(1-i\epsilon)}|0\rangle$$
$$= e^{-iE_0T}\Big[|\langle\Omega|0\rangle|^2 + \sum_{\{n\neq\Omega\}} e^{-i(E_n-E_0)T}|\langle n|0\rangle|^2\Big] = |\langle\Omega|0\rangle|^2e^{-iE_0T} = |\langle\Omega|0\rangle|^2e^{-i\mathcal{V}\mathcal{E}_0}$$

where $E_0 = \int d^3x\ \mathcal{E}_0(x) = L^3\mathcal{E}_0$ and $2TL^3 = \mathcal{V}$. Thus, the shift of vacuum state energy does not affect the Green functions (and hence the cross sections due to LSZ theorem). That is why it is consistent to set $\hat H_0|0\rangle = 0$: if $\hat H_0|0\rangle = E_0'$, the contribution $e^{-2E'T}$ will be cancelled in the ratio in l.h.s of Eq.

9.4 Feynman rules for ϕ^4 theory in the coordinate space

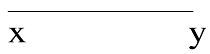
Feynman rules for the n-point Green function in the coordinate representation

$$G(x_1, x_2, \ldots x_n) \tag{9.69}$$
$$\equiv \langle\Omega|\mathrm{T}\{\hat\phi(x_1)\hat\phi(x_2)\ldots\hat\phi(x_n)\}|\Omega\rangle = \frac{\langle 0|\mathrm{T}\big\{e^{-i\int d^4z\ \frac{\lambda}{4!}\hat\phi_I^4(z)}\hat\phi_I(x_1)\hat\phi_I(x_2)\ldots\hat\phi(x_n)\big\}|0\rangle}{\langle 0|\mathrm{T}\big\{e^{-i\int d^4z\ \frac{\lambda}{4!}\hat\phi_I^4(z)}\big\}||0\rangle}$$

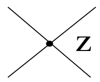
are:

1. Propagator: $\quad x\underline{\hspace{2cm}}y = \overline{\hat\phi_I(z)\hat\phi_I}(z') = D_F(x-y)$
2. Vertex: $\quad -i\lambda\int d^4z$
3. Divide by symmetry coefficient

Feynman rules in the coordinate space.

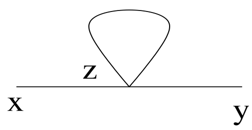
1. Propagator 

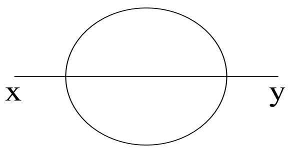
$= \phi(x)\,\phi(y) = D_F(x-y)$

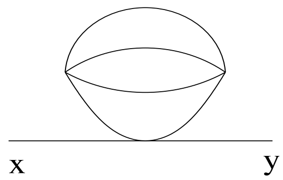
2. Vertex 

$= -i\lambda \int d^4z$

3. Divide by symmetry factor S:







S=2 S=6 S=12

9.5 Feynman rules for $\lambda\phi^4$ theory in the momentum space

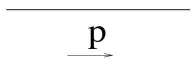
In the momentum representation

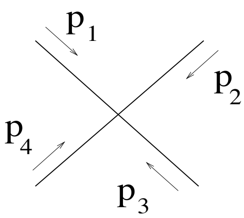
$$G(p_1, p_2, ...p_n) \equiv \int d^4x_1 d^4x_2 ... d^4x_n e^{ip_1x_1 + ip_2x_2 + ... ip_nx_n} G(x_1, x_2, ...x_n)$$

The set of Feynman rules for the Green function $G(p_1, p_2, ...p_n)$ in the momentum space is as follows:

I. Draw all possible (but different!) diagrams with proper symmetry combinatorial factors.
II. Put $G_0(p) = \frac{1}{i(m^2 - p^2 - i\epsilon)}$ for each line with momentum p.
III. Put $-i\lambda(2\pi)^4\delta(\sum p_j)$ in each vertex (where p_j are the momenta flowing into this vertex).
IV. Integrate over the momenta of internal lines (an internal line is any line that is not the tail). Each integration over momenta comes with $(2\pi)^4$ in the denominator.

Feynman rules in the momentum space.

1. Propagator  $= \dfrac{-i}{m^2 - p^2 - i\varepsilon}$

2. Vertex  $= -i\lambda\,(2\pi)^4\,\delta^{(4)}(p_1 + p_2 + p_3 + p_4)$

3. Integrate over all momenta $\prod_i \int \dfrac{d^4 p_i}{(2\pi)^4}$

4. Divide by symmetry factor

9.5.1 About the limit T $\to \infty$

At each vertex we get

$$(2\pi)^4 \delta(p_1 + p_2 + p_3 + p_4) \;=\; \int d^4z\; e^{i(p_1+p_2+p_3+p_4)z} \tag{9.70}$$

Before the limit $T \to \infty$ we had

$$\int_{-\tau(1-i\epsilon)}^{\tau(1-i\epsilon)} dz_0 \int d^3z\; e^{iz_0 \sum_i p_{i0} - i\vec{z}\cdot\sum_i \vec{p}_i} \;=\; (2\pi)^3 \delta\Big(\sum_i \vec{p}_i\Big) \int_{-\tau(1-i\epsilon)}^{\tau(1-i\epsilon)} dz_0\; e^{iz_0 \sum_i p_{i0}} \tag{9.71}$$

To ensure convergence of the integral over z_0 in the r.h.s. of Eq. (9.72) we can take $p_0 = (\text{real}) \times (1 + i\epsilon)$ and therefore in Feynman rules in the momentum space we must integrate over slightly imaginary $p_0 = (\text{real}) \times (1 + i\epsilon)$. This is equivalent to taking poles $p_0 = \pm E_p$ slightly off the real axis as shown in Fig. 7 We get

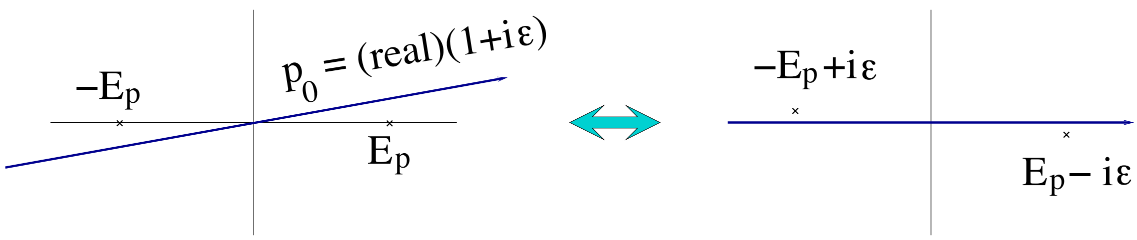


Figure 7. Shift of integration contour

$$\begin{aligned}
\int_{-\tau(1-i\epsilon)}^{\tau(1-i\epsilon)} dz_0\; e^{iz_0(1+i\epsilon)\sum_i p_{i0}} \;&=\; \int_{-\tau}^{\tau} dz_0\; e^{iz_0(1-i\epsilon)(1+i\epsilon)\sum_i p_{i0}} \\
\simeq \int_{-\tau}^{\tau} dz_0\; e^{iz_0 \sum_i p_{i0}} \overset{\tau\to\infty}{\to} \int_{-\tau}^{\tau} dz_0\; e^{iz_0 \sum_i p_{i0}} \;&=\; 2\pi\delta\Big(\sum_i p_{i0}\Big)
\end{aligned} \tag{9.72}$$

Thus, one should use Feynman propagators with poles at $\pm(E_p - i\epsilon)$ and write down the full integral $\int dz_0$ (or $\int dp_0$) over the real axis.

9.5.2 Feynman rules for reduced Green functions

Feynman rules in the momentum space can be simplified even more by performing the integration using δ-functions coming from momentum conservation in each vertex. After taking into account the momentum conservation in each vertex there is only non-trivial integrations corresponding to *loop*s. On the other hand, if one considers the so-called *tree* diagrams ($\equiv$ without loops) the value of these diagrams in momentum representation is actually already fixed by simply drawing the diagram with taking into account the momentum conservation in each vertex.

Let us formulate the final set of rules for calculation of so-called reduced Green function in the momentum representation. The definition of reduced Green function $\mathcal{G}(p_1, p_2, ...p_n)$ has the form

$$\begin{aligned} G(x_1, x_2, ...x_n) &= \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}...\frac{d^4p_n}{(2\pi)^4}e^{-ip_1x_1-ip_2x_2...-ip_nx_n}G(p_1,p_2,...p_n) \\ G(p_1,p_2,...p_n) &= (-i)^{n-1}(2\pi)^4\delta(p_1+p_2+...+p_n)\mathcal{G}(p_1,p_2,...p_n) \end{aligned} \tag{9.73}$$

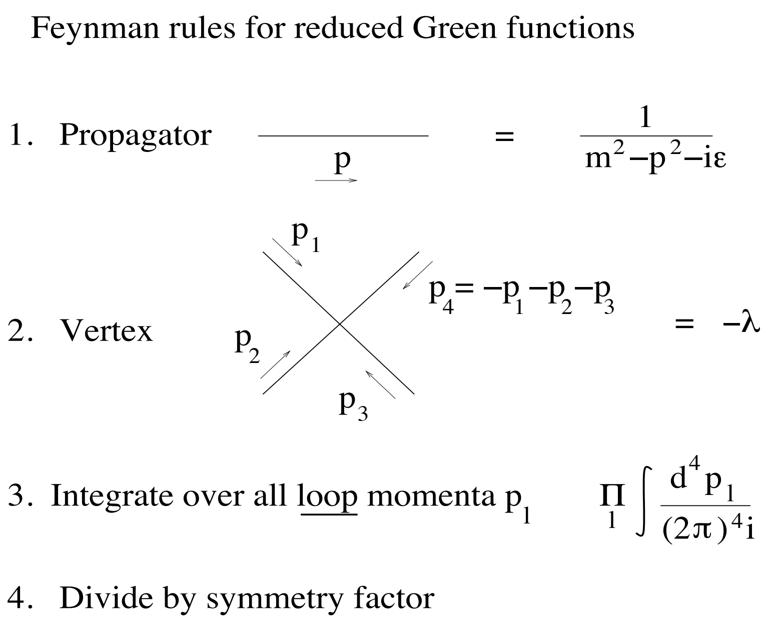
The set of Feynman rules for $\mathcal{G}(p_1, p_2, ...p_n)$ is:

I. Draw all different connected diagrams taking into account the symmetry (combinatorial) factors.

II. Draw momenta flow for each diagram taking into account conservation of the momentum in each vertex.

III. Each line with momentum p brings factor $\mathcal{G}_0(p) = \frac{1}{m^2-p^2-i\epsilon}$ and each vertex factor $(-\lambda)$

IV. There is an integration $\int \frac{d^4k}{(2\pi)^4 i}$ for each loop.



9.5.3 Reduced Green functions and invariant martix elements $\mathcal{M}(p_i \to p_j)$

LSZ theorem (9.74) states that

$$\begin{aligned} S\big(p_1, p_1', ...p_1^{(m)} &\to p_2, p_2', ...p_2^{(n)}\big) \\ &= i^{m+n} \lim_{p_i^2 \to m^2} \Pi(m^2 - p_i^2)\ G\big(p_1, p_1', ...p_1^{(m)} \to p_2, p_2', ...p_2^{(n)}\big) \end{aligned} \tag{9.74}$$

The relation between S-martix and invariant transition matrix $\mathcal{M}\big(p_1, p_1', ...p_1^{(m)} \to p_2, p_2', ...p_2^{(n)}\big)$ is (see AQM course)

$$S\big(p_1, p_1', ...p_1^{(m)} \to p_2, p_2', ...p_2^{(n)}\big) = \{1\} + (2\pi)^4 i\delta\big(\sum p_1^{(i)} - \sum p_2^{(j)}\big)\mathcal{M}\big(p_1, p_1', ...p_1^{(m)} \to p_2, p_2', ...p_2^{(n)}\big)$$

where $\{1\}$ denotes combination of $\delta^{(3)}\big(p_1^{(i)} - \sum p_2^{(j)}\big)$ corresponding to process without scattering (if $m = n$). Looking at the relation (9.73) between G and reduced function $\mathcal{G}$ we see that

$$\mathcal{M}\big(p_1, p_1', ...p_1^{(m)} \to p_2, p_2', ...p_2^{(n)}\big) = \lim_{p_i^2 \to m^2} \Pi(m^2 - p_i^2)\ \mathcal{G}\big(p_1, p_1', ...p_1^{(m)} \to p_2, p_2', ...p_2^{(n)}\big) \tag{9.75}$$

Part IX

10 Feynman diagrams for S-matrix

Consider two-particle elastic scattering

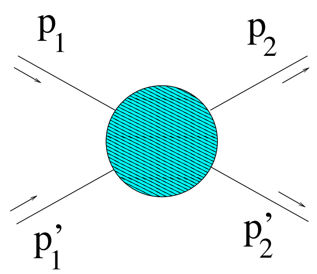


Figure 8. Two-particle elastic scattering

10.1 First order of perturbation theory

LSZ theorem for two-particle scattering (8.11):

$$\begin{aligned} S(p_1, p_1' \ \to\ p_2, p_2') &= i^4 \lim_{p_i^2 \to m^2} (m^2 - p_1^2)\ (m^2 - p_2^2)\ (m^2 - p'^2_1)\ (m^2 - p'^2_2) \\ &\quad \times \int dx dx' dy dy'\ e^{-ip_1x_1 - ip_1'x' + ip_2y + ip_2'y'}\ G(x, x'; y, y') \\ = \lim_{p_i^2 \to m^2} (m^2 - p_1^2)\ (m^2 - p_2^2)\ (m^2 - p'^2_1)\ (m^2 - p'^2_2) & G(p_1, p_1' \to p_2, p_2') \equiv G^{\mathrm{amp}}(p_1, p_1' \to p_2, p_2') \end{aligned} \tag{10.1}$$

(here we use better-looking notation $G(p_1,p_1'\to p_2,p_2')\equiv G(-p_1,-p_1',p_2,p_2')$)
where

$$
\begin{aligned}
G(x,x';y,y') &= \frac{\langle 0|\mathrm{T}\{e^{-i\int d^4z\ \frac{\lambda}{4!}\hat\phi_I^4(z)}\hat\phi_I(x)\hat\phi_I(x')\hat\phi(y)\hat\phi(y')\}|0\rangle}{\langle 0|\mathrm{T}\{e^{-i\int d^4z\ \frac{\lambda}{4!}\hat\phi_I^4(z)}\}||0\rangle}\\
&= \text{set of connected Feynman diagrams with four tails}
\end{aligned}
\tag{10.2}
$$

In the trivial order in perturbation theory we get

so

$$
\int dzdz'\ e^{-iqz+ikz'}\frac{1}{m^2-p_1^2-i\epsilon} = \frac{(2\pi)^4\delta^{(4)}(q-k)}{m^2-q^2-i\epsilon} = 0 \tag{10.3}
$$

if $p_1,p_1'\neq p_2,p_2'$. If momenta are equal LSZ theorem is not applicable since we assumed $p_1,p_1'\neq p_2,p_2'$ throughout the proof in Sect. 8.

$$
S(p_1,p_1'\ \to\ p_2,p_2') = (2\pi)^3\delta(\vec p_1-\vec p_2)(2\pi)^3\delta(\vec p_1'-\vec p_2')+(2\pi)^3\delta(\vec p_1-\vec p_2')(2\pi)^3\delta(\vec p_1'-\vec p_2)) \tag{10.4}
$$

The first non-trivial contribution to the r.h.s. of Eq. (10.2) is

$$
\begin{aligned}
G^{(1)}(x,x',y.y') &= -\frac{i\lambda}{4!}\int d^4z\ \langle 0|\mathrm{T}\{\hat\phi_I(x)\hat\phi_I(x')\hat\phi_I(y)\hat\phi_I(y')\hat\phi_I^4(z)\}|0\rangle\\
&-i\lambda\int d^4z\ \widehat{\hat\phi_I(x)\hat\phi_I}(z)\widehat{\hat\phi_I(x')\hat\phi_I}(z)\widehat{\hat\phi_I(y)\hat\phi_I}(z)\widehat{\hat\phi_I(y')\hat\phi_I}(z)\\
&= -i\lambda\int d^4z\ D_F(x-z)D_F(x'-z)D_F(y-z)D_F(y'-z)
\end{aligned}
\tag{10.5}
$$

$$
\begin{aligned}
\Rightarrow\ G^{(1)}(p_1,p_1'\to p_2,p_2')) &= \int dxdx'dydy'\ e^{-ip_1x_1-ip_1'x'+ip_2y+ip_2'y'}\ G^{(1)}(x,x';y,y')\\
&-i\lambda\int d^4z\ \widehat{\hat\phi_I(x)\hat\phi_I}(z)\widehat{\hat\phi_I(x')\hat\phi_I}(z)\widehat{\hat\phi_I(y)\hat\phi_I}(z)\widehat{\hat\phi_I(y')\hat\phi_I}(z)\\
&= -i\lambda\frac{(2\pi)^4\delta(p_1+p_1'-p_2-p_2')}{(m^2-p_1^2-i\epsilon)(m^2-p_1'^2-i\epsilon)(m^2-p_2^2-i\epsilon)(m^2-p_2'^2-i\epsilon)}
\end{aligned}
\tag{10.6}
$$

From Eq. (10.1) we get

$$
S^{(1)}(p_1,p_1'\to p_2,p_2') = -i\lambda(2\pi)^4\delta(p_1+p_1'-p_2-p_2') \tag{10.7}
$$

From the AQM course we know that

$$
\begin{aligned}
S(p_1,p_1'\to p_2,p_2') &= (2\pi)^6[\delta(\vec p_1-\vec p_2)\delta(\vec p_1'-\vec p_2')+(\vec p_2'\leftrightarrow\vec p_2)]\\
&+(2\pi)^4 i\delta(p_1+p_1'-p_2-p_2')M(p_1,p_1'\to p_2,p_2')\\
&\Rightarrow\ M^{(1)}(p_1,p_1'\to p_2,p_2') = -\lambda
\end{aligned}
\tag{10.8}
$$

This can be derived directly from the Feynman rules for the reduced Green functions in Sect. (9.5.3) and formula (10.14).

Finally, the cross section of meson-meson scattering in this model is given by the standard formula

$$
\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{64\pi^2 s} = \frac{\lambda^2}{64\pi^2 s}+O(\lambda^4) \tag{10.9}
$$

and the total cross section is

$$\sigma_{\text{tot}} \;=\; \frac{1}{2}\int d\Omega \frac{d\sigma}{d\Omega} \;=\; \frac{\lambda^2}{32\pi s} \;+\; O(\lambda^4) \tag{10.10}$$

($\frac{1}{2}$ is due to identical particles in the final state, see the AQM course).

10.2 Second order of perturbation theory

$$\begin{aligned} & G^{(2)}(x,x',y,y') \\ = & \; -\frac{1}{2}\left(\frac{\lambda}{4!}\right)^2 \int d^4z d^4z' \langle 0|\mathrm{T}\{\hat{\phi}_I(x)\hat{\phi}_I(x')\hat{\phi}_I(y)\hat{\phi}_I(y')\hat{\phi}_I^4(z)\hat{\phi}_I^4(z')\}|0\rangle_{\text{connected}} \end{aligned} \tag{10.11}$$

The set of (connected) Feynman diagrams is shown in Fig. **??**.

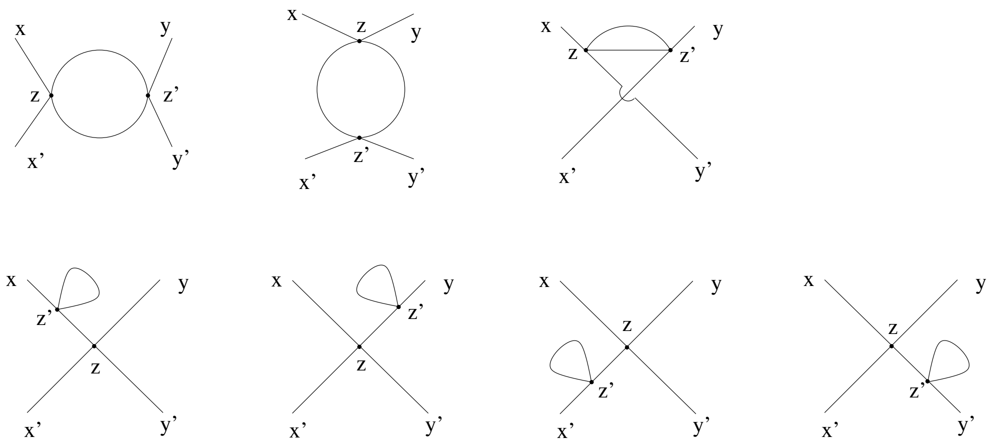


Figure 9. Set of second-order Feynman diagrams for two-particle scattering

In the momentum space these diagrams look like so the set of corresponding amputated

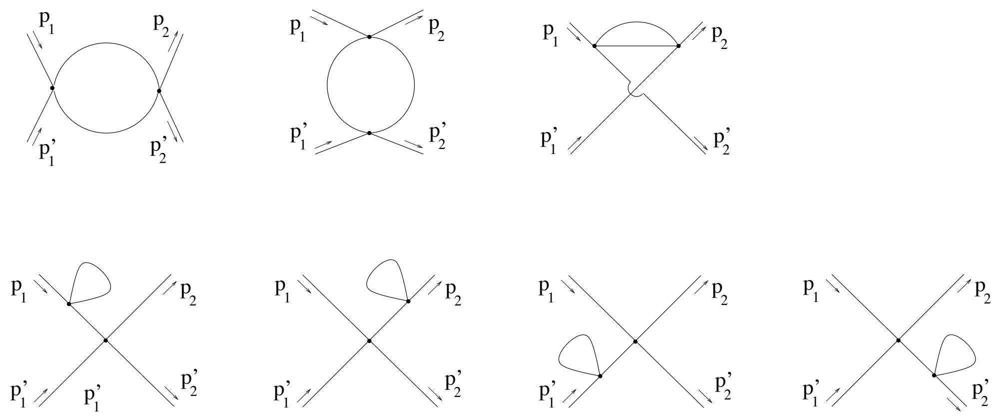


Figure 10. Second-order Feynman diagrams for two-particle scattering in the momentum space

diagrams is

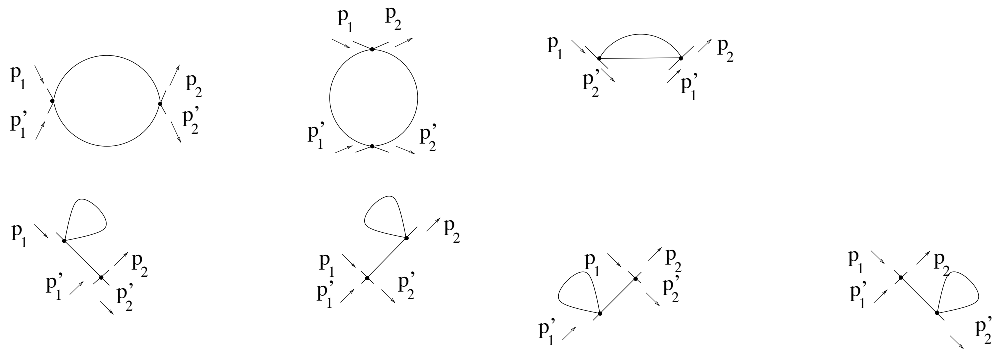
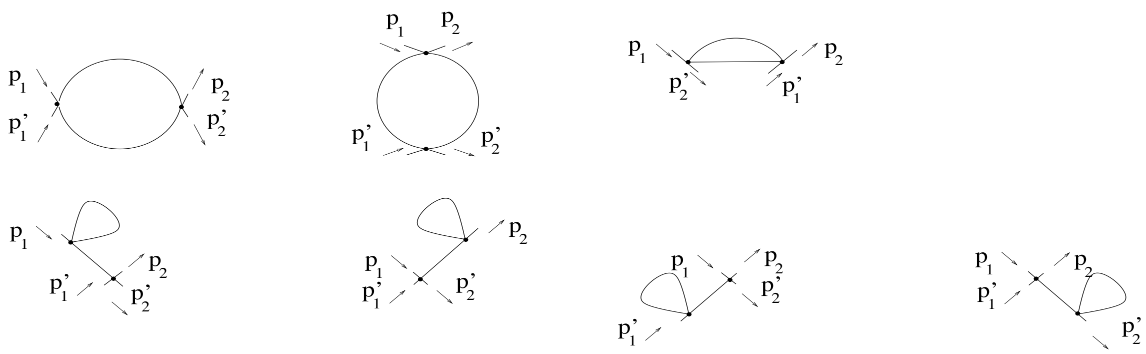


Figure 11. Amputated Feynman diagrams for two-particle scattering



$$
\begin{aligned}
G^{(2)}_{\rm amp}(p_1, p'_1 \to p_2, p'_2) \;&=\; \int \frac{\bar{d}^k}{i} \frac{1}{(m^2 - k^2 - i\epsilon)(m^2 - (p_1 + p'_1 - k)^2 - i\epsilon} && (10.12)\\
+ \int \frac{\bar{d}^k}{i} \frac{1}{(m^2 - k^2 - i\epsilon)(m^2 - (p_1 - p_2 - k)^2 - i\epsilon} &+ \int \frac{\bar{d}^k}{i} \frac{1}{(m^2 - k^2 - i\epsilon)(m^2 - (p_1 - p'_2 - k)^2 - i\epsilon} \\
+ \frac{1}{m^2 - p_1^2 - i\epsilon} \int \frac{\bar{d}k}{i} \frac{1}{m^2 - k^2 - i\epsilon} &+ \frac{1}{m^2 - p_1'^2 - i\epsilon} \int \frac{\bar{d}^k}{i} \frac{1}{m^2 - k^2 - i\epsilon} \\
+ \frac{1}{m^2 - p_2^2 - i\epsilon} \int \frac{\bar{d}k}{i} \frac{1}{m^2 - k^2 - i\epsilon} &+ \frac{1}{m^2 - p_2'^2 - i\epsilon} \int \frac{\bar{d}^k}{i} \frac{1}{m^2 - k^2 - i\epsilon} && (10.13)
\end{aligned}
$$

First three terms are OK but we have a problem with last four ones: when we calculate the cross section

$$
\mathcal{M}(p_1, p'_1 \to p_2, p'_2) \;=\; \lim_{p_i^2 \to m^2} \Pi(m^2 - p_i^2)\; \mathcal{G}(p_1, p'_1 \to p_2, p'_2) \tag{10.14}
$$

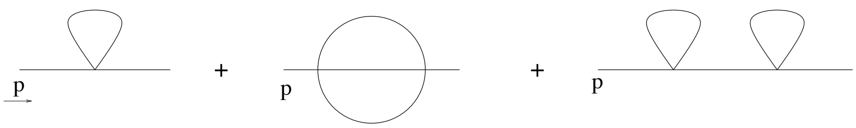
we have, for example, the contibution

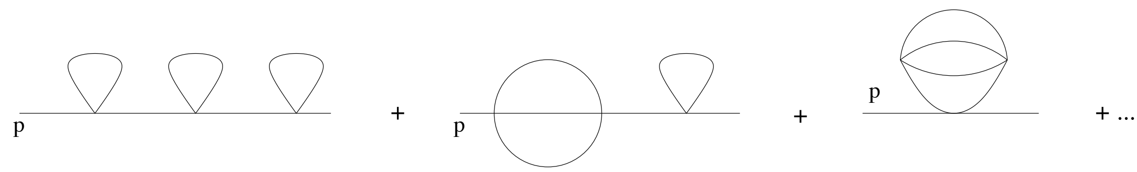
$$
\lim_{p_1^2 \to m^2} \frac{1}{m^2 - p_1^2 - i\epsilon} \int \frac{\bar{d}k}{i} \frac{1}{m^2 - k^2 - i\epsilon}
$$

which is infinite as $p_1^2 \to m^2$!
Q: What happened?
A: mass renormalization

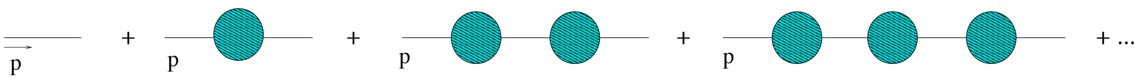
10.3 Renormalization of the mass of scalar particle

Let us draw the (connected) diagrams for the 2-point Green function

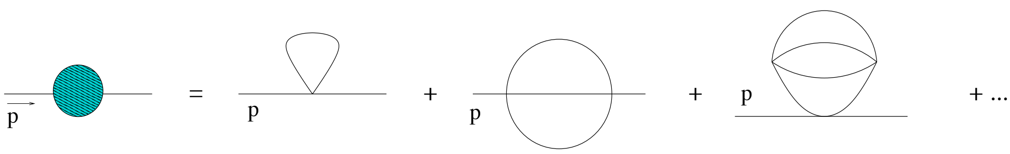




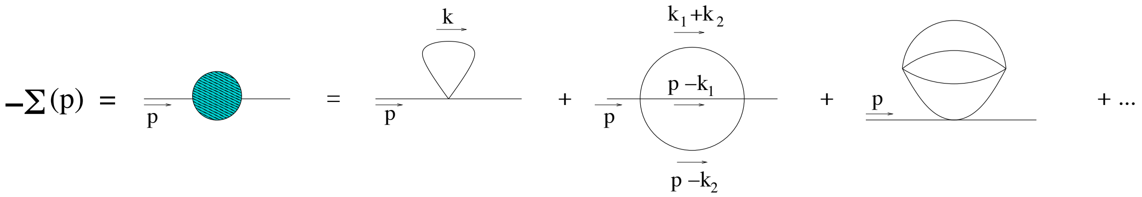
and let us rewrite this sum of diagrams as follows:



where



is called a one-particle irreducible (1PI) part. Let us denote it it $-\Sigma(p^2)$



$$\Sigma(p^2) \;=\; \lambda\int\frac{d\!\!{}^-{}^4k}{i}\frac{1}{m^2-k^2-i\epsilon} \tag{10.15}$$

$$+\;\frac{\lambda^2}{6}\int\frac{d\!\!{}^-{}^4k_1}{i}\frac{d\!\!{}^-{}^4k_2}{i}\frac{1}{[m^2-(k_1+k_2)^2-i\epsilon][m^2-(p-k_1)^2-i\epsilon][m^2-(p-k_2)^2-i\epsilon]}\;+\;\ldots$$

We get

$$\mathcal{G}(p^2) \;=\; \frac{1}{p^2}-\frac{1}{m^2-p^2}\Sigma(p^2)\frac{1}{m^2-p^2}+\frac{1}{m^2-p^2}\Sigma(p^2)\frac{1}{m^2-p^2}\Sigma(p^2)\frac{1}{m^2-p^2}+\ldots\;=\;\frac{1}{m^2-p^2+\Sigma(p^2)} \tag{10.16}$$

We see that $\mathcal{G}(p^2)$ no longer has a pole at $p^2=m^2$. Indeed,

$$\mathcal{G}(p^2)\big|_{p^2=m^2} \;=\; \frac{1}{m^2-p^2+\Sigma(p^2)}\bigg|_{p^2=m^2} \;=\; \frac{1}{\Sigma(m^2)} \;=\; \text{finite} \tag{10.17}$$

Instead, it has a pole at some other value p^2 where

$$m^2-p^2+\Sigma(p^2) \;=\; 0 \tag{10.18}$$

Let us denote the solution of this equation $m^2_{\rm ph}$, then

$$m^2-p^2+\Sigma(p^2) \;=\; m^2_{\rm ph}-p^2+\Sigma(p^2)-\Sigma(m^2_{\rm ph}) \;\overset{p^2\to m^2_{\rm ph}}{\simeq}\; (m^2_{\rm ph}-p^2)\Big(1-\frac{d\Sigma}{dp^2}\Big|_{p^2=m^2_{\rm ph}}\Big) \tag{10.19}$$

The factor $1-\frac{d\Sigma}{dp^2}\Big|_{p^2=m^2_{\rm ph}}$ is denoted as Z^{-1}

$$Z^{-1} \;\equiv\; 1-\frac{d\Sigma}{dp^2}\Big|_{p^2=m^2_{\rm ph}} \tag{10.20}$$

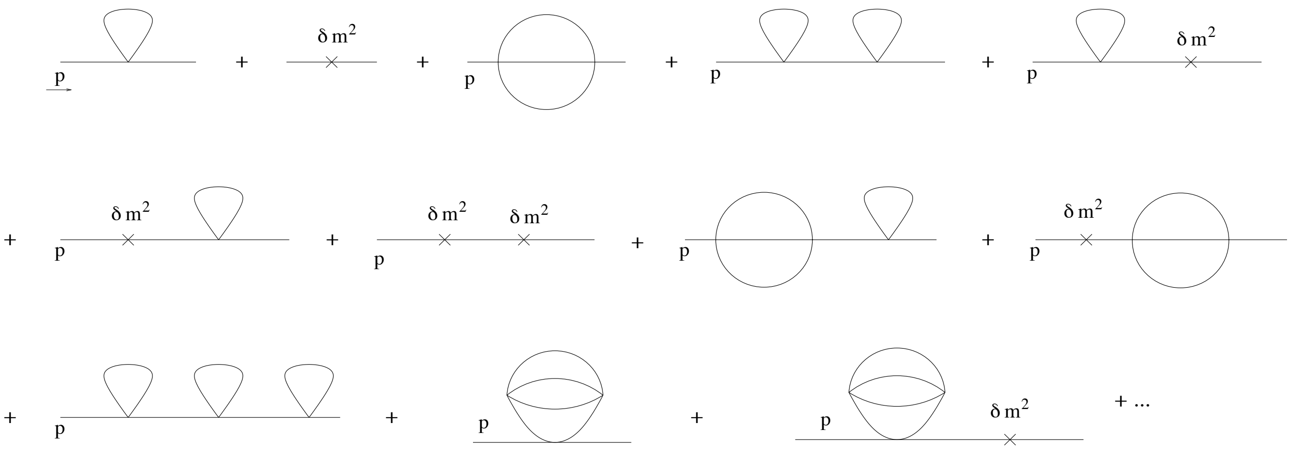
Let us denote

$$\delta m^2 \;\equiv\; m^2_{\rm ph}-m^2 \;=\; \Sigma(m^2_{\rm ph}) \qquad \text{mass counterterm} \tag{10.21}$$

(the last "=" is due to Eq. (10.18)). Next, we introduce free propagator with physical mass

$$\tilde{\mathcal{G}}_0(p^2) \;\equiv\; \frac{1}{m^2_{\rm ph}-p^2-i\epsilon} \tag{10.22}$$

and rewrite the exact propagator (10.16) as



$$= \quad \frac{1}{m_{\rm ph}^2 - p^2 + (\Sigma(p^2) - \delta m^2) - i\epsilon} \tag{10.23}$$

As $p^2 \to m_{\rm ph}^2$ we get

$$\begin{aligned}
\Sigma(p^2) - \delta m^2 \simeq \Sigma(m^2) - \delta m^2 + (p^2 - m_{\rm ph}^2)\frac{d\Sigma}{dp^2}\bigg|_{p^2=m_{\rm ph}^2} + O(p^2 - m_{\rm ph}^2)^2 \\
= \quad (p^2 - m_{\rm ph}^2)\frac{d\Sigma}{dp^2}\bigg|_{p^2=m_{\rm ph}^2} + O(p^2 - m_{\rm ph}^2)^2
\end{aligned} \tag{10.24}$$

(recall that $\delta m^2 = \Sigma(m_{\rm ph}^2)$) and therefore

$$\begin{aligned}
\frac{1}{m_{\rm ph}^2 - p^2 + \Sigma(p^2) - \delta m^2} \quad &\overset{p^2\to m_{\rm ph}^2}{\to} \quad \frac{1}{m_{\rm ph}^2 - p^2 + (p^2 - m_{\rm ph}^2)\frac{d\Sigma}{dp^2}\Big|_{p^2=m_{\rm ph}^2}} \quad = \quad \frac{Z}{m_{\rm ph}^2 - p^2}, \\
Z^{-1} \quad \equiv \quad 1 - (p^2 - m_{\rm ph}^2)\frac{d\Sigma}{dp^2}\bigg|_{p^2=m_{\rm ph}^2} \quad &= \quad 1 + z_1\lambda + z_2\lambda^2 + \ldots
\end{aligned} \tag{10.25}$$

Let us demonstrate that $m_{\rm ph}$ is a physical mass of the scalar boson. Consider the two-point Green function at large time

$$\begin{aligned}
G(t, \vec{x}) \quad \overset{t\to\infty}{=} \quad Z\int \frac{đp}{i}\frac{e^{-ip_0t+i\vec{p}\cdot\vec{x}}}{m_{\rm ph}^2 - p^2} \quad &= \quad Z\int \frac{đp}{i}\frac{e^{-ip_0t+i\vec{p}\cdot\vec{x}}}{m_{\rm ph}^2 + \vec{p}^2 - p_0^2 - i\epsilon} \\
= \quad Z\int \frac{đ^3p}{2E_p}e^{-iE_pt+i\vec{p}\cdot\vec{x}}, \qquad E_p \quad &= \quad \sqrt{m_{\rm ph}^2 + \vec{p}^2}
\end{aligned} \tag{10.26}$$

$\Rightarrow$ $m_{\rm ph}$ is a mass of the particle.

Let us prove now that the same Z-factor relates $\hat{\phi}$ to $\hat{\phi}_{\rm in}$ and $\hat{\phi}_{\rm out}$ (see the footnote at p.40)

$$\hat{\phi}(x) \overset{t\to-\infty}{\to} Z^{\frac{1}{2}}\hat{\phi}_{\rm in}(x), \qquad \hat{\phi}(x) \overset{t\to\infty}{\to} Z^{\frac{1}{2}}\hat{\phi}_{\rm out}(x) \tag{10.27}$$

Proof: suppose $\hat{\phi}(x) \overset{t\to-\infty}{\to} c_1\hat{\phi}_{\rm in}(x)$ and $\hat{\phi}(x) \overset{t\to\infty}{\to} c_2\hat{\phi}_{\rm out}(x)$ with some constants c_1 and c_2. First, from time reversal invariance of the theory one sees that $c_1 = c_2 = c$. Second, consider the two-point Green function $G(x-y) = \langle\Omega|\mathrm{T}\{\hat\phi(x)\hat\phi(y)\}|\Omega\rangle$ as $x_0\to\infty$ and $y_0\to-\infty$

$$G(x-y) = \langle\Omega|\mathrm{T}\{\hat\phi(x)\hat\phi(y)\}|\Omega\rangle = \sum_{\{n\}=\text{all in-states}}\langle\Omega|\hat\phi(x)|n_{\rm in}\rangle\langle n_{\rm in}|\hat\phi(y)|\Omega\rangle \tag{10.28}$$

$$= \int\frac{d\!\!\!{}^-{}^3p}{2E_p}\langle\Omega|\hat\phi(x)|p_{\rm in}\rangle\langle p_{\rm in}|\hat\phi(y)|\Omega\rangle + \sum_{\{n\}=2+\text{ particles states}}\langle\Omega|\hat\phi(x)|n\rangle\langle n\hat\phi(y)|\Omega\rangle$$

$$\to c^2\int\frac{d\!\!\!{}^-{}^3p}{2E_p}\langle\Omega|\hat\phi_{\rm out}(x)|p\rangle\langle p|\hat\phi_{\rm in}(y)|\Omega\rangle + \sum_{\{n\}=2+\text{ particles states}}\langle\Omega|\hat\phi_{\rm out}(x)|n_{\rm in}\rangle\langle n_{\rm in}|\hat\phi_{\rm in}(y)|\Omega\rangle$$

(recall that one-particle state is the same for "ins" and "outs": $|p_{\rm in}\rangle = |p_{\rm out}\rangle = |p\rangle$). Now, since

$$\hat\phi_{\rm in}(x) = \int\frac{d\!\!\!{}^-{}^3p}{\sqrt{2E_p}}\left[\hat a_p^{\rm in}e^{-ipx} + \hat a_p^{\dagger{\rm in}}e^{ipx}\right]\Big|_{p_0=E_p} \tag{10.29}$$

we see that

$${}_{\rm in}\langle p_2,p_2'|\hat\phi_{\rm in}|\Omega\rangle \sim \int\frac{d\!\!\!{}^-{}^3p}{\sqrt{2E_p}}\langle\Omega|a^{\rm in}_{p_2}a^{\rm in}_{p_2'}\left[\hat a_p^{\rm in}e^{-ipx} + \hat a_p^{\dagger{\rm in}}e^{ipx}\right]|\Omega\rangle = 0 \tag{10.30}$$

and similarly for ${}_{\rm in}\langle p_2,p_2',p_2''|$ and states with larger number of particles. Thus, as $x_0\to\infty$ and $y_0\to-\infty$

$$G(x-y) \to c^2\int\frac{d\!\!\!{}^-{}^3p}{2E_p}\langle\Omega|\hat\phi_{\rm out}(x)|p\rangle\langle p|\hat\phi_{\rm in}(y)|\Omega\rangle = c^2\int\frac{d\!\!\!{}^-{}^3p}{2E_p}\,e^{-iE_p(x_0-y_0)+i\vec p(\vec x-\vec y)} \tag{10.31}$$

Comparing this to Eq. (10.26) we see that $c^2 = Z$ so $c = Z^{\frac12}$, Q.E.D.

Let us now revisit LSZ theorem in terms of physical mass. Because of Eq. (10.27) the formula (8.11) will look like

$$S(p_1,p_1'\to p_2,p_2') = \left(\frac{i}{\sqrt Z}\right)^4\lim_{p_i^2\to m^2}(m^2_{\rm ph}-p_1^2)\,(m^2_{\rm ph}-p_2^2)\,(m^2_{\rm ph}-p'^2_1)\,(m^2_{\rm ph}-p'^2_2)$$

$$\times\int dx\,dx'\,dy\,dy'\;e^{-ip_1x_1-ip_1'x'+ip_2y+ip_2'y'}\;\langle\Omega|\mathrm{T}\{\hat\phi(x)\hat\phi(x')\hat\phi(y)\hat\phi(y')\}|\Omega\rangle$$

$$= \left(\frac{i}{\sqrt Z}\right)^4\lim_{p_i^2\to m^2}(m^2_{\rm ph}-p_1^2)(m^2_{\rm ph}-p_2^2)(m^2_{\rm ph}-p'^2_1)(m^2_{\rm ph}-p'^2_2)$$

$$\times\frac{Z}{m^2_{\rm ph}-p_1^2}\frac{Z}{m^2_{\rm ph}-p'^2_1}\frac{Z}{m^2_{\rm ph}-p_2^2}\frac{Z}{m^2_{\rm ph}-p'^2_2}G^{\rm 1PI}_{\rm amp}(p_1,p_1'\to p_2,p_2')$$

$$= (\sqrt Z)^4 G^{\rm 1PI}_{\rm amp}(p_1,p_1'\to p_2,p_2') \tag{10.32}$$

where $G^{\rm 1PI}$ is a one-particle irreducible diagram (1PI diagram is a diagram which cannot be reduced to two disconnected parts by removing one propagator).

Let us ignore for now the factor $\left(\frac{i}{\sqrt Z}\right)^4$ (it will lead to proper renormalization of the coupling constant λ) and summarize the final rule for matrix elements of the transition matrix

$$\mathcal{M}(p_1,p_1'\to p_2,p_2') = \mathcal{G}^{\rm amp}_{\rm 1PI}(p_1,p_1'\to p_2,p_2')\big|_{p_i^2\to m^2_{\rm ph}} \tag{10.33}$$

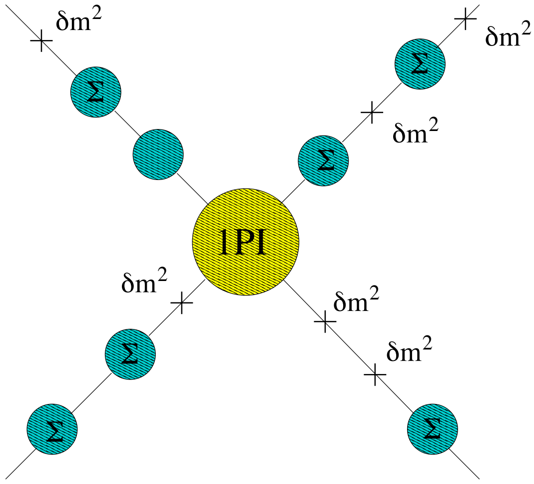


Figure 12. Amputated Feynman diagrams for two-particle scattering

10.4 Peskin's mnemonic rule

In Peskin's textbook there is a handy mnemonic rule for calculation of matrix elements of S-matrix

$$\begin{aligned} S(p_1,p_1',...p_1^{(m)}\to p_2,p_2',...p_2^{(n)}) \;&\equiv\; {}_{\rm out}\langle p_2,p_2',...p_2^{(n)}|p_1,p_1',...p_1^{(n)}\rangle_{\rm in} \\ =\; \langle p_2,p_2',...p_2^{(n)}|{\rm T}\Big\{e^{i\int d^4z\ \mathcal{L}_I(\hat\phi(z))}\Big\}|p_1,p_1',...p_1^{(m)}\rangle^{\rm 1PI}_{\rm connected} \end{aligned} \tag{10.34}$$

where in the r.h.s. everything is in the interaction representation:

$$\begin{aligned} \langle p_2,p_2',...p_2^{(n)} \;&=\; \langle 0|\hat a_{p_2}\hat a_{p_2'}...\hat a_{p_2^{(n)}}, \qquad |p_1,p_1',...p_1^{(m)}\rangle \;=\; \hat a^\dagger_{p_1}\hat a^\dagger_{p_1'}...a^\dagger_{p_1^{(m)}}|0\rangle, \\ \mathcal{L}_I(\hat\phi(z)) \;&=\; -\frac{\lambda}{4!}\hat\phi_I^4(z), \qquad \hat\phi_I(z) \;=\; \int\frac{d\!\!{}^-{}^3p}{\sqrt{2E_p}}\left[\hat a_p e^{-ipz}+\hat a^\dagger_p e^{ipz}\right]\Big|_{p_0=E_p} \end{aligned} \tag{10.35}$$

and contractions of ladder operators with the field operators are defined as follows

$$\begin{aligned} \widehat{\hat a_p\hat\phi_I}(z) \;&\equiv\; \langle 0|\hat a_p\hat\phi_I(z)|0\rangle \;=\; \langle 0|\hat a_p\int\frac{d\!\!{}^-{}^3p'}{\sqrt{2E_{p'}}}\left[\hat a_{p'}e^{-ip'z}+\hat a^\dagger_{p'}e^{ip'z}\right]\Big|_{p_0'=E_{p'}}|0\rangle \;=\; \frac{e^{ipz}}{\sqrt{2E_p}}, \\ \widehat{\hat\phi(z)\hat a^\dagger_p} \;&\equiv\; \langle 0|\hat\phi(z)\hat a^\dagger_p|0\rangle \;=\; \langle 0|\int\frac{d\!\!{}^-{}^3p'}{\sqrt{2E_{p'}}}\left[\hat a_{p'}e^{-ip'z}+\hat a^\dagger_{p'}e^{ip'z}\right]\Big|_{p_0'=E_{p'}}\hat a^\dagger_p|0\rangle \;=\; \frac{e^{-ipz}}{\sqrt{2E_p}} \end{aligned} \tag{10.36}$$

This mnemonic rule can be justified by the LSZ theorem. Let us illustrate this rule for the elastic two-particle scattering in the first order in λ

$$\begin{aligned} S^{(1)}(p_1,p_1'\to p_2,p_2') \;&=\; 4\sqrt{E_{p_2}E_{p_2'}E_{p_1}E_{p_1'}}\langle 0|\hat a_{p_2}\hat a_{p_2'}\frac{(-i\lambda)}{4!}\int d^4z\ \hat\phi_I^4(z)\hat a^\dagger_{p_1}\hat a^\dagger_{p_1'}|0\rangle \\ =\; &-4i\lambda\sqrt{E_{p_2}E_{p_2'}E_{p_1}E_{p_1'}}\int d^4z\ \widehat{\hat a_{p_2}\hat\phi_I}(z)\widehat{\hat a_{p_2'}\hat\phi_I}(z)\widehat{\hat\phi(z)\hat a^\dagger_{p_1}}\widehat{\hat\phi(z)\hat a^\dagger_{p_1'}} \\ =\; &-i\lambda\int d^4z\ e^{-ip_1z-ip_1'z+ip_2z+ip_2'z} \;=\; -i\lambda(2\pi^4)\delta^{(4)}(p_1+p_1'-p_2-p_2') \end{aligned} \tag{10.37}$$

and therefore we get the same result (10.8) for the matrix element of transition matrix.

Part X

11 Complex Klein-Gordon field

Consider a set of two non-interacting scalar Klein-Gordon fields $\phi_1(x)$ and $\phi_2(x)$ and define the complex KG field $\phi(x) \equiv \frac{1}{\sqrt{2}}[\phi_1(x) + i\phi_2(x)]$ satisfying the KG equation

$$(\partial^2 + m^2)\phi(x) = 0 \tag{11.1}$$

Obviously, the complex conjugate field $\phi^*(x)$ satisfies the same KG equation. It turns out that for the description of interactions with the electromagnetic fields it is convenient to consider $\phi(x)$ and $\phi^*(x)$ as independent canonical coordinates (instead of $\phi_1(x)$ and $\phi_2(x)$). The Lagrangian for the free complex KG field is

$$\begin{aligned} L(t) &= \int d^3x\ \mathcal{L}(\vec{x}, t) \\ \mathcal{L}(x) &= \partial_\mu \phi^*(x)\partial^\mu \phi(x) - m^2 \phi^*(x)\phi(x) \end{aligned} \tag{11.2}$$

Note that the Largangian density for the set of non-interacting fields ϕ_1 and ϕ_2 can be written as

$$\mathcal{L}(\phi_1) + \mathcal{L}(\phi_2) = \frac{1}{2}\partial^\mu(\phi_1 - i\phi_2)\partial_\mu(\phi_1 + i\phi_2) - \frac{m^2}{2}(\phi_1 - i\phi_2)(\phi_1 + i\phi_2)$$

which coincides with $\mathcal{L}(\phi(x))$ given by Eq. (11.2) so one free complex KG field describes two free real KG fields which do not interact with each other.

The canonical momenta for complex KG field are

$$\begin{aligned} \pi(t,x) &\equiv \frac{\partial \mathcal{L}}{\partial \dot\phi}(t,x) = \dot\phi^*(t,x) \\ \pi^*(t,x) &\equiv \frac{\partial \mathcal{L}}{\partial \dot\phi^*}(t,x) = \dot\phi(t,x) \end{aligned} \tag{11.3}$$

so the classical Hamiltonian for the KG field takes the form

$$\begin{aligned} H &= \int d^3x \big[\pi(t,\vec{x})\dot\phi(t,\vec{x}) + \pi^*(t,\vec{x})\phi^*(t,\vec{x})\big] \\ &- \int d^3x \big[\dot\phi^*(t,\vec{x})\dot\phi^*(t,\vec{x}) - \vec\nabla\phi^*(t,\vec{x})\cdot\vec\nabla\phi(t,\vec{x}) - m^2\phi^*(t,\vec{x})\phi(t,\vec{x})\big] \\ &= \int d^3x \big[\pi(t,x)\dot\phi(t,x) + \pi^*(t,x)\dot\phi^*(t,x) - \dot\phi^*(t,\vec{x})\dot\phi^*(t,\vec{x}) + \vec\nabla\phi^*(t,\vec{x})\cdot\vec\nabla\phi(t,\vec{x}) + m^2\phi^*(t,\vec{x})\phi(t,\vec{x})\big] \\ &= \int d^3x \big[\pi^*(t,\vec{x})\pi(t,\vec{x}) + \vec\nabla\phi^*(t,\vec{x})\cdot\vec\nabla\phi(t,\vec{x}) + m^2\phi^*(t,\vec{x})\phi(t,\vec{x})\big] \end{aligned} \tag{11.4}$$

Again, it is east to see that $H(\phi, \phi^*; \pi, \pi^*) = H(\phi_1; \pi_1) + H(\phi_2; \pi_2)$ (note that $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ but $\pi = \frac{1}{\sqrt{2}}(\pi_1 - i\pi_2)$ according to our definition (11.3)).

The classical energy-momentum tensor has the form (cf. Eq. (11.7)

$$T_{\mu\nu} \;=\; \partial_\mu\phi\frac{\partial\mathcal{L}}{\partial\partial^\nu\phi}+\partial_\mu\phi^*\frac{\partial\mathcal{L}}{\partial\partial^\nu\phi^*}-g_{\mu\nu}\mathcal{L} \;=\; \partial_\mu\phi^*\partial_\nu\phi+\partial_\mu\phi\partial_\nu\phi^*-g_{\mu\nu}(\partial^\alpha\phi^*\partial_\alpha\phi-m^2\phi^2) \tag{11.5}$$

and the classical momentum of complex KG field can be written as

$$P_i \;\equiv\; \int d^3x\; T_{0i}(t,\vec{x}) \;=\; \int d^3x\;\left[\pi(t,\vec{x})\partial_i\phi(t,\vec{x})+\pi^*(t,\vec{x})\partial_i\phi^*(t,\vec{x})\right] \tag{11.6}$$

11.1 Quantization of the complex KG field

As usual, we promote classical coordinates ϕ,ϕ^* and classical momenta π,π^* to operators $\hat\phi(\vec{x}),\hat\phi^\dagger(\vec{x})$ and $\hat\pi(\vec{x}),\hat\pi^\dagger(\vec{x})$ satisfying the canonical commutation relations [5]

$$[\hat\phi(\vec{x}),\hat\pi(\vec{y})] \;=\; [\hat\phi^\dagger(\vec{x}),\hat\pi^\dagger(\vec{y})] \;=\; i\delta(\vec{x}-\vec{y}),\quad \text{all other commutators vanish} \tag{11.7}$$

The corresponding quantum Hamiltonian has the form

$$\hat H \;=\; \int d^3x\big[\hat\pi^\dagger(\vec{x})\hat\pi(\vec{x})+\vec\nabla\hat\phi^\dagger(\vec{x})\cdot\vec\nabla\hat\phi(\vec{x})+m^2\hat\phi^\dagger(\vec{x})\hat\phi(\vec{x})\big] \tag{11.8}$$

Again, it is easy to check that

$$\hat H(\hat\phi,\hat\phi^\dagger;\hat\pi,\hat\pi^\dagger) \;=\; \hat H(\hat\phi_1,\hat\pi_1)+\hat H(\hat\phi_2,\hat\pi_2) \tag{11.9}$$

Now we can construct Heisenberg picture of quantization following usual rules in Sect. 6: First, we define vacuum in Heisenberg picture as Schrödinger vacuumat $t=0$. Due to Eq. (11.7) this vacuum state is a direct product of vacuum states $|0\rangle_1$ and $|0\rangle_2$

$$|0\rangle \;\equiv\; |0\rangle_1|0\rangle_2 \tag{11.10}$$

The explicit form is the product of wave functionals (4.55) which can be written as

$$\langle\{\phi(\vec{x}),\phi^*(\vec{x})\}|\Psi\rangle \;=\; e^{-\int d^3x\phi^*(\vec{x})\hat W\phi(\vec{x})} \tag{11.11}$$

where the differential operator W is defined in Eq. (**??**)

Next, we define the time-dependent operators

$$\begin{aligned}\hat\phi(t,\vec{x}) \;&\equiv\; e^{i\hat Ht}\hat\phi(\vec{x})e^{-i\hat Ht}, \qquad \hat\phi^\dagger(t,\vec{x}) \;\equiv\; e^{i\hat Ht}\hat\phi^\dagger(\vec{x})e^{-i\hat Ht},\\ \hat\pi(t,\vec{x}) \;&\equiv\; e^{i\hat Ht}\hat\pi(\vec{x})e^{-i\hat Ht}, \qquad \hat\pi^\dagger(t,\vec{x}) \;\equiv\; e^{i\hat Ht}\hat\pi^\dagger(\vec{x})e^{-i\hat Ht}\end{aligned} \tag{11.12}$$

satisfying Heisenberg equations

$$\begin{aligned}\frac{\partial\hat\phi(t,\vec{x})}{\partial t} \;&=\; i[\hat H,\hat\phi(t,\vec{x})], \qquad \frac{\partial\hat\phi^\dagger(t,\vec{x})}{\partial t} \;=\; i[\hat H,\hat\phi^\dagger(t,\vec{x})],\\ \frac{\partial\hat\pi(t,\vec{x})}{\partial t} \;&=\; i[\hat H,\hat\pi(t,\vec{x})], \qquad \frac{\partial\hat\pi^\dagger(t,\vec{x})}{\partial t} \;=\; i[\hat H,\hat\pi^\dagger(t,\vec{x})]\end{aligned} \tag{11.13}$$

[5] These commutation relations are in agreement with $[\hat\phi_1(\vec{x}),\hat\pi_1(\vec{y})] \;=\; [\hat\phi_2(\vec{x}),\hat\pi_2(\vec{y})] \;=\; i\delta(\vec{x}-\vec{y})$, for example $[\hat\phi(\vec{x}),\hat\pi(\vec{y})] \;=\; \frac{1}{2}[\hat\phi_1(\vec{x})+i\hat\phi_2(\vec{x}),\hat\pi_1(\vec{x})-i\hat\pi_2(\vec{x})] \;=\; i\delta(\vec{x}-\vec{y})$.

Similarly to Eq. (6.33) one can prove the equal-time commutation relations

$$[\hat\phi(t,\vec x),\hat\pi(t,\vec y)] \;=\; [\hat\phi^\dagger(t,\vec x),\hat\pi^\dagger(t,\vec y)] \;=\; i\delta(\vec x-\vec y), \tag{11.14}$$
$$[\hat\phi(t,\vec x),\hat\phi(t,\vec y)] \;=\; [\hat\phi(t,\vec x),\hat\pi^\dagger(t,\vec y)] \;=\; [\hat\pi(t,\vec x),\hat\pi(t,\vec y)] \;=\; [\hat\pi(t,\vec x),\hat\phi^\dagger(t,\vec y)] \;=\; 0$$

Let us consrtuct the expansion of field operators in ladder operators. Since the complex field is defined as $\hat\phi(x) = \frac{1}{\sqrt2}\big(\hat\phi_1(x)+i\hat\phi_2(x)\big)$ (and $\hat\phi^\dagger(x) = \frac{1}{\sqrt2}\big(\hat\phi_1^\dagger(x)-i\hat\phi_2^\dagger(x)\big)$, $\hat\pi(x) = \frac{1}{\sqrt2}\big(\hat\pi_1(x)-i\hat\pi_2(x)\big)$, $\hat\pi^\dagger(x) = \frac{1}{\sqrt2}\big(\hat\pi_1^\dagger(x)+i\hat\pi_2^\dagger(x)\big)$) we can use ladder expansion (6.9) for $\hat\phi_1$ and $\hat\phi_2$

$$\hat\phi_1(x) \;=\; \int\frac{\bar{d}p}{\sqrt{2E_p}}\,(\hat a_{1\vec p}e^{-ipx}+\hat a^\dagger_{1\vec p}e^{ipx}),\quad \hat\pi_1(x) \;=\; -i\int\frac{\bar{d}p}{\sqrt{2E_p}}\,E_{\vec p}(\hat a_{1_{\vec p}}e^{-ipx}-\hat a^\dagger_{1\vec p}e^{ipx}),$$
$$\hat\phi_2(x) \;=\; \int\frac{\bar{d}p}{\sqrt{2E_p}}\,(\hat a_{2\vec p}e^{-ipx}+\hat a^\dagger_{2\vec p}e^{ipx}),\quad \hat\pi_2(x) \;=\; -i\int\frac{\bar{d}p}{\sqrt{2E_p}}\,E_{\vec p}(\hat a_{2_{\vec p}}e^{-ipx}-\hat a^\dagger_{2\vec p}e^{ipx}) \tag{11.15}$$

and get

$$\hat\phi(x) \;=\; \int\frac{\bar{d}p}{\sqrt{2E_p}}\,(\hat a_{\vec p}e^{-ipx}+\hat b^\dagger_{\vec p}e^{ipx}),\quad \hat\pi(x) \;=\; -i\int\frac{\bar{d}p}{\sqrt{2E_p}}\,E_{\vec p}(\hat b_{\vec p}e^{-ipx}-\hat a^\dagger_{\vec p}e^{ipx}),$$
$$\hat\phi^\dagger(x) \;=\; \int\frac{\bar{d}p}{\sqrt{2E_p}}\,(\hat b_{\vec p}e^{-ipx}+\hat a^\dagger_{\vec p}e^{ipx}),\quad \hat\pi^\dagger(x) \;=\; -i\int\frac{\bar{d}p}{\sqrt{2E_p}}\,E_{\vec p}(\hat a_{\vec p}e^{-ipx}-\hat b^\dagger_{\vec p}e^{ipx}) \tag{11.16}$$

where

$$\hat a_{\vec p} \;\equiv\; \frac{1}{\sqrt2}(\hat a_{1_{\vec p}}+i\hat a_{2\vec p}),\qquad \hat b_{\vec p} \;\equiv\; \frac{1}{\sqrt2}(\hat a_{1_{\vec p}}-i\hat a_{2\vec p}) \tag{11.17}$$

(and $\hat a^\dagger_{\vec p} = \frac{1}{\sqrt2}(\hat a^\dagger_{1_{\vec p}}-i\hat a^\dagger_{2\vec p})$, $\hat b^\dagger_{\vec p} = \frac{1}{\sqrt2}(\hat a^\dagger_{1_{\vec p}}+i\hat a^\dagger_{2\vec p})$). The commutation relations between a's and b's follow from commutation relations between a_i and $a_i^\dagger$:

$$[\hat a_{\vec p},\hat a^\dagger_{\vec p'}] \;=\; [\hat b_{\vec p},\hat b^\dagger_{\vec p'}] \;=\; (2\pi)^3\delta(\vec p-\vec p') \tag{11.18}$$
$$[\hat a_{\vec p},\hat a_{\vec p'}] \;=\; [\hat b_{\vec p},\hat b_{\vec p'}] \;=\; [\hat a^\dagger_{\vec p},\hat a^\dagger_{\vec p'}] \;=\; [\hat b^\dagger_{\vec p},\hat b^\dagger_{\vec p'}] \;=\; [\hat a_{\vec p},\hat b^\dagger_{\vec p'}] \;=\; [\hat b_{\vec p},\hat a^\dagger_{\vec p'}] \;=\; 0$$

Let us check that the commutation relations (11.18) for ladder operators lead to CCR (11.7).

$$[\hat\phi(\vec x),\hat\pi(\vec y)] \;=\; -\frac{i}{2}\int\bar{d}^3p\,\bar{d}^3p'\sqrt{\frac{E_{p'}}{E_p}}\,[\hat a_{\vec p}e^{i\vec p\vec x}+\hat b^\dagger_{\vec p'}e^{-i\vec p\vec x},\hat b_{\vec p}e^{i\vec p\vec y}-\hat a^\dagger_{\vec p'}e^{-i\vec p\vec y}] \;=$$
$$=\; \frac{i}{2}\int\bar{d}^3p\,\left(e^{i\vec p(\vec x-\vec y)}+e^{-i\vec p(\vec x-\vec y)}\right) \;=\; i\delta(\vec x-\vec y)$$
$$[\hat\phi^\dagger(\vec x),\hat\pi^\dagger(\vec y)] \;=\; -\frac{i}{2}\int\bar{d}^3p\,\bar{d}^3p'\sqrt{\frac{E_{p'}}{E_p}}\,[\hat b_{\vec p}e^{-i\vec p\vec x}+\hat a^\dagger_{\vec p'}e^{i\vec p\vec x},\hat a_{\vec p}e^{-i\vec p\vec y}-\hat b^\dagger_{\vec p'}e^{i\vec p\vec y}] \;=$$
$$=\; \frac{i}{2}\int\bar{d}^3p\,\left(e^{-i\vec p(\vec x-\vec y)}+e^{i\vec p(\vec x-\vec y)}\right) \;=\; i\delta(\vec x-\vec y) \tag{11.19}$$

The Hamiltonian in terms of ladder operators looks like

$$\hat{H} \ = \ \frac{1}{2}\int đ^3p\ E_p(\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}} + \hat{a}_{\vec{p}}\hat{a}^\dagger_{\vec{p}} + \hat{b}^\dagger_{\vec{p}}\hat{b}_{\vec{p}} + \hat{b}_{\vec{p}}\hat{b}^\dagger_{\vec{p}}) \ = \ \int đ^3p\ E_p(\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}} + \hat{b}^\dagger_{\vec{p}}\hat{b}_{\vec{p}}) \tag{11.20}$$

in agreement with Eq. (5.18) and the fact that $\hat{H} \ = \ \hat{H}_1 + \hat{H}_2$. (As usual, we throw away the infinite constant, see the discussion after Eq. (5.17)).

Note that both $\hat{a}_{\vec{p}}$ and $\hat{b}_{\vec{p}}$ annihilate the vacuum state (11.10). Indeed, since $a_{1\vec{p}}|0\rangle \ = \ a_{1\vec{p}}|0\rangle_1|0\rangle_2 \ = \ 0$ and $a_{2\vec{p}}|0\rangle \ = \ |0\rangle_1 a_{2\vec{p}}|0\rangle_2 \ = \ 0$, we have

$$\hat{a}_{\vec{p}}\ |0\rangle \ = \ \hat{b}_{\vec{p}}|0\rangle \ = \ 0 \tag{11.21}$$

(which, as we discussed in Sect. , can serve as a definition of vacuum state $|0\rangle$).

Now we can promote the classical momentum (11.23) to the momentum operator

$$\hat{P}_i \ = \ \int d^3x\ \left[\hat{\pi}(\vec{x})\partial_i\hat{\phi}(\vec{x}) + \hat{\pi}^\dagger(\vec{x})\partial_i\hat{\phi}^\dagger(\vec{x})\right] \tag{11.22}$$

and check that in terms of ladder operators it reads

$$\hat{P}_i \ = \ \int đ^3p\ p_i(a^\dagger_{\vec{p}}a_{\vec{p}} + b^\dagger_{\vec{p}}b_{\vec{p}}) \tag{11.23}$$

Similarly to Eq. (6.15) we can represent the quantum operator of 4-momentum as

$$\hat{P}^\mu \ \equiv \ (\hat{H}, \hat{P}^i) \ = \ \int đ^3p\ p^\mu(a^\dagger_{\vec{p}}a_{\vec{p}} + b^\dagger_{\vec{p}}b_{\vec{p}}) \tag{11.24}$$

and prove formulas

$$\hat{\phi}(x+a) \ = \ e^{i\hat{P}a}\hat{\phi}(x)e^{-i\hat{P}a}, \qquad \hat{\phi}^\dagger(x+a) \ = \ e^{i\hat{P}a}\hat{\phi}^\dagger(x)e^{-i\hat{P}a} \tag{11.25}$$

which are in agreement with the representation (11.16) in terms of ladder operators.

From the explicit form of momentum operator (11.24) we see that

$$[\hat{P}^\mu, \hat{a}^\dagger_p] \ = \ p^\mu\hat{a}^\dagger_p, \quad [\hat{P}^\mu, \hat{b}^\dagger_p] \ = \ p^\mu\hat{b}^\dagger_p, \quad [\hat{P}^\mu, \hat{a}_p] \ = \ -p^\mu\hat{a}_p, \quad [\hat{P}^\mu, \hat{b}_p] \ = \ -p^\mu\hat{b}_p, \tag{11.26}$$

Using these commutators it is easy to show that $\hat{a}^\dagger_p|0\rangle$ and $\hat{b}^\dagger_p|0\rangle$ are one-particle states

$$|p,-\rangle \ \equiv \ \sqrt{2E_p}\hat{a}^\dagger_p|0\rangle, \qquad |p,+\rangle \ \equiv \ \sqrt{2E_p}\hat{b}^\dagger_p|0\rangle \tag{11.27}$$

with the same mass m (since $E_p = \sqrt{m^2+\vec{p}^2}$ for both of them). These states will correspond to states of particle and antiparticle with same masses and opposite charges. [6] One can formally introduce the charge operator

$$\hat{Q} \ \equiv \ \frac{i}{2}\int d^3x\ \left(\hat{\pi}(\vec{x})\hat{\phi}(\vec{x}) - \hat{\phi}^\dagger(\vec{x})\hat{\pi}^\dagger(\vec{x})\right) \tag{11.28}$$

[6] The name "charge" is formal here since in a free theory there is no physical notion of charge but in the theory of quantum electrodynamics of scalar particles these states will correspond to states of particle and antiparticle with same masses and opposite charges like π^+ and π^- mesons

In terms of ladder operators it looks like

$$\hat{Q} \;=\; \frac{1}{2}\int d^3x đ^3pđ^3p'\left[\frac{\sqrt{E_p}}{\sqrt{E_{p'}}}(\hat{b}_{\vec{p}}e^{i\vec{p}\vec{x}} - \hat{a}^\dagger_{\vec{p}}e^{-i\vec{p}\vec{x}})(\hat{a}_{\vec{p}}e^{i\vec{p}\vec{x}} + \hat{b}^\dagger_{\vec{p}}e^{-i\vec{p}\vec{x}})\right.$$
$$\left. - \frac{\sqrt{E_{p'}}}{\sqrt{E_p}}(\hat{b}_{\vec{p}}e^{i\vec{p}\vec{x}} + \hat{a}^\dagger_{\vec{p}}e^{-i\vec{p}\vec{x}})(\hat{a}_{\vec{p}}e^{i\vec{p}'\vec{x}} - \hat{b}^\dagger_{\vec{p}}e^{-i\vec{p}'\vec{x}})\right]$$
$$= \int đ^3p\,(\hat{b}_{\vec{p}}\hat{b}^\dagger_{\vec{p}} - \hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}}) \;=\; \int đ^3p\,(\hat{b}^\dagger_{\vec{p}}\hat{b}_{\vec{p}} - \hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}}) \tag{11.29}$$

where again we dropped the infinite commutator term so the vacuum will have charge zero $\hat{Q}|0\rangle = 0$ (alternatively one may just define the charge operator $\hat{Q}$ as the r.h.s. of the above equation). The commutators of operator $\hat{Q}$ with ladder operators are

$$[\hat{Q}, a^\dagger_{\vec{p}}] \;=\; a^\dagger_{\vec{p}}, \quad [\hat{Q}, a_{\vec{p}}] \;=\; -\,a_{\vec{p}}; \qquad [\hat{Q}, b^\dagger_{\vec{p}}] \;=\; -\,b^\dagger_{\vec{p}}, \quad [\hat{Q}, b_{\vec{p}}] \;=\; b_{\vec{p}} \tag{11.30}$$

From these commutators it is easy to see that the operator $\hat{Q}$ counts the number of (+) particles minus the number of (−) particles in an (m+n)-particle state

$$\hat{Q}|p_1, p_2, ...p_m; k_1, k_2, ...k_n\rangle \;=\; \Pi_m\sqrt{2E_i}\Pi_n\sqrt{2E_i}\hat{Q}\hat{a}^\dagger_{\vec{p}_1}\hat{a}^\dagger_{\vec{p}_2}...\hat{a}^\dagger_{\vec{p}_m}\hat{b}^\dagger_{\vec{k}_1}\hat{b}^\dagger_{\vec{k}_2}...\hat{b}^\dagger_{\vec{k}_n}|0\rangle \tag{11.31}$$
$$= \Pi_m\sqrt{2E_i}\Pi_n\sqrt{2E_i}(m-n)\hat{a}^\dagger_{\vec{p}_1}\hat{a}^\dagger_{\vec{p}_2}...\hat{a}^\dagger_{\vec{p}_m}\hat{b}^\dagger_{\vec{k}_1}\hat{b}^\dagger_{\vec{k}_2}...\hat{b}^\dagger_{\vec{k}_n}|0\rangle \;=\; (m-n)|p_1, p_2, ...p_m; k_1, k_2, ...k_n\rangle$$

From expansion of field operators in ladder operators (11.16) it is easy to get the Feynman propagator for complex KG field

$$\langle 0|\mathrm{T}\{\hat{\phi}(x)\hat{\phi}^\dagger(y)\} \;=\; \theta(x_0 - y_0)\int\frac{đ^3pđ^3p'}{2\sqrt{E_pE_{p'}}}\,\langle 0|(\hat{a}_{\vec{p}}e^{-ipx} + \hat{b}^\dagger_{\vec{p}}e^{ipx})(\hat{b}_{\vec{p}'}e^{-ip'y} + \hat{a}^\dagger_{\vec{p}'}e^{ip'y})|0\rangle$$
$$+\;\theta(y_0 - x_0)\int\frac{đ^3pđ^3p'}{2\sqrt{E_pE_{p'}}}\,\langle 0|(\hat{b}_{\vec{p}'}e^{-ip'y} + \hat{a}^\dagger_{\vec{p}'}e^{ip'y})(\hat{a}_{\vec{p}}e^{-ipx} + \hat{b}^\dagger_{\vec{p}}e^{ipx})|0\rangle$$
$$=\; \theta(x_0 - y_0)D(x-y) \;+\; \theta(y_0 - x_0)D(y-x) \;=\; D_F(x-y)$$

which is the same as the propagator for real ("neutral") KG field. One can also demonstrate that

$$\langle 0|\mathrm{T}\{\hat{\phi}(x)\hat{\phi}(y)\}|0\rangle \;=\; \theta(x_0 - y_0)\int\frac{đ^3pđ^3p'}{2\sqrt{E_pE_{p'}}}\,\langle 0|(\hat{a}_{\vec{p}}e^{-ipx} + \hat{b}^\dagger_{\vec{p}}e^{ipx})(\hat{a}_{\vec{p}'}e^{-ip'y} + \hat{b}^\dagger_{\vec{p}'}e^{ip'y})|0\rangle$$
$$+\;\theta(y_0 - x_0)\int\frac{đ^3pđ^3p'}{2\sqrt{E_pE_{p'}}}\,\langle 0|(\hat{a}_{\vec{p}'}e^{-ip'y} + \hat{b}^\dagger_{\vec{p}'}e^{ip'y})(\hat{a}_{\vec{p}}e^{-ipx} + \hat{b}^\dagger_{\vec{p}}e^{ipx})|0\rangle \;=\; 0 \tag{11.32}$$

and similarly $\langle 0|\mathrm{T}\{\hat{\phi}^\dagger(x)\hat{\phi}^\dagger(y)\}|0\rangle$. In accordance with these formulas the propagator of complex KG field is depicted as a line with an arrow going from $\hat{\phi}(x)$ to $\hat{\phi}^\dagger(y)$

11.2 Complex KG field with self-interaction

As an example of self-interacting KG field, consider theory with Lagrangian

$$\mathcal{L} \;=\; \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi - \frac{\lambda}{4}(\phi^*\phi)^2 \tag{11.33}$$

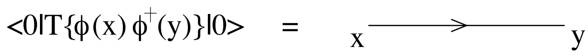


Figure 13. Feynman propagator for complex KG field

and independent canonical coordinates $\phi(x)$ and $\phi^*(x)$. The Euler-Lagrange equations are

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\phi} &= \partial^\mu\frac{\partial\mathcal{L}}{\partial\partial^\mu\phi} \;\Rightarrow\; (\partial^2+m^2)\phi^* &= -\frac{\lambda}{2}\phi^{*2}\phi,\\
\frac{\partial\mathcal{L}}{\partial\phi^*} &= \partial^\mu\frac{\partial\mathcal{L}}{\partial\partial^\mu\phi^*} \;\Rightarrow\; (\partial^2+m^2)\phi &= -\frac{\lambda}{2}\phi^*\phi^2
\end{aligned}\tag{11.34}$$

The canonical momenta are the same as in Eq. (11.3

$$\begin{aligned}
\pi(t,x) &\equiv \frac{\partial\mathcal{L}}{\partial\dot\phi}(t,x) = \dot\phi^*(t,x)\\
\pi^*(t,x) &\equiv \frac{\partial\mathcal{L}}{\partial\dot\phi^*}(t,x) = \dot\phi(t,x)
\end{aligned}\tag{11.35}$$

and the classical Hamiltonian for the KG field takes the form (cf. Eq. (11.4))

$$H = \int d^3x\big[\pi^*(t,\vec x)\pi(t,\vec x) + \vec\nabla\phi^*(t,\vec x)\cdot\vec\nabla\phi(t,\vec x) + m^2\phi^*(t,\vec x)\phi(t,\vec x) + \frac{\lambda}{4}\phi^{*2}(t,\vec x)\phi^2(t,\vec x)\big]\tag{11.36}$$

11.2.1 Quantization

For quantization, we repeat the same steps as we did for real KG field (see Sect. 7). As usually, we promote ϕ,ϕ^* and π,π^* to operators

$$\phi(t,\vec x) \to \hat\phi(\vec x), \quad \pi(t,\vec x) \to \hat\pi(\vec x),\tag{11.37}$$

satisfying the canonical commutation relations (11.7)

$$[\hat\phi(\vec x),\hat\pi(\vec y)] = [\hat\phi^\dagger(\vec x),\hat\pi^\dagger(\vec y)] = i\delta(\vec x-\vec y), \quad \text{all other commutators vanish}\tag{11.38}$$

The quantum Hamiltonian is (11.8) plus the interaction term

$$\begin{aligned}
\hat H &= \hat H_0 + \hat H_{\rm int},\\
\hat H_0 &\equiv \int d^3x\big[\hat\pi^\dagger(\vec x)\hat\pi(\vec x) + \vec\nabla\hat\phi^\dagger(\vec x)\cdot\vec\nabla\hat\phi(\vec x) + m^2\hat\phi^\dagger(\vec x)\hat\phi(\vec x)\big], \quad \hat H_{\rm int} = \frac{\lambda}{4}\hat\phi^{\dagger 2}(\vec x)\hat\phi^2(\vec x)
\end{aligned}\tag{11.39}$$

As usual, we define vacuum state $|\Omega\rangle$ as an eigenstate of $\hat H$ with the lowest energy (and suppose it is non-degenerate).

$$\hat H|\Omega\rangle = E_{\rm vac}|\Omega\rangle \quad - \quad \text{stationary Schrodinger equation}\tag{11.40}$$

As we saw in Sect. 9.3, the vacuum energy is a sum of (divergent) vacuum bubbles which cancels in the expressions for physical cross sections

The Heisenberg picture of quantization is constructed in the same way as for real KG field in Sect. 9: we start with $\Psi(\{\phi(\vec{x})\}) = \Psi_{\rm Schro}(t=0,\{\phi(\vec{x})\})$ and define time-dependent canonical operators

$$\begin{aligned}\hat\phi(t,\vec{x}) &\equiv e^{i\hat{H}t}\hat\phi(\vec{x})e^{-i\hat{H}t}, \quad \hat\phi^\dagger(x) \equiv e^{i\hat{H}t}\hat\phi^\dagger(\vec{x})e^{-i\hat{H}t},\\ \hat\pi(t,\vec{x}) &\equiv e^{i\hat{H}t}\hat\pi(\vec{x})e^{-i\hat{H}t}, \quad \hat\pi^\dagger(x) \equiv e^{i\hat{H}t}\hat\pi^\dagger(\vec{x})e^{-i\hat{H}t}\end{aligned} \tag{11.41}$$

satisfying Heisenberg equations

$$\begin{aligned}\frac{d\hat\phi(t,\vec{x})}{dt} &= i[\hat{H},\hat\phi(t,\vec{x})], \qquad \frac{d\hat\pi(t,\vec{x})}{dt} = i[\hat{H},\hat\pi(t,\vec{x})]\\ \frac{d\hat\phi^\dagger(t,\vec{x})}{dt} &= i[\hat{H},\hat\phi^\dagger(t,\vec{x})], \qquad \frac{d\hat\pi^\dagger(t,\vec{x})}{dt} = i[\hat{H},\hat\pi^\dagger(t,\vec{x})])\end{aligned} \tag{11.42}$$

Similarly to Eq. (7.14) one can show that the equal-time commutators are the same as in free theory (see Eq. (11.43)

$$[\hat\phi(t,\vec{x}),\hat\pi(t,\vec{y})] = [\hat\phi^\dagger(t,\vec{x}),\hat\pi^\dagger(t,\vec{y})] = i\delta(\vec{x}-\vec{y}), \tag{11.43}$$
$$[\hat\phi(t,\vec{x}),\hat\phi(t,\vec{y})] = [\hat\phi(t,\vec{x}),\hat\pi^\dagger(t,\vec{y})] = [\hat\pi(t,\vec{x}),\hat\pi(t,\vec{y})] = [\hat\pi(t,\vec{x}),\hat\phi^\dagger(t,\vec{y})] = 0$$

Also, following logic of Sect 7.1.1 one can prove that the quantum operators $\hat\phi(x)$ and $\hat\phi^\dagger(x)$ (given by Eq. (11.41)) satisfy the same non-linear KG equations (11.34) as their classical counterparts

$$(\partial^2+m^2)\hat\phi(x) = -\frac{\lambda}{2}\hat\phi^\dagger(x)\hat\phi^2(x), \qquad (\partial^2+m^2)\hat\phi^\dagger(x) = -\frac{\lambda}{2}\hat\phi^{\dagger 2}(x)\hat\phi(x) \tag{11.44}$$

11.2.2 LSZ theorem

The proof of LSZ theorem repeats Sect. 8. We define set of free in- and out- complex KG fields

$$\begin{aligned}\hat\phi_{\rm in}(x) &\equiv \hat\phi(x) + \frac{i\lambda}{2}\int d^4z\ \ G_R^0(x-z)\ \hat\phi^2(z)\hat\phi^\dagger(z)\\ \hat\phi^\dagger_{\rm in}(x) &\equiv \hat\phi^\dagger(x) + \frac{i}{2}\int d^4z\ \ G_R^0(x-z)\ \hat\phi(z)\hat\phi^{\dagger 2}(z)\\ \hat\phi_{\rm out}(x) &\equiv \hat\phi(x)\ +\ i\frac{\lambda}{2}\int\ d^4z\ G_A^0(x-z)\hat\phi^2(z)\hat\phi^\dagger(z)\\ \hat\phi^\dagger_{\rm out}(x) &\equiv \hat\phi^\dagger(x) + \frac{i\lambda}{2}\int d^4z\ \ G_A^0(x-z)\ \hat\phi(z)\hat\phi^{\dagger 2}(z)\end{aligned} \tag{11.45}$$

and define a set of $|p_1+,p'_1+,...p_1^{(m_+)}+,p_1-,p'_1-,...p_1^{(m_-)}-\rangle_{\rm in}$ states and a set of ${}_{\rm out}\langle p_2+,p'_2+,...p_2^{(n_+)}+,p_2-,p'_2-,...p_2^{(n_-)}-|$ states. Let us consider for example the two-particle scattering of one (+)-particle and one (-)-particle

$$S(p_1+,p'_1-\to p_2+,p'_2) \equiv {}_{\rm out}\langle p_2+,p'_2-|p_1+,p'_1-\rangle_{\rm in} \tag{11.46}$$

Using formulas

$$\sqrt{2E_p}\ \hat{a}_{\rm out}(p) = i\int d^3x\ \ e^{-ip\vec{x}+iE_pt}\ \overleftrightarrow{\partial_0}\ \hat\phi_{\rm out}(x), \qquad \sqrt{2E_p}\ \hat{a}^\dagger_{\rm in}(p) = -i\int d^3x\ \ e^{ip\vec{x}-iE_pt}\ \overleftrightarrow{\partial_0}\ \hat\phi_{\rm in}(x) \tag{11.47}$$

and repeating the derivation from Sect. 8.2 we get

$$
\begin{aligned}
&{}_{\rm out}\langle p_2+,p_2'-|p_1+,p_1'-\rangle_{\rm in} \ = \qquad (11.48)\\
&= \lim_{p_1^2,p_2^2\to m^2}(m^2-p_1^2)(m^2-p_2^2)\ i^2\int d^4x\ d^4y\ e^{-ip_1x+ip_2y}\ {}_{\rm out}\langle p_2'+|{\rm T}\{\hat\phi(x)\hat\phi(y)\}|p_1'+\rangle_{\rm in}
\end{aligned}
$$

To reduce this formula to T-product of four currents we need to use

$$
\begin{aligned}
{}_{\rm out}\langle p_2'+| \ &= \ \sqrt{2E_{p_2'}}\langle 0|\hat b^{\rm out}_{\vec p_2}, \qquad |p_1'+\rangle_{\rm in} \ = \ \sqrt{2E_{p_2'}}(\hat b^{\rm in}_{p_2})^\dagger|0\rangle \qquad (11.49)\\
\sqrt{2E_p}\ \hat b^{\rm out}_{\vec p} \ &= \ i\int d^3x\ e^{-i\vec p\vec x+iE_pt}\ \overleftrightarrow{\partial_0}\ \hat\phi^\dagger_{\rm out}(x) \qquad \sqrt{2E_p}\ \hat b^{\dagger{\rm in}}_{\vec p} \ = \ -i\int d^3x\ e^{i\vec p\vec x-iE_pt}\ \overleftrightarrow{\partial_0}\ \hat\phi^\dagger_{\rm in}(x)
\end{aligned}
$$

and repeat the derivation or Sect. 8.2 again. We get

$$
\begin{aligned}
&{}_{\rm out}\langle p_2'+|{\rm T}\{\hat\phi(x)\hat\phi(y)\}|p_1'+\rangle_{\rm in} \qquad (11.50)\\
&= \lim_{p'^2_1,p'^2_2\to m^2}(m^2-p'^2_1)(m^2-p'^2_2)\ i^2\int d^4x\ d^4y\ e^{-ip_1'x'+ip_2'y'}\ \langle\Omega|{\rm T}\{\hat\phi(x)\hat\phi^\dagger(x')\hat\phi(y)\hat\phi^\dagger(y')\}|\Omega\rangle
\end{aligned}
$$

so the final form of LSZ theorem fot two-particle scattering reads

$$
\begin{aligned}
{}_{\rm out}\langle p_2+,p_2'-|p_1+,p_1'-\rangle_{\rm in} \ &= \ \lim_{p_1^2,p_2^2,p_1'^2,p_2'^2\to m^2}(m^2-p_1^2)(m^2-p_2^2)(m^2-p'^2_1)(m^2-p'^2_2)\\
&\times\ i^2\int d^4x\ d^4y\ e^{-ip_1x-ip_1'x'+ip_2y+ip_2'y'}\ \langle\Omega|{\rm T}\{\hat\phi(x)\hat\phi^\dagger(x')\hat\phi(y)\hat\phi^\dagger(y')\}|\Omega\rangle \qquad (11.51)
\end{aligned}
$$

Thus, the only difference with LSZ for real KG field is that $|p-\rangle$ and $\langle p-|$ states correspond to $\phi^\dagger$ operators (rather than ϕ operators) in Green functions.

For a general scattering process the LSZ formula gives

$$
\begin{aligned}
&S\big(p_1+,p_1'+,...p_1^{(m_+)}+,q_1-,q_1'-,...q_1^{(m_-)}- \ \to \ p_2+,p_2'+,...p_2^{(n_+)}+,q_2-,q_2'-,...q_2^{(n_-)}-\big) \qquad (11.52)\\
&= \ {}_{\rm out}\langle p_2+,p_2'+,...p_2^{(n_+)}+,p_2-,p_2'-,...p_2^{(n_-)}-|p_1+,p_1'+,...p_1^{(m_+)}+,p_1-,p_1'-,...p_1^{(m_-)}-\rangle_{\rm in} =\\
&= \ i^{m_++m_-+n_++n_-}\lim_{p_i^2\to m^2}\Pi(m^2-p_i^2)\int\Pi dx_1^{(i)}\ \Pi dx_2^{(j)}\ \Pi dy_1^{(k)}\ \Pi dy_2^{(l)}\ e^{-i\sum p_1^{(i)}x_1^{(i)}-i\sum q_1^{(k)}y_1^{(k)}+i\sum p_2^{(j)}x_2^{(j)}+i\sum q_2^{(l)}y_2^{(l)}}\\
&\qquad\times\ \langle\Omega|T\{\hat\phi(x_1)\ ...\ \hat\phi(x_1^{(m_+)})\hat\phi^\dagger(y_1)\ ...\ \hat\phi^\dagger(y_1^{(m_-)})\hat\phi(x_2)\ ...\ \hat\phi(x_2^{(n_+)})\hat\phi^\dagger(y_2)\ ...\ \hat\phi^\dagger(y_2^{(n_-)})\}|\Omega\rangle
\end{aligned}
$$

11.2.3 Interaction representation and Feynman diagrams

The construction of interaction representation repeats Sect. 9.1. For example,

$$
\begin{aligned}
\langle\Omega|{\rm T}\{\hat\phi(x)\hat\phi^\dagger(y)\}|\Omega\rangle \ &= \ \lim_{\epsilon\to 0}\lim_{\tau\to\infty}\frac{\langle 0|{\rm T}\big\{e^{i\int_{-T}^{T}dt\int d^3z\ \hat{\mathcal L}(t,\vec z)}\hat\phi_I(x)\hat\phi_I^\dagger(y)\big\}|0\rangle}{\langle 0|{\rm T}\big\{e^{i\int_{-T}^{T}dt\int d^3z\ \hat{\mathcal L}(t,\vec z)}\big\}|0\rangle}\Bigg|_{T=\tau(1-i\epsilon)}\\
&\overset{\text{Feynman poles}}{=}\ \frac{\langle 0|{\rm T}\big\{e^{i\int d^4z\ \hat{\mathcal L}(z)}\hat\phi_I(x)\hat\phi_I^\dagger(y)\big\}|0\rangle}{\langle 0|{\rm T}\big\{e^{i\int d^4z\ \hat{\mathcal L}(z)}\big\}|0\rangle} \qquad (11.53)
\end{aligned}
$$

where

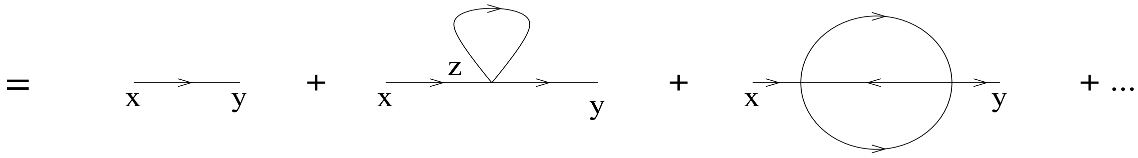
$$
\hat{\mathcal L}_I(z) \ = \ -\frac{\lambda}{4}[\hat\phi_I^\dagger(z)\hat\phi_I(z)]^2 \qquad (11.54)
$$

is the interaction term in the Lagrangian. Wick's theorem is also the same with the only exception: the contraction between $\hat{\phi}$ and $\hat{\phi}$ and between $\hat{\phi}^\dagger$ and $\hat{\phi}^\dagger$ vanishes, see Eq. (11.32)

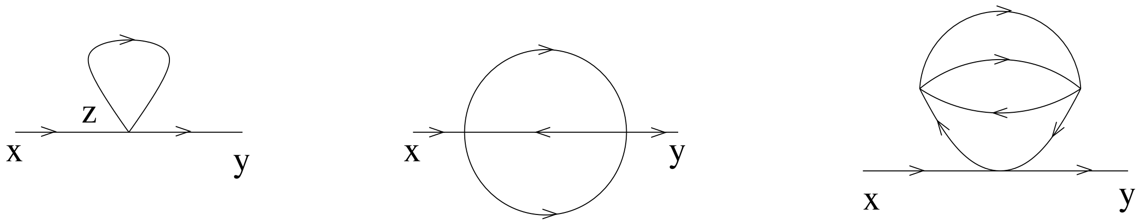
$$\widehat{\hat{\phi}_I(x)\hat{\phi}_I^\dagger}(y) \;=\; D_F(x-y,) \qquad \widehat{\hat{\phi}_I(x)\hat{\phi}_I}(y) \;=\; \widehat{\hat{\phi}_I^\dagger(x)\hat{\phi}_I^\dagger}(y) \;=\; 0 \tag{11.55}$$

As we mentioned, the Green function $\widehat{\hat{\phi}_I(x)\hat{\phi}_I^\dagger}(y)$ is depicted by a line with an arrow, see Fig. 17. The relavant diagrams for 2-point Green function (11.53) look the same as in Eq. (9.67)

$$\langle\Omega|\mathrm{T}\{\hat{\phi}(x)\hat{\phi}^\dagger(y)\}|\Omega\rangle \;=\; \frac{\langle 0|\mathrm{T}\big\{e^{-i\int d^4z\;\frac{\lambda}{4}[\hat{\phi}_I^\dagger(z)\hat{\phi}_I(z)]^2}\,\hat{\phi}_I(x)\hat{\phi}_I^\dagger(y)\big\}|0\rangle}{\langle 0|\mathrm{T}\big\{e^{-i\int d^4z\;\frac{\lambda}{4}[\hat{\phi}_I^\dagger(z)\hat{\phi}_I(z)]^2}\big\}||0\rangle} \tag{11.56}$$



with the exception of symmetry coefficients



S=1 S=2 S=2

Part XI

12 Lorentz transformations

Reminder from E & M:

$$\text{Metric tensor} \quad g^{\mu\nu} \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{12.1}$$

$$\begin{aligned} x'^\mu \;&=\; \Lambda^\mu{}_\nu x^\nu, \qquad \Lambda^\mu{}_\nu \;=\; 4\times 4 \text{ matrix}, \qquad \Lambda_\mu{}^\nu \;\equiv\; g^{\mu\alpha}g_{\nu\beta}\Lambda^\alpha{}_\beta \\ x'^2 \;&=\; x^2 \;\Rightarrow\; \Lambda^\mu{}_\alpha\Lambda_\mu{}^\beta x^\alpha x_\beta \;=\; x^\alpha x_\beta \delta^\alpha_\beta \;\Rightarrow\; \Lambda^\mu{}_\alpha\Lambda_\mu{}^\beta \;=\; \delta^\alpha_\beta \end{aligned} \tag{12.2}$$

In terms of matrices $\Lambda^\mu{}_\nu \;\equiv\; \mathbf{\Lambda}$, $\Lambda_\mu{}^\nu \;\equiv\; \mathbf{g\Lambda g}$

$$g^{\mu\alpha}g_{\nu\beta}\Lambda^\alpha{}_\beta \;\Rightarrow\; \mathbf{\Lambda g\Lambda g} \;=\; \mathbf{1}, \qquad |\det\mathbf{g}| = 1 \;\Rightarrow\; (\det\mathbf{\Lambda})^2 = 1 \;\Rightarrow\; \det\mathbf{\Lambda} = \pm 1 \tag{12.3}$$

Lorentz transformations: rotations, boosts, and 3 discrete Lorentz transformations (P, T, and PT).

12.1 Relativistic invariance in a classical field theory

12.1.1 Scalar field

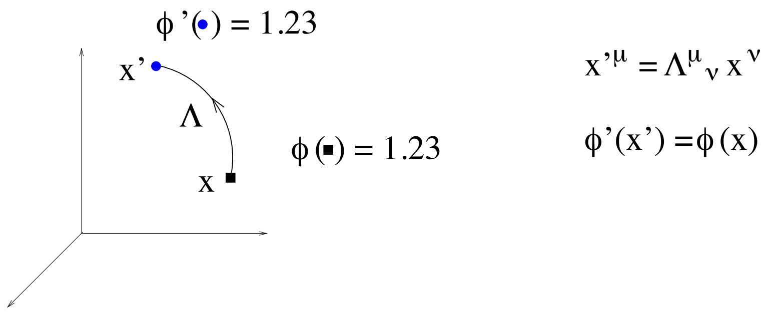


Figure 14. Active Lorentz transformation of a scalar field

Scalar field: $\phi'(x') \;=\; \phi(x)$ (example: temperature in the room)

$$\phi'(x) \;=\; \phi(\mathbf{\Lambda^{-1}}x) \qquad \text{active rotation} \tag{12.4}$$

Action must be relativistic invariant

$$\int d^4x' \; \mathcal{L}(\phi'(x')) \;=\; \int d^4x \; \mathcal{L}(\phi(x)) \tag{12.5}$$

We know that

$$x' \;=\; \mathbf{\Lambda}x \quad \Rightarrow \quad \int d^4x' \;=\; |\det \mathbf{\Lambda}| \int d^4x \;=\; \int d^4x \tag{12.6}$$

so

$$\mathcal{L}(\phi'(x')) \;=\; \mathcal{L}(\phi(x)) \tag{12.7}$$

⇒ Lagrangian density is relativistic invariant.

Let us check relativistic invariance for a classical KG Lagrangian (3.2)

$$\mathcal{L}(\vec{x},t) \;=\; \mathcal{L}\big(\phi(t,\vec{x})\partial_\mu\phi(t,\vec{x})\big) \;=\; \frac{1}{2}\partial_\mu\phi\partial^\mu\phi(x) - \frac{m^2}{2}\phi^2(x) \tag{12.8}$$

For the mass term the invariance is trivial.

For the kinetic term

$$\begin{aligned}\frac{\partial}{\partial x'_\mu}\phi'(x')\frac{\partial}{\partial x'^\mu}\phi'(x') \;&=\; \Big(\Lambda^\mu{}_\alpha\frac{\partial}{\partial x_\alpha}\phi'(x')\Big)\Lambda_\mu{}^\beta\frac{\partial}{\partial x^\beta}\phi'(x') \;=\; \Big(\Lambda^\mu{}_\alpha\frac{\partial}{\partial x_\alpha}\phi(x)\Big)\Lambda_\mu{}^\beta\frac{\partial}{\partial x^\beta}\phi(x) \\ &=\; (\Lambda^\mu{}_\alpha\Lambda_\mu{}^\beta)\frac{\partial}{\partial x_\alpha}\phi(x)\frac{\partial}{\partial x^\beta}\phi(x) \;=\; \delta^\beta_\alpha\frac{\partial}{\partial x_\alpha}\phi(x)\frac{\partial}{\partial x^\beta}\phi(x) \;=\; \frac{\partial}{\partial x_\mu}\phi(x)\frac{\partial}{\partial x^\mu}\phi(x)\end{aligned} \tag{12.9}$$

where we used

$$\frac{\partial}{\partial x'_\mu} \;=\; \Lambda^\mu{}_\nu\frac{\partial}{\partial x_\nu}, \qquad \frac{\partial}{\partial x'^\mu} \;=\; \Lambda_\mu{}^\nu\frac{\partial}{\partial x^\nu} \tag{12.10}$$

which can be proved as follows

$$\begin{aligned}
\frac{\partial}{\partial x'_\mu} x'_\nu &= \Lambda^\mu{}_\alpha \frac{\partial}{\partial x_\alpha}(\Lambda_\nu{}^\beta x_\beta) = \Lambda^\mu{}_\alpha \Lambda_\nu{}^\alpha = \delta^\mu_\nu \\
\frac{\partial}{\partial x'^\mu} x'^\nu &= \Lambda_\mu{}^\alpha \frac{\partial}{\partial x^\alpha}(\Lambda^\nu{}_\beta x^\beta) = \Lambda_\mu{}^\alpha \Lambda^\nu{}_\alpha = \delta^\nu_\mu
\end{aligned} \tag{12.11}$$

(The self-interacting KG Lagrangian $\frac{1}{2}\partial_\mu\phi\partial^\mu\phi(x) - \frac{m^2}{2}\phi^2(x) - \frac{\lambda}{4!}\phi^4(x)$ is also evidently relativistic invariant.)

Let us also check explicitly the relativistic invariance of classical KG equation:

$$\frac{\partial}{\partial x'_\mu}\frac{\partial}{\partial x'^\mu}\phi'(x') = \Lambda^\mu{}_\alpha \frac{\partial}{\partial x_\alpha}\Lambda_\mu{}^\beta \frac{\partial}{\partial x^\beta}\phi'(x') = \Lambda^\mu{}_\alpha \frac{\partial}{\partial x_\alpha}\Lambda_\mu{}^\beta \frac{\partial}{\partial x^\beta}\phi(x) \tag{12.12}$$

$$= \Lambda^\mu{}_\alpha \Lambda_\mu{}^\beta \frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x^\beta}\phi(x) = \frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x^\alpha}\phi(x) \quad \Rightarrow \quad (\partial'^2 + m^2)\phi'(x') = (\partial^2 + m^2)\phi(x)$$

$\Rightarrow$ form of the equation does not depend on frame.

12.1.2 Generators of Lorentz transformations

We will demonstrate that the differential operator

$$e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}, \qquad J^{\mu\nu} \equiv i(x^\mu \frac{\partial}{\partial x_\nu} - \mu \leftrightarrow \nu) \tag{12.13}$$

generates Lorentz transformations

$$e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}\phi(x) = \phi'(x) = \phi(\mathbf{\Lambda^{-1}}x) \tag{12.14}$$

where the matrix of Lorentz transformation $\mathbf{\Lambda}$ is completely determined by 6 parameters $\omega_{\mu\nu}$ corresponding to three boosts and three rotations.

Let us demonstrate that for boosts using the boost in x direction as an example. The matrix of this boost is [7]

$$\mathbf{\Lambda}(\theta) = \begin{pmatrix} \cosh\theta & \sinh\theta & 0 & 0 \\ \sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \mathbf{\Lambda^{-1}}(\theta) = \mathbf{\Lambda}(-\theta) = \begin{pmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{12.15}$$

Let us make an (educated) guess: $\omega_{01} = \theta, \omega_{10} = -\theta$, all other elements vanish

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & \theta & 0 & 0 \\ -\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{12.16}$$

We must prove that

$$e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}\phi(x) = e^{-i\omega_{01}J^{01}}\phi(x) = \phi(x^0\cosh\theta - x^1\sinh\theta, -x^0\sinh\theta + x^1\cosh\theta, x^2, x^3) \tag{12.17}$$

[7] This is the matrix of an active Lorentz boost in positive x direction. Note that all textbooks on Special Relativity present the matrix of passive Lorentz boost ($\equiv$ Lorentz boost of a frame) which differs in sign of θ from Eq. (12.15).

Proof: let us rewrite this equation as

$$e^{\theta\left(x^0\frac{\partial}{\partial x_1}-x^1\frac{\partial}{\partial x_0}\right)}\phi(x) \;=\; \phi(x^0\cosh\theta - x^1\sinh\theta, -x^0\sinh\theta + x^1\cosh\theta, x^2, x^3), \qquad (12.18)$$

differentiate l.h.s. and r.h.s. with respect to θ, and compare the differential equations.

$$\frac{d}{d\theta}(\text{l.h.s.}) \;=\; \big(x^0\frac{\partial}{\partial x_1} - x^1\frac{\partial}{\partial x_0}\big)(\text{l.h.s.}) \qquad (12.19)$$

$$\begin{aligned}
\frac{d}{d\theta}(\text{r.h.s.}) \;&=\; \frac{d}{d\theta}\phi(x^0\cosh\theta - x^1\sinh\theta, -x^0\sinh\theta + x^1\cosh\theta, x^2, x^3) \qquad (12.20)\\
&= \frac{d}{d\theta}(x^0\cosh\theta - x^1\sinh\theta)\frac{\partial}{\partial X^0}\;\phi(X^0,X^1,x^2,x^3)\big|_{X^0=x^0\cosh\theta-x^1\sinh\theta,X^1=-x^0\sinh\theta+x^1\cosh\theta}\\
&+ \frac{d}{d\theta}(-x^0\sinh\theta + x^1\cosh\theta)\frac{\partial}{\partial X^1}\;\phi(X^0,X^1,x^2,x^3)\big|_{X^0=x^0\cosh\theta-x^1\sinh\theta,X^1=-x^0\sinh\theta+x^1\cosh\theta}\\
&= (x^0\sinh\theta - x^1\cosh\theta)\frac{\partial}{\partial X^0}\;\phi(X^0,X^1,x^2,x^3)\big|_{X^0=x^0\cosh\theta-x^1\sinh\theta,X^1=-x^0\sinh\theta+x^1\cosh\theta}\\
&+ (-x^0\cosh\theta + x^1\sinh\theta)\frac{\partial}{\partial X^1}\;\phi(X^0,X^1,x^2,x^3)\big|_{X^0=x^0\cosh\theta-x^1\sinh\theta,X^1=-x^0\sinh\theta+x^1\cosh\theta}
\end{aligned}$$

Now

$$\begin{aligned}
\frac{d}{dx^0}(\text{r.h.s.}) \;&=\; \frac{d}{dx^0}\phi(x^0\cosh\theta - x^1\sinh\theta, -x^0\sinh\theta + x^1\cosh\theta, x^2, x^3) \qquad (12.21)\\
&= \frac{d}{dx^0}(x^0\cosh\theta - x^1\sinh\theta)\frac{\partial}{\partial X^0}\;\phi(X^0,X^1,x^2,x^3)\big|_{X^0=x^0\cosh\theta-x^1\sinh\theta,X^1=-x^0\sinh\theta+x^1\cosh\theta}\\
&+ \frac{d}{dx^0}(-x^0\sinh\theta + x^1\cosh\theta)\frac{\partial}{\partial X^1}\;\phi(X^0,X^1,x^2,x^3)\big|_{X^0=x^0\cosh\theta-x^1\sinh\theta,X^1=-x^0\sinh\theta+x^1\cosh\theta}\\
&= \cosh\theta\frac{\partial}{\partial X^0}\;\phi(X^0,X^1,x^2,x^3)\big|_{X^0=x^0\cosh\theta-x^1\sinh\theta,X^1=-x^0\sinh\theta+x^1\cosh\theta}\\
&- \sinh\theta\frac{\partial}{\partial X^1}\;\phi(X^0,X^1,x^2,x^3)\big|_{X^0=x^0\cosh\theta-x^1\sinh\theta,X^1=-x^0\sinh\theta+x^1\cosh\theta}
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dx_1}(\text{r.h.s.}) \;&=\; -\frac{d}{dx^1}(\text{r.h.s.}) \;=\; -\frac{d}{dx^1}\phi(x^0\cosh\theta - x^1\sinh\theta, -x^0\sinh\theta + x^1\cosh\theta, x^2, x^3)\\
&= \sinh\theta\frac{\partial}{\partial X^0}\;\phi(X^0,X^1,x^2,x^3)\big|_{X^0=x^0\cosh\theta-x^1\sinh\theta,X^1=-x^0\sinh\theta+x^1\cosh\theta}\\
&- \cosh\theta\frac{\partial}{\partial X^1}\;\phi(X^0,X^1,x^2,x^3)\big|_{X^0=x^0\cosh\theta-x^1\sinh\theta,X^1=-x^0\sinh\theta+x^1\cosh\theta} \qquad (12.22)
\end{aligned}$$

and therefore

$$
\begin{aligned}
& x^0 \frac{d}{dx_1}(\text{r.h.s.}) - x^1 \frac{\partial}{\partial x^0}(\text{r.h.s.}) \ = \\
&= \ x^0 \Big(\sinh\theta \frac{\partial}{\partial X^0}\ \phi(X^0, X^1, x^2, x^3)\big|_{X^0, X^1 = \dots} - \cosh\theta \frac{\partial}{\partial X^1}\ \phi(X^0, X^1, x^2, x^3)\Big)\Big|_{X^0, X^1 = \dots} \\
&- \ x^1 \Big(\cosh\theta \frac{\partial}{\partial X^0}\ \phi(X^0, X^1, x^2, x^3)\big|_{X^0, X^1 = \dots} - \ \sinh\theta \frac{\partial}{\partial X^1}\ \phi(X^0, X^1, x^2, x^3)\Big)\Big|_{X^0, X^1 = \dots} \\
&= \ (x^0 \sinh\theta - x^1 \cosh\theta) \frac{\partial}{\partial X^0}\ \phi(X^0, X^1, x^2, x^3)\big|_{X^0, X^1 = \dots} \\
& \qquad\qquad + (-x^0 \cosh\theta + x^1 \sinh\theta) \frac{\partial}{\partial X^1}\ \phi(X^0, X^1, x^2, x^3)\big|_{X^0, X^1 = \dots} \\
&= \ \text{r.h.s. of Eq. (12.20)} \ = \ \frac{d}{d\theta}(\text{r.h.s.})
\end{aligned}
\tag{12.23}
$$

We see that the differential equations (12.19) for l.h.s. and (12.23) for r.h.s. coincide. Also, the initial conditions at $\theta = 0$ are equal, and therefore the l.h.s. of Eq. (12.18) is equal to the r.h.s.

Let us now consider rotations, for example the rotation around z axis with the matrix

$$
\mathbf{\Lambda}(\varphi) \ = \ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi & 0 \\ 0 & \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ \Rightarrow \ \mathbf{\Lambda^{-1}}(\varphi) \ = \ \mathbf{\Lambda}(-\varphi) \ = \ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\tag{12.24}
$$

In a similar way one can prove that the matrix

$$
\omega_{\mu\nu} \ = \ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \varphi & 0 \\ 0 & -\varphi & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\tag{12.25}
$$

generates rotation on angle φ around z axis:

$$
\begin{aligned}
& e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}\phi(x) \ = \ e^{-i\omega_{12}J^{12}}\phi(x) \ = \ e^{\varphi\left(x^1 \frac{\partial}{\partial x_2} - x^2 \frac{\partial}{\partial x_1}\right)}\phi(x) \ = \ e^{\varphi\left(x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}\right)}\phi(x) \\
& = \ \phi(x^0, x^1\cos\varphi + x^2\sin\varphi, -x^1\sin\varphi + x^2\cos\varphi, x^3)
\end{aligned}
\tag{12.26}
$$

For an arbitrary Lorentz transformation formula (12.14) still holds true but the relation between $\omega_{\mu\nu}$ and $\Lambda^{\mu}{}_{\nu}$ is more complicated .

Commutation relations between generators of Lorentz transformations

$$
\begin{aligned}
[P^\mu, J^{\lambda\rho}] \ &= \ ig^{\mu\lambda}P^\rho - ig^{\mu\rho}P^\lambda \\
[J^{\mu\nu}, J^{\lambda\rho}] \ &= \ -i\big(g^{\mu\lambda}J^{\nu\rho} - g^{\mu\rho}J^{\nu\lambda} - g^{\nu\lambda}J^{\mu\rho} + g^{\nu\rho}J^{\mu\lambda}\big)
\end{aligned}
\tag{12.27}
$$

$(P^\mu\phi(x) \;=\; i\partial^\mu\phi(x)$, see Eq. (6.29)) Proof:

$$
\begin{aligned}
&[J^{\mu\nu}, J^{\lambda\rho}] \\
&= i^2\Big(x^\mu\frac{\partial}{\partial x_\nu}x^\lambda\frac{\partial}{\partial x_\rho} - x^\lambda\frac{\partial}{\partial x_\rho}x^\mu\frac{\partial}{\partial x_\nu}\Big) \;-\; \mu\leftrightarrow\nu \;-\; \lambda\leftrightarrow\rho \\
&= \;-\Big(x^\mu x^\lambda\frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x_\rho} + g^{\nu\lambda}x^\mu\frac{\partial}{\partial x_\rho} - x^\mu x^\lambda\frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x_\rho} - g^{\mu\rho}x^\lambda\frac{\partial}{\partial x_\nu}\Big) \;-\; \mu\leftrightarrow\nu \;-\; \lambda\leftrightarrow\rho \\
&= g^{\mu\rho}x^\lambda\frac{\partial}{\partial x_\nu} - g^{\nu\lambda}x^\mu\frac{\partial}{\partial x_\rho} \;-\; \mu\leftrightarrow\nu \;-\; \lambda\leftrightarrow\rho \\
&= g^{\mu\rho}x^\lambda\frac{\partial}{\partial x_\nu} - g^{\nu\lambda}x^\mu\frac{\partial}{\partial x_\rho} - g^{\nu\rho}x^\lambda\frac{\partial}{\partial x_\mu} + g^{\mu\lambda}x^\nu\frac{\partial}{\partial x_\rho} - g^{\mu\lambda}x^\rho\frac{\partial}{\partial x_\nu} + g^{\nu\rho}x^\mu\frac{\partial}{\partial x_\lambda} + g^{\nu\lambda}x^\rho\frac{\partial}{\partial x_\mu} - g^{\mu\rho}x^\nu\frac{\partial}{\partial x_\lambda} \\
&= ig^{\mu\lambda}i\big(x^\rho\frac{\partial}{\partial x_\nu} - x^\nu\frac{\partial}{\partial x_\rho}\big) + ig^{\mu\rho}i\big(x^\nu\frac{\partial}{\partial x_\lambda} - x^\lambda\frac{\partial}{\partial x_\nu}\big) + ig^{\nu\lambda}i\big(x^\mu\frac{\partial}{\partial x_\rho} - x^\rho\frac{\partial}{\partial x_\mu}\big) + ig^{\nu\rho}i\big(x^\lambda\frac{\partial}{\partial x_\mu} - x^\mu\frac{\partial}{\partial x_\lambda}\big) \\
&= \;-ig^{\mu\lambda}J^{\nu\rho} + ig^{\mu\rho}J^{\nu\lambda} + ig^{\nu\lambda}J^{\mu\rho} - ig^{\nu\rho}J^{\mu\lambda}
\end{aligned}
\tag{12.28}
$$

12.2 Vector field

Vector field $V^\mu(x)$: the field which transforms like the coordinate. Example: $V^\mu(x) = x^\mu\Phi(x)$ where $\Phi(x)$ is a scalar field.

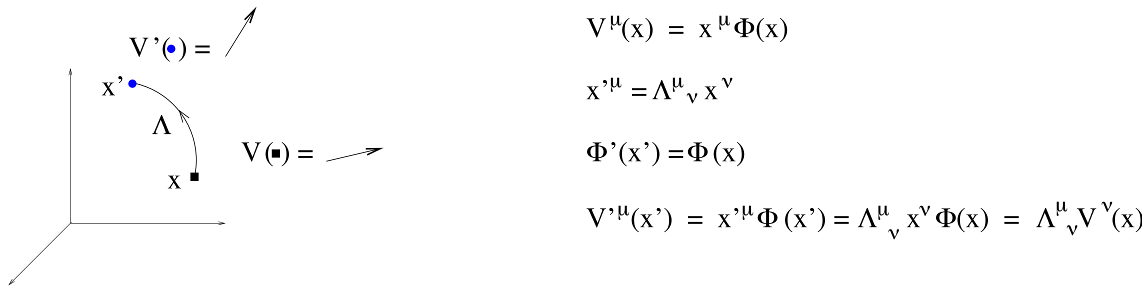


Figure 15. Active Lorentz transformation of vector field $V^\mu(x) = x^\mu\Phi(x)$

In general

$$V'^\mu(x') \;=\; \Lambda^\mu{}_\nu V^\nu(x), \qquad \text{cf.} \qquad x'^\mu \;=\; \Lambda^\mu{}_\nu x^\nu \tag{12.29}$$

Equivalent representation

$$V'^\mu(x) \;=\; \Lambda^\mu{}_\nu V^\nu(\mathbf{\Lambda}^{-1}x) \tag{12.30}$$

Example: electromagnetic field $A^\mu(x)$.

Let us prove that Maxwell's equations

$$\frac{\partial}{\partial x_\mu}F_{\mu\nu}(x) \;=\; j_\nu(x), \qquad F_{\mu\nu} \;=\; \frac{\partial}{\partial x^\mu}A_\nu \;-\; \mu\leftrightarrow\nu \tag{12.31}$$

are relativistic invariant, i.e. in after Lorentz transformation

$$A'^\mu(x') \;=\; \Lambda^\mu{}_\nu A^\nu(x) \tag{12.32}$$

they look the same:

$$\frac{\partial}{\partial x'_\mu} F_{\mu\nu}(x') \;=\; j_\nu(x'). \tag{12.33}$$

First,let us rewrite Eq. (12.10)

$$\frac{\partial}{\partial x'_\mu} \;=\; \Lambda^\mu{}_\nu \frac{\partial}{\partial x_\nu}, \qquad \frac{\partial}{\partial x'^\mu} \;=\; \Lambda_\mu{}^\nu \frac{\partial}{\partial x^\nu} \tag{12.34}$$

From this equation and Eq. (12.32) we see that

$$\begin{aligned} \frac{\partial}{\partial x'_\mu} A'{}_\nu(x') \;&=\; \Lambda_\mu{}^\lambda \frac{\partial}{\partial x_\lambda} \Lambda_\nu{}^\rho A_\rho(x) \;=\; \Lambda_\mu{}^\lambda \Lambda_\nu{}^\rho \frac{\partial}{\partial x_\lambda} A_\rho(x) \\ \Rightarrow\; F_{\mu\nu}(x') \;&=\; \Lambda_\mu{}^\lambda \Lambda_\nu{}^\rho F_{\lambda\rho}(x) \\ \Rightarrow\; \frac{\partial}{\partial x'_\mu} F_{\mu\nu}(x') \;&=\; \Lambda^\mu{}_\sigma \Lambda_\mu{}^\lambda \Lambda_\nu{}^\rho \frac{\partial}{\partial x_\sigma} F_{\lambda\rho}(x) \;=\; \Lambda_\nu{}^\rho \frac{\partial}{\partial x_\lambda} F_{\lambda\rho}(x) \end{aligned} \tag{12.35}$$

The current $j_\mu(x)$ is also 4-vector so it transforms like Eq. (12.32)

$$j'_\nu(x') \;=\; \Lambda_\nu{}^\rho j_\rho(x) \tag{12.36}$$

and therefore

$$\frac{\partial}{\partial x'_\mu} F_{\mu\nu}(x') - j'_\nu(x') \;=\; \Lambda_\nu{}^\rho \Big[\frac{\partial}{\partial x_\lambda} F_{\lambda\rho}(x) - j_\rho(x)\Big] \;=\; 0 \tag{12.37}$$

so Maxwell's equations retain the form (12.33) after Lorentz transformation.

12.2.1 Generators of Lorentz transformations for vector field

Let us prove that

$$V'^\mu(x') \;=\; \big(e^{-\frac{i}{2}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta}}\big)^\mu{}_\nu V^\nu(x) \tag{12.38}$$

where

$$\big(\mathcal{J}^{\alpha\beta}\big)_{\mu\nu} \;=\; i\big(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu\big) \quad\Leftrightarrow\quad \big(\mathcal{J}^{\alpha\beta}\big)^\mu{}_\nu \;=\; i\big(g^{\mu\alpha}\delta^\beta_\nu - g^{\mu\beta}\delta^\alpha_\nu\big) \tag{12.39}$$

Let us illustrate that for the boost in x direction. The vector field transforms like the coordinate (see Eq. (12.15))

$$x'^\mu \;=\; \Lambda^\mu{}_\nu x^\nu \quad\Leftrightarrow\quad \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} \;=\; \begin{pmatrix} \cosh\theta & \sinh\theta & 0 & 0 \\ \sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{12.40}$$

(μ=column, ν=row) so the boost of vector field looks like

$$V'^\mu(x') \;=\; \Lambda^\mu{}_\nu V^\nu(x) \quad\Leftrightarrow\quad \begin{pmatrix} V'^0(x') \\ V'^1(x') \\ V'^2(x') \\ V'^3(x') \end{pmatrix} \;=\; \begin{pmatrix} \cosh\theta & \sinh\theta & 0 & 0 \\ \sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V^0(x) \\ V^1(x) \\ V^2(x) \\ V^3(x) \end{pmatrix} \tag{12.41}$$

Let us check that the Lorentz boost (12.41) is described by matrix (12.39) with ω given by Eq. (13.44) ($\omega_{01}=\theta, \omega_{10}=-\theta$, all other elements vanish)

$$
\begin{aligned}
&(e^{-\frac{i}{2}(\omega_{01}\mathcal{J}^{01}+\omega_{10}\mathcal{J}^{10})})^{\mu}{}_{\nu} ~=~ (e^{-i\omega_{01}\mathcal{J}^{01}})^{\mu}{}_{\nu} ~=~ (1-i\omega_{01}\mathcal{J}^{01}-\frac{1}{2}\omega_{01}^2\mathcal{J}^{01}\mathcal{J}^{01}-\frac{i^3}{6}\omega_{01}^3.\mathcal{J}^{01}\mathcal{J}^{01}\mathcal{J}^{01} ~+...)^{\mu}{}_{\nu} \\
&=~ \delta^{\nu}_{\mu}-i\theta(\mathcal{J}^{01})^{\mu}{}_{\nu}-\frac{\theta^2}{2}(\mathcal{J}^{01})^{\mu}{}_{\alpha}(\mathcal{J}^{01})^{\alpha}{}_{\nu}+\frac{i\theta^3}{6}(\mathcal{J}^{01})^{\mu}{}_{\alpha}(\mathcal{J}^{01})^{\alpha}{}_{\beta}(\mathcal{J}^{01})^{\beta}{}_{\nu} ~+~ ...
\end{aligned}
\tag{12.42}
$$

From Eq. (12.39) we see that

$$
(\mathcal{J}^{01})^{\mu}{}_{\alpha}(\mathcal{J}^{01})^{\alpha}{}_{\nu} ~=~ -\big(g^{0\mu}\delta^1_\alpha-g^{1\mu}\delta^0_\alpha\big)\big(g^{0\alpha}\delta^1_\nu-g^{1\alpha}\delta^0_\nu\big) ~=~ -g^{0\mu}\delta^0_\nu+g^{1\mu}\delta^1_\nu ~=~ -\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}
\tag{12.43}
$$

so the matrices $\big((\mathcal{J}^{01})^n\big)^{\mu}{}_{\nu}$ are

$$
\mathcal{J}^{\mathbf{01}} ~=~ i\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \quad (\mathcal{J}^{\mathbf{01}})^2 ~=~ -\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \quad (\mathcal{J}^{\mathbf{01}})^3 ~=~ -i\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}
\tag{12.44}
$$

and therefore

$$
\mathbf{e}^{-\mathbf{i}\omega_{\mathbf{01}}\mathcal{J}^{\mathbf{01}}} ~=~ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}+\begin{pmatrix}0&\theta&0&0\\\theta&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}+\begin{pmatrix}\frac{\theta^2}{2}&0&0&0\\0&\frac{\theta^2}{2}&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}+\begin{pmatrix}0&\frac{\theta^3}{6}&0&0\\\frac{\theta^3}{6}&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}+... ~=~ \begin{pmatrix}\cosh\theta&\sinh\theta&0&0\\\sinh\theta&\cosh\theta&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}
\tag{12.45}
$$

Thus,

$$
V'^{\mu}(x') ~=~ \big(e^{-i\omega_{01}\mathcal{J}^{01}}\big)^{\mu}{}_{\nu}V^{\nu}(x) \quad\Leftrightarrow\quad \begin{pmatrix}V'^0\\V'^1\\V'^2\\V'^3\end{pmatrix} ~=~ \begin{pmatrix}\cosh\theta&\sinh\theta&0&0\\\sinh\theta&\cosh\theta&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}V^0\\V^1\\V^2\\V^3\end{pmatrix}
\tag{12.46}
$$

which is a Lorentz boost of a vector field (12.41).

NB: We have proved that for the Lorentz boost in x direction

$$
(\mathbf{\Lambda})^{\mu}{}_{\nu} ~=~ \begin{pmatrix}\cosh\theta&\sinh\theta&0&0\\\sinh\theta&\cosh\theta&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} ~=~ \Big(\mathbf{e}^{-\mathbf{i}\omega_{\mathbf{01}}\mathcal{J}^{\mathbf{01}}}\Big)^{\mu}{}_{\nu}
\tag{12.47}
$$

Similarly, one can prove that in general

$$
(\mathbf{\Lambda})^{\mu}{}_{\nu} ~=~ \Big(\mathbf{e}^{-\frac{\mathbf{i}}{\mathbf{2}}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta}}\Big)^{\mu}{}_{\nu}
\tag{12.48}
$$

but the relation between matrix elements of $\mathbf{\Lambda}$ and $\omega_{\alpha\beta}$ is more complicated.

Combining the formulas (12.14) and (12.48) we get

$$
V'^{\mu}(x) ~=~ \Lambda^{\mu}_{\ \nu}V^{\nu}(\mathbf{\Lambda}^{-1}x) ~=~ \big(e^{-\frac{i}{2}\omega_{\lambda\rho}\mathcal{J}^{\lambda\rho}}\big)^{\mu}{}_{\nu}V^{\nu}(\mathbf{\Lambda}^{-1}x) ~=~ \big(e^{-\frac{i}{2}\omega_{\lambda\rho}\mathcal{J}^{\lambda\rho}}\big)^{\mu}{}_{\nu}e^{-\frac{i}{2}\omega_{\lambda\rho}J^{\lambda\rho}}V^{\nu}(x)
\tag{12.49}
$$

The first (differential) operator shifts the argument x of vector field to $\mathbf{\Lambda^{-1}}x$ and the second (matrix) operator makes boost or rotation (or combination thereof) of vector field.

12.2.2 Lorentz transformation for a general field

For a general field Φ_a (a is some index) the generator of Lorentz transformation is

$$\Phi'_a(x') \;=\; \mathbf{M}_{ab}(\Lambda)\Phi_b(x) \quad\Leftrightarrow\quad \Phi'_a(x) \;=\; \mathbf{M}_{ab}(\Lambda)\Phi_b(\mathbf{\Lambda^{-1}}x) \tag{12.50}$$

Mathematically, $M_{ab}(\Lambda)$ is a representation of Lorentz group. Indeed, let us consider two successive Lorentz transformations Λ_1 and Λ_2:

$$\begin{aligned}
x' = \mathbf{\Lambda_1}x \quad&\Rightarrow\quad \Phi'_b(x') \;=\; \mathbf{M}_{bc}(\mathbf{\Lambda_1})\Phi_c(x)\\
x'' = \mathbf{\Lambda_2}x' \quad&\Rightarrow\quad \Phi''_a(x') \;=\; \mathbf{M}_{bc}(\mathbf{\Lambda_2})\Phi'_c(x')\\
x'' = \mathbf{\Lambda_2}\mathbf{\Lambda_1}x \quad&\Rightarrow\quad \Phi''_a(x) \;=\; \mathbf{M}_{ab}(\mathbf{\Lambda_2})\mathbf{M}_{bc}(\mathbf{\Lambda_1})\Phi_c(x)
\end{aligned}\tag{12.51}$$

On the other hand

$$x'' = \mathbf{\Lambda_2}\mathbf{\Lambda_1}x \quad\Rightarrow\quad \Phi''_a(x) \;=\; \mathbf{M}_{ac}(\mathbf{\Lambda_2}\mathbf{\Lambda_1})\Phi_c(x) \tag{12.52}$$

and therefore

$$\mathbf{M}_{ac}(\mathbf{\Lambda_2}\mathbf{\Lambda_1}) \;=\; \mathbf{M}_{ab}(\mathbf{\Lambda_2})\mathbf{M}_{bc}(\mathbf{\Lambda_1}) \tag{12.53}$$

so matrices $\mathbf{M}_{ab}(\mathbf{\Lambda})$ form a representation of the Lorentz group.

In terms of generators

$$\mathbf{M}(\mathbf{\Lambda}) \;=\; e^{-\frac{i}{2}\omega_{\alpha\beta}\mathbf{M}^{\alpha\beta}} \tag{12.54}$$

where matrices $\mathbf{M}^{\alpha\beta}$ are the generators of this representation (and, as we discussed, the 6 papameters $\omega_{\alpha\beta}$ define Lorentz transformation Λ). Mathematically, the matrices $\mathbf{M}^{\alpha\beta}$ form Lie algebra of the Lie group $\mathbf{M}_{ab}(\mathbf{\Lambda})$.

Commutation relations between generators are the same of all representations

$$\left[\mathbf{M}^{\mu\nu},\mathbf{M}^{\lambda\rho}\right] \;=\; -i\big(g^{\mu\lambda}\mathbf{M}^{\nu\rho} - g^{\nu\lambda}\mathbf{M}^{\mu\rho} - g^{\mu\rho}\mathbf{M}^{\nu\lambda} + g^{\nu\rho}\mathbf{M}^{\mu\lambda}\big) \tag{12.55}$$

For example, it is easy to check that

$$\left[\mathcal{J}^{\mu\nu},\mathcal{J}^{\lambda\rho}\right] \;=\; -i\big(g^{\mu\lambda}\mathcal{J}^{\nu\rho} - g^{\nu\lambda}\mathcal{J}^{\mu\rho} - g^{\mu\rho}\mathcal{J}^{\nu\lambda} + g^{\nu\rho}\mathcal{J}^{\mu\lambda}\big) \tag{12.56}$$

Check:

$$\begin{aligned}
&\left[\mathcal{J}^{\mu\nu},\mathcal{J}^{\lambda\rho}\right]^{\xi}{}_{\eta} \;=\; (\mathcal{J}^{\mu\nu})^{\xi}{}_{\sigma}(\mathcal{J}^{\lambda\rho})^{\sigma}{}_{\eta} - (\mathcal{J}^{\lambda\rho})^{\xi}{}_{\sigma}(\mathcal{J}^{\mu\nu})^{\sigma}{}_{\eta}\\
&=\; -\,(g^{\mu\xi}\delta^{\nu}_{\sigma} - \mu\leftrightarrow\nu)(g^{\lambda\sigma}\delta^{\rho}_{\eta} - \lambda\leftrightarrow\rho) + (g^{\lambda\xi}\delta^{\rho}_{\sigma} - \lambda\leftrightarrow\rho)(g^{\mu\sigma}\delta^{\nu}_{\eta} - \mu\leftrightarrow\nu)\\
&=\; g^{\mu\rho}g^{\lambda\xi}\delta^{\eta}_{\nu} - g^{\mu\xi}g^{\nu\lambda}\delta^{\rho}_{\eta} \;-\; \mu\leftrightarrow\nu \;-\; \lambda\leftrightarrow\rho \;=\; g^{\mu\rho}g^{\lambda\xi}\delta^{\eta}_{\nu} - g^{\nu\xi}g^{\mu\rho}\delta^{\lambda}_{\eta} \;-\; \mu\leftrightarrow\nu \;-\; \lambda\leftrightarrow\rho\\
&=\; ig^{\mu\rho}i\big(-g^{\lambda\xi}\delta^{\eta}_{\nu} + g^{\nu\xi}\delta^{\lambda}_{\eta}\big) \;-\; \mu\leftrightarrow\nu \;-\; \lambda\leftrightarrow\rho\\
&=\; i\big(g^{\mu\rho}\mathcal{J}^{\nu\lambda}\big)^{\xi}{}_{\eta} \;-\; \mu\leftrightarrow\nu \;-\; \lambda\leftrightarrow\rho \;=\; \text{r.h.s. of Eq. (12.56)}
\end{aligned}\tag{12.57}$$

For the proof of general Eq. (12.55) we need a mathematical formula: for any two operators A and B

$$e^Ae^B \;=\; e^{A+B+\frac{1}{2}[A,B]}(1 + O([A,[A,B]] + O([B,[B,A]]) \tag{12.58}$$

Proof of Eq. (12.55): consider two successive infinitesimal Lorentz transformations with matrices $\mathbf{\Lambda}_1 \simeq 1 + \delta\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2 \simeq 1 + \delta\mathbf{\Lambda}_2$, compute

$$\mathbf{M}(\mathbf{\Lambda}_1\mathbf{\Lambda}_2) - \mathbf{M}(\mathbf{\Lambda}_2\mathbf{\Lambda}_1) \; = \; \mathbf{M}(\mathbf{\Lambda}_1)\mathbf{M}(\mathbf{\Lambda}_2) - \mathbf{M}(\mathbf{\Lambda}_2)\mathbf{M}(\mathbf{\Lambda}_1) \tag{12.59}$$

and compare the l.h.s. and the r.h.s of this equation.

For infinitesimal Lorentz transformations

$$\begin{aligned}(\Lambda_1)^\mu_{\;\;\nu} \; &= \; \big(e^{-\frac{i}{2}\omega_1^{\alpha\beta}\mathcal{J}_{\alpha\beta}}\big)^\mu_{\;\;\nu} \; \simeq \; \delta^\mu_\nu - \frac{i}{2}\omega_1^{\alpha\beta}\big(\mathcal{J}_{\alpha\beta}\big)^\mu_{\;\;\nu} \\ (\Lambda_2)^\mu_{\;\;\nu} \; &= \; \big(e^{-\frac{i}{2}\omega_2^{\alpha\beta}\mathcal{J}_{\alpha\beta}}\big)^\mu_{\;\;\nu} \; \simeq \; \delta^\mu_\nu - \frac{i}{2}\omega_2^{\alpha\beta}\big(\mathcal{J}_{\alpha\beta}\big)^\mu_{\;\;\nu}\end{aligned} \tag{12.60}$$

we get (we keep terms up to ω^2)

$$\begin{aligned}(\Lambda_1\Lambda_2)^\mu_{\;\;\nu} \; &= \; \big(e^{-\frac{i}{2}\omega_1^{\alpha\beta}\mathcal{J}_{\alpha\beta}}\big)^\mu_{\;\;\xi}\big(e^{-\frac{i}{2}\omega_2^{\lambda\rho}\mathcal{J}_{\lambda\rho}}\big)^\xi_{\;\;\nu} \; = \; \big(e^{-\frac{i}{2}(\omega_1+\omega_2)^{\alpha\beta}\mathcal{J}_{\alpha\beta}-\frac{1}{8}\omega_1^{\alpha\beta}\omega_2^{\lambda\rho}[\mathcal{J}_{\alpha\beta},\mathcal{J}_{\lambda\rho}]}\big)^\mu_{\;\;\nu}\big(1+O(\omega^3)\big) \\ = \; &\big(e^{-\frac{i}{2}(\omega_1+\omega_2)^{\alpha\beta}\mathcal{J}_{\alpha\beta}+\frac{i}{2}g_{\alpha\lambda}\omega_1^{\alpha\beta}\omega_2^{\lambda\rho}\mathcal{J}_{\beta\rho}}\big)^\mu_{\;\;\nu}\big(1+O(\omega^3)\big) \; = \; e^{-\frac{i}{2}(\omega_1^{\alpha\beta}+\omega_2^{\alpha\beta}-g_{\xi\eta}\omega_1^{\alpha\xi}\omega_2^{\beta\eta})\mathcal{J}_{\alpha\beta}}\big(1+O(\omega^3)\big)\end{aligned} \tag{12.61}$$

From this equation we see that

$$\begin{aligned}\mathbf{M}(\mathbf{\Lambda}_1\mathbf{\Lambda}_2) \; &= \; e^{-\frac{i}{2}(\omega_1^{\alpha\beta}+\omega_2^{\alpha\beta}-g_{\xi\eta}\omega_1^{\alpha\xi}\omega_2^{\beta\eta})\mathbf{M}_{\alpha\beta}}\big(1+O(\omega^3)\big) \qquad (12.62) \\ = \; 1 &- \frac{i}{2}(\omega_1^{\alpha\beta}+\omega_2^{\alpha\beta}-g_{\xi\eta}\omega_1^{\alpha\xi}\omega_2^{\beta\eta})\mathbf{M}_{\alpha\beta} - \frac{1}{8}(\omega_1+\omega_2)^{\alpha\beta}\mathbf{M}_{\alpha\beta}(\omega_1+\omega_2)^{\lambda\rho}\mathbf{M}_{\lambda\rho} \; + \; O(\omega^3)\end{aligned}$$

and therefore

$$\mathbf{M}(\mathbf{\Lambda}_1\mathbf{\Lambda}_2)-\mathbf{M}(\mathbf{\Lambda}_2\mathbf{\Lambda}_1) \; = \; \frac{i}{2}g_{\xi\eta}\omega_1^{\alpha\xi}\omega_2^{\beta\eta}\mathbf{M}_{\alpha\beta}-\frac{i}{2}g_{\xi\eta}\omega_2^{\alpha\xi}\omega_1^{\beta\eta}\mathbf{M}_{\alpha\beta} \; = \; ig_{\xi\eta}\omega_1^{\alpha\xi}\omega_2^{\beta\eta}\mathbf{M}_{\alpha\beta}+O(\omega^3) \tag{12.63}$$

Now tet us turn our attention to the r.h.s. of Eq (12.59). We get

$$\begin{aligned}\mathbf{M}(\mathbf{\Lambda}_1) \; &= \; \exp\big(-\frac{i}{2}\omega_1^{\alpha\beta}\mathbf{M}_{\alpha\beta}\big) \; \simeq \; 1-\frac{i}{2}\omega_1^{\alpha\beta}\mathbf{M}_{\alpha\beta} \; - \; \frac{1}{8}\omega_1^{\alpha\beta}\mathbf{M}_{\alpha\beta}\omega_1^{\alpha\beta}\mathbf{M}_{\alpha\beta} \; + \; O(\omega_1^3) \\ \mathbf{M}(\mathbf{\Lambda}_2) \; &= \; \exp\big(-\frac{i}{2}\omega_2^{\alpha\beta}\mathbf{M}_{\alpha\beta}\big) \; \simeq \; 1-\frac{i}{2}\omega_2^{\alpha\beta}\mathbf{M}_{\alpha\beta} \; - \; \frac{1}{8}\omega_2^{\alpha\beta}\mathbf{M}_{\alpha\beta}\omega_2^{\alpha\beta}\mathbf{M}_{\alpha\beta} \; + \; O(\omega_2^3)\end{aligned} \tag{12.64}$$

and therefore

$$\begin{aligned}\Rightarrow \; \mathbf{M}(\mathbf{\Lambda}_1)\mathbf{M}(\mathbf{\Lambda}_2) - \mathbf{M}(\mathbf{\Lambda}_2)\mathbf{M}(\mathbf{\Lambda}_1) \; &\simeq \; -\frac{1}{4}\omega_1^{\alpha\beta}\omega_2^{\lambda\rho}\big(\mathbf{M}_{\alpha\beta}\mathbf{M}_{\lambda\rho}-\mathbf{M}_{\lambda\rho}\mathbf{M}_{\alpha\beta}\big) \; + \; O(\omega^3) \\ = \; -\frac{1}{4}\omega_1^{\alpha\beta}\omega_2^{\lambda\rho}\big[\mathbf{M}_{\alpha\beta},\mathbf{M}_{\lambda\rho}\big] \; + \; O(\omega^3)\end{aligned} \tag{12.65}$$

Comparing Eqs. (12.63) and (12.65) we see that

$$\begin{aligned}-\frac{1}{4}\omega_1^{\alpha\beta}\omega_2^{\lambda\rho}\big[\mathbf{M}_{\alpha\beta},\mathbf{M}_{\lambda\rho}\big] \; &= \; ig_{\xi\eta}\omega_1^{\alpha\xi}\omega_2^{\beta\eta}\mathbf{M}_{\alpha\beta} \Rightarrow \; \omega_1^{\alpha\beta}\omega_2^{\lambda\rho}\big(-\frac{1}{4}\big[\mathbf{M}_{\alpha\beta},\mathbf{M}_{\lambda\rho}\big]\big) \; = \; i\omega_1^{\alpha\beta}\omega_2^{\lambda\rho}g_{\beta\rho}\mathbf{M}_{\alpha\lambda} \\ \Rightarrow \; \omega_1^{\alpha\beta}\omega_2^{\lambda\rho}\big[\mathbf{M}_{\alpha\beta},\mathbf{M}_{\lambda\rho}\big] \; &= \; \omega_1^{\alpha\beta}\omega_2^{\lambda\rho}\big(-ig_{\beta\rho}\mathbf{M}_{\alpha\lambda}+ig_{\beta\lambda}\mathbf{M}_{\alpha\rho}+ig_{\alpha\rho}\mathbf{M}_{\beta\lambda}-ig_{\alpha\lambda}\mathbf{M}_{\beta\rho}\big)\end{aligned} \tag{12.66}$$

Since $\omega_1^{\alpha\beta}$ and $\omega_2^{\alpha\beta}$ are arbitrary we get the commutation relation (12.55).

Part XII

13 Dirac field and Dirac equation

13.1 Spinor representations of Lorentz group

We must find a representation of Lorentz group describing particles with spin $\frac{1}{2}$ (electron is historically the first example of such particle). From general formulas (12.50) and (12.54) we see that we that the fermion field transforms as

$$\psi'_\xi(x') \;=\; \left(e^{-\frac{i}{2}\omega_{\mu\nu}\mathbf{S}^{\mu\nu}}\right)_{\xi\eta}\psi_\eta(x) \tag{13.1}$$

with some matrices $\mathbf{S}$ satisfying the commutation relations (12.55)

$$\left[\mathbf{S}^{\mu\nu},\mathbf{S}^{\alpha\beta}\right] \;=\; -\,ig^{\mu\alpha}\mathbf{S}^{\nu\beta} \;-\; \mu\leftrightarrow\nu \;-\; \alpha\leftrightarrow\beta \tag{13.2}$$

From QM we know that spin-$\frac{1}{2}$ particle is described by two-dimensional spinors, and there exist 2×2 matrices satisfying the commutational relations (13.2):

- Solution #1

$$\mathbf{\Sigma}_{\mu\nu} \;\equiv\; \frac{i}{2}\sigma^\mu\bar{\sigma}^\nu - \mu\leftrightarrow\nu \tag{13.3}$$

- Solution #2

$$\bar{\mathbf{\Sigma}}_{\mu\nu} \;\equiv\; \frac{i}{2}\bar{\sigma}^\mu\sigma^\nu - \mu\leftrightarrow\nu \tag{13.4}$$

where σ^μ and $\bar{\sigma}^\mu$ are 4-dimensional Pauli matrices:

$$\sigma^\mu \;\equiv\; (1,\vec{\sigma}), \qquad \bar{\sigma}^\mu \;\equiv\; (1,-\vec{\sigma}), \qquad \vec{\sigma} \;=\; (\sigma_x,\sigma_y,\sigma_z) \tag{13.5}$$

The explicit form is

$$\sigma^0 \;=\; \begin{pmatrix}1&0\\0&1\end{pmatrix}, \qquad \sigma^1 \;=\; \begin{pmatrix}0&1\\1&0\end{pmatrix}, \qquad \sigma^0 \;=\; \begin{pmatrix}0&-i\\i&0\end{pmatrix}, \qquad \sigma^3 \;=\; \begin{pmatrix}1&0\\0&-1\end{pmatrix} \tag{13.6}$$

and

$$\bar{\sigma}^0 \;=\; \begin{pmatrix}1&0\\0&1\end{pmatrix}, \qquad \bar{\sigma}^1 \;=\; \begin{pmatrix}0&-1\\-1&0\end{pmatrix}, \qquad \bar{\sigma}^0 \;=\; \begin{pmatrix}0&i\\i-&0\end{pmatrix}, \qquad \bar{\sigma}^3 \;=\; \begin{pmatrix}-1&0\\0&1\end{pmatrix} \tag{13.7}$$

Later we will need the anticommutation relations for these matrices

$$\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu \;=\; 2g^{\mu\nu}, \qquad \bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu \;=\; 2g^{\mu\nu} \tag{13.8}$$

Thus, one may consider fermion fields $\nu_\xi(x)$ and $\bar{\nu}_\xi(x)$ which transform as

$$\nu'_\xi(x') \;=\; \left(e^{-\frac{i}{2}\omega_{\alpha\beta}\mathbf{\Sigma}^{\alpha\beta}}\right)_{\xi\eta}\nu_\eta(x), \qquad \bar{\nu}'_\xi(x') \;=\; \left(e^{-\frac{i}{2}\omega_{\alpha\beta}\bar{\mathbf{\Sigma}}^{\alpha\beta}}\right)_{\xi\eta}\bar{\nu}_\eta(x) \tag{13.9}$$

It turns out that these fields satisfy Weyl equations

$$\sigma^\mu \frac{\partial}{\partial x^\mu}\nu(x) \;=\; 0, \qquad \bar{\sigma}^\mu \frac{\partial}{\partial x^\mu}\bar{\nu}(x) \;=\; 0 \tag{13.10}$$

These field are not parity-even: one turns into the other under the parity inversion. Indeed, let us define the field $\tilde{\nu}(x) \;\equiv\; \nu(x^0, -\vec{x})$. The first of equations (13.10) can be rewritten as

$$\sigma^0 \frac{\partial}{\partial x^0}\nu(x^0,\vec{x}) + \vec{\sigma}\cdot\nabla\nu(x^0,\vec{x}) \;=\; 0 \tag{13.11}$$

so the field $\tilde{\nu}(x)$ satisfies the equation

$$\sigma^0 \frac{\partial}{\partial x^0}\tilde{\nu}(x^0,-\vec{x}) + \vec{\sigma}\cdot\nabla\tilde{\nu}(x^0,-\vec{x}) \;=\; 0 \;\stackrel{\vec{x}\leftrightarrow-\vec{x}}{\rightarrow}\; \sigma^0 \frac{\partial}{\partial x^0}\tilde{\nu}(x^0,\vec{x}) - \vec{\sigma}\cdot\nabla\tilde{\nu}(x^0,\vec{x}) \;=\; 0 \tag{13.12}$$

which is the second of equations (13.10). Thus, Weyl equations are not invariant under parity transformation and therefore are not applicable to parity-even electrons. (Nowadays we know that they describe neutrino and antineutrino fields.)

Historically, Dirac found the parity-even representation of the Lorentz group realized by 4×4 Dirac matrices γ^μ satisfying the anticommutation relation

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \;=\; 2g^{\mu\nu} \tag{13.13}$$

The solution of this equation can be written as [8]

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \;\Leftrightarrow\; \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix} \tag{13.14}$$

where 1 stands for the 2×2 unit matrix and σ's are 2×2 Pauli matrices, see Eq. (13.5). Later we will need also the matrix

$$\gamma_5 \;\equiv\; i\gamma^0\gamma^1\gamma^2\gamma^3 \;=\; \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{13.15}$$

Note that γ_5 anticommutes with all γ-matrices $\gamma^\mu\gamma_5 = -\gamma^5\gamma^\mu$.

Using the equation (13.14) is easy to see that if we define

$$\mathbf{S}_{\mu\nu} \;=\; \frac{i}{4}[\gamma^\mu,\gamma^\nu] \tag{13.16}$$

the matrices $\mathbf{S}$ satisfy the commutation relations (12.55)

$$[\mathbf{S}_{\mu\nu},\mathbf{S}_{\lambda\rho}] \;=\; -\,ig_{\mu\lambda}\mathbf{S}_{\nu\rho} \;-\; \mu\leftrightarrow\nu \;-\; \lambda\leftrightarrow\rho \tag{13.17}$$

so they can serve as generators of Lorentz transformations, so

$$\psi'_\xi(x') \;=\; \left(e^{-\frac{i}{2}\omega^{\mu\nu}\mathbf{S}_{\mu\nu}}\right)_{\xi\eta}\psi_\eta(x) \tag{13.18}$$

is a representation of Lorentz group. It remains to be seen that the representation (13.18) describes particles with spin $\frac{1}{2}$ and we will prove this later.

[8] There are different representations of γ matrices related by unitary transformations. The matrices in Eq. (13.14) correspond to so-called spinor representation used in Peskin's textbook. Another common form of γ-matrices is called standard representation, see e.g. textbook by Bjorken and Drell

13.2 Dirac equation

Dirac's question: suppose $\psi_\xi(x)$ is a 4-component field which transforms according to Eq. (13.18). How the classical equation for such field may look like?

First, it is clear that the equation of a KG type

$$(\partial^2 + m^2)\psi_\xi(x) \;=\; 0$$

is possible. Indeed,

$$\begin{aligned}(\frac{\partial}{\partial x'_\mu}\frac{\partial}{\partial x'^\mu} + m^2)\psi'_\xi(x') \;&=\; (\frac{\partial}{\partial x'_\mu}\frac{\partial}{\partial x'^\mu} + m^2)\Big(e^{-\frac{i}{2}\omega^{\mu\nu}\mathbf{S}_{\mu\nu}}\Big)_{\xi\eta}\psi_\eta(x) \\ = \;\Big(e^{-\frac{i}{2}\omega^{\mu\nu}\mathbf{S}_{\mu\nu}}\Big)_{\xi\eta}(\frac{\partial}{\partial x'_\mu}\frac{\partial}{\partial x'^\mu} + m^2)\psi_\eta(x) \;&=\; \Big(e^{-\frac{i}{2}\omega^{\mu\nu}\mathbf{S}_{\mu\nu}}\Big)_{\xi\eta}(\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\mu} + m^2)\psi_\eta(x) \;=\; 0\end{aligned} \tag{13.19}$$

since we can repeat the proof from Eq. (12.12) for each component $\psi_\eta(x)$.

However, Dirac wanted the first-order differential equation [9], and his guess was the famous Dirac equation

$$(i\gamma^\mu\partial_\mu - m)_{\xi\eta}\psi_\eta(x) \;=\; 0 \quad\Leftrightarrow\quad i(\gamma^\mu)_{\xi\eta}\frac{\partial\psi_\eta(x)}{\partial x^\mu} \;=\; m\psi_\xi(x) \tag{13.20}$$

Let us prove that Eq. (13.20) is relativistic invariant. Introduce the notation

$$\mathbf{\Lambda}_{\frac{1}{2}\,\xi\eta} \;\equiv\; \Big(e^{-\frac{i}{2}\omega^{\mu\nu}\mathbf{S}_{\mu\nu}}\Big)_{\xi\eta} \tag{13.21}$$

First, we will prove that

$$(\mathbf{\Lambda}^{-1}_{\frac{1}{2}})_{\xi\lambda}(\gamma^\mu)_{\lambda\rho}(\mathbf{\Lambda}_{\frac{1}{2}})_{\rho\eta} \;\equiv\; \mathbf{\Lambda}^\mu{}_\nu(\gamma^\nu)_{\xi\eta} \tag{13.22}$$

or, in explicit form

$$\Big(e^{\frac{i}{2}\omega^{\mu\nu}\mathbf{S}_{\mu\nu}}\Big)_{\xi\lambda}(\gamma^\mu)_{\lambda\rho}\Big(e^{-\frac{i}{2}\omega^{\mu\nu}\mathbf{S}_{\mu\nu}}\Big)_{\rho\eta} \;=\; \Big(e^{-\frac{i}{2}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta}}\Big)^\mu{}_\nu(\gamma^\nu)_{\xi\eta} \tag{13.23}$$

(recall that $\mathbf{\Lambda}$ is given by Eq. (12.48)).

To prove Eq. (13.23) we expand both sides in powers of ω using formula

$$e^A B e^{-A} \;=\; \sum_{n=0}^{\infty}\frac{1}{n!}[A,[A,[A...[A,B]]]...] \tag{13.24}$$

The l.h.s. of Eq. (13.23) takes the form

$$\Big(e^{\frac{i}{2}\omega^{\mu\nu}\mathbf{S}_{\mu\nu}}\Big)_{\xi\lambda}(\gamma^\mu)_{\lambda\rho}\Big(e^{-\frac{i}{2}\omega^{\mu\nu}\mathbf{S}_{\mu\nu}}\Big)_{\rho\eta} \;=\; \sum_{n=0}^{\infty}\frac{(i/2)^n}{n!}[\omega\mathbf{S},[\omega\mathbf{S}...[\omega\mathbf{S},\gamma^\mu]]]...] \tag{13.25}$$

[9]He thought that, similarly to the Schrödinger equation, the first-order differential equation for $\psi(x)$ would allow probabilistic interpretation at the one-particle level. This proved to be wrong - we know for the relativistic description we need both electrons and positrons (which were predicted by Dirac!).

Let us calculate these commutators (spinor indices of Dirac matrices are implied)

$$[\gamma^\mu, S^{\xi\eta}] \;=\; \left(\mathcal{J}^{\xi\eta}\right)^\mu{}_\nu \gamma^\nu \;\Rightarrow\; [\omega_{\xi\eta}S^{\xi\eta}, \gamma_\mu] \;=\; \left(\omega_{\xi\eta}\mathcal{J}^{\xi\eta}\right)^\mu{}_\nu \gamma^\nu \;\Leftrightarrow\; [\omega\mathbf{S}, \gamma^\mu] \;=\; -(\omega\mathcal{J})^\mu{}_\nu\gamma^\nu \tag{13.26}$$

We get

$$\begin{aligned}
[\omega\mathbf{S}, [\omega\mathbf{S}\gamma^\mu]] \;&=\; -(\omega\mathcal{J})^\mu{}_\nu[\omega\mathbf{S}, \gamma^\nu] \;=\; (\omega\mathcal{J})^\mu{}_\nu(\omega\mathcal{J})^\nu{}_\lambda\gamma^\lambda \;=\; \left((\omega\mathcal{J})^2\right)^\mu{}_\lambda\gamma^\lambda \qquad (13.27)\\
[\omega\mathbf{S}, [\omega\mathbf{S}, [\omega\mathbf{S}\gamma^\mu]]] \;&=\; -(\omega\mathcal{J})^\mu{}_\nu[\omega\mathbf{S}, [\omega\mathbf{S}, \gamma^\nu]] \;=\; (\omega\mathcal{J})^\mu{}_\nu(\omega\mathcal{J})^\nu{}_\lambda[\omega\mathbf{S}, \gamma^\lambda]]\\
&=\; -(\omega\mathcal{J})^\mu{}_\nu(\omega\mathcal{J})^\nu{}_\lambda(\omega\mathcal{J})^\lambda{}_\rho\gamma^\rho \;=\; -\left((\omega\mathcal{J})^3\right)^\mu{}_\rho\gamma^\rho\\
&\ldots\\
[\omega\mathbf{S}, [\omega\mathbf{S}...[\omega\mathbf{S}, \gamma^\mu]]]...] \;&=\; (-1)^n\left((\omega\mathcal{J})^n\right)^\mu{}_\nu\gamma^\nu\\
\Rightarrow \sum_{n=0}^{\infty}\frac{(i/2)^n}{n!}[\omega\mathbf{S}, [\omega\mathbf{S}...[\omega\mathbf{S}, \gamma^\mu]]]...] \;&=\; \sum_{n=0}^{\infty}\frac{(-i/2)^n}{n!}\left((\omega\mathcal{J})^n\right)^\mu{}_\nu\gamma^\nu \;=\; \left(e^{-\frac{i}{2}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta}}\right)^\mu{}_\nu\gamma^\nu
\end{aligned}$$

Now we can prove that Dirac equation is relativistic invariant. Using $\frac{\partial}{\partial x'^\mu} = \Lambda_\mu{}^\nu\frac{\partial}{\partial x^\nu}$ and Eq. (13.22) we get

$$\begin{aligned}
&\left(i(\gamma^\mu)_{\xi\eta}\frac{\partial\psi'_\eta(x')}{\partial x'^\mu} - m\psi'(x')\right) \;=\; \left(i\gamma^\mu\Lambda^\mu{}_\alpha\frac{\partial}{\partial x^\alpha} - m\right)_{\xi\rho}(\mathbf{\Lambda}_{\frac{1}{2}})_{\rho\eta}\psi_\eta(x) \qquad (13.28)\\
&=\; i(\gamma^\mu)_{\xi\rho}\Lambda^\mu{}_\alpha\frac{\partial}{\partial x^\alpha}(\mathbf{\Lambda}_{\frac{1}{2}})_{\rho\eta}\psi_\eta(x) - m(\mathbf{\Lambda}_{\frac{1}{2}})_{\xi\eta}\psi_\eta(x)\\
&=\; (\mathbf{\Lambda}_{\frac{1}{2}})_{\xi\rho}\left\{i(\mathbf{\Lambda}_{\frac{1}{2}}^{-1})_{\rho\sigma}(\gamma^\mu)_{\sigma\phi}(\mathbf{\Lambda}_{\frac{1}{2}})_{\phi\eta}\Lambda_\mu{}^\alpha\frac{\partial}{\partial x^\alpha} - m\delta_{\rho\eta}\right\}\psi_\eta(x)\\
&=\; (\mathbf{\Lambda}_{\frac{1}{2}})_{\xi\rho}\left\{i(\gamma^\nu)_{\rho\eta}\Lambda^\mu{}_\nu\Lambda_\mu{}^\alpha\frac{\partial}{\partial x^\alpha} - m\delta_{\rho\eta}\right\}\psi_\eta(x) = (\mathbf{\Lambda}_{\frac{1}{2}})_{\xi\rho}\left\{i(\gamma^\alpha)_{\rho\eta}\frac{\partial}{\partial x^\alpha} - m\delta_{\rho\eta}\right\}\psi_\eta(x) \;=\; 0
\end{aligned}$$

Note that Dirac field satisfies also the KG equation:

$$\begin{aligned}
(i\gamma^\mu\partial_\mu + m)\psi(x) \;&=\; 0 \;\;\Rightarrow\;\; (i\gamma^\nu\partial_\nu - m)(i\gamma^\mu\partial_\mu + m)\psi(x) \;=\; 0\\
&=\; (-\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - m^2)\psi(x) \;=\; -(\partial^2 + m^2)\psi(x) \;=\; 0
\end{aligned} \tag{13.29}$$

where we used Eq. (13.13).

13.3 Lagrangian of the Dirac field

The Dirac equation (13.20) must follow from some Lagrangian. The Lagrangian is a scalar, so how we make a scalar combination of spinor fields?

The simplest invariant is $\bar\psi(x)\psi(x)$ where $\bar\psi \equiv \psi^\dagger\gamma^0$. Check:

$$\psi'_\xi(x') \;=\; \left(e^{-\frac{i}{2}\omega^{\mu\nu}\mathbf{S}_{\mu\nu}}\right)_{\xi\eta}\psi_\eta(x) \;\Rightarrow\; \psi'^\dagger_\xi(x')\gamma^0_{\xi\rho} \;=\; \psi^\dagger_\eta(x)\left(e^{\frac{i}{2}\omega^{\mu\nu}\mathbf{S}^\dagger_{\mu\nu}}\right)_{\eta\xi}\gamma^0_{\xi\rho} \tag{13.30}$$

(recall that $\omega_{\mu\nu}$ are real numbers). Now we use formula

$$\gamma^0\gamma^\dagger_\mu\gamma^0 \;=\; \gamma_\mu \tag{13.31}$$

so

$$\gamma_0(\omega\mathbf{S}^\dagger)\gamma_0 \;\equiv\; \omega^{\mu\nu}\gamma_0\mathbf{S}^\dagger{}_{\mu\nu}\gamma_0 \;=\; \omega_{\mu\nu}\gamma_0\Big(\frac{i}{4}[\gamma^\mu,\gamma^\nu]\Big)^\dagger\gamma_0 \tag{13.32}$$
$$=\; \omega_{\mu\nu}\gamma_0\frac{i}{4}[\gamma^{\mu\dagger},\gamma^{\nu\dagger}]\gamma_0 \;=\; \omega_{\mu\nu}\frac{i}{4}[\gamma_0\gamma^{\mu\dagger}\gamma_0,\gamma_0\gamma^{\nu\dagger}\gamma_0] \;=\; \omega_{\mu\nu}\frac{i}{4}[\gamma^\mu,\gamma^\nu] \;=\; \omega_{\mu\nu}\mathbf{S}^{\mu\nu} \;\equiv\; (\omega\mathbf{S})$$

and we get

$$\psi'^\dagger_\xi(x')(\gamma_0)_{\xi\rho} \;=\; \psi^\dagger_\eta(x)\gamma^0_{\eta\phi}\gamma^0_{\phi\sigma}\big(e^{\frac{i}{2}\omega^{\mu\nu}\mathbf{S}^\dagger_{\mu\nu}}\big)_{\sigma\xi}\gamma^0_{\xi\rho} \;=\; \psi^\dagger_\eta(x)\gamma^0_{\eta\varsigma}\big(e^{\frac{i}{2}\omega^{\mu\nu}\mathbf{S}_{\mu\nu}}\big)_{\varsigma\xi} \;=\; \bar\psi_\varsigma\big(e^{\frac{i}{2}\omega^{\mu\nu}\mathbf{S}_{\mu\nu}}\big)_{\varsigma\xi} \tag{13.33}$$

so $\bar\psi\psi$ is a scalar:

$$\bar\psi'_\xi(x')\psi'_\xi(x') \;=\; \bar\psi_\varsigma(x')\big(e^{\frac{i}{2}(\omega\mathbf{S})}\big)_{\varsigma\xi}\big(e^{-\frac{i}{2}(\omega\mathbf{S})}\big)_{\xi\eta}\psi_\eta(x) \;=\; \bar\psi_\eta(x)\psi_\eta(x) \tag{13.34}$$

In a similar way one can prove that $j^\mu \;=\; \bar\psi\gamma^\mu\psi$ is a 4-vector (recall notation $\mathbf{\Lambda}_{\frac{1}{2}} \equiv e^{-\frac{i}{2}(\omega\mathbf{S})}$):

$$j'^\mu(x') \;=\; \bar\psi'\gamma^\mu\psi'(x') \;\equiv\; \bar\psi'_\xi(x')(\gamma^\mu)_{\xi\eta}\psi'_\eta(x') \;=\; \bar\psi_\xi(x)\big(\mathbf{\Lambda}_{\frac{1}{2}}\big)_{\xi\varsigma}(\gamma^\mu)_{\varsigma\sigma}(\mathbf{\Lambda}^\dagger_{\frac{1}{2}})_{\sigma\eta}\psi_\eta(x)$$
$$=\; \Lambda^\mu{}_\nu\bar\psi_\xi(x)(\gamma^\mu)_{\xi\eta}\psi_\eta(x) \;=\; \Lambda^\mu{}_\nu\bar\psi\gamma^\mu\psi(x) \;=\; \Lambda^\mu{}_\nu j^\nu(x) \tag{13.35}$$

Now, if $\bar\psi\gamma^\mu\psi(x)$ is a vector and ∂_μ is a (co)vector, their product must be a scalar:

$$\bar\psi'\gamma^\mu\partial'_\mu\psi'(x') \;\equiv\; \bar\psi'\gamma^\mu\frac{\partial}{\partial x'^\mu}\psi'(x') \;=\; \bar\psi_\xi(x)\big(\mathbf{\Lambda}^{-1}_{\frac{1}{2}}\big)_{\xi\varsigma}(\gamma^\mu)_{\varsigma\sigma}(\mathbf{\Lambda}_{\frac{1}{2}})_{\sigma\eta}\Lambda_\mu{}^\lambda\frac{\partial}{\partial x^\lambda}\psi_\eta(x)$$
$$=\; \Lambda^\mu{}_\nu\Lambda_\mu{}^\lambda\bar\psi_\xi(x)(\gamma^\nu)_{\xi\eta}\frac{\partial}{\partial x^\lambda}\psi_\eta(x) \;=\; \bar\psi_\xi(x)(\gamma^\nu)_{\xi\eta}\frac{\partial}{\partial x^\nu}\psi_\eta(x) \;=\; \bar\psi\gamma^\mu\partial_\mu\psi(x) \tag{13.36}$$

where we used Eq. (13.22).

Let us demonstrate that the scalar

$$\mathcal{L}_D(x) \;=\; \bar\psi\big(i\gamma^\mu\partial_\mu - m\big)\psi(x) \tag{13.37}$$

may serve as a Dirac Lagrangian. We need to check that Euler-Lagrange equations for the Lagrangian (13.37) reduce to Dirac equation

$$\frac{\partial\mathcal{L}}{\partial\bar\psi} \;=\; i\big(\gamma^\mu\partial_\mu - m\big)\psi, \quad \frac{\partial\mathcal{L}}{\partial\partial^\mu\bar\psi} \;=\; 0 \quad\Rightarrow\quad \partial^\mu\frac{\partial\mathcal{L}}{\partial\partial^\mu\psi} \;=\; \frac{\partial\mathcal{L}}{\partial\psi} \quad\Leftrightarrow\quad i\big(\gamma^\mu\partial_\mu - m\big)\psi \;=\; 0$$
$$\frac{\partial\mathcal{L}}{\partial\psi} \;=\; -m\bar\psi, \quad \frac{\partial\mathcal{L}}{\partial\partial^\mu\psi} \;=\; i\bar\psi\gamma_\mu \quad\Rightarrow\quad \partial^\mu\frac{\partial\mathcal{L}}{\partial\partial^\mu\psi} \;=\; \frac{\partial\mathcal{L}}{\partial\psi} \quad\Leftrightarrow\quad i\partial^\mu\bar\psi\gamma_\mu \;=\; -m\bar\psi \tag{13.38}$$

We see that the first equation is Dirac equation (13.20) and the second is its complex conjugate.

Part XIII

13.4 Plane wave solutions of Dirac equation

13.4.1 Dirac equation in spinor representation of γ-matrices

Dirac bispinor

$$\psi \;=\; \binom{\psi_L}{\psi_R} \tag{13.39}$$

ψ_L, ψ_R - Weyl spinors (two-component spinors)

Dirac equation

$$i\gamma^\mu\partial_\mu\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} = m\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} \quad\Rightarrow\quad i\begin{pmatrix}0 & \sigma^\mu\\ \bar\sigma^\mu & 0\end{pmatrix}\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} = m\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} \quad\Rightarrow\quad \begin{array}{rcl} i\sigma^\mu\partial_\mu\psi_R &=& m\psi_L\\ i\bar\sigma^\mu\partial_\mu\psi_L &=& m\psi_R\end{array} \tag{13.40}$$

If $m = 0$ - two Weyl equations for neutrino and antineutrino fields $\bar\nu(x) = \psi_R(x)$ and $\nu(x) = \psi_L(x)$

$$i\sigma^\mu\partial_\mu\bar\nu(x) = 0, \qquad i\bar\sigma^\mu\partial_\mu\nu(x) = 0 \tag{13.41}$$

13.5 Plane waves

Ansatz: $\psi_p(x) = u(p)e^{-ipx}$. Dirac equation (common notation $\not a \equiv a^\mu\gamma_\mu$):

$$i\gamma^\mu\partial_\mu\psi_p(x) = 0 \quad\Rightarrow\quad (\not p - m)u(p) = 0 \tag{13.42}$$

We start with particle at rest $p_r = (m, 0, 0, 0)$.

$$(m\gamma^0 - m)u(p_r) = 0 \quad\Rightarrow\quad m\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}u(p_r) = 0 \quad\Rightarrow\quad u(p_r) = \sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix} \tag{13.43}$$

$\sqrt{m}$ is introduced for convenient normalization (we assume $\xi^\dagger\xi = 1$)

Next, we boost particle is z direction. The martix of this boost is given by Eq. (12.15) and the corresponding matrix $\boldsymbol{\omega}$ by Eq. (13.44)

$$\Lambda^\mu{}_\nu(\theta) = \begin{pmatrix}\cosh\theta & 0 & 0 & \sinh\theta\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \sinh\theta & 0 & 0 & \cosh\theta\end{pmatrix} \qquad \omega_{\mu\nu} = \begin{pmatrix}0 & 0 & 0 & \theta\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -\theta & 0 & 0 & 0\end{pmatrix} \tag{13.44}$$

where $\cosh\theta = \frac{p^0}{m}$ and $\sinh\theta = \frac{|\vec p|}{m}$. The matrix of Lorentz boost for spinors is given by Eq. (13.22). In our case $\omega_{03} = -\omega_{30} = \theta$ and

$$\mathbf{S}^{03} = \frac{i}{4}[\gamma^0, \gamma^3] = -\frac{i}{2}\begin{pmatrix}\sigma_z & 0\\ 0 & -\sigma_z\end{pmatrix} \tag{13.45}$$

so

$$\begin{aligned}\boldsymbol{\Lambda}_{\frac{1}{2}\,\xi\eta}(\theta) &\equiv \left(e^{-\frac{i}{2}\omega_{\mu\nu}\mathbf{S}^{\mu\nu}}\right)_{\xi\eta} = \left(e^{-i\omega_{03}\mathbf{S}^{03}}\right)_{\xi\eta} = \exp\left\{-\frac{\theta}{2}\begin{pmatrix}\sigma_z & 0\\ 0 & -\sigma_z\end{pmatrix}\right\}\\ &= \begin{pmatrix}\cosh\frac{\theta}{2} - \sinh\frac{\theta}{2}\sigma_z & 0\\ 0 & \cosh\frac{\theta}{2} + \sinh\frac{\theta}{2}\sigma_z\end{pmatrix}\end{aligned} \tag{13.46}$$

Since $\cosh\frac{\theta}{2} = \sqrt{\frac{p^0+m}{2m}}$ and $\sinh\frac{\theta}{2} = \sqrt{\frac{p^0-m}{2m}}$

$$u(p) = \boldsymbol{\Lambda}_{\frac{1}{2}\,\xi\eta}(\theta)\sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix} \tag{13.47}$$

$$= \sqrt{m}\begin{pmatrix}\cosh\frac{\theta}{2} - \sinh\frac{\theta}{2}\sigma_z & 0\\ 0 & \cosh\frac{\theta}{2} + \sinh\frac{\theta}{2}\sigma_z\end{pmatrix}\begin{pmatrix}\xi\\ \xi\end{pmatrix} = \frac{1}{\sqrt{2(p_0+m)}}\begin{pmatrix}(p_0 + m - p_z\sigma_z)\xi\\ (p_0 + m + p_z\sigma_z)\xi\end{pmatrix}$$

We see that although bispinor has 4 components, the two lower components are determined by the two upper components. One may object that if the lower components of bispinor are defined in an unique way by the upper ones, why is this doubling of writing? The answer is that this double-writing is convenient since under spatial reflection the upper and lower components of the bispinor (13.47) simply trade places so if we arrange our formalism to be symmetric under exchange of upper structures and lower structures, such formalism will be explicitly parity-even.

For arbitrary boost

$$u(p) \;=\; \frac{1}{\sqrt{2(p_0+m)}}\begin{pmatrix}(p_0+m-\vec{p}\cdot\vec{\sigma})\xi\\(p_0+m+\vec{p}\cdot\vec{\sigma})\xi\end{pmatrix} \tag{13.48}$$

This is the plane wave with definite direction of the spin in c.m. frame ($\xi = \binom{1}{0}$ spin up or $\xi = \binom{0}{1}$ spin down)

Peskin's notations

$$\sqrt{p\sigma} \;=\; \frac{p_0+m-\vec{p}\cdot\vec{\sigma}}{\sqrt{2(p_0+m)}}, \qquad \sqrt{p\bar{\sigma}} \;=\; \frac{p_0+m+\vec{p}\cdot\vec{\sigma}}{\sqrt{2(p_0+m)}} \tag{13.49}$$

Check (see the definition (13.5)):

$$\begin{aligned}\Big[\frac{1}{\sqrt{2(p_0+m)}}(p_0+m-\vec{p}\cdot\vec{\sigma})\Big]^2 \;&=\; \frac{(p_0+m)^2-2(p_0+m)\vec{p}\cdot\vec{\sigma}+|\vec{p}|^2}{2(p_0+m)} \;=\; p_0-\vec{p}\cdot\vec{\sigma} \;=\; p\sigma\\ \Big[\frac{1}{\sqrt{2(p_0+m)}}(p_0+m+\vec{p}\cdot\vec{\sigma})\Big]^2 \;&=\; \frac{(p_0+m)^2+2(p_0+m)\vec{p}\cdot\vec{\sigma}+|\vec{p}|^2}{2(p_0+m)} \;=\; p_0+\vec{p}\cdot\vec{\sigma} \;=\; p\bar{\sigma}\end{aligned} \tag{13.50}$$

and therefore

$$u(p) \;=\; \begin{pmatrix}\sqrt{p\sigma}\xi\\\sqrt{p\bar{\sigma}}\xi\end{pmatrix} \tag{13.51}$$

Plane waves with definite helicity

Sometimes it is convenient to specify not the spin of the particle in the rest frame but the helicity (=projection of the spin on the direction of motion) of the particle in a certain frame [10]. The spinors of definite helicity have the form [11]:

$$u^{[\frac{1}{2}]}(p) \;=\; \frac{1}{\sqrt{2(p_0+m)}}\begin{pmatrix}(p_0+m-|\vec{p}|)\omega^{(1)}\\(p_0+m+|\vec{p}|)\omega^{(1)}\end{pmatrix}, \qquad u^{[-\frac{1}{2}]}(p) \;=\; \frac{1}{\sqrt{2(p_0+m)}}\begin{pmatrix}(p_0+m-|\vec{p}|)\omega^{(2)}\\(p_0+m+|\vec{p}|)\omega^{(2)}\end{pmatrix} \tag{13.52}$$

[10] The helicity of the massive particle depends on the frame of reference, since one can always boost to a frame in which its momentum is in the opposite direction (but spin is unchanged). For a massless particle, which travels at the speed of light, one cannot perform such a boost so helicity is an inherent property of a massless particle.

[11] In order to distinguish these spinors with helicity $\pm\frac{1}{2}$ from the spinors with z-component of the spin equal to $\pm\frac{1}{2}$ we put the helicity $\pm\frac{1}{2}$ in square brackets.

where (θ and ϕ are polar and azimuthal angles of momentum $\vec{p}$)

$$\omega^{(1)} = \begin{pmatrix} e^{-i\phi}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}, \qquad \omega^{(2)} = \begin{pmatrix} -e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix} \tag{13.53}$$

The spinors (13.52) are eigenstates of the helicity operator

$$\mathbf{h} \equiv \frac{1}{2|\vec{p}|}\begin{pmatrix} \vec{p}\cdot\vec{\sigma} & 0 \\ 0 & \vec{p}\cdot\vec{\sigma} \end{pmatrix} \tag{13.54}$$

Check

$$\vec{p}\cdot\vec{\sigma} = |\vec{p}|\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & \cos\theta \end{pmatrix} \quad\Rightarrow\quad \vec{p}\cdot\vec{\sigma}\omega^{[(1)} = |\vec{p}|\omega^{[(1)}, \quad \vec{p}\cdot\vec{\sigma}\omega^{[(2)} = -|\vec{p}|\omega^{[(2)} \tag{13.55}$$

and therefore

$$\mathbf{h}u^{[\frac{1}{2}]}(p) = \frac{1}{2\sqrt{2(p_0+m)}}\begin{pmatrix} (p_0+m-|\vec{p}|)\vec{p}\cdot\vec{\sigma}\omega^{(1)} \\ (p_0+m+|\vec{p}|)\vec{p}\cdot\vec{\sigma}\omega^{(1)} \end{pmatrix} = \frac{1/2}{\sqrt{2(p_0+m)}}\begin{pmatrix} (p_0+m-|\vec{p}|)\omega^{(1)} \\ (p_0+m+|\vec{p}|)\omega^{(1)} \end{pmatrix} = \frac{1}{2}u^{(1)}(p)$$

$$\mathbf{h}u^{[-\frac{1}{2}]}(p) = \frac{1}{2\sqrt{2(p_0+m)}}\begin{pmatrix} (p_0+m-|\vec{p}|)\vec{p}\cdot\vec{\sigma}\omega^{(2)} \\ (p_0+m+|\vec{p}|)\vec{p}\cdot\vec{\sigma}\omega^{(2)} \end{pmatrix} = \frac{-1/2}{\sqrt{2(p_0+m)}}\begin{pmatrix} (p_0+m-|\vec{p}|)\omega^{(2)} \\ (p_0+m+|\vec{p}|)\omega^{(2)} \end{pmatrix} = -\frac{1}{2}u^{(2)}(p) \tag{13.56}$$

where $h = \pm\frac{1}{2}$.

13.5.1 Negative-frequency plane waves

Ansatz: $\psi_p(x) = v(p)e^{ipx}$. Dirac equation:

$$i\gamma^\mu\partial_\mu\psi_p(x) = 0 \quad\Rightarrow\quad (\not{p}+m)v(p) = 0 \tag{13.57}$$

Solutions

$$v(p) = \frac{1}{\sqrt{2(p_0+m)}}\begin{pmatrix} (p_0+m-\vec{p}\cdot\vec{\sigma})\eta \\ (-p_0-m-\vec{p}\cdot\vec{\sigma})\eta \end{pmatrix} = \begin{pmatrix} \sqrt{p\sigma}\eta \\ -\sqrt{p\bar{\sigma}}\eta \end{pmatrix} \tag{13.58}$$

Explicit form of spinors $v(p)$ with definite z-projection in rest frame and definite helicity is given in by Eqs. (25.12) - (25.21) from the Appendix.

13.5.2 Orthogonality and completeness of spinors

Using the explicit formulas for the Dirac spinors u and v it is easy to check the orthogonality conditions

$$\begin{gathered} \bar{u}^\lambda(p)u^{\lambda'}(p) = 2m\delta_{\lambda\lambda'} = -\bar{v}^\lambda(p)v^{\lambda'}(p) \\ \bar{u}^\lambda(p)\gamma^\mu u^{\lambda'}(p) = \bar{v}^\lambda(p)\gamma^\mu v^{\lambda'}(p) = 2p^\mu\delta_{\lambda\lambda'} \\ \bar{u}^\lambda(p)v^{\lambda'}(p) = 0 = \bar{v}^\lambda(p)u^{\lambda'}(p) \end{gathered} \tag{13.59}$$

and the conditions of completeness

$$\begin{gathered} \sum_{\lambda=1,2}\Big(u^\lambda_\alpha(p)\bar{u}^\lambda_\beta(p) - v^\lambda_\alpha(p)\bar{v}^\lambda_\beta(p)\Big) = 2m\delta_{\alpha\beta} \\ \sum_{\lambda=1,2} u^\lambda_\alpha(p)\bar{u}^\lambda_\beta(p) = (m+\not{p})_{\alpha\beta} \\ \sum_{\lambda=1,2} v^\lambda_\alpha(p)\bar{v}^\lambda_\beta(p) = (\not{p}-m)_{\alpha\beta} \end{gathered} \tag{13.60}$$

Here $\lambda = \pm\frac{1}{2}$ can be either z-component of the spin in the c.m. frame or helicity. We will need two more formulas (proven in the Appendix, see Eq. (25.26)

$$\bar{v}(\vec{p},s)\gamma_0 u(-\vec{p},s') \;=\; v^\dagger(\vec{p},s)u(-\vec{p},s') \;=\; 0, \qquad \bar{u}(\vec{p},s)\gamma_0 v(-\vec{p},s') \;=\; u^\dagger(\vec{p},s)v(-\vec{p},s') \;=\; 0 \tag{13.61}$$

13.5.3 4-vector of spin

It is instructive to write down the relativistic invariant generalization of the operator of the spin of the electron in the rest frame. Suppose the spin of the electron in the rest frame is given by the vector

$$\vec{s} = \kappa^\dagger \vec{\sigma}\kappa \tag{13.62}$$

where κ is our spinor in the rest frame. Let us intoduce formally the four-vector s^μ which coincides with $(0,\vec{s})$ in the rest frame [12]. Note that $s^2 = -1$ and $s\cdot p = 0$.

With this notation the non-relativistic equation

$$\frac{1}{2}\vec{\sigma}\cdot\vec{s}\kappa^{(\vec{s})} = \frac{1}{2}\kappa^{(\vec{s})} \tag{13.63}$$

for the spinor $\kappa^{(\vec{s})}$ is generalized to

$$\frac{1}{2}\gamma_5\gamma_\mu s^\mu u(p,s) = \frac{1}{2}u(p,s) \tag{13.64}$$

where the spinor $u(p,s) \equiv u^{(\vec{s})}(p)$ is given by usual Lorentz transformation of the (bi)spinor $\begin{pmatrix}\kappa^{(\vec{s})}\\ \kappa^{(\vec{s})}\end{pmatrix}$

$$u(p,s) = \frac{1}{\sqrt{2(p_0+m)}}\begin{pmatrix}(m+p_0-\vec{p}\cdot\vec{\sigma})\kappa^{(\vec{s})}\\ (m+p_0+\vec{p}\cdot\vec{\bar{\sigma}})\kappa^{(\vec{s})}\end{pmatrix} \tag{13.65}$$

Indeed,

$$\gamma_5\gamma^\mu s_\mu = \begin{pmatrix}-I & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}0 & \sigma^\mu\\ \bar{\sigma}^\mu & 0\end{pmatrix}s_\mu = \begin{pmatrix}0 & -\sigma\cdot s\\ \bar{\sigma}\cdot s & 0\end{pmatrix} \tag{13.66}$$

so in the rest frame the eq. (13.64) reduces to:

$$\frac{1}{2}\begin{pmatrix}0 & \vec{\sigma}\cdot\vec{s}\\ \vec{\psi}\cdot\vec{s} & 0\end{pmatrix}\sqrt{m}\begin{pmatrix}\kappa^{(\vec{s})}\\ \kappa^{(\vec{s})}\end{pmatrix} = \lambda\sqrt{m}\begin{pmatrix}\kappa^{(\vec{s})}\\ \kappa^{(\vec{s})}\end{pmatrix} \Leftrightarrow \frac{1}{2}(\vec{\sigma}\cdot\vec{s})\kappa^{(\vec{s})} = \frac{1}{2}\kappa^{(\vec{s})} \tag{13.67}$$

which coincides with the eq. (13.63).

So, the two equations

$$\begin{aligned} p^\mu\gamma_\mu u(p,s) &= m u(p,s)\\ \gamma_5\gamma_\mu s^\mu u(p,s) &= u(p,s) \end{aligned} \tag{13.68}$$

fix the Dirac spinor unambigously.

[12] For example, $s = (0,1,0,0)$ for $\kappa = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$, $s = (0,0,1,0)$ for $\kappa = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$, and $s = (0,0,0,1)$ for $\kappa = \begin{pmatrix}1\\0\end{pmatrix}$, $s = (0,0,0,-1)$ for $\kappa = \begin{pmatrix}0\\1\end{pmatrix}$ as we already know.

13.5.4 Dirac field bilinears and Fierz identities

$\bar\psi\psi$ is a scalar, $\bar\psi\gamma^\mu\psi$ is a vector, $\bar\psi\gamma^\mu\gamma^\nu....\gamma^\lambda\psi$ is a what?

Complete set of γ-matrices

$$\begin{array}{lcl r} 1 & & & 1 \\ \gamma^\mu & & & 4 \\ \sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu,\gamma^\nu] & = & i\gamma^\mu\gamma^\nu - ig^{\mu\nu} & 6 \\ \gamma_5 & & & 1 \\ \gamma^\mu\gamma_5 & & & 4 \end{array} \tag{13.69}$$

Overall, 16 independent matrices. The rest is expressed in terms of matrices (13.71) with the help of formula

$$\gamma^\mu\gamma^\nu\gamma^\lambda = g^{\mu\nu}\gamma^\lambda + g^{\nu\lambda}\gamma^\mu - g^{\mu\nu}\gamma^\lambda - i\epsilon^{\mu\nu\lambda\rho}\gamma_\rho\gamma_5 \tag{13.70}$$

where $\epsilon^{\mu\nu\lambda\rho}$ is a totally antisymmetric tensor with $\epsilon^{0123} = -1$.

NB: I use sign convention from Bjorken & Drell. In Peskin's book the sign is different: $\epsilon^{0123}_{\text{Peskin}} = 1$

Dirac bilinears

$$\begin{array}{ll} \bar\psi\psi & \text{scalar} \\ \bar\psi\gamma^\mu\psi & \text{vector} \\ \bar\psi\sigma^{\mu\nu}\psi & \text{antisymmetric tensor of rank 2} \\ \bar\psi\gamma_5\psi & \text{pseudoscalar} \\ \bar\psi\gamma^\mu\gamma_5\psi & \text{pseudovector} \end{array} \tag{13.71}$$

For example (see Eq. (13.22)):

$$\begin{aligned} \bar\psi'(x')\sigma^{\mu\nu}\psi'(x') &= \frac{i}{2}\bar\psi'(x')\gamma^\mu\gamma^\nu\psi'(x') - \mu\leftrightarrow\nu = \frac{i}{2}\bar\psi(x)\mathbf{\Lambda}^{-1}_{\frac12}\gamma^\mu\gamma^\nu\mathbf{\Lambda}_{\frac12}\psi(x) - \mu\leftrightarrow\nu \\ &= \frac{i}{2}\bar\psi(x)\mathbf{\Lambda}^{-1}_{\frac12}\gamma^\mu\mathbf{\Lambda}_{\frac12}\mathbf{\Lambda}^{-1}_{\frac12}\gamma^\nu\mathbf{\Lambda}_{\frac12}\psi(x) - \mu\leftrightarrow\nu = \frac{i}{2}\Lambda^\mu{}_\xi\Lambda^\nu{}_\eta\bar\psi(x)\gamma^\xi\gamma^\eta\psi - \frac{i}{2}\Lambda^\nu{}_\xi\Lambda^\mu{}_\eta\bar\psi(x)\gamma^\xi\gamma^\eta\psi \\ &= \frac{i}{2}\Lambda^\mu{}_\xi\Lambda^\nu{}_\eta\bar\psi(x)\gamma^\xi\gamma^\eta\psi - \frac{i}{2}\Lambda^\nu{}_\eta\Lambda^\mu{}_\xi\bar\psi(x)\gamma^\eta\gamma^\xi\psi = \Lambda^\mu{}_\xi\Lambda^\nu{}_\eta\bar\psi(x)\sigma^{\xi\eta}\psi(x) \end{aligned} \tag{13.72}$$

$\Rightarrow$ $\bar\psi(x)\sigma^{\xi\eta}\psi(x)$ is an (antisymmetric) tensor of rank 2.

Set of matrices (13.71) is complete $\Rightarrow$ one can expand an arbitrary 4×4 matrix as

$$\Gamma_{\xi\eta} = \frac14\delta_{\xi\eta}\mathrm{Tr}\{\Gamma\} + \frac14\gamma^\mu_{\xi\eta}\mathrm{Tr}\{\Gamma\gamma_\mu\} + \frac14(\gamma_5)_{\xi\eta}\mathrm{Tr}\{\Gamma\gamma_5\} - \frac14(\gamma^\mu\gamma_5)_{\xi\eta}\mathrm{Tr}\{\Gamma\gamma_\mu\gamma_5\} + \frac18(\sigma^{\mu\nu})_{\xi\eta}\mathrm{Tr}\{\Gamma\sigma_{\mu\nu}\} \tag{13.73}$$

$\Rightarrow$ Fierz identities.

Example:

$$\begin{aligned} \bar\psi\gamma^\mu\psi\bar\chi\gamma_\mu\chi &= \bar\psi\gamma^\mu(\psi\bar\chi\equiv\Gamma)\gamma_\mu\chi = \bar\psi\gamma^\mu\Gamma\gamma_\mu\chi \\ &= \bar\psi\chi\mathrm{Tr}\{\Gamma\} + \frac14\bar\psi\gamma^\mu\gamma^\alpha\gamma_\mu\chi\mathrm{Tr}\{\Gamma\gamma_\alpha\} - \bar\psi\gamma_5\chi\mathrm{Tr}\{\Gamma\gamma_5\} - \frac14\bar\psi\gamma^\mu\gamma^\alpha\gamma_5\gamma_\mu\chi\mathrm{Tr}\{\Gamma\gamma_\alpha\gamma_5\} + \frac18\bar\psi\gamma^\mu\sigma^{\alpha\beta}\gamma_\mu\chi\mathrm{Tr}\{\Gamma\sigma_{\alpha\beta}\} \\ &= \bar\psi\chi\mathrm{Tr}\{\Gamma\} - \frac12\bar\psi\gamma^\alpha\chi\mathrm{Tr}\{\Gamma\gamma_\alpha\} - \bar\psi\gamma_5\chi\mathrm{Tr}\{\Gamma\gamma_5\} - \frac12\bar\psi\gamma^\alpha\gamma_5\chi\mathrm{Tr}\{\Gamma\gamma_\alpha\gamma_5\} \\ &= (\bar\psi\chi)(\bar\chi\psi) - \frac12(\bar\psi\gamma^\alpha\chi)(\bar\chi\gamma_\alpha\psi) - (\bar\psi\gamma_5\chi)(\bar\chi\gamma_5\psi) - \frac12(\bar\psi\gamma^\alpha\gamma_5\chi)(\bar\chi\gamma_\alpha\gamma_5\psi) \end{aligned} \tag{13.74}$$

where we used formulas $\gamma^\mu\gamma^\alpha\gamma^\mu \;=\; -2\gamma^\alpha$ and $\gamma^\mu\gamma_\alpha\gamma_\beta\gamma_\mu \;=\; 4g^{\alpha\beta}$.

Part XIV

14 Quantization of the Dirac field

14.0.5 Classical Dirac field

Now we can write down the total expression for the classical fermion field satisfying the Dirac equation:

$$\psi(x) = \sum_s \int \frac{d\!\!\bar{}^3p}{\sqrt{2E_p}}\big[(u(\vec{p},s)e^{-ipx}a^s_{\vec{p}} + v(\vec{p},s)e^{+ipx}b^{s*}_{\vec{p}}\big] \tag{14.1}$$

where s may be z-projections of the spin in a rest frame or helicities (or any other two independent spin states) and $a^s_{\vec{p}}$ and $b^{s*}_{\vec{p}}$ are some numerical functions. The complex conjugate field looks like

$$\psi^\dagger(x) = \sum_s \int \frac{d\!\!\bar{}^3p}{\sqrt{2E_p}}\big[v^\dagger(\vec{p},s)b^s_{\vec{p}}e^{-ipx} + u^\dagger(\vec{p},s)a^{s*}_{\vec{p}}e^{ipx}\big] \tag{14.2}$$

and the Dirac conjugate $\bar{\psi}(x) = \psi^\dagger(x)\gamma_0$ has the form

$$\bar{\psi}(x) = \sum_s \int \frac{d\!\!\bar{}^3p}{\sqrt{2E_p}}\big[\bar{v}(\vec{p},s)b^s_{\vec{p}}e^{-ipx} + \bar{u}(\vec{p},s)a^{s*}_{\vec{p}}e^{ipx}\big] \tag{14.3}$$

The classical fields ψ given by (14.1) and $\bar{\psi}$ given by (14.3) satisfy the Dirac equation

$$\begin{aligned} i\gamma^\mu\frac{d}{dx^\mu}\psi(x) &= m\psi(x) \\ -i\frac{d}{dx^\mu}\bar{\psi}(x)\gamma^\mu &= m\bar{\psi}(x) \end{aligned} \tag{14.4}$$

(note that the second line is a hermitian conjugation of the first line).

To quantize the Dirac field, let us choose $\psi(x)$ as a canonical coordinate. The Dirac Lagrangian is given by $\mathcal{L} \;=\; \bar{\psi}(i\not{\partial} - m)\psi$ so the canonical momentum is

$$\pi(x) \;=\; \frac{\partial\mathcal{L}}{\partial\partial_0\psi} \;=\; i\psi^\dagger(x) \tag{14.5}$$

and the Hamiltonian $H \equiv \sum p\dot{q} - L$ takes the form

$$\begin{aligned} H \;&=\; \int d^3x\ \pi(t,\vec{x})\frac{\partial}{\partial t}\psi(t,\vec{x}) - \int d^3x\ \mathcal{L}(t,\vec{x}) \\ &=\; i\int d^3x\ \psi^\dagger(t,\vec{x})\frac{\partial}{\partial t}\psi(t,\vec{x}) - \int d^3x\ \psi^\dagger(t,\vec{x})\gamma^0\big[i\gamma^0\frac{\partial}{\partial t} + i\vec{\gamma}\cdot\vec{\nabla} - m\big]\psi(t,\vec{x}) \\ &=\; \int d^3x\ \psi^\dagger(t,\vec{x})[-i\gamma^0\vec{\gamma}\cdot\vec{\nabla} + m\gamma^0]\psi(t,\vec{x}) \end{aligned} \tag{14.6}$$

Let us try to quantize Dirac field in the same way as Klein-Gordon field: promote ψ and π to operators satisfying canonical commutation relations similar to Eq. (11.43) (Spoiler: we will face trouble pretty soon)

14.0.6 Attempt to quantize Dirac field with canonical commutation relations

As usual, we promote $\psi(t=0,\vec{x})$,$\pi(t=0,\vec{x})$ to operators $\hat{\psi}(\vec{x})$, $\hat{\pi}(\vec{x})$, then similarly to Eq. (11.16) $a^s_{\vec{p}}$ and $b^s_{\vec{p}}$ get promoted to operators $\hat{a}^s_{\vec{p}}$ and $\hat{b}^s_{\vec{p}}$

$$\hat{\psi}(\vec{x}) = \sum_s \int \frac{d\!\!\bar{}^3p}{\sqrt{2E_p}} \big[u(\vec{p},s)e^{i\vec{p}\vec{x}}\hat{a}^s_{\vec{p}} + v(\vec{p},s)e^{-i\vec{p}\vec{x}}\hat{b}^{s\dagger}_{\vec{p}}\big]$$
$$\hat{\bar{\psi}}(\vec{x}) = \sum_s \int \frac{d\!\!\bar{}^3p}{\sqrt{2E_p}} \big[\bar{v}(\vec{p},s)e^{i\vec{p}\vec{x}}\hat{b}^s_{\vec{p}} + \bar{u}(\vec{p},s)e^{-i\vec{p}\vec{x}}\hat{a}^{s\dagger}_{\vec{p}}\big] \tag{14.7}$$

If we impose canonical commutation relations (CCRs) on the operators on $\hat{\psi}(\vec{x})$ and $\hat{\pi}(\vec{x})$, then similarly to Eq. (11.18) the ladder operators will satisfy the following CCRs:

$$[\hat{a}^s_{\vec{p}}, \hat{a}^{s'\dagger}_{\vec{p}'}] = (2\pi)^3\delta(\vec{p}-\vec{p}')\delta_{ss'}, [\hat{b}^s_{\vec{p}}, \hat{b}^{s'\dagger}_{\vec{p}'}] = -(2\pi)^3\delta(\vec{p}-\vec{p}')\delta_{ss'} \tag{14.8}$$
$$[\hat{a}^s_{\vec{p}}, \hat{a}^{s'}_{\vec{p}'}] = [\hat{b}^s_{\vec{p}}, \hat{b}^{s'}_{\vec{p}'}] = [\hat{a}^{s\dagger}_{\vec{p}}, \hat{a}^{s'\dagger}_{\vec{p}'}] = [\hat{b}^{s\dagger}_{\vec{p}}, \hat{b}^{s'\dagger}_{\vec{p}'}] = [\hat{a}^s_{\vec{p}}, \hat{b}^{s'\dagger}_{\vec{p}'}] = [\hat{b}^s_{\vec{p}}, \hat{a}^{s'\dagger}_{\vec{p}'}] = 0$$

Proof:

$$\begin{aligned}
&[\hat{\psi}(\vec{x}), \hat{\psi}^\dagger(\vec{y})] \\
&= \sum_{s,s'}\int \frac{d\!\!\bar{}^3p\, d\!\!\bar{}^3p'}{2\sqrt{E_pE_{p'}}}\big[u(\vec{p},s)e^{i\vec{p}\vec{x}}\hat{a}^s_{\vec{p}} + v(\vec{p},s)e^{-i\vec{p}\vec{x}}\hat{b}^{s\dagger}_{\vec{p}}, \bar{v}(\vec{p},s')e^{i\vec{p}'\vec{y}}\hat{b}^{s'}_{\vec{p}'} + \bar{u}(\vec{p},s')e^{-i\vec{p}'\vec{y}}\hat{a}^{s'\dagger}_{\vec{p}'}\big]\gamma_0 = \\
&= \sum_{s,s'}\int \frac{d\!\!\bar{}^3p\, d\!\!\bar{}^3p'}{2\sqrt{E_pE_{p'}}}\big(u(\vec{p},s)\bar{u}(\vec{p},s')e^{i\vec{p}\vec{x}-i\vec{p}'\vec{y}}[\hat{a}^s_{\vec{p}}, \hat{a}^{s'\dagger}_{\vec{p}'}] + v(\vec{p},s)\bar{v}(\vec{p},s')e^{-i\vec{p}\vec{x}+i\vec{p}'\vec{y}}[\hat{b}^{s\dagger}_{\vec{p}}, \hat{b}^{s'}_{\vec{p}'}]\big)\gamma_0 = \\
&= \sum_{s,s'}\int \frac{d\!\!\bar{}^3p\, d\!\!\bar{}^3p'}{2\sqrt{E_pE_{p'}}}(2\pi)^3\delta(\vec{p}-\vec{p}')\delta_{ss'}\big(u(\vec{p},s)\bar{u}(\vec{p},s)e^{i\vec{p}\vec{x}-i\vec{p}'\vec{y}} + v(\vec{p},s)\bar{v}(\vec{p},s)e^{-i\vec{p}\vec{x}+i\vec{p}'\vec{y}}\big)\gamma_0 = \\
&= \sum_s\int \frac{d\!\!\bar{}^3p}{2E_p}\big(u(\vec{p},s)\bar{u}(\vec{p},s)e^{i\vec{p}(\vec{x}-\vec{y})} + v(\vec{p},s)\bar{v}(\vec{p},s)e^{-i\vec{p}(\vec{x}-\vec{y})}\big)\gamma_0 = \int \frac{d\!\!\bar{}^3p}{2E_p}\big[(\not{p}+m)e^{i\vec{p}(\vec{x}-\vec{y})} + (\not{p}-m)e^{-i\vec{p}(\vec{x}-\vec{y})}\big]\gamma_0 \\
&= \int \frac{d\!\!\bar{}^3p}{2E_p}\big[(\gamma_0E_p - \vec{\gamma}\cdot\vec{p}+m)e^{i\vec{p}(\vec{x}-\vec{y})} + (\gamma_0E_p - \vec{\gamma}\cdot\vec{p}-m)e^{-i\vec{p}(\vec{x}-\vec{y})}\big]\gamma_0 = \int d\!\!\bar{}^3p\ e^{i\vec{p}(\vec{x}-\vec{y})} = \delta^3(\vec{x}-\vec{y})
\end{aligned} \tag{14.9}$$

In terms of ladder operators, the quantum Hamiltonian reads

$$\begin{aligned}
\hat{H} &= \int d^3x\ \hat{\psi}^\dagger(\vec{x})[-i\gamma^0\vec{\gamma}\cdot\vec{\nabla} + m\gamma^0]\hat{\psi}(\vec{x}) = \int d^3x\ \hat{\bar{\psi}}(\vec{x})[-i\vec{\gamma}\cdot\vec{\nabla} + m]\hat{\psi}(\vec{x}) \\
&= \sum_{s,s'}\int d^3x \int \frac{d\!\!\bar{}^3p\, d\!\!\bar{}^3p'}{2\sqrt{E_pE_{p'}}}\{\bar{v}(\vec{p},s)e^{i\vec{p}\vec{x}}\hat{b}^s_{\vec{p}} + \bar{u}(\vec{p},s)e^{-i\vec{p}\vec{x}}\hat{a}^{s\dagger}_{\vec{p}}\}\{(\vec{\gamma}\cdot\vec{p}'+m)u(\vec{p}',s')e^{i\vec{p}'\vec{x}}\hat{a}^{s'}_{\vec{p}'} - (\vec{\gamma}\cdot\vec{p}'-m)v(\vec{p}',s')e^{-i\vec{p}'\vec{x}}\hat{b}^{s'\dagger}_{\vec{p}'}\} \\
&= \sum_{s,s'}\int d^3x \int \frac{d\!\!\bar{}^3p\, d\!\!\bar{}^3p'}{2\sqrt{E_pE_{p'}}}\{\bar{v}(\vec{p},s)e^{i\vec{p}\vec{x}}\hat{b}^s_{\vec{p}} + \bar{u}(\vec{p},s)e^{-i\vec{p}\vec{x}}\hat{a}^{s\dagger}_{\vec{p}}\}\{E_{p'}\gamma_0u(\vec{p}',s')e^{i\vec{p}'\vec{x}}\hat{a}^{s'}_{\vec{p}'} - E_{p'}\gamma_0v(\vec{p}',s')e^{-i\vec{p}'\vec{x}}\hat{b}^{s'\dagger}_{\vec{p}'}\} \\
&= \frac{1}{2}\sum_{s,s'}\int d\!\!\bar{}^3p\{\bar{v}(\vec{p},s)\gamma_0u(-p,s')\hat{b}^s_{\vec{p}}\hat{a}^{s'}_{-\vec{p}} + \bar{u}(\vec{p},s)\gamma_0u(\vec{p},s')\hat{a}^{s\dagger}_{\vec{p}}\hat{a}^{s'}_{\vec{p}} - \bar{u}(\vec{p},s)\gamma_0v(-\vec{p},s')\hat{a}^{s\dagger}_{\vec{p}}\hat{b}^{s'\dagger}_{-\vec{p}} - \bar{v}(\vec{p},s)\gamma_0v(\vec{p},s')\hat{b}^s_{\vec{p}}\hat{b}^{s'\dagger}_{\vec{p}'}\} \\
&= \frac{1}{2}\sum_{s,s'}\int d\!\!\bar{}^3p\{v^\dagger(\vec{p},s)u(-p,s')\hat{b}^s_{\vec{p}}\hat{a}^{s'}_{-\vec{p}} + \bar{u}(\vec{p},s)\gamma_0u(\vec{p},s')\hat{a}^{s\dagger}_{\vec{p}}\hat{a}^{s'}_{\vec{p}} - u^\dagger(\vec{p},s)v(-\vec{p},s')\hat{a}^{s\dagger}_{\vec{p}}\hat{b}^{s'\dagger}_{-\vec{p}} - \bar{v}(\vec{p},s)\gamma_0v(\vec{p},s')\hat{b}^s_{\vec{p}}\hat{b}^{s'\dagger}_{\vec{p}'}\} \\
&= \sum_s\int d\!\!\bar{}^3p\ E_p(\hat{a}^{s\dagger}_{\vec{p}}\hat{a}^s_{\vec{p}} - \hat{b}^s_{\vec{p}}\hat{b}^{s\dagger}_{\vec{p}'}) = \sum_s\int d\!\!\bar{}^3p\ E_p(\hat{a}^{s\dagger}_{\vec{p}}\hat{a}^s_{\vec{p}} - \hat{b}^{s\dagger}_{\vec{p}'}\hat{b}^s_{\vec{p}} + (2\pi)^3\delta(0))
\end{aligned} \tag{14.10}$$

where we used the orthogonality properties $\bar{u}(\vec{p},s)\gamma_0 u(\vec{p},s') = \bar{v}(\vec{p},s)\gamma_0 v(\vec{p},s') = 2p_0\delta_{ss'}$ and $v^\dagger(\vec{p},s)u(-p,s') = u^\dagger(\vec{p},s)v(-\vec{p},s') = 0$ (see Eq. (3.65) in Peskin's textbook). If we disregard (as usual) the infinite constant, we get

$$\hat{H} = \sum_s \int đ^3p\ E_p(\hat{a}^{s\dagger}_{\vec{p}}\hat{a}^s_{\vec{p}} - \hat{b}^{s\dagger}_{\vec{p}}\hat{b}^s_{\vec{p}}) \tag{14.11}$$

The - sign means that we have Hamiltonian with the spectrum unbounded from below. Indeed, if we calculate the commutators of $\hat{H}$ with creation operators we will get

$$[\hat{H},\hat{a}^{s\dagger}_{\vec{p}}] = E_p\hat{a}^{s\dagger}_{\vec{p}}, \quad \text{but} \quad [\hat{H},\hat{b}^{s\dagger}_{\vec{p}}] = -\,\hat{b}^{s\dagger}_{\vec{p}} \tag{14.12}$$

Now, if we define the vacuum as a state $|0\rangle$ annihilated by $\hat{a}^s_{\vec{p}}$ and $\hat{b}^s_{\vec{p}}$

$$\hat{H}\hat{a}^{s\dagger}_{\vec{p}}|0\rangle = E_p\hat{a}^{s\dagger}_{\vec{p}}|0\rangle, \quad \text{but} \quad \hat{H}\hat{b}^{s\dagger}_{\vec{p}}|0\rangle = -\,E_p\hat{b}^{s\dagger}_{\vec{p}}|0\rangle \tag{14.13}$$

so the state $\hat{b}^{s\dagger}_{\vec{p}}|0\rangle$ has energy $-E_p$. Similarly, the state $\hat{b}^{s\dagger}_{\vec{p}}\hat{b}^{s'\dagger}_{\vec{p}'}|0\rangle$ will have energy $-E_p - E_{p'}$ etc. This is a sick Hamiltonian (e.g. no state(s) with lowest energy $\Rightarrow$ no consistent vacuum state). We must try some other way to quantize Dirac field.

14.1 Correct quantization of Dirac field

Correct quantization: same story as above, only we impose canonical anticommutation relations:

$$\left.\begin{array}{lcl} \{\psi(\vec{x}),\psi^\dagger(\vec{y})\} &=& \delta(\vec{x}-\vec{y}) \\ \{\psi(\vec{x}),\psi(\vec{y})\} &=& 0 \\ \{\psi^\dagger(\vec{x}),\psi^\dagger(\vec{y})\} &=& 0 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{lcl} \{a^s_{\vec{p}},a^{s'\dagger}_{\vec{p}'}\} &=& (2\pi)^3\delta(\vec{p}-\vec{p}')\delta_{ss'} \\ \{b^s_{\vec{p}},b^{s'\dagger}_{\vec{p}'}\} &=& (2\pi)^3\delta(\vec{p}-\vec{p}')\delta_{ss'} \\ \{a^s_{\vec{p}},a^{s'}_{\vec{p}'}\} &=& \{a^{s\dagger}_{\vec{p}},a^{s'\dagger}_{\vec{p}'}\} = \{b^s_{\vec{p}},b^{s'}_{\vec{p}'}\} = \{b^{s\dagger}_{\vec{p}},b^{s'\dagger}_{\vec{p}'}\} = 0 \end{array}\right. \tag{14.14}$$

(the conventional notation is $\{\hat{A},\hat{B}\} \equiv \hat{A}\hat{B} + \hat{B}\hat{A}$). With this anticommutation relations, instead of Eq. (14.11) we get

$$\hat{H} = \int d^3x\ \hat{\psi}^\dagger(\vec{x})[-i\gamma^0\vec{\gamma}\cdot\vec{\nabla} + m\gamma^0]\hat{\psi}(\vec{x}) = \int d^3x\ \hat{\bar{\psi}}(\vec{x})[-i\vec{\gamma}\cdot\vec{\nabla} + m]\hat{\psi}(\vec{x}) \tag{14.15}$$

$$= \sum_s \int đ^3p\ E_p(\hat{a}^{s\dagger}_{\vec{p}}\hat{a}^s_{\vec{p}} - \hat{b}^s_{\vec{p}}\hat{b}^{s\dagger}_{\vec{p}'}) = \sum_s \int đ^3p\ E_p(\hat{a}^{s\dagger}_{\vec{p}}\hat{a}^s_{\vec{p}} + \hat{b}^{s\dagger}_{\vec{p}'}\hat{b}^s_{\vec{p}} + (2\pi)^3\delta(0)) \to \sum_s \int đ^3p\ E_p(\hat{a}^{s\dagger}_{\vec{p}}\hat{a}^s_{\vec{p}} + \hat{b}^{s\dagger}_{\vec{p}'}\hat{b}^s_{\vec{p}})$$

and therefore

$$\hat{H} = \sum_s \int đ^3p\ E_p(\hat{a}^{s\dagger}_{\vec{p}}\hat{a}^s_{\vec{p}} + \hat{b}^{s\dagger}_{\vec{p}'}\hat{b}^s_{\vec{p}}) \tag{14.16}$$

similarly to Eq. (11.20). Note that Hamiltonian is hermitian: $\hat{H}^\dagger = \hat{H}$.

Commutators of $\hat{H}$ with ladder operators are now OK:

$$\begin{array}{lcllcl} [\hat{H},\hat{a}^{s\dagger}_{\vec{p}}] &=& E_p\hat{a}^{s\dagger}_{\vec{p}}, & [\hat{H},\hat{a}_{\vec{p}}] &=& -\,E_p\hat{a}_{\vec{p}} \\ [\hat{H},\hat{b}^{s\dagger}_{\vec{p}}] &=& E_p\hat{b}^{s\dagger}_{\vec{p}}, & [\hat{H},\hat{b}_{\vec{p}}] &=& -\,E_p\hat{b}_{\vec{p}} \end{array} \tag{14.17}$$

Quantization in Heisenberg picture

$$\hat{\psi}(t,\vec{x}) \equiv e^{i\hat{H}t}\hat{\psi}(\vec{x})e^{-i\hat{H}t}, \qquad \hat{\bar{\psi}}(t,\vec{x}) \equiv e^{i\hat{H}t}\hat{\bar{\psi}}(\vec{x})e^{-i\hat{H}t} \tag{14.18}$$

From Eq. (14.17) we see that

$$e^{i\hat{H}t}\hat{a}^s_{\vec{p}}e^{-i\hat{H}t} = \sum_{n=0}^{\infty} i^n\frac{t^n}{n!}[\hat{H},[\hat{H},....[\hat{H},\hat{a}^s_{\vec{p}}]]] = \sum_{n=0}^{\infty} i^n\frac{t^n}{n!}(-E_p)^n\hat{a}_{\vec{p}} = \hat{a}^s_{\vec{p}}e^{-iE_pt}$$
$$e^{i\hat{H}t}\hat{a}^{s\dagger}_{\vec{p}}e^{-i\hat{H}t} = \sum_{n=0}^{\infty} i^n\frac{t^n}{n!}[\hat{H},[\hat{H},....[\hat{H},\hat{a}^{s\dagger}_{\vec{p}}]]] = \sum_{n=0}^{\infty} i^n\frac{t^n}{n!}(E_p)^n\hat{a}_{\vec{p}} = \hat{a}^{s\dagger}_{\vec{p}}e^{iE_pt} \tag{14.19}$$

(cf. Eq. (6.10)) and similarly for $\hat{b}$ operators:

$$e^{i\hat{H}t}\hat{b}^s_{\vec{p}}e^{-i\hat{H}t} = \hat{b}^s_{\vec{p}}e^{-iE_pt}, \quad e^{i\hat{H}t}\hat{b}^{s\dagger}_{\vec{p}}e^{-i\hat{H}t} = \hat{b}^{s\dagger}_{\vec{p}}e^{iE_pt} \tag{14.20}$$

and therefore ($x = (t,\vec{x})$)

$$\hat{\psi}(x) = e^{i\hat{H}t}\hat{\psi}(\vec{x})e^{-i\hat{H}t} = \sum_s\int\frac{d\!\!\!{}^-{}^3p}{\sqrt{2E_p}}\big[u(\vec{p},s)e^{-ipx}\hat{a}^s_{\vec{p}} + v(\vec{p},s)e^{ipx}\hat{b}^{s\dagger}_{\vec{p}}\big]$$
$$\hat{\bar{\psi}}(\vec{x}) = e^{i\hat{H}t}\hat{\bar{\psi}}(\vec{x})e^{-i\hat{H}t} = \sum_s\int\frac{d\!\!\!{}^-{}^3p}{\sqrt{2E_p}}\big[\bar{v}(\vec{p},s)e^{-ipx}\hat{b}^s_{\vec{p}} + \bar{u}(\vec{p},s)e^{ipx}\hat{a}^{s\dagger}_{\vec{p}}\big] \tag{14.21}$$

where we used decomposition of $\hat{\psi}(\vec{x})$ and $\hat{\psi}(\vec{x})$ in ladder operators (14.7). Now, from this equation and Dirac equations for spinors $\not{p}u(p) = mu(p)$, $\not{p}v(p) = -mu(p)$ one can easily see that quantum operators $\hat{\psi}(x)$ and $\hat{\bar{\psi}}(x)$ satisfy the same Dirac equations

$$i\gamma^\mu\frac{d}{dx^\mu}\hat{\psi}(x) = m\hat{\psi}(x)$$
$$-i\frac{d}{dx^\mu}\hat{\bar{\psi}}(x)\gamma^\mu = m\hat{\bar{\psi}}(x) \tag{14.22}$$

as their classical counterparts, see Eq. (14.4).

Let us now prove equal-time (anti)commutators

$$\{\hat{\psi}(t,\vec{x}),\hat{\psi}^\dagger(t,\vec{y})\} = \delta(\vec{x}-\vec{y}), \quad \{\hat{\psi}(t,\vec{x}),\hat{\psi}(t,\vec{y})\} = \{\hat{\psi}^\dagger(t,\vec{x}),\hat{\psi}^\dagger(t,\vec{y})\} = 0 \tag{14.23}$$

Proof:

$$\begin{aligned}
&\{\hat{\psi}_\xi(t,\vec{x}),\hat{\psi}^\dagger_\eta(t,\vec{y})\}\\
&= \int\frac{d\!\!\!{}^-{}^3p\,d\!\!\!{}^-{}^3p'}{2\sqrt{E_pE_{p'}}}\sum_{s,s'}\{u_\xi(\vec{p},s)e^{-ipx}\hat{a}^s_{\vec{p}} + v_\xi(\vec{p},s)e^{ipx}\hat{b}^{s\dagger}_{\vec{p}},\bar{v}_\zeta(\vec{p}',s')e^{-ip'y}\hat{b}^{s'}_{\vec{p}'} + \bar{u}_\zeta(\vec{p}',s')e^{ip'y}\hat{a}^{s'\dagger}_{\vec{p}'}\}(\gamma_0)_{\zeta\eta}\\
&= \int\frac{d\!\!\!{}^-{}^3p\,d\!\!\!{}^-{}^3p'}{2\sqrt{E_pE_{p'}}}\sum_{s,s'}\big(u_\xi(\vec{p},s)\bar{u}_\zeta(\vec{p}',s')e^{-ipx+ip'y}\{\hat{a}^s_{\vec{p}},\hat{a}^{s'\dagger}_{\vec{p}'}\} + v_\xi(\vec{p},s)\bar{v}_\zeta(\vec{p}',s')e^{ipx-ip'y}\{\hat{b}^s_{\vec{p}},\hat{b}^{s'\dagger}_{\vec{p}'}\}\big)(\gamma_0)_{\zeta\eta}\\
&= \int\frac{d\!\!\!{}^-{}^3p\,d\!\!\!{}^-{}^3p'}{2\sqrt{E_pE_{p'}}}\sum_{s,s'}\big(u_\xi(\vec{p},s)\bar{u}_\zeta(\vec{p}',s')e^{-i(E_p-E_{p'})t+i\vec{p}\vec{x}-i\vec{p}'\vec{y}}\{\hat{a}^s_{\vec{p}},\hat{a}^{s'\dagger}_{\vec{p}'}\} + v_\xi(\vec{p},s)\bar{v}_\zeta(\vec{p}',s')e^{i(E_p-E_{p'})t-i\vec{p}\vec{x}+i\vec{p}'\vec{y}}\{\hat{b}^s_{\vec{p}},\hat{b}^{s'\dagger}_{\vec{p}'}\}\big)(\gamma_0)_{\zeta\eta}\\
&= \int\frac{d\!\!\!{}^-{}^3p}{2E_p}\sum_s\big(u_\xi(\vec{p},s)\bar{u}_\zeta(\vec{p},s)e^{i\vec{p}(\vec{x}-\vec{y})} + v_\xi(\vec{p},s)\bar{v}_\zeta(\vec{p},s)e^{-i\vec{p}(\vec{x}-\vec{y})}\big)(\gamma_0)_{\zeta\eta}\\
&= \int\frac{d\!\!\!{}^-{}^3p}{2E_p}\big[(\not{p}+m)\gamma_0e^{i\vec{p}(\vec{x}-\vec{y})} + (\not{p}-m)\gamma_0e^{-i\vec{p}(\vec{x}-\vec{y})}\big]_{\xi\eta}\\
&= \int\frac{d\!\!\!{}^-{}^3p}{2E_p}\big[(E_p-\vec{p}\cdot\vec{\gamma}\gamma_0+m\gamma_0)e^{i\vec{p}(\vec{x}-\vec{y})} + (E_p+\vec{p}\cdot\vec{\gamma}\gamma_0-m\gamma_0)e^{+i\vec{p}(\vec{x}-\vec{y})}\big]_{\xi\eta} = \delta(\vec{x}-\vec{y})\delta_{\xi\eta}
\end{aligned} \tag{14.24}$$

Vacuum and one-particle states

By definition, vacuum $|0\rangle$ is a state annihilated by both $\hat{a}^s_{\vec{p}}$ and $\hat{b}^s_{\vec{p}}$:

$$\hat{a}^s_{\vec{p}}|0\rangle \;=\; \hat{b}^s_{\vec{p}}|0\rangle \;=\; 0 \tag{14.25}$$

Similarly to Eqs. (11.27) we can try

$$a^{s\dagger}_{\vec{p}}|0\rangle \quad \text{and} \quad b^{s\dagger}_{\vec{p}}|0\rangle \tag{14.26}$$

as a candidates for one-particle states. From Eq. (14.17) we see that they are eigenstates of the Dirac Hamiltonian with eigenvalue E_p

$$\hat{H}\hat{a}^{s\dagger}_{\vec{p}}|0\rangle \;=\; [\hat{H},\hat{a}^{s\dagger}_{\vec{p}}]|0\rangle \;=\; E_p\hat{a}^{s\dagger}_{\vec{p}}|0\rangle, \quad \hat{H}\hat{b}^{s\dagger}_{\vec{p}}|0\rangle \;=\; [\hat{H},\hat{b}^{s\dagger}_{\vec{p}}]|0\rangle \;=\; E_p\hat{b}^{s\dagger}_{\vec{p}}|0\rangle \tag{14.27}$$

To finish the proof of one-particle interpretation of states (14.39) we need to prove that they are the eigenstates of the momentum operator. Thus, the next step is construction of quantum operator of momentum for Dirac field.

14.2 Momentum operator in the Dirac theory

14.2.1 Momentum of classical Dirac field

The general definition of classical energy-momentum tensor is $T_{\mu\nu} \;=\; \partial_\mu\Phi\frac{\partial\mathcal{L}}{\partial\partial^\nu\Phi} - \mathcal{L}g_{\mu\nu}$. For Dirac field it gives

$$T_{\mu\nu} \;=\; \frac{\partial\mathcal{L}}{\partial\partial^\nu\psi}\partial_\mu\psi + \partial_\mu\bar{\psi}\frac{\partial\mathcal{L}}{\partial\partial^\nu\bar{\psi}} - \mathcal{L}g_{\mu\nu} \;=\; i\bar{\psi}\gamma_\nu\partial_\mu\psi - g_{\mu\nu}\bar{\psi}(i\not{\partial}-m)\psi \;=\; i\bar{\psi}\gamma_\nu\partial_\mu\psi \tag{14.28}$$

Up to a total derivative it can be rewritten in a symmetric form

$$\begin{aligned} T_{\mu\nu}(x) \;&=\; \frac{i}{4}\Big(\bar{\psi}(x)\gamma_\mu\frac{\partial\psi(x)}{\partial x^\nu} - \frac{\partial\bar{\psi}(x)}{\partial x^\nu}\gamma_\mu\psi(x) + \bar{\psi}(x)\gamma_\nu\frac{\partial\psi(x)}{\partial x^\mu} - \frac{\partial\bar{\psi}(x)}{\partial x^\mu}\gamma_\nu\psi(x)\Big) \\ &\equiv\; \frac{i}{4}\bar{\psi}(x)\big(\gamma_\mu \frac{\overleftrightarrow{\partial}}{\partial x^\nu} + \gamma_\nu \frac{\overleftrightarrow{\partial}}{\partial x^\mu}\big)\psi(x) \end{aligned} \tag{14.29}$$

where $A(x) \frac{\overleftrightarrow{\partial}}{\partial x^\nu} B(x) \equiv A(x)\frac{\partial}{\partial x^\nu}B(x) - \big(\frac{\partial}{\partial x^\nu}A(x)\big)B(x)$.

Similarly to the case of KG field

$$\begin{aligned} \partial^\mu T_{\mu\nu}(x) \;&=\; \frac{i}{4}\frac{\partial}{\partial x_\mu}\Big(\bar{\psi}(x)\gamma_\mu\frac{\partial\psi(x)}{\partial x^\nu} - \frac{\partial\bar{\psi}(x)}{\partial x^\nu}\gamma_\mu\psi(x) + \bar{\psi}(x)\gamma_\nu\frac{\partial\psi(x)}{\partial x^\mu} - \frac{\partial\bar{\psi}(x)}{\partial x^\mu}\gamma_\nu\psi(x)\Big) \\ &=\; \frac{i}{4}\big[\partial^\mu\bar{\psi}(x)\gamma_\mu\partial_\nu\psi(x) + \bar{\psi}(x)\gamma_\mu\partial^\mu\partial_\nu\psi(x) - \partial_\nu\bar{\psi}(x)\gamma_\mu\partial^\mu\psi(x) - \partial^\mu\partial_\nu\bar{\psi}(x)\gamma_\mu\psi(x) \\ &+ \partial^\mu\bar{\psi}(x)\gamma_\nu\partial_\mu\psi(x) + \bar{\psi}(x)\gamma_\nu\partial^2\psi(x) - \partial^2\bar{\psi}(x)\gamma_\nu\psi(x) - \partial_\mu\bar{\psi}(x)\gamma_\nu\partial^\mu\psi(x)\big] \\ &=\; \frac{1}{4}\big[-m\bar{\psi}(x)\partial_\nu\psi(x) + m\bar{\psi}(x)\partial_\nu\psi(x) - m\partial_\nu\bar{\psi}(x)\psi(x) + m\partial_\nu\bar{\psi}(x)\psi(x) \\ &+ \partial^\mu\bar{\psi}(x)\gamma_\nu\partial_\mu\psi(x) - im^2\bar{\psi}(x)\gamma_\nu\psi(x) + im^2\bar{\psi}(x)\gamma_\nu\psi(x) - \partial_\mu\bar{\psi}(x)\gamma_\nu\partial^\mu\psi(x)\big] \;=\; 0 \end{aligned} \tag{14.30}$$

where we used Eqs. (13.20) and (13.29).

The equation (14.30) leads to conservation of the momentum:

$$\partial_\mu T^{\mu i} = 0 \quad\Rightarrow\quad \int d^3x \big(\partial_0 T^{0i} + \partial_k T^{ki}\big) = 0 \quad\Rightarrow\quad \frac{d}{dt}\int d^3x \; T^{0i}(t,\vec{x}) = 0 \qquad (14.31)$$

where we use integration by parts to get rid of the second term. Now, similarly to the KG case (see Eq. (5.31)), we can interpret the integral in the r.h.s of this equation as the momentum of classical Dirac field:

$$P^i = \int d^3x \; T^{0i}(t,\vec{x}) = \frac{i}{4}\int d^3x \; \big[\bar{\psi}(t,\vec{x})\gamma_0 \overleftrightarrow{\partial^i} \psi(t,\vec{x}) + \bar{\psi}(t,\vec{x})\gamma^i \overleftrightarrow{\partial_t} \psi(t,\vec{x})\big] \qquad (14.32)$$

14.2.2 Momentum operator in Dirac theory

As usual, to get quantum momentum operator we take the expression for classical momentum and promote canonical coordinates and canonical momenta (ψ and $\psi^\dagger$ in our case) to operators

$$\hat{P}^i = \int d^3x \; T^{0i}(t,\vec{x}) = \frac{i}{4}\int d^3x \; \big[\hat{\bar{\psi}}(t,\vec{x})\gamma_0 \overleftrightarrow{\partial^i} \hat{\psi}(t,\vec{x}) + \hat{\bar{\psi}}(t,\vec{x})\gamma^i \overleftrightarrow{\partial_t} \hat{\psi}(t,\vec{x})\big] \qquad (14.33)$$

In terms of ladder operators it takes the form

$$\hat{P}^i = \sum_s \int d\!\!\bar{}^3p \; p^i \big(\hat{a}^{s\dagger}_{\vec{p}}\hat{a}^s_{\vec{p}} + \hat{b}^{s\dagger}_{\vec{p}}\hat{b}^s_{\vec{p}}\big) \qquad (14.34)$$

where we used formulas

$$\begin{aligned}
\bar{u}(\vec{p},s)\gamma_0 u(\vec{p},s') &= \bar{v}(\vec{p},s)\gamma_0 v(\vec{p},s') = 2E_p\delta_{ss'}, \qquad \bar{u}(\vec{p},s)\gamma^i u(\vec{p},s') = \bar{v}(\vec{p},s)^i v(\vec{p},s') = 2p^i\delta_{ss'},\\
\bar{v}(\vec{p},s)\gamma_0 u(-\vec{p},s') &= v^\dagger(\vec{p},s)u(-\vec{p},s') = 0, \qquad \bar{u}(\vec{p},s)\gamma_0 v(-\vec{p},s') = u^\dagger(\vec{p},s)v(-\vec{p},s') = 0
\end{aligned} \qquad (14.35)$$

HW 4: prove Eq. (14.33) ⇒ Eq. (14.37).

Proof:

$$\begin{aligned}
\hat{P}^i &= \frac{i}{4}\int d^3x \sum_{s,s'}\int \frac{d\!\!\bar{}^3p\, d\!\!\bar{}^3p'}{2\sqrt{E_pE_{p'}}}\big[\bar{v}_\zeta(-\vec{p},s)\hat{b}^s_{-\vec{p}}e^{-iE_pt} + \bar{u}_\zeta(\vec{p},s)\hat{a}^{s\dagger}_{\vec{p}}e^{iE_pt}\big]e^{-i\vec{p}\cdot\vec{x}}\big(\gamma_0 \overleftrightarrow{\partial^i} + \gamma^i \overleftrightarrow{\partial^0}\big)_{\zeta\xi} \qquad (14.36)\\
&\times \big[u_\xi(\vec{p}',s')\hat{a}^{s'}_{\vec{p}'}e^{-iE_{p'}t} + v_\xi(-\vec{p}',s')\hat{b}^{s'\dagger}_{-\vec{p}'}e^{iE_{p'}t}\big]e^{i\vec{p}'\cdot\vec{x}}\\
&= \frac{1}{4}\int d^3x \sum_{s,s'}\int \frac{d\!\!\bar{}^3p\, d\!\!\bar{}^3p'}{2\sqrt{E_pE_{p'}}}\Big\{\big[\bar{v}(-\vec{p},s)\hat{b}^s_{-\vec{p}}e^{-iE_pt} + \bar{u}(\vec{p},s)\hat{a}^{s\dagger}_{\vec{p}}e^{iE_pt}\big]e^{-i\vec{p}\cdot\vec{x}}\gamma_0(p+p')^i\\
&\qquad\qquad \times \big[u(\vec{p}',s')\hat{a}^{s'}_{\vec{p}'}e^{-iE_{p'}t} + v(-\vec{p}',s')\hat{b}^{s'\dagger}_{-\vec{p}'}e^{iE_{p'}t}\big]e^{i\vec{p}'\cdot\vec{x}}\\
&+ \big[\bar{v}(-\vec{p},s)\hat{b}^s_{-\vec{p}}e^{-iE_pt} + \bar{u}(\vec{p},s)\hat{a}^{s\dagger}_{\vec{p}}e^{iE_pt}\big]e^{-i\vec{p}\cdot\vec{x}}\gamma^i\big[E_{p'}u(\vec{p}',s')\hat{a}^{s'}_{\vec{p}'}e^{-iE_{p'}t} - E_{p'}v(-\vec{p}',s')\hat{b}^{s'\dagger}_{-\vec{p}'}e^{iE_{p'}t}\big]e^{i\vec{p}'\cdot\vec{x}}\\
&+ \big[-E_p\bar{v}(-\vec{p},s)\hat{b}^s_{-\vec{p}}e^{-iE_pt} + E_p\bar{u}(\vec{p},s)\hat{a}^{s\dagger}_{\vec{p}}e^{iE_pt}\big]e^{-i\vec{p}\cdot\vec{x}}\gamma^i\big[E_{p'}u(\vec{p}',s')\hat{a}^{s'}_{\vec{p}'}e^{-iE_{p'}t} + v(-\vec{p}',s')\hat{b}^{s'\dagger}_{-\vec{p}'}e^{iE_{p'}t}\big]e^{i\vec{p}'\cdot\vec{x}}
\end{aligned}$$

$$= \frac{1}{8}\sum_{s,s'}\int đ^3p\Big\{\frac{2p^i}{E_p}\big[\bar{v}(-\vec{p},s)\hat{b}^s_{-\vec{p}}e^{-iE_pt} + \bar{u}(\vec{p},s)\hat{a}^{s\dagger}_{\vec{p}}e^{iE_pt}\big]\gamma_0\big[u(\vec{p},s')\hat{a}^{s'}_{\vec{p}}e^{-iE_pt} + v(-\vec{p},s')\hat{b}^{s'\dagger}_{-\vec{p}}e^{iE_pt}\big]$$
$$+ \big[\bar{v}(-\vec{p},s)\hat{b}^s_{-\vec{p}}e^{-iE_pt} + \bar{u}(\vec{p},s)\hat{a}^{s\dagger}_{\vec{p}}e^{iE_pt}\big]\gamma^i\big[u(\vec{p},s')\hat{a}^{s'}_{\vec{p}}e^{-iE_pt} - v(-\vec{p},s')\hat{b}^{s'\dagger}_{-\vec{p}}e^{iE_pt}\big]$$
$$+ \big[-\bar{v}(-\vec{p},s)\hat{b}^s_{-\vec{p}}e^{-iE_pt} + \bar{u}(\vec{p},s)\hat{a}^{s\dagger}_{\vec{p}}e^{iE_pt}\big]\gamma^i\big[u(\vec{p},s')\hat{a}^{s'}_{\vec{p}}e^{-iE_pt} + v(-\vec{p},s')\hat{b}^{s'\dagger}_{-\vec{p}}e^{iE_pt}\big]\Big\}$$
$$= \frac{1}{4}\sum_{s,s'}\int đ^3p\ \Big\{\frac{p^i}{E_p}\big[\bar{v}(-\vec{p},s)\gamma_0 u(\vec{p},s')\hat{b}^s_{-\vec{p}}\hat{a}^{s'}_{\vec{p}}e^{-2iE_pt} + \bar{u}(\vec{p},s)\gamma_0 u(\vec{p},s')\hat{a}^{s\dagger}_{\vec{p}}\hat{a}^{s'}_{\vec{p}}$$
$$+ \bar{v}(-\vec{p},s)\gamma_0 v(-\vec{p},s')\hat{b}^s_{-\vec{p}}\hat{b}^{s'\dagger}_{-\vec{p}} + \bar{u}(\vec{p},s)\gamma_0 v(-\vec{p},s')\hat{a}^{s\dagger}_{\vec{p}}\hat{b}^{s'\dagger}_{-\vec{p}}\ e^{2iE_pt}$$
$$+ \big[\bar{u}(\vec{p},s)\gamma^i u(\vec{p},s')\hat{a}^{s\dagger}_{\vec{p}}\hat{a}^{s'}_{\vec{p}} - \ \bar{v}(-\vec{p},s)\gamma^i v(-\vec{p},s')\hat{b}^s_{-\vec{p}}\hat{b}^{s'\dagger}_{-\vec{p}}\big]\Big\}$$

Now we use formulas (14.35) and get

$$\hat{P}^i \ = \ \frac{1}{2}\sum_s\int đ^3p\ \{p^i[\hat{a}^{s\dagger}_{\vec{p}}\hat{a}^s_{\vec{p}} + \hat{b}^s_{-\vec{p}}\hat{b}^{s\dagger}_{-\vec{p}}] + \ [p^i\hat{a}^{s\dagger}_{\vec{p}}\hat{a}^s_{\vec{p}} + p^i\hat{b}^s_{-\vec{p}}\hat{b}^{s\dagger}_{-\vec{p}}]\}$$
$$= \ \sum_s\int đ^3p\ p^i(\hat{a}^{s\dagger}_{\vec{p}}\hat{a}^s_{\vec{p}} - \hat{b}^s_{\vec{p}}\hat{b}^{s\dagger}_{\vec{p}}) \ = \ \sum_s\int đ^3p\ p^i(\hat{a}^{s\dagger}_{\vec{p}}\hat{a}^s_{\vec{p}} + \hat{b}^{s\dagger}_{\vec{p}}\hat{b}^s_{\vec{p}}), \qquad \text{Q.E.D.} \qquad (14.37)$$

The commutators of the momentum operator with ladder operators are

$$[\hat{P}^i,\hat{a}^{s\dagger}_{\vec{p}}] \ = \ p^i\hat{a}^{s\dagger}_{\vec{p}}, \quad [\hat{P}^i,\hat{b}^{s\dagger}_{\vec{p}}] \ = \ p^i\hat{b}^{s\dagger}_{\vec{p}}, \quad [\hat{P}^i,\hat{a}^s_{\vec{p}}] \ = \ -p^i\hat{a}^s_{\vec{p}}, \quad [\hat{P}^i,\hat{b}^s_{\vec{p}}] \ = \ -p^i\hat{b}^s_{\vec{p}} \qquad (14.38)$$

and it is easy to see now that the states (14.39) are eigenstates of the momentum operator (14.37)

$$\hat{P}^i a^{s\dagger}_{\vec{p}}|0\rangle \ = \ [\hat{P}^i, a^{s\dagger}_{\vec{p}}]|0\rangle \ = \ p^i a^{s\dagger}_{\vec{p}}|0\rangle, \qquad \hat{P}^i b^{s\dagger}_{\vec{p}}|0\rangle \ = \ [\hat{P}^i, b^{s\dagger}_{\vec{p}}]|0\rangle \ = \ p^i b^{s\dagger}_{\vec{p}}|0\rangle \qquad (14.39)$$

with eigenvalues p^i. Thus, the states

$$|p,s,-\rangle \ \equiv \ \sqrt{2E_p}a^{s\dagger}_{\vec{p}}|0\rangle \quad \text{and} \quad |p,s,+\rangle \ \equiv \ \sqrt{2E_p}b^{s\dagger}_{\vec{p}}|0\rangle \qquad (14.40)$$

are one-particle fermion and antifermion states (in QED they will be one- electron and one-positron states).

Now we can define the operator of 4-momentum:

$$\hat{P}^\mu \ = \ (\hat{H},\hat{P}^i) \ = \ \sum_s\int đ^3p\ p^\mu(\hat{a}^{s\dagger}_{\vec{p}}\hat{a}^s_{\vec{p}} + \hat{b}^{s\dagger}_{\vec{p}}\hat{b}^s_{\vec{p}}) \qquad (14.41)$$

(in the r.h.s. $p^0 \equiv E_p$). The commutators of the operator of 4-momentum with ladder operators (14.27) and (14.28) can be combined as

$$[\hat{P}^\mu,\hat{a}^{s\dagger}_{\vec{p}}] \ = \ p^\mu\hat{a}^{s\dagger}_{\vec{p}}, \quad [\hat{P}^\mu,\hat{b}^{s\dagger}_{\vec{p}}] \ = \ p^\mu\hat{b}^{s\dagger}_{\vec{p}}, \quad [\hat{P}^i,\hat{a}^s_{\vec{p}}] \ = \ -p^\mu\hat{a}^s_{\vec{p}}, \quad [\hat{P}^\mu,\hat{b}^s_{\vec{p}}] \ = \ -p^\mu\hat{b}^s_{\vec{p}} \quad (14.42)$$

Similarly to the KG case one can prove that $\hat{P}$ generates shifts in the coordinate space:

$$e^{i\hat{P}a}\hat{\psi}(x)e^{i\hat{P}a} = \ \hat{\psi}(x+a) \qquad (14.43)$$

Interesting question: which quantum operator generates Lorentz transformations (rotations and boosts)?

15 Quantum generators of Lorentz transformations

The classical Dirac field is Lorentz-transformed according to Eq. (13.18)

$$\psi'_\xi(x') \;=\; \mathbf{\Lambda}_{\frac{1}{2}}\psi(x) \;=\; \left(e^{-\frac{i}{2}\omega^{\mu\nu}\mathbf{S}_{\mu\nu}}\right)_{\xi\eta}\psi_\eta(x) \tag{15.1}$$

If we are interested in the field at the same point after Lorentz transformation, the formula is a combination of Eq. (13.18) and Eq. (12.14)

$$\psi'(x) \;=\; \mathbf{\Lambda}_{\frac{1}{2}}\psi(\mathbf{\Lambda}^{-1}x) \;=\; \left(e^{-\frac{i}{2}\omega^{\mu\nu}\mathbf{S}_{\mu\nu}}\right)_{\xi\eta}e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}\psi_\eta(x) \tag{15.2}$$

similarly to Eq. (12.49) for vectors. Our aim is to find a quantum operator $\hat{M}^{\mu\nu}$ which generates this transformation

$$\hat{\psi}'_\xi(x) \;=\; e^{\frac{i}{2}\omega_{\alpha\beta}\hat{M}^{\alpha\beta}}\hat{\psi}_\xi(x)e^{-\frac{i}{2}\omega_{\alpha\beta}\hat{M}^{\alpha\beta}} \;=\; \left(e^{-\frac{i}{2}\omega^{\mu\nu}\mathbf{S}_{\mu\nu}}\right)_{\xi\eta}e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}\hat{\psi}_\eta(x) \tag{15.3}$$

Similarly to the operator of momentum (14.33) we may try an ansatz

$$\hat{M}^{\alpha\beta} \;=\; \int d^3x\hat{\bar{\psi}}(t,\vec{x})\Gamma^{\alpha\beta}\hat{\psi}(t,\vec{x}) \tag{15.4}$$

where $\Gamma^{\alpha\beta}$ is some matrix and/or differential operator [13]. To find this operator we can take small $\omega_{\alpha\beta}$ and expand both sides of Eq. (15.3) up to the first nontrivial order in $\omega_{\alpha\beta}$. We get

$$\left.\begin{array}{lcc}\text{l.h.s.} & = & \hat{\psi}_\xi(x)+\frac{i}{2}\omega_{\alpha\beta}[\hat{M}^{\alpha\beta},\hat{\psi}_\xi(x)] \\ \text{r.h.s.} & = & \hat{\psi}_\xi(x)-\frac{i}{2}\omega_{\alpha\beta}(\mathbf{S}^{\alpha\beta}_{\xi\eta}+\delta_{\xi\eta}J^{\alpha\beta})\hat{\psi}_\xi(x)\end{array}\right\} \;\Rightarrow\; [\hat{M}^{\alpha\beta},\hat{\psi}_\xi(x)] \;=\; -(\mathbf{S}^{\alpha\beta}_{\xi\eta}+\delta_{\xi\eta}J^{\alpha\beta})\hat{\psi}_\xi(x) \tag{15.5}$$

Substituting the ansatz (15.4) to the l.h.s. of this equation, we obtain

$$\int d^3z\big[\hat{\bar{\psi}}(t',\vec{z})\Gamma^{\alpha\beta}\hat{\psi}(t',\vec{z}),\hat{\psi}(t,\vec{x})\big] \;=\; -\left(\mathbf{S}^{\alpha\beta}_{\xi\eta}+\delta_{\xi\eta}J^{\alpha\beta}\right)\hat{\psi}_\xi(t,\vec{x}) \tag{15.6}$$

We will first consider this equation at $t = t' = 0$, find $\Gamma^{\alpha\beta}$, and then prove the above equation at arbitrary t and t'.

First, at $t = t' = 0$ we get

$$\begin{aligned}(\mathbf{S}^{\alpha\beta}_{\xi\eta}+\delta_{\xi\eta}J^{\alpha\beta})\hat{\psi}_\xi(\vec{x}) \;&=\; -\int d^3z\big[\hat{\bar{\psi}}(\vec{z})_\zeta\Gamma^{\alpha\beta}_{\zeta\eta}\hat{\psi}_\zeta(\vec{z}),\hat{\psi}_\xi(\vec{x})\big] \;=\; \int d^3z\{\hat{\bar{\psi}}(\vec{z})_\zeta,\hat{\psi}_\xi(\vec{x})\}\Gamma^{\alpha\beta}_{\zeta\eta}\hat{\psi}_\eta(\vec{z}) \\ =\; (\gamma_0)_{\xi\zeta}\int d^3z\delta(\vec{x}-\vec{z})\Gamma^{\alpha\beta}_{\zeta\eta}\hat{\psi}_\eta(\vec{z}) \;&=\; (\gamma_0\Gamma^{\alpha\beta})_{\xi\eta}\hat{\psi}_\eta(\vec{x}) \quad\Rightarrow\quad \Gamma^{\alpha\beta}_{\xi\eta} \;=\; \left(\gamma_0\mathbf{S}^{\alpha\beta}+\gamma_0J^{\alpha\beta}\right)_{\xi\eta}\end{aligned} \tag{15.7}$$

so our candidate for $\hat{M}^{\alpha\beta}$ is

$$\hat{M}^{\alpha\beta} \;=\; \int d^3z\hat{\bar{\psi}}(z_0,\vec{z})\gamma_0\big[\mathbf{S}^{\alpha\beta}+i(z^\alpha\frac{\partial}{\partial z_\beta}-\alpha\leftrightarrow\beta)\big]\hat{\psi}(z_0,\vec{z}) \tag{15.8}$$

[13] It looks like a bad ansatz since it appears that $\hat{M}^{\alpha\beta}$ depends on t but we will show below that for our solution (15.8) it actually does not depend on t

Let us first prove that the r.h.s. of this equation does not depend on $z_0 \equiv t$

$$\begin{aligned}
&\frac{d}{dt}\int d^3z\hat{\bar{\psi}}(t,\vec{z})\gamma_0\big[\mathbf{S}^{\alpha\beta}+i(z^\alpha\frac{\partial}{\partial z_\beta}-\alpha\leftrightarrow\beta)\big]\hat{\psi}(t,\vec{z})\\
&= \int d^3z\int d^3z\big(\frac{\partial}{\partial t}\hat{\bar{\psi}}(t,\vec{z})\gamma_0+\hat{\bar{\psi}}(t,\vec{z})\gamma_0\frac{\partial}{\partial t}\big)\big[\mathbf{S}^{\alpha\beta}+i(z^\alpha\frac{\partial}{\partial z_\beta}-\alpha\leftrightarrow\beta)\big]\hat{\psi}(t,\vec{z}) \qquad (15.9)
\end{aligned}$$

From Eq. (14.22) we see that $\frac{\partial}{\partial t}\bar{\psi}(z)\gamma^0 = -\frac{\partial}{\partial z^i}\bar{\psi}(z)\gamma^i + im\bar{\psi}(z)$ and therefore

$$\begin{aligned}
\text{r.h.s. of Eq. 15.9} \;=\; &\int d^3z\,\big(im\hat{\bar{\psi}}(t,\vec{z})-\frac{\partial}{\partial z^i}\hat{\bar{\psi}}(t,\vec{z})\gamma^i+\hat{\bar{\psi}}(t,\vec{z})\gamma_0\frac{\partial}{\partial t}\big)\big[\mathbf{S}^{\alpha\beta}+i(z^\alpha\frac{\partial}{\partial z_\beta}-\alpha\leftrightarrow\beta)\big]\hat{\psi}(t,\vec{z})\\
\overset{\text{by parts}}{=} &\int d^3z\;\hat{\bar{\psi}}(t,\vec{z})\big(\gamma_0\frac{\partial}{\partial t}+\gamma^i\frac{\partial}{\partial z^i}+im\big)\big[\mathbf{S}^{\alpha\beta}+i(z^\alpha\frac{\partial}{\partial z_\beta}-\alpha\leftrightarrow\beta)\big]\hat{\psi}(t,\vec{z})\\
= &\int d^3z\;\hat{\bar{\psi}}(t,\vec{z})\big(\gamma^\mu\frac{\partial}{\partial z^\mu}+im\big)\big[\mathbf{S}^{\alpha\beta}+i(z^\alpha\frac{\partial}{\partial z_\beta}-\alpha\leftrightarrow\beta)\big]\hat{\psi}(t,\vec{z})\\
= &\int d^3z\;\hat{\bar{\psi}}(t,\vec{z})\Big\{\big[\mathbf{S}^{\alpha\beta}+i(z^\alpha\frac{\partial}{\partial z_\beta}-\alpha\leftrightarrow\beta)\big]\big(\gamma^\mu\frac{\partial}{\partial z^\mu}+im\big)+[\gamma^\mu,\mathbf{S}^{\alpha\beta}]\frac{\partial}{\partial z^\mu}+i(\gamma^\alpha\frac{\partial}{\partial z_\beta}-\alpha\leftrightarrow\beta)\hat{\psi}(t,\vec{z})\Big\}\\
= &\int d^3z\;\hat{\bar{\psi}}(t,\vec{z})\Big(\frac{i}{2}[\gamma^\mu,\gamma^\alpha\gamma^\beta]\frac{\partial}{\partial z^\mu}+i(\gamma^\alpha\frac{\partial}{\partial z_\beta}-\alpha\leftrightarrow\beta)\hat{\psi}(t,\vec{z})\Big) \;=\; 0 \qquad (15.10)
\end{aligned}$$

Finally, let us prove Eq. (15.11) for arbitrry t and t'. Since out $M^{\alpha\beta}$ does not depend on time one may put $t'=t$ in Eq. (15.11)

$$\int d^3z\big[\hat{\bar{\psi}}(t,\vec{z})\Gamma^{\alpha\beta}\hat{\psi}(t,\vec{z}),\hat{\psi}(t,\vec{x})\big] \;=\; -\big(\mathbf{S}^{\alpha\beta}_{\xi\eta}+\delta_{\xi\eta}J^{\alpha\beta}\big)\hat{\psi}_\xi(t,\vec{x}) \qquad (15.11)$$

which is evident since equal-time commutators look exactly like commutators at $t=0$ so one can repeat the derivation of Eq. (15.7) at arbitrary t.

Thus, we have proved that with $\hat{M}^{\alpha\beta}$ given by Eq. (15.8) the l.h.s. and r.h.s of Eq. (15.3) coincide for small $\omega^{\alpha\beta}$. By expanding both sides of Eq. (15.3) in Taylor series and using repeatedly the equation

$$[\hat{M}^{\alpha\beta},\hat{\psi}_\xi(x)] \;=\; -\big(\mathbf{S}^{\alpha\beta}_{\xi\eta}+\delta_{\xi\eta}J^{\alpha\beta}\big)\hat{\psi}_\xi(x) \qquad (15.12)$$

one can prove Eq. (15.3) for arbitrary $\omega^{\alpha\beta}$.

We have constructed quantum tensor operator $\hat{M}^{\alpha\beta}$ (15.8) which is a generator of arbitrary Lorentz transformations. Spatial components of this tensor govern the rotations and thus should determine the quantum operator of angular momentum.

Part XV

15.1 Operator of angular momentum in Dirac theory

Define

$$\hat{J}_i \;\equiv\; \frac{1}{2}\epsilon_{ijk}\hat{M}^{jk} \;=\; \epsilon_{ijk}\int d^3x\hat{\psi}^\dagger(\vec{x})\big(ix^j\frac{\partial}{\partial x_k}+\frac{1}{4}\sigma^{jk}\big)\hat{\psi}(\vec{x}) \qquad (15.13)$$

Since

$$\epsilon_{ijk}\sigma^{jk} = \frac{i}{2}\epsilon_{ijk}[\gamma^j,\gamma^k] = \frac{i}{2}\epsilon_{ijk}\begin{pmatrix} -[\sigma^j,\sigma^k] & 0 \\ 0 & -[\sigma^j,\sigma^k] \end{pmatrix} = 2\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \tag{15.14}$$

the operator (15.13) takes the form

$$\hat{J}_i \equiv \frac{1}{2}\epsilon_{ijk}\hat{M}^{jk} = \int d^3x\ \hat{\psi}^\dagger(\vec{x})\big(i\epsilon_{ijk}x^j\frac{\partial}{\partial x_k} + \frac{1}{2}\Sigma_i\big)\hat{\psi}(\vec{x}) \tag{15.15}$$

where

$$\Sigma^i \stackrel{\text{def}}{=} \begin{pmatrix} \vec{\sigma}_i & 0 \\ 0 & \vec{\sigma}_i \end{pmatrix} \equiv \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \tag{15.16}$$

We will demonstrate that the first and the second terms in the r.h.s. of the equation (15.15) are orbital angular momentum and spin operators, respectively:

$$\begin{aligned} \hat{J}_i &= \hat{L}_i + \hat{S}_i \\ \hat{L}_i &= i\int d^3x\ \hat{\psi}^\dagger(\vec{x})\epsilon_{ijk}x^j\frac{\partial}{\partial x_k}\hat{\psi}(\vec{x}), \qquad \hat{S}_i = \frac{1}{2}\int d^3x\ \hat{\psi}^\dagger(\vec{x})\Sigma_i\hat{\psi}(\vec{x}) \end{aligned} \tag{15.17}$$

To identify $\hat{L}_i$ with orbital angular momentum we will use analogy with the classical orbital momentum. In classical mechanics

$$\vec{L} = \sum \vec{r}\times\vec{p} \ \rightarrow\ \int d^3x\ \vec{x}\times\vec{p}(t,\vec{x}) \tag{15.18}$$

where $\vec{p}(t,\vec{x})$ is a density of the momentum of classical field. For the Dirac field the momentum is given by Eq. (14.33)

$$\begin{aligned} P^i &= \int d^3x\ T^{0i}(t,\vec{x}) = \frac{i}{4}\int d^3x\ \big[\bar{\psi}(t,\vec{x})\gamma_0 \overset{\leftrightarrow}{\partial^i} \psi(t,\vec{x}) + \bar{\psi}(t,\vec{x})\gamma^i \overset{\leftrightarrow}{\partial_t} \psi(t,\vec{x})\big] \\ &= i\int d^3x\ \bar{\psi}^\dagger(t,\vec{x})i\partial^i\psi(t,\vec{x}) \ +\ \text{total derivatives} \end{aligned} \tag{15.19}$$

We see that the density of the momentum is $\vec{p}(t,\vec{x}) = \bar{\psi}^\dagger(t,\vec{x})i\vec{\nabla}\psi(t,\vec{x})$ so the angular momentum of classical Dirac field is

$$\vec{L} = \int d^3x\ \vec{x}\times\bar{\psi}^\dagger(t,\vec{x})i\vec{\nabla}\psi(t,\vec{x}) \tag{15.20}$$

Now, when we promote Dirac fields to operators $\psi(t,\vec{x}) \to \bar{\psi}(\vec{x})$ we get

$$\hat{\vec{L}} = \int d^3x\ \vec{x}\times\hat{\bar{\psi}}^\dagger(t,\vec{x})i\vec{\nabla}\hat{\psi}(t,\vec{x}) \tag{15.21}$$

which coincides with $\hat{L}$ given by Eq. (15.17).

To demonstrate that the second term in Eq. (15.17) is a spin operator is a bit more difficult since spin has no analogy in classical mechanics.

We will do it in a different way: consider particle at rest and calculate the average of the operator $\hat{\vec{S}}$ from Eq. (15.17).

Dirac particle at rest is given by Eq. (14.40)

$$|\vec{0},s,-\rangle \;=\; \sqrt{2m}\hat{a}^{s\dagger}_{\vec{0}}|0\rangle \tag{15.22}$$

The orbital momentum of such particle should be zero:

$$\begin{aligned}\langle\vec{0},s,-|\vec{\hat{L}}|\vec{0},s,-\rangle \;&=\; 2m\int d^3x\;\langle 0|\hat{a}^s_{\vec{0}}\hat{\psi}^\dagger(\vec{x})\vec{x}\times i\vec{\nabla}\hat{\psi}(\vec{x})\hat{a}^{s\dagger}_{\vec{0}}|0\rangle \qquad (15.23)\\ = -m\int d^3x\sum_{\tilde{s},s'}\int\frac{đ^3pđ^3p'}{\sqrt{E_pE_{p'}}}&\langle 0|\hat{a}^s_{\vec{0}}\big[\bar{v}(\vec{p},\tilde{s})e^{i\vec{p}\vec{x}}\hat{b}^{\tilde{s}}_{\vec{p}}+\bar{u}(\vec{p},\tilde{s})e^{-i\vec{p}\vec{x}}\hat{a}^{\tilde{s}\dagger}_{\vec{p}}\big]\vec{x}\times\vec{p}'\big[u(\vec{p},s)e^{i\vec{p}'\vec{x}}\hat{a}^{s'}_{\vec{p}'}-v(\vec{p}',s')e^{-i\vec{p}'\vec{x}}\hat{b}^{s'\dagger}_{\vec{p}'}\big]\hat{a}^{s\dagger}_{\vec{0}}|0\rangle\end{aligned}$$

Let us prove it. First, note that the term $\sim\hat{a}^{s'}_{\vec{p}'}$ in the second square bracket does not contribute since $\{\hat{a}^{s'}_{\vec{p}'},\hat{a}^{s\dagger}_{\vec{0}}\}\;\sim\;\delta(\vec{p}')$ and we get

$$\begin{aligned}&\langle\vec{0},s,-|\vec{\hat{L}}|\vec{0},s,-\rangle \qquad (15.24)\\ &=\; m\int d^3x\sum_{\tilde{s},s'}\int\frac{đ^3pđ^3p'}{\sqrt{E_pE_{p'}}}\langle 0|\hat{a}^s_{\vec{0}}\big[\bar{v}(\vec{p},\tilde{s})e^{i\vec{p}\vec{x}}\hat{b}^{\tilde{s}}_{\vec{p}}+\bar{u}(\vec{p},\tilde{s})e^{-i\vec{p}\vec{x}}\hat{a}^{\tilde{s}\dagger}_{\vec{p}}\big]\vec{x}\times\vec{p}'v(\vec{p}',s')e^{-i\vec{p}'\vec{x}}\hat{b}^{s'\dagger}_{\vec{p}'}\big]\hat{a}^{s\dagger}_{\vec{0}}|0\rangle\end{aligned}$$

Second, by replacing $\vec{x}\times\vec{p}'\to i\vec{x}\times\vec{\nabla}$ and integration by parts one obtains

$$\begin{aligned}&\langle\vec{0},s,-|\vec{\hat{L}}|\vec{0},s,-\rangle \qquad (15.25)\\ &=\; m\int d^3x\sum_{\tilde{s},s'}\int\frac{đ^3pđ^3p'}{\sqrt{E_pE_{p'}}}\langle 0|\hat{a}^s_{\vec{0}}\big[\bar{v}(\vec{p},\tilde{s})e^{i\vec{p}\vec{x}}\hat{b}^{\tilde{s}}_{\vec{p}}-\bar{u}(\vec{p},\tilde{s})e^{-i\vec{p}\vec{x}}\hat{a}^{\tilde{s}\dagger}_{\vec{p}}\big]\vec{x}\times\vec{p}v(\vec{p}',s')e^{-i\vec{p}'\vec{x}}\hat{b}^{s'\dagger}_{\vec{p}'}\hat{a}^{s\dagger}_{\vec{0}}|0\rangle\end{aligned}$$

Repeating the argument that $\{\hat{a}^s_{\vec{0}},\hat{a}^{\tilde{s}\dagger}_{\vec{p}}\}\sim\delta(\vec{p})$ we reduce the r.h.s. of this equation to

$$\langle\vec{0},s,-|\vec{\hat{L}}|\vec{0},s,-\rangle = \;m\int d^3x\sum_{\tilde{s},s'}\int\frac{đ^3pđ^3p'}{\sqrt{E_pE_{p'}}}e^{i\vec{p}\vec{x}-i\vec{p}'\vec{x}}\bar{v}(\vec{p},\tilde{s})v(\vec{p}',s')\vec{x}\times\vec{p}\langle 0|\hat{a}^s_{\vec{0}}\hat{b}^{\tilde{s}}_{\vec{p}}\hat{b}^{s'\dagger}_{\vec{p}'}\hat{a}^{s\dagger}_{\vec{0}}|0\rangle \tag{15.26}$$

The last step is to note that $\hat{b}^{\tilde{s}}_{\vec{p}}\hat{b}^{s'\dagger}_{\vec{p}'}$ can be replaced by the anticommutator $\{\hat{b}^{\tilde{s}}_{\vec{p}},\hat{b}^{s'\dagger}_{\vec{p}'}\}\;=\;\delta_{\tilde{s}s'}(2\pi)^3\delta(\vec{p}-\vec{p}')$ so one gets

$$\begin{aligned}&\langle\vec{0},s,-|\vec{\hat{L}}|\vec{0},s,-\rangle \qquad (15.27)\\ &=\;m\sum_{s'}\int d^3x\;\vec{x}\times\int\frac{đ^3p}{E_p}\vec{p}\;\bar{v}(\vec{p},s')v(\vec{p},s')\langle 0|\hat{a}^s_{\vec{0}}\;\hat{a}^{s\dagger}_{\vec{0}}|0\rangle \;=\; -m\langle\vec{0},s,-|\vec{0},s,-\rangle\int d^3x\;\vec{x}\times\int\frac{đ^3p}{E_p}\vec{p}\;=\;0\end{aligned}$$

since each of the integrals over $\vec{x}$ and over $\vec{p}$ vanishes. Thus, as one may suspect, the orbital angular momentum of a particle at rest vanishes.

Let us apply now the operator $\vec{\hat{S}}$ to the state $|\vec{0},s,-\rangle$. For example, take $\hat{S}_z$

$$\begin{aligned}\hat{S}_z \;&=\; \int d^3x\;\bar{\psi}^\dagger(\vec{x})\frac{\Sigma_z}{2}\hat{\psi}(\vec{x}) \qquad (15.28)\\ &=\; \sum_{r,r'}\int\frac{đ^3p}{2E_p}\big(u^\dagger(\vec{p},r')\hat{a}^{r'\dagger}_{\vec{p}}+v^\dagger(-\vec{p},r')\hat{b}^{r'}_{-\vec{p}}\big)\frac{\Sigma_z}{2}\big(u(\vec{p},r)\hat{a}^r_{\vec{p}}+v(-\vec{p},r)\hat{b}^{r\dagger}_{-\vec{p}}\big)\end{aligned}$$

We get

$$\langle\vec{0},s,-|\hat{S}_z|\vec{0},s,-\rangle \;=\; 2m\langle 0|a^s_{\vec{0}}\hat{S}_z a^{s\dagger}_{\vec{0}}|0\rangle \tag{15.29}$$
$$=\; 2m\langle 0|a^s_{\vec{0}}[\hat{S}_z,a^{s\dagger}_{\vec{0}}]|0\rangle + 2m\langle 0|a^s_{\vec{0}}a^{s\dagger}_{\vec{0}}\hat{S}_z|0\rangle \;=\; 2m\langle 0|a^s_{\vec{0}}[\hat{S}_z,a^{s\dagger}_{\vec{0}}]|0\rangle + 2m\mathcal{V}\langle 0|\hat{S}_z|0\rangle$$

where $\mathcal{V}=(2\pi)^3\delta(0)$ is the total volume of space and $\langle 0|\hat{S}_z|0\rangle$ is the spin of the vacuum. The factor $2m\mathcal{V}=(2\pi)^3\delta(0)$ appears due to our normalization of states $\langle p|p'\rangle \;=\; 2p_0(2\pi)^3\delta(\vec{p}-\vec{p}')$ so our one-particle state $|p\rangle$ describes actually a plane wave with momentum $\vec{p}$ filling the whole space. A true one-particle state in a box with side L would be described by the wavefunction $L^{-3/2}e^{-ipx}$, see tghe AQM course.

Because vacuum is Lorentz invariant it should have no spin so $\langle 0|\hat{S}_z|0\rangle=0$ [14] and we are left with the first term in the r.h.s. of this equation

$$\langle\vec{0},s,-|\hat{S}_z|\vec{0},s,-\rangle \;=\; 2m\int d^3x\;\langle 0|a^s_{\vec{0}}[\bar{\psi}^\dagger(\vec{x})\frac{\Sigma_z}{2}\hat{\psi}(\vec{x}),a^{s\dagger}_{\vec{0}}]|0\rangle$$
$$=\; m\sum_{r,r'}\int\frac{d\!\!{}^-{}^3p}{2E_p}\langle 0|a^s_{\vec{0}}\big[\big(u^\dagger(\vec{p},r')\hat{a}^{r'\dagger}_{\vec{p}}+v^\dagger(-\vec{p},r')\hat{b}^{r'}_{-\vec{p}}\big)\Sigma_z\big(u(\vec{p},r)\hat{a}^r_{\vec{p}}+v(-\vec{p},r)\hat{b}^{r\dagger}_{-\vec{p}}\big),a^{s\dagger}_{\vec{0}}\big]|0\rangle$$
$$=\; m\sum_{r,r'}\int\frac{d\!\!{}^-{}^3p}{2E_p}(2\pi)^3\delta(\vec{p})\delta_{rs}\langle 0|a^s_{\vec{0}}u^\dagger(\vec{p},r')\Sigma_z u(\vec{p},r)\hat{a}^{r'\dagger}_{\vec{p}}|0\rangle$$
$$=\; \sum_{r'}u^\dagger(\vec{0},r')\frac{\Sigma_z}{2}u(\vec{0},s)\langle 0|a^s_{\vec{0}}\hat{a}^{r'\dagger}_{\vec{0}}|0\rangle \;=\; \mathcal{V}u^\dagger(\vec{0},s)\frac{\Sigma_z}{2}u(\vec{0},s)$$

As we discussed above, the factor $2m\mathcal{V}$ is due to our plane-wave normalization so the spin for one Dirac fermion is

$$s_z \;=\; \frac{1}{2m}u^\dagger(\vec{0},s)\frac{\Sigma_z}{2}u(\vec{0},s) \tag{15.30}$$

For the Dirac fermion at rest $u^\lambda(\vec{0}) \;=\; \sqrt{m}\binom{\xi^\lambda}{\xi^\lambda}$ so

$$s_z \;=\; (\xi^\lambda)^\dagger\sigma_z\xi^\lambda \quad\Rightarrow\quad s_z=\frac{1}{2}\;\;\text{for }\xi^{(\frac{1}{2})}\;=\;\begin{pmatrix}1\\0\end{pmatrix}\quad\text{and}\quad s_z=-\frac{1}{2}\;\;\text{for }\xi^{(-\frac{1}{2})}\;=\;\begin{pmatrix}0\\1\end{pmatrix} \tag{15.31}$$

This means our guess (15.28) for $\hat{S}_z$ has correct quantum-mechanical interpretation as a spin of the particle.

[14] If one computes the v.e.v. of the operator (15.28) there will be an (infinite) contribution coming from anticommutator $\{\hat{b}^{r'}_{-\vec{p}},\hat{b}^{r\dagger}_{-\vec{p}}\} \;=\; \delta(\vec{0})$. This is related to the problem of ordering of quantum operators at the same point. When we promote some classical quantities like spin to operators $\mathcal{O}(x) \;\to\; \hat{\mathcal{O}}(x)$ we face a dilemma: classical fields always (anti)commute but quantum operators do not so we sometimes get an uncertainty proportional to $\delta(\vec{x}-\vec{x})=\delta(0)$. The way is to avoid it is to define the operator $\hat{\mathcal{O}}$ as a normal product $\hat{\mathcal{O}} \;\equiv\; :\hat{\mathcal{O}}:$ so the v.e.v. of the operator defined in such a way will always vanish.

Note that for the antifermion

$$\frac{1}{2m\mathcal{V}}\langle\vec{0},s,+|\hat{S}_z|\vec{0},s,+\rangle \;=\; 2m\int d^3x\ \langle 0|b^s_{\vec{0}}[\hat{\psi}^\dagger(\vec{x})\frac{\Sigma_z}{2}\hat{\psi}(\vec{x}),b^{s\dagger}_{\vec{0}}]|0\rangle$$

$$= \frac{1}{2\mathcal{V}}\sum_{r,r'}\int\frac{d\!\!{}^-{}^3p}{2E_p}\langle 0|b^s_{\vec{0}}\big[\big(u^\dagger(\vec{p},r')\hat{a}^{r'\dagger}_{\vec{p}}+v^\dagger(-\vec{p},r')\hat{b}^{r'}_{-\vec{p}}\big)\Sigma_z\big(u(\vec{p},r)\hat{a}^r_{\vec{p}}+v(-\vec{p},r)\hat{b}^{r\dagger}_{-\vec{p}}\big),b^{s\dagger}_{\vec{0}}\big]|0\rangle$$

$$= -\frac{1}{2\mathcal{V}}\sum_{r,r'}\int\frac{d\!\!{}^-{}^3p}{2E_p}(2\pi)^3\delta(\vec{p})\delta_{r's}\langle 0|v^\dagger(-\vec{p},r')\Sigma_z v(-\vec{p},r)b^s_{\vec{0}}\hat{b}^{r\dagger}_{-\vec{p}}|0\rangle$$

$$= -\frac{1}{4m\mathcal{V}}\sum_r v^\dagger(\vec{0},s)\frac{\Sigma_z}{2}v(\vec{0},r)\langle 0|b^s_{\vec{0}}\hat{b}^{r\dagger}_{\vec{0}}|0\rangle \;=\; -\frac{1}{2m}v^\dagger(\vec{0},s)\frac{\Sigma_z}{2}v(\vec{0},s)$$

From Eq. (25.13) we get

$$v^{(\frac{1}{2})}(p) \;=\; \sqrt{m}\begin{pmatrix}0\\-1\\0\\1\end{pmatrix}, \qquad v^{(-\frac{1}{2})}(p) \;=\; \sqrt{m}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} \tag{15.32}$$

$$\Rightarrow\quad v^{(\frac{1}{2})\dagger}(\vec{0})\frac{\Sigma_z}{2}v^{(\frac{1}{2})}(\vec{0}) \;=\; -m, \qquad v^{(-\frac{1}{2})\dagger}(\vec{0})\frac{\Sigma_z}{2}v^{(-\frac{1}{2})}(\vec{0}) \;=\; m \tag{15.33}$$

and therefore

$$\frac{1}{2m\mathcal{V}}\langle\vec{0},\frac{1}{2},+|\hat{S}_z|\vec{0},\frac{1}{2},+\rangle \;=\; \frac{1}{2}, \qquad \frac{1}{2m\mathcal{V}}\langle\vec{0},-\frac{1}{2},+|\hat{S}_z|\vec{0},-\frac{1}{2},+\rangle \;=\; -\frac{1}{2}, \tag{15.34}$$

which is a self-consistency check for our formulas (25.13) for antifermion spinors, i.e. that unlike the fermion case, the spin $\frac{1}{2}$ up corresponds to non-relativistic spinor $\binom{0}{1}$ and spin $-\frac{1}{2}$ to spinor $\binom{1}{0}$.

15.2 Charge operator

From Dirac equations (13.20) it is easy to see that

$$\partial^\mu\hat{j}_\mu(x) \;=\; \partial^\mu\big(\hat{\bar{\psi}}(x)\gamma_\mu\hat{\psi}(x)\big) \;=\; \big(\partial^\mu\hat{\bar{\psi}}(x)\big)\gamma_\mu\hat{\psi}(x)+\hat{\bar{\psi}}(x)\gamma_\mu\partial^\mu\hat{\psi}(x) \;=\; 0 \tag{15.35}$$

so the operator

$$\hat{Q}(t) \;\equiv\; e_{\rm el}\int d^3x\ \hat{\psi}^\dagger(t,\vec{x})\hat{\psi}(t,\vec{x}) \tag{15.36}$$

is conserved: $\frac{d}{dt}\hat{Q}(t)\ = 0$ (here $e_{\rm el}$ is the negative electron charge). Indeed,

$$\frac{1}{e_{\rm el}}\frac{d}{dt}\hat{Q}(t) \;=\; \int d^3x\ \frac{\partial}{\partial t}\hat{\psi}^\dagger(t,\vec{x})\hat{\psi}(t,\vec{x}) \;=\; \int d^3x\ \big[\partial_\mu\hat{\bar{\psi}}(t,\vec{x})\gamma^\mu\hat{\psi}(t,\vec{x})-\vec{\nabla}\cdot\hat{\bar{\psi}}(t,\vec{x})\vec{\gamma}\hat{\psi}(t,\vec{x})\big] \;=\; 0 \tag{15.37}$$

where the first term vanishes due to Eq. (15.35) and second after integration by parts. In terms of ladder operators

$$\hat{Q} \;=\; e_{\rm el}\int d^3x\ \sum_{s,s'}\int\frac{d\!\!{}^-{}^3p\,d\!\!{}^-{}^3p'}{2\sqrt{E_pE_{p'}}}\big[v^\dagger(\vec{p},s)e^{i\vec{p}\vec{x}}\hat{b}^s_{\vec{p}}\ +u^\dagger(\vec{p},s)e^{-i\vec{p}\vec{x}}\hat{a}^{s\dagger}_{\vec{p}}\big]\big[u(\vec{p}',s')e^{i\vec{p}'\vec{x}}\hat{a}^s_{\vec{p}}+v(\vec{p}',s')e^{-i\vec{p}'\vec{x}}\hat{b}^{s'\dagger}_{\vec{p}}\big]$$

$$= e_{\rm el}\sum_{s,s'}\int\frac{d\!\!{}^-{}^3p}{2E_p}\big[v^\dagger(\vec{p},s)u(-\vec{p},s')\hat{b}^s_{\vec{p}}\hat{a}^s_{-\vec{p}}+u^\dagger(\vec{p},s)u(\vec{p},s')\hat{a}^{s\dagger}_{\vec{p}}\hat{a}^{s'}_{\vec{p}}\ +\ v^\dagger(\vec{p},s)v(\vec{p},s')\hat{b}^s_{\vec{p}}\hat{b}^{s'\dagger}_{\vec{p}}+u^\dagger(\vec{p},s)v(-\vec{p},s')\hat{a}^{s\dagger}_{\vec{p}}\hat{b}^{s'\dagger}_{-\vec{p}}\big]$$

$$= e_{\rm el}\sum_s\int d^3p\big(\hat{a}^{s\dagger}_{\vec{p}}\hat{a}^s_{\vec{p}}+\hat{b}^s_{\vec{p}}\hat{b}^{s\dagger}_{\vec{p}}\big) \tag{15.38}$$

where we used formulas (13.59) and (13.61). Discarding the infinite constant $\sim\int đ^3p\{\hat{b}^s_{\vec{p}}, \hat{b}^{s\dagger}_{\vec{p}}\}$ to avoid vacuum charge (see the footnote at page 105), we obtain the charge operator in the form

$$\hat{Q} \;=\; e_{\rm el}\sum_s\int đ^3p\;\big(\hat{a}^{s\dagger}_{\vec{p}}\hat{a}^s_{\vec{p}} - \hat{b}^{s\dagger}_{\vec{p}}\hat{b}^s_{\vec{p}}\big) \tag{15.39}$$

It is clear that the charge of state $|p,s,-\rangle$ is $e_{\rm el}$ and the charge of state $|p,s,+\rangle$ is $-e_{\rm el}$

$$\begin{aligned}\hat{Q}|p,s,-\rangle \;&=\; \sqrt{2E_p}[\hat{Q},a^{s\dagger}_{\vec{p}}]|0\rangle \;=\; e_{\rm el}\sqrt{2E_p}a^{s\dagger}_{\vec{p}}|0\rangle \;=\; e_{\rm el}|p,s,-\rangle\\ \hat{Q}|p,s,+\rangle \;&=\; \sqrt{2E_p}[\hat{Q},b^{s\dagger}_{\vec{p}}]|0\rangle \;=\; -\,e_{\rm el}\sqrt{2E_p}b^{s\dagger}_{\vec{p}}|0\rangle \;=\; -\,e_{\rm el}|p,s,+\rangle\end{aligned} \tag{15.40}$$

16 Dirac propagator and Wick's theorem for fermions

16.1 Feynman propagator of Dirac particle

T-product of fermionic operators is defined with respect to their anticommuting properties

$$\begin{aligned}\mathrm{T}\{\hat{\psi}_\xi(x)\hat{\bar{\psi}}_\eta(y)\} \;&\equiv\; \theta(x_0-y_0)\hat{\psi}_\xi(x)\hat{\bar{\psi}}_\eta(y) - \theta(y_0-x_0)\hat{\bar{\psi}}_\eta(y)\hat{\psi}_\xi(x)\\ \mathrm{T}\{\hat{\psi}_\xi(x)\hat{\psi}_\eta(y)\} \;&\equiv\; \theta(x_0-y_0)\hat{\psi}_\xi(x)\hat{\psi}_\eta(y) - \theta(y_0-x_0)\hat{\psi}_\eta(y)\hat{\psi}_\xi(x)\\ \mathrm{T}\{\hat{\bar{\psi}}_\xi(x)\hat{\bar{\psi}}_\eta(y)\} \;&\equiv\; \theta(x_0-y_0)\hat{\bar{\psi}}_\xi(x)\hat{\bar{\psi}}_\eta(y) - \theta(y_0-x_0)\hat{\bar{\psi}}_\eta(y)\hat{\bar{\psi}}_\xi(x)\end{aligned} \tag{16.1}$$

From Eq. (14.21) it is clear that

$$\langle 0|\mathrm{T}\{\hat{\psi}_\xi(x)\hat{\psi}_\eta(y)\}|0\rangle \;=\; \langle 0|\mathrm{T}\{\hat{\bar{\psi}}_\xi(x)\hat{\bar{\psi}}_\eta(y)\}|0\rangle \;=\; 0 \tag{16.2}$$

and $\langle 0|\mathrm{T}\{\hat{\psi}_\xi(x)\hat{\bar{\psi}}_\eta(y)\}|0\rangle$ represents the Feynman propagator of a Dirac fermion

$$S^F_{\xi\eta}(x-y) \;\equiv\; \langle 0|\mathrm{T}\{\hat{\psi}_\xi(x)\hat{\bar{\psi}}_\eta(y)\}|0\rangle = \int\frac{đ^4p}{i}\,\frac{m+\not{p}}{m^2-p^2-i\epsilon}\;e^{-ip(x-y)} \tag{16.3}$$

Let us prove this formula

$$
\begin{aligned}
\langle 0|\mathrm{T}\{\hat\psi_\xi(x)\hat{\bar\psi}_\eta(y)\}|0\rangle \;&\equiv\; \theta(x_0-y_0)\langle 0|\hat\psi_\xi(x)\hat{\bar\psi}_\eta(y)|0\rangle - \theta(y_0-x_0)\langle 0|\hat{\bar\psi}_\eta(y)\hat\psi_\xi(x)|0\rangle \qquad (16.4)\\
=\;& \theta(x_0-y_0)\sum_{s,s'}\int\frac{\text{đ}^3p\,\text{đ}^3p'}{2\sqrt{E_pE_{p'}}}\langle 0|\big[u_\xi(\vec p,s)e^{-ipx}\hat a^s_{\vec p}+v_\xi(\vec p,s)e^{ipx}\cancel{\hat b^{s\dagger}_{\vec p}}\big]\big[\bar v_\eta(\vec p',s')e^{-ip'y}\cancel{\hat b^{s'}_{\vec p'}}\;+\bar u_\eta(\vec p',s')e^{ip'y}\hat a^{s'\dagger}_{\vec p'}\big]|0\rangle\\
-\;& \theta(y_0-x_0)\sum_{s,s'}\int\frac{\text{đ}^3p\,\text{đ}^3p'}{2\sqrt{E_pE_{p'}}}\langle 0|\big[\bar v_\eta(\vec p',s')e^{-ip'y}\hat b^{s'}_{\vec p'}\;+\bar u_\eta(\vec p',s')e^{ip'y}\cancel{\hat a^{s'\dagger}_{\vec p'}}\big]\big[u_\xi(\vec p,s)e^{-ipx}\cancel{\hat a^s_{\vec p}}+v_\xi(\vec p,s)e^{ipx}\hat b^{s\dagger}_{\vec p}\big]|0\rangle\\
=\;& \theta(x_0-y_0)\sum_{s,s'}\int\frac{\text{đ}^3p\,\text{đ}^3p'}{2\sqrt{E_pE_{p'}}}u_\xi(\vec p,s)\bar u_\eta(\vec p',s')e^{-ipx+ip'y}\langle 0|\hat a^s_{\vec p}\hat a^{s'\dagger}_{\vec p'}|0\rangle\\
-\;& \theta(y_0-x_0)\sum_{s,s'}\int\frac{\text{đ}^3p}{2\sqrt{E_pE_{p'}}}e^{ipx-ip'y}v_\xi(\vec p,s)\bar v_\eta(\vec p',s')\langle 0|\hat b^{s'}_{\vec p'}\hat b^{s\dagger}_{\vec p}|0\rangle\\
=\;& \theta(x_0-y_0)\sum_{s}\int\frac{\text{đ}^3p}{2E_p}u_\xi(\vec p,s)\bar u_\eta(\vec p,s)e^{-ipx+ipy} - \;\theta(y_0-x_0)\sum_{s}\int\frac{\text{đ}^3p}{2E_p}e^{ipx-ipy}v_\xi(\vec p,s)\bar v_\eta(\vec p',s')\\
=\;& \int\frac{\text{đ}^3p}{2E_p}\big[\theta(x_0-y_0)(\not p+m)_{\xi\eta}e^{-ip(x-y)} - \;\theta(y_0-x_0)(\not p-m)_{\xi\eta}e^{ip(x-y)}\big]\\
=\;& \int\frac{\text{đ}^3p}{2E_p}\big[\theta(x_0-y_0)(m+E_p\gamma_0-\vec p\cdot\vec\gamma)_{\xi\eta}e^{-iE_p(x-y)_0+i\vec p(\vec x-\vec y)} + \;\theta(y_0-x_0)(m-E_p\gamma_0+\vec p\cdot\vec\gamma)_{\xi\eta}e^{iE_p(x-y)_0-i\vec p(\vec x-\vec y)}\big]\\
=\;& \int\frac{\text{đ}^3p}{2E_p}\big[\theta(x_0-y_0)(m+E_p\gamma_0-\vec p\cdot\vec\gamma)_{\xi\eta}e^{-iE_p(x-y)_0+i\vec p(\vec x-\vec y)} + \;\theta(y_0-x_0)(m-E_p\gamma_0-\vec p\cdot\vec\gamma)_{\xi\eta}e^{iE_p(x-y)_0+i\vec p(\vec x-\vec y)}\big]\\
=\;& \theta(x_0-y_0)\int\frac{\text{đ}^3p}{2E_p}\;(m+p^0\gamma_0-\vec p\cdot\vec\gamma)_{\xi\eta}e^{-ip_0(x-y)_0+i\vec p(\vec x-\vec y)}\Big|_{p_0=E_p}\\
&\qquad\qquad + \theta(y_0-x_0)\int\frac{\text{đ}^3p}{2E_p}\;(m+p^0\gamma_0-\vec p\cdot\vec\gamma)_{\xi\eta}e^{-ip^0(x-y)_0+i\vec p(\vec x-\vec y)}\Big|_{p_0=-E_p}\\
=\;& \theta(x_0-y_0)\int\frac{\text{đ}^3p}{2E_p}\;(m+\not p)_{\xi\eta}e^{-ip(x-y)}\Big|_{p_0=E_p} + \;\theta(y_0-x_0)\int\frac{\text{đ}^3p}{2E_p}\;(m+\not p)_{\xi\eta}e^{-ip(x-y)}\Big|_{p_0=-E_p}\\
=\;& \int\frac{\text{đ}^4p}{i}\;\frac{m+\not p}{m^2-p^2-i\epsilon}\;e^{-ip(x-y)} \qquad (16.5)
\end{aligned}
$$

16.1.1 Wick's theorem for Dirac fermions

The T-product for fermion operators is defined in the same way as for boson operators (operators are arranged according to their times) but with the factor (-1) for any exchange of fermion operators. For example

$$
\begin{aligned}
&\mathrm{T}\{\hat\psi(x)\hat{\bar\psi}(y)\hat\psi(z)\} \qquad (16.6)\\
&=\;\theta(x_0>y_0>z_0)\hat\psi(x)\hat{\bar\psi}(y)\hat\psi(z)-\theta(y_0>x_0>z_0)\hat{\bar\psi}(y)\hat\psi(x)\hat\psi(z)+\theta(y_0>z_0>x_0)\hat{\bar\psi}(y)\hat\psi(z)\hat\psi(x)\\
&-\theta(x_0>z_0>y_0)\hat\psi(x)\hat\psi(z)\hat{\bar\psi}(y)+\theta(z_0>x_0>y_0)\hat\psi(z)\hat\psi(x)\hat{\bar\psi}(y)-\theta(z_0>y_0>x_0)\hat\psi(z)\hat{\bar\psi}(y)\hat\psi(x)
\end{aligned}
$$

Normal product for fermions: the recipe is again "same way as for boson operators plus (-1) for any exchanged fermion operators".

Examples:

$$
\begin{aligned}
:\hat a_{\vec p}\hat a_{\vec p'}\hat a^\dagger_{\vec q}: \;&=\; (-1)^2\hat a^\dagger_{\vec q}\hat a_{\vec p}\hat a_{\vec p'} \;=\; \hat a^\dagger_{\vec q}\hat a_{\vec p}\hat a_{\vec p'} \;(= -\hat a^\dagger_{\vec q}\hat a_{\vec p'}\hat a_{\vec p}) \\
:\hat a_{\vec p}\hat a_{\vec p'}\hat a^\dagger_{\vec q}\hat a^\dagger_{\vec q'}: \;&=\; :\hat a^\dagger_{\vec q}\hat a_{\vec p}\hat a_{\vec p'}\hat a^\dagger_{\vec q'}: \;=\; :\hat a^\dagger_{\vec q}\hat a^\dagger_{\vec q'}\hat a_{\vec p}\hat a_{\vec p'}: \;=\; \hat a^\dagger_{\vec q}\hat a^\dagger_{\vec q'}\hat a_{\vec p}\hat a_{\vec p'} \;(= -\hat a^\dagger_{\vec q'}\hat a^\dagger_{\vec q}\hat a_{\vec p'}\hat a_{\vec p}) \text{ etc} \\
:\hat a_{\vec p}\hat a_{\vec p'}\hat a^\dagger_{\vec q}\hat a_{\vec p''}\hat a^\dagger_{\vec q'}: \;&=\; :\hat a^\dagger_{\vec q}\hat a_{\vec p}\hat a_{\vec p'}\hat a_{\vec p''}\hat a^\dagger_{\vec q'}: \;=\; -:\hat a^\dagger_{\vec q}\hat a^\dagger_{\vec q'}\hat a_{\vec p}\hat a_{\vec p'}\hat a_{\vec p''}: \;=\; -\hat a^\dagger_{\vec q}\hat a^\dagger_{\vec q'}\hat a_{\vec p}\hat a_{\vec p'}\hat a_{\vec p''}
\end{aligned}
\tag{16.7}
$$

Contractions of fermion operators

$$
\overbrace{\hat\psi(x)\hat{\bar\psi}(y)} \;\equiv\; \mathrm{T}\{\hat\psi(x)\hat{\bar\psi}(y)\} \;-\; :\hat\psi(x)\hat{\bar\psi}(y):, \qquad \overbrace{\hat{\bar\psi}(y)\hat\psi(x)} \;=\; -\,\overbrace{\hat\psi(x)\hat{\bar\psi}(y)} \tag{16.8}
$$

Taking v.e.v. of this equation we get

$$
\overbrace{\hat\psi(x)\hat{\bar\psi}(y)} \;\equiv\; \langle 0|\mathrm{T}\{\hat\psi(x)\hat{\bar\psi}(y)\}|0\rangle \;=\; S_F(x-y), \tag{16.9}
$$

There is no contraction of two $\hat\psi$'s or two $\hat{\bar\psi}$'s because by definition $\overbrace{\hat\psi(x)\hat{\bar\psi}(y)} \;=\; \mathrm{T}\{\hat\psi(x)\hat\psi(y)\} - :\hat\psi(x)\hat{\bar\psi}(y):$ and

$$
\overbrace{\hat\psi(x)\hat{\bar\psi}(y)} \;=\; \langle 0|\mathrm{T}\{\hat\psi(x)\hat\psi(y)\} \;-\; :\hat\psi(x)\hat{\bar\psi}(y):|0\rangle \;=\; \langle 0|\mathrm{T}\{\hat\psi(x)\hat\psi(y)\}|0\rangle \;=\; 0 \tag{16.10}
$$

due to Eq. (16.2). Similarly, $\overbrace{\hat{\bar\psi}(x)\hat{\bar\psi}(y)} \;=\; 0$. Due to these properties, it is convenient to depict a contraction by a line with an arrow, same as in complex KG case Because of

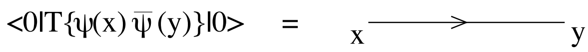


Figure 16. Feynman propagator for complex KG field

the sign rule, it is convenient before replacing $...\bar\psi(x)....\hat{\bar\psi}(y)...$ with a contraction to make several jumps to put these operators in a nearby position $...\bar\psi(x)\hat{\bar\psi}(y)...$, for example

$$
:\overbrace{\hat\psi(x)\hat\psi(y)\hat{\bar\psi}}(z): \;=\; -:\overbrace{\hat\psi(x)\hat{\bar\psi}(z)}\hat\psi(y): \;=\; -\,S_F(x-y)\hat\psi(y) \tag{16.11}
$$

Finally, Wick's theorem for fermions is the same as for the bosons:

$$
\begin{aligned}
&\mathrm{T}\{\hat\psi(x_1)\hat\psi(x_2)\hat{\bar\psi}(x_3)\hat\psi(x_4)\hat{\bar\psi}(x_5)....\hat\psi(x_n)\} \\
&= \;:\hat\psi(x_1)\hat\psi(x_2)\hat{\bar\psi}(x_3)\hat\psi(x_4)\hat{\bar\psi}(x_5)....\hat\psi(x_n): \;+\; \text{all possible contractions}
\end{aligned}
\tag{16.12}
$$

NB: when replacing contractions $...\overbrace{\hat\psi(x_i)....\hat{\bar\psi}(x_k)}...$ by $...S_F(x_i - x_k)...$ do not forget to make necessary permutations to put the operators $\hat\psi(x_i)$ and $\hat{\bar\psi}(x_k)$ in adjacent positions $...\overbrace{\hat\psi(x_i)\hat{\bar\psi}(x_k)}...!$

Part XVI

17 Yukawa theory

The Yukawa theory is a theory of interacting Dirac and (real) KG fields with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{M^2}{2}\phi^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi(x) - g\phi\bar{\psi}\psi \tag{17.1}$$

where the first two terms in the r.h.s. are the free KG Lagrangian (3.2), the third term is a Dirac Lagrangian (13.37) and the third term is an interaction Lagrangian describing the would-be fermion-fermion-scalar vertex with the coupling constant $\sim g$ (as usual $\bar{\psi}\psi \equiv \bar{\psi}_\xi\psi_\xi$).

Quantization: similar to self-interacting scalar theory

$$\hat{H} = \hat{H}_0 + \hat{H}_{\rm int}, \qquad \hat{H}_0 = \hat{H}_{\rm KG} + \hat{H}_{\rm D}, \qquad \hat{H}_{\rm int} = g\int d^3x\ \hat{\phi}(\vec{x})\hat{\bar{\psi}}(\vec{x})\hat{\psi}(\vec{x}) \tag{17.2}$$

Interaction representation:

$$\begin{aligned}
\hat{\phi}_I(z) &= e^{i\hat{H}_0 t}\hat{\phi}(\vec{z})e^{-i\hat{H}_0 t} = e^{i\hat{H}_{\rm KG} t}\hat{\phi}(\vec{z})e^{-i\hat{H}_{\rm KG} t}\\
\hat{\psi}_I(z) &= e^{i\hat{H}_0 t}\hat{\psi}(\vec{z})e^{-i\hat{H}_0 t} = e^{i\hat{H}_{\rm D} t}\hat{\psi}(\vec{z})e^{-i\hat{H}_{\rm D} t}\\
\hat{\bar{\psi}}_I(z) &= e^{i\hat{H}_0 t}\hat{\bar{\psi}}(\vec{z})e^{-i\hat{H}_0 t} = e^{i\hat{H}_{\rm D} t}\hat{\bar{\psi}}(\vec{z})e^{-i\hat{H}_{\rm D} t}
\end{aligned} \tag{17.3}$$

From Eqs. (6.11) and (14.21) we get

$$\begin{aligned}
\hat{\phi}_{\rm I}(x) &= e^{i\hat{H}t}\hat{\psi}(\vec{x})e^{-i\hat{H}t} = \int\frac{d\!\!{}^-{}^3p}{\sqrt{2E_p}}\big[e^{-ipx}\hat{a}_{\vec{p}} + e^{ipx}\hat{a}^\dagger_{\vec{p}}\big]\Big|_{p_0=E_p=\sqrt{M^2+\vec{p}^2}} \qquad (17.4)\\
\hat{\psi}_{\rm I}(x) &= e^{i\hat{H}t}\hat{\psi}(\vec{x})e^{-i\hat{H}t} = \sum_s\int\frac{d\!\!{}^-{}^3p}{\sqrt{2E_p}}\big[u(\vec{p},s)e^{-ipx}\hat{a}^s_{\vec{p}} + v(\vec{p},s)e^{ipx}\hat{b}^{s\dagger}_{\vec{p}}\big]\Big|_{p_0=E_p=\sqrt{m^2+\vec{p}^2}}\\
\hat{\bar{\psi}}_{\rm I}(\vec{x}) &= e^{i\hat{H}t}\hat{\bar{\psi}}(\vec{x})e^{-i\hat{H}t} = \sum_s\int\frac{d\!\!{}^-{}^3p}{\sqrt{2E_p}}\big[\bar{v}(\vec{p},s)e^{-ipx}\hat{b}^s_{\vec{p}} + \bar{u}(\vec{p},s)e^{ipx}\hat{a}^{s\dagger}_{\vec{p}}\big]\Big|_{p_0=E_p=\sqrt{m^2+\vec{p}^2}}
\end{aligned}$$

Similarly to the KG case (see Eq. (9.69 and (11.56)) we have an interaction representation formula for Green functions

$$\begin{aligned}
&\langle\Omega|\mathrm{T}\{\hat{\psi}(x_1)\hat{\bar{\psi}}(x_2)\hat{\bar{\psi}}(x_3)....\hat{\psi}(x_m)\hat{\phi}(y_1)...\hat{\phi}(y_n)\}|\Omega\rangle \qquad (17.5)\\
&= \frac{\langle 0|\mathrm{T}\{\hat{\psi}_I(x_1)\hat{\bar{\psi}}_I(x_2)\hat{\bar{\psi}}_I(x_3)....\hat{\psi}_I(x_m)\hat{\phi}_I(y_1)...\hat{\phi}_I(y_n)e^{-ig\int d^4z\hat{\phi}_I(z)\hat{\bar{\psi}}_I(z)\hat{\psi}_I(z)}\}|0\rangle}{\langle 0|\mathrm{T}\{e^{-ig\int d^4z\hat{\phi}_I(z)\hat{\bar{\psi}}_I(z)\hat{\psi}_I(z)}\}|0\rangle}
\end{aligned}$$

where $|\Omega\rangle$ is the "true vacuum" of the interacting theory (lowest-energy eigenstate of the Hamiltonian $\hat{H}$ (17.2)) and $|0\rangle$ is the "perturbative vacuum" (direct product of KG and Dirac perturbative vacuums $|0\rangle_{\rm KG}|0\rangle_{\rm D}$).

This formula is prepared for perturbative calculations: one should expand the exponentials in the r.h.s. in powers of $g\int d^4z\hat{\phi}(z)\hat{\bar{\psi}}(z)\hat{\psi}(z)$ and go ahead using the ladder representations (17.4) of operators $\hat{\phi}, \hat{\psi}$ and $\hat{\bar{\psi}}$. Because

$$a_{\vec{p}}|0\rangle = a^s_{\vec{p}}|0\rangle = b^s_{\vec{p}}|0\rangle = 0 \quad \text{and} \quad \langle 0|a^\dagger_{\vec{p}} = \langle 0|a^{s\dagger}_{\vec{p}} = \langle 0|b^{s\dagger}_{\vec{p}} = 0 \tag{17.6}$$

one should push all creation operators to the left and annihilation ones to the right using (anti)commutators

$$\begin{aligned}
[\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{p}'}] &= (2\pi)^3\delta(\vec{p}-\vec{p}') \\
\{\hat{a}^s_{\vec{p}}, \hat{a}^{s'\dagger}_{\vec{p}'}\} &= \{\hat{b}^s_{\vec{p}}, \hat{b}^{s'\dagger}_{\vec{p}'}\} = (2\pi)^3\delta(\vec{p}-\vec{p}')
\end{aligned} \tag{17.7}$$

As we discussed above, the result of this operations is given by Wick's theorem: v.e.v. of T-product of operators in the interaction representation is given by a sum of all possible contractions with each contraction being either KG propagator (6.43) or the DIrac one (16.3). As a result, one gets a set of connected Feynman diagrams (the vacuum bubbles are canceled between the numerator and the denominator in Eq. (17.5) as discussed in Sect. 9.3).

For calculation of cross sections we need LSZ theorem. It reads (up to renormalization Z-factors to be discussed later)

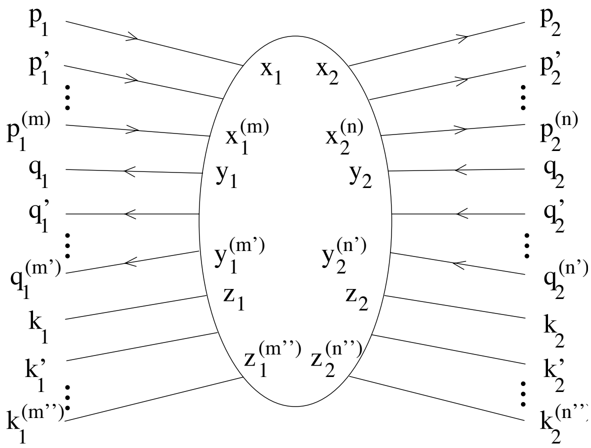


Figure 17. LSZ theorem for Yukawa theory

$$\begin{aligned}
&{}_{\text{out}}\langle p_2, s_2; p_2', s_2'; ...p_2^{(n)}, s_2^{(n)}; q_2, r_2; ...q_2^{(n')}, r_2^{(n')}; k_2, ...k_2^{(n'')} | p_1, s_1; p_1', s_1'; ...p_1^{(m)}, s_1^{(m)}; q_1, r_1; ...q_1^{(m')}, r_1^{(m')}; k_1, ...k_1^{(m'')}\rangle_{\text{in}} \\
&= \lim_{k_1^{(k)2}\to M^2}\lim_{k_2^{(n)2}\to M^2}\lim_{p_1^{(i)2}\to m^2}\lim_{p_2^{(l)2}\to m^2}\lim_{q_1^{(j)2}\to m^2}\lim_{q_2^{(m)2}\to m^2}\int\prod_{i=1}^{m}dx_1^{(i)}\prod_{j=1}^{m'}dy_1^{(j)}\prod_{k=1}^{m''}dz_1^{(k)}\prod_{i=1}^{l}dx_2^{(l)}\prod_{m=1}^{n'}dy_2^{(m)}\prod_{n=1}^{n''}dz_1^{(n)} \\
&\times\ \exp\Big\{-i\sum_{i=1}^{m}p_1^{(i)}x_1^{(i)} - i\sum_{j=1}^{m'}q_1^{(j)}y_1^{(j)} - i\sum_{k=1}^{m''}k_1^{(k)}z_1^{(k)} + i\sum_{l=1}^{n}p_2^{(l)}x_2^{(l)} + i\sum_{m=1}^{n'}q_2^{(m)}y_2^{(m)} + i\sum_{n=1}^{n''}k_2^{(n)}z_2^{(n)}\Big\} \\
&\times\ i^{m+n-m'-n'+m''+n''}\langle\Omega|\mathrm{T}\{\prod_{l=1}^{n}\bar{u}_\xi(p_2^{(l)}, s_2^{(l)})(m-\not{p}_2^{(l)})_{\xi\eta}\hat{\psi}_\eta(x_2^{(l)})\prod_{m=1}^{n'}\hat{\bar{\psi}}_\omega(y_2^{(m)})(m+\not{q}_2^{(m)})_{\omega\chi}v_\chi(q_2^{(m)}, r_2^{(m)})\prod_{n=1}^{n''}(M^2-k_2^{(n)2})\hat{\phi}(z_2^{(n)}) \\
&\times\prod_{l=1}^{m}\hat{\bar{\psi}}_\lambda(x_1^{(i)})(m-\not{p}_1^{(l)})_{\lambda\rho}u_\rho(p_1^{(i)}, r_1^{(i)})\prod_{j=1}^{m'}\bar{v}_\zeta(q_1^{(j)}, r_1^{(j)})(m+\not{q}_1^{(j)})_{\zeta\sigma}\hat{\psi}_\sigma(y_1^{(j)})\prod_{k=1}^{m''}(M^2-k_1^{(k)2})\hat{\phi}(z_1^{(k)})\}|\Omega\rangle
\end{aligned} \tag{17.8}$$

Proof: similar to Sect. 8.2 (see textbook by Bjorken & Drell)

As usual, Eq. (17.8) means that the matrix element of S-matrix is an amputated Green function (in the momentum space) on the mass shell.

$\Rightarrow$

Result: matrix element of the transition matrix $\mathcal{M}$ is an amputated reduced Green function on the mass shell multiplied by $\bar{u}(p,s)$ for each outgoing fermion, $u(p,s)$ for each incoming fermion, $v(p,s)$ for each outgoing antifermion, and $\bar{v}(p,s)$ for each incoming antifermion

LSZ $\Leftrightarrow$ Peskin's mnemonic rule

$$\begin{aligned}
&{}_{\rm out}\langle p_2,s_2;p_2',s_2';...p_2^{(n)},s_1^{(m)};q_2,r_2;...q_2^{(n')},r_2^{(n')};k_2,...k_2^{(n'')}|p_1,s_1;p_1',s_1';...p_1^{(m)},s_1^{(m)};q_1,r_1;...q_1^{(m')},r_1^{(m')};k_1,...k_1^{(m'')}\rangle_{\rm in}\\
&= \langle p_2,s_2;p_2',s_2';...p_2^{(n)},s_1^{(m)};q_2,r_2;...q_2^{(n')},r_2^{(n')};k_2,...k_2^{(n'')}|\\
&\qquad \times\ \mathrm{T}\{e^{-ig\int d^4z\hat{\phi}_I(z)\hat{\bar{\psi}}_I(z)\hat{\psi}_I(z)}\}|p_1,s_1;p_1',s_1';...p_1^{(m)},s_1^{(m)};q_1,r_1;...q_1^{(m')},r_1^{(m')};k_1,...k_1^{(m'')}\rangle_{\rm connected}
\end{aligned} \tag{17.9}$$

where

$$\begin{aligned}
|p_1,s_1;p_1',s_1';...p_1^{(m)},s_1^{(m)};q_1,r_1;...q_1^{(m')},r_1^{(m')};k_1,...k_1^{(m'')}\rangle &= \prod_{i=1}^{m}\sqrt{2E_p^{(i)}}\prod_{j=1}^{m'}\sqrt{2E_q^{(j)}}\prod_{k=1}^{m''}\sqrt{2E_k^{(i)}}\ \hat{a}_{\vec{p}_1^{(i)}}^{s_1^{(i)}\dagger}b_{\vec{q}_1^{(j)}}^{r_1^{(j)}\dagger}\hat{a}^\dagger_{\vec{k}_1^{(k)}}|0\rangle,\\
\langle p_2,s_2;p_2',s_2';...p_2^{(n)},s_1^{(m)};q_2,r_2;...q_2^{(n')},r_2^{(n')};k_2,...k_2^{(n'')}| &= \prod_{l=1}^{n}\sqrt{2E_p^{(l)}}\prod_{m=1}^{n'}\sqrt{2E_q^{(m)}}\prod_{n=1}^{n''}\sqrt{2E_k^{(n)}}\ \langle 0|\hat{a}_{\vec{p}_2^{(l)}}^{s_2^{(l)}}b_{\vec{q}_2^{(m)}}^{r_2^{(m)}}\hat{a}_{\vec{k}_2^{(n)}}
\end{aligned} \tag{17.10}$$

are the states in the interaction representation.

As an example, consider fermion-fermion scattering in the lowest order in perturbation theory

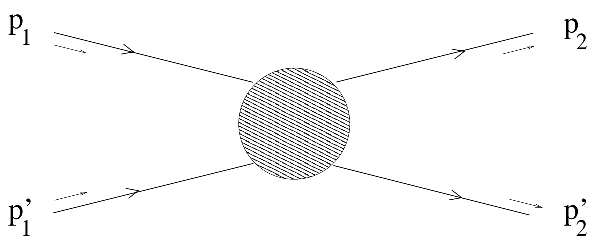


Figure 18. Fermion-fermion scattering in Yukawa theory

$$\begin{aligned}
&S(p_1,s_1;p_1',s_1'\to p_2,s_2;p_2',s_2')\\
&= \langle p_2,s_2;p_2',s_2'|\mathrm{T}\{e^{-ig\int d^4z\hat{\phi}_I(z)\hat{\bar{\psi}}_I(z)\hat{\psi}_I(z)}\}|p_1,s_1;p_1',s_1'\rangle_{\rm connected}\\
&= 4\sqrt{E_{p_1}E_{p_1'}E_{p_2}E_{p_2'}}\langle 0|\hat{a}_{\vec{p}_2}^{s_2}\hat{a}_{\vec{p}_2'}^{s_2'}\mathrm{T}\{e^{-ig\int d^4z\hat{\phi}_I(z)\hat{\bar{\psi}}_I(z)\hat{\psi}_I(z)}\}\hat{a}_{\vec{p}_1}^{s_1}\hat{a}_{\vec{p}_1'}^{s_1'}|0\rangle
\end{aligned} \tag{17.11}$$

First-order term of the expansion in powers of g vanishes

$$-4ig\sqrt{E_{p_1}E_{p_1'}E_{p_2}E_{p_2'}}\langle 0|\hat{a}_{\vec{p}_2}^{s_2}\hat{a}_{\vec{p}_2'}^{s_2'}\int d^4z\hat{\phi}_I(z)\hat{\bar{\psi}}_I(z)\hat{\psi}_I(z)\hat{a}_{\vec{p}_1}^{s_1\dagger}\hat{a}_{\vec{p}_1'}^{s_1'\dagger}|0\rangle \tag{17.12}$$

$$= 4\sqrt{E_{p_1}E_{p_1'}E_{p_2}E_{p_2'}}\langle 0|\hat{a}_{\vec{p}_2}^{s_2}\hat{a}_{\vec{p}_2'}^{s_2'}\int d^4z\int\frac{d\!\!{}^-{}^3p}{\sqrt{2E_p}}\big[e^{-ipx}\hat{a}_{\vec{p}}+e^{ipx}\hat{a}^\dagger_{\vec{p}}\big]\hat{\bar{\psi}}_I(z)\hat{\psi}_I(z)\hat{a}_{\vec{p}_1}^{s_1\dagger}\hat{a}_{\vec{p}_1'}^{s_1'\dagger}|0\rangle\ =\ 0$$

since boson operators a and $a^\dagger$ commute with all fermion operators so the first nontrivial term of the expansion of r.h.s of Eq. (17.11) is the second-order term

$$-2g^2\sqrt{E_{p_1}E_{p_1'}E_{p_2}E_{p_2'}}\langle 0|\hat{a}^{s_2}_{\vec{p}_2}\hat{a}^{s_2'}_{\vec{p}_2'}\int d^4z d^4z' \mathrm{T}\{\hat{\phi}_I(z)\hat{\bar{\psi}}_I(z)\hat{\psi}_I(z)\hat{\phi}_I(z')\hat{\bar{\psi}}_I(z')\hat{\psi}_I(z')\}\hat{a}^{s_1\dagger}_{\vec{p}_1}\hat{a}^{s_1'\dagger}_{\vec{p}_1'}|0\rangle \tag{17.13}$$

$$= \; -2g^2\sqrt{E_{p_1}E_{p_1'}E_{p_2}E_{p_2'}}\langle 0|\hat{a}^{s_2}_{\vec{p}_2}\hat{a}^{s_2'}_{\vec{p}_2'}\int d^4z d^4z' \big(:\hat{\phi}_I(z)\hat{\bar{\psi}}_I(z)\hat{\psi}_I(z)\hat{\phi}_I(z')\hat{\bar{\psi}}_I(z')\hat{\psi}_I(z'): +\text{contractions}\big)\hat{a}^{s_1}_{\vec{p}_1\dagger}\hat{a}^{s_1'\dagger}_{\vec{p}_1'}|0\rangle$$

$$= \; -2g^2\sqrt{E_{p_1}E_{p_1'}E_{p_2}E_{p_2'}}\langle 0|\hat{a}^{s_2}_{\vec{p}_2}\hat{a}^{s_2'}_{\vec{p}_2'}\int d^4z d^4z' \widehat{\phi_I(z)\hat{\phi}}_I(z') :\hat{\bar{\psi}}_I(z)\hat{\psi}_I(z)\hat{\bar{\psi}}_I(z')\hat{\psi}_I(z'): \; \hat{a}^{s_1\dagger}_{\vec{p}_1}\hat{a}^{s_1'\dagger}_{\vec{p}_1'}|0\rangle$$

$$= \; -2g^2 D_F(z-z')\sqrt{E_{p_1}E_{p_1'}E_{p_2}E_{p_2'}}\int d^4z d^4z' \langle 0|\hat{a}^{s_2}_{\vec{p}_2}\hat{a}^{s_2'}_{\vec{p}_2'} :\hat{\bar{\psi}}_I(z)\hat{\psi}_I(z)\hat{\bar{\psi}}_I(z')\hat{\psi}_I(z'): \; \hat{a}^{s_1\dagger}_{\vec{p}_1}\hat{a}^{s_1'\dagger}_{\vec{p}_1'}|0\rangle$$

$$= \; 4g^2 D_F(z-z')\sqrt{E_{p_1}E_{p_1'}E_{p_2}E_{p_2'}}\int d^4z d^4z' \langle 0|\hat{a}^{s_2'}_{\vec{p}_2'} \; \widehat{\hat{a}^{s_2}_{\vec{p}_2} : \hat{\bar{\psi}}}_{I\xi}(z)\hat{\bar{\psi}}_{I\eta}(z')\hat{\psi}_{I\eta}(z')\widehat{\hat{\psi}_{I\xi}(z) : \hat{a}^{s_1\dagger}_{\vec{p}_1}}\hat{a}^{s_1'\dagger}_{\vec{p}_1'}|0\rangle$$

$$+ \; 4g^2 D_F(z-z')\sqrt{E_{p_1}E_{p_1'}E_{p_2}E_{p_2'}}\int d^4z d^4z' \langle 0|\hat{a}^{s_2'}_{\vec{p}_2'} \; \widehat{\hat{a}^{s_2}_{\vec{p}_2} : \hat{\bar{\psi}}}_{I\xi}(z)\hat{\psi}_{I\xi}(z)\hat{\bar{\psi}}_{I\eta}(z')\widehat{\hat{\psi}_{I\eta}(z') : \hat{a}^{s_1\dagger}_{\vec{p}_1}}\hat{a}^{s_1'\dagger}_{\vec{p}_1'}|0\rangle$$

Now we use

$$\widehat{\hat{a}^{s_2}_{\vec{p}_2}\hat{\bar{\psi}}}_{I\xi}(z) \;\stackrel{\text{def}}{\equiv}\; \{\hat{a}^{s_2}_{\vec{p}_2},\hat{\bar{\psi}}_{I\xi}(z)\} \;=\; \sum_s\int\frac{d\!\!\bar{}^3p}{\sqrt{2E_p}}\{\hat{a}^{s_2}_{\vec{p}_2},\big[\bar{v}_\xi(\vec{p},s)e^{-ipz'}\hat{b}^s_{\vec{p}} + \bar{u}_\xi(\vec{p},s)e^{ipz}\hat{a}^{s\dagger}_{\vec{p}}\big]\} \;=\; \frac{e^{ip_2z}}{\sqrt{2E_{p_2}}}\bar{u}_\xi(\vec{p}_2,s_2)$$

$$\widehat{\hat{\psi}_{I\eta}(z')\hat{a}^{s_1\dagger}_{\vec{p}_1}} \;\stackrel{\text{def}}{\equiv}\; \{\hat{\psi}_{I\eta}(z'),\hat{a}^{s_1\dagger}_{\vec{p}_1}\} \;=\; \sum_s\int\frac{d\!\!\bar{}^3p}{\sqrt{2E_p}}\{\big[u_\eta(\vec{p},s)e^{-ipz'}\hat{a}^s_{\vec{p}} + v_\eta(\vec{p},s)e^{ipz'}\hat{b}^{s\dagger}_{\vec{p}}\big],\hat{a}^{s_1\dagger}_{\vec{p}_1}\} \;=\; \frac{e^{-ip_1z'}}{\sqrt{2E_{p_1}}}u_\eta(\vec{p}_1,s_1), \tag{17.14}$$

and get

$$= \; 2g^2\sqrt{E_{p_1'}E_{p_2'}}\bar{u}_\xi(\vec{p}_2,s_2)u_\xi(\vec{p}_1,s_1)\int d^4z d^4z' \; D_F(z-z')e^{ip_2z-ip_1z}\langle 0|\hat{a}^{s_2'}_{\vec{p}_2'} \; :\hat{\bar{\psi}}_{I\eta}(z')\hat{\psi}_{I\eta}(z'): \hat{a}^{s_1'\dagger}_{\vec{p}_1'}|0\rangle \tag{17.15}$$

$$+ \; 2g^2\sqrt{E_{p_1'}E_{p_2'}}\bar{u}_\xi(\vec{p}_2,s_2)u_\eta(\vec{p}_1,s_1)\int d^4z d^4z' \; D_F(z-z')e^{ip_2z-ip_1z'}\langle 0|\hat{a}^{s_2'}_{\vec{p}_2'} :\hat{\psi}_{I\xi}(z)\hat{\bar{\psi}}_{I\eta}(z'): \hat{a}^{s_1'\dagger}_{\vec{p}_1'}|0\rangle$$

$$= \; 2g^2\sqrt{E_{p_1'}E_{p_2'}}\bar{u}_\xi(\vec{p}_2,s_2)u_\xi(\vec{p}_1,s_1)\int d^4z d^4z' \; D_F(z-z')e^{i(p_2-p_1)z}\widehat{\hat{a}^{s_2'}_{\vec{p}_2'}\hat{\bar{\psi}}}_{I\eta}(z')\widehat{\hat{\psi}_{I\eta}(z')\hat{a}^{s_1'\dagger}_{\vec{p}_1'}}$$

$$-2g^2\bar{u}_\xi(\vec{p}_2,s_2)u_\eta(\vec{p}_1,s_1) \; \sqrt{E_{p_1'}E_{p_2'}}\int d^4z d^4z' \; D_F(z-z')e^{ip_2z-ip_1z'} \; \widehat{\hat{a}^{s_2'}_{\vec{p}_2'} \; \hat{\bar{\psi}}}_{I\eta}(z')\widehat{\hat{\psi}_{I\xi}(z)\hat{a}^{s_1'\dagger}_{\vec{p}_1'}}$$

$$= \; g^2\bar{u}_\xi(\vec{p}_2,s_2)u_\xi(\vec{p}_1,s_1)\int d^4z d^4z' \; D_F(z-z')e^{i(p_2-p_1)z+i(p_2'-p_1')z'} \; \bar{u}_\eta(\vec{p}_2',s_2')u_\eta(\vec{p}_1',s_1')$$

$$- \; g^2\bar{u}_\xi(\vec{p}_2,s_2)u_\eta(\vec{p}_1,s_1) \; \int d^4z d^4z' \; D_F(z-z')e^{i(p_2-p_1')z-i(p_1-p_2')z'} \; \bar{u}_\eta(\vec{p}_2',s_2')u_\xi(\vec{p}_1',s_1')$$

$$= \; g^2\bar{u}_\xi(\vec{p}_2,s_2)u_\xi(\vec{p}_1,s_1)\bar{u}_\eta(\vec{p}_2',s_2')u_\eta(\vec{p}_1',s_1')\int\frac{d\!\!\bar{}^4p}{i}\frac{e^{-ip(z-z')}}{M^2-p^2-i\epsilon}e^{i(p_2-p_1)z+i(p_2'-p_1')z'}$$

$$- \; g^2\bar{u}_\xi(\vec{p}_2,s_2)u_\xi(\vec{p}_1',s_1')\bar{u}_\eta(\vec{p}_2',s_2')u_\eta(\vec{p}_1,s_1) \; \int d^4z d^4z'\int\frac{d\!\!\bar{}^4p}{i}\frac{e^{-ip(z-z')}}{M^2-p^2-i\epsilon}e^{i(p_2-p_1')z-i(p_1-p_2')z'}$$

$$= \; -ig^2(2\pi)^4\delta(p_1+p_1'-p_2-p_2')\Big[\frac{\bar{u}_\xi(\vec{p}_2,s_2)u_\xi(\vec{p}_1,s_1)\bar{u}_\eta(\vec{p}_2',s_2')u_\eta(\vec{p}_1',s_1')}{M^2-(p_1-p_2)^2-i\epsilon} - \frac{\bar{u}_\xi(\vec{p}_2,s_2)u_\xi(\vec{p}_1',s_1')\bar{u}_\eta(\vec{p}_2',s_2')u_\eta(\vec{p}_1,s_1)}{M^2-(p_1-p_2')^2-i\epsilon}\Big]$$

so we got two diagrams shown in Fig. 19

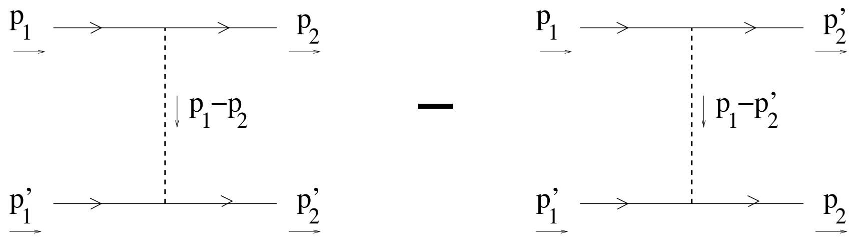


Figure 19. Fermion-fermion scattering in Yukawa theory in the leading order in g^2. Scalar boson is depicted by a dashed line

NB: Note the relative sign (-) between two amplitudes! (the absolute sign is not important).

This was the matrix element of the S-matrix, so

$$\begin{aligned}
&\mathcal{M}(p_1,s_1;p_1',s_1'\to p_2,s_2;p_2',s_2') \qquad (17.16)\\
=&\ -g^2\Big[\frac{\bar{u}_\xi(\vec{p}_2,s_2)u_\xi(\vec{p}_1,s_1)\bar{u}_\eta(\vec{p}_2',s_2')u_\eta(\vec{p}_1',s_1')}{M^2-t-i\epsilon}-\frac{\bar{u}_\xi(\vec{p}_2,s_2)u_\xi(\vec{p}_1',s_1')\bar{u}_\eta(\vec{p}_2',s_2')u_\eta(\vec{p}_1,s_1)}{M^2-u-i\epsilon}\Big]
\end{aligned}$$

which is amputated reduced Green function on the mass shell multiplied by spinors $\bar{u}(\vec{p}_2,s_2)$, $\bar{u}(\vec{p}_2',s_2')$ $u(\vec{p}_1,s_1)$ and $u(\vec{p}_1',s_1')$ in accordance with the general rule on p. 111. Hereafter we will use Mandelstam variables for $2\to 2$ particle scattering:

$$s\ \equiv\ (p_1+p_1')^2=(p_2+p_2')^2,\quad t\ =\ (p_1-p_2)^2=(p_1'-p_2')^2,\quad u\ =\ (p_1-p_2')^2=(p_1'-p_2)^2 \qquad (17.17)$$

Note that $s+t+u=4m^2$ (for particles with different masses $s+t+u=\sum_1^4 m^2$).

18 Set of Feynman rules for Yukawa theory

Lagrangian (density) of Yukawa theory

$$\mathcal{L}(x)\ =\ \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x)-\frac{M^2}{2}\phi^2(x)+\bar{\psi}(x)\big(i\gamma^\mu\partial_\mu-m\big)\psi(x)-g\phi(x)\bar{\psi}(x)\psi(x) \qquad (18.1)$$

Green functions in the momentum space are defined as [15]

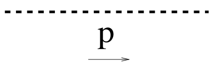
$$\begin{aligned}
&G\big(p_1,s_1;p_1',s_1';...p_1^{(m)},s_1^{(m)};q_1,r_1;...q_1^{(m')},r_1^{(m')};k_1,...k_1^{(m'')}\to p_2,s_2;p_2',s_2';...p_2^{(n)},s_1^{(m)};q_2,r_2;...q_2^{(n')},r_2^{(n')};k_2,...k_2^{(n'')}\big)\\
=&\int\prod_{i=1}^m dx_1^{(i)}\prod_{j=1}^{m'}dy_1^{(j)}\prod_{k=1}^{m''}dz_1^{(k)}\prod_{i=1}^{l}dx_2^{(l)}\prod_{m=1}^{n'}dy_2^{(m)}\prod_{n=1}^{n''}dz_1^{(n)}G\big(x_1,...x_1^{(m)},y_1,...y_1^{(m')},z_1,...z_1^{(m'')},x_2,...x_2^{(n)},y_2,...y_2^{(n')},z_2,...z_2^{(n'')}\big)\\
&\times\ \exp\Big\{-i\sum_{i=1}^m p_1^{(i)}x_1^{(i)}-i\sum_{j=1}^{m'}q_1^{(j)}y_1^{(j)}-i\sum_{k=1}^{m''}k_1^{(k)}z_1^{(k)}+i\sum_{l=1}^{n}p_2^{(l)}x_2^{(l)}+i\sum_{m=1}^{n'}q_2^{(m)}y_2^{(m)}+i\sum_{n=1}^{n''}k_2^{(n)}z_2^{(n)}\Big\} \qquad (18.2)
\end{aligned}$$

[15]Here label "1" denotes the incoming momenta, label "2" the outgoing ones, and an arrow separates those two sets of momenta. This is again an unconventional notation (usually $G(p_1,...p_n)$ is defined with all momenta either incoming or outgoing) but I find this notation very convenient for subsequent calculation of matrix elements of $\mathcal{M}$ -matrix.

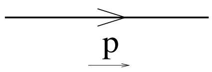
where $G\big(x_1,...x_1^{(m)}, y_1,...y_1^{(m')}, z_1,...z_1^{(m'')}, x_2,...x_2^{(n)}, y_2,...y_2^{(n')}, z_2,...z_2^{(n'')}\big)$ is a Green function in the coordinate space given by Eq. (17.5)

$$\begin{aligned}
&G\big(x_1,...x_1^{(m)}, y_1,...y_1^{(m')}, z_1,...z_1^{(m'')}, x_2,...x_2^{(n)}, y_2,...y_2^{(n')}, z_2,...z_2^{(n'')}\big) \qquad (18.3)\\
&= \langle\Omega|\mathrm{T}\{\prod_{l=1}^{n}\hat\psi(x_2^{(l)})\prod_{m=1}^{n'}\hat{\bar\psi}(y_2^{(m)})\prod_{n=1}^{n''}\hat\phi(z_2^{(n)})\prod_{l=1}^{m}\hat{\bar\psi}(x_1^{(i)})\prod_{j=1}^{m'}\hat\psi(y_1^{(j)})\prod_{k=1}^{m''}\hat\phi(z_1^{(k)})\}|\Omega\rangle\\
&= \frac{\langle 0|\mathrm{T}\{\prod_{l=1}^{n}\hat\psi_I(x_2^{(l)})\prod_{m=1}^{n'}\hat{\bar\psi}_I(y_2^{(m)})\prod_{n=1}^{n''}\hat\phi_I(z_2^{(n)})\prod_{l=1}^{m}\hat{\bar\psi}_I(x_1^{(i)})\prod_{j=1}^{m'}\hat\psi_I(y_1^{(j)})\prod_{k=1}^{m''}\hat\phi_I(z_1^{(k)})\}e^{-ig\int d^4z\hat\phi_I(z)\hat{\bar\psi}_I(z)\hat\psi_I(z)}\}|0\rangle}{\langle 0|\mathrm{T}\{e^{-ig\int d^4z\hat\phi_I(z)\hat{\bar\psi}_I(z)\hat\psi_I(z)}\}|0\rangle}
\end{aligned}$$

Feynman rules for Green functions in Yukawa theory

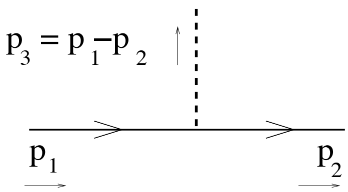
1. Scalar boson propagator 

$= \dfrac{-i}{M^2 - p^2 - i\varepsilon}$

2. Dirac fermion propagator 

$= \dfrac{-i(m + \not p)}{m^2 - p^2 - i\varepsilon}$

(Arrow on the fermion line in the direction of the flow of negative charge)

3. Vertex 

$= -ig\,(2\pi)^4\,\delta(p_1 - p_2 - p_3)$

4. Integrate over all momenta p_k of internal lines $\prod_k \int \dfrac{d^4 p_k}{(2\pi)^4}$

5. Extra factor (-1) for each fermion loop

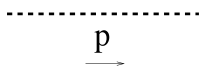
6. Negative relative sign between two amplitudes obtained by permutation of identical external lines corresponding to fermions

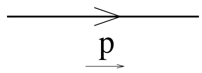
7. No symmetry factors in this theory

The reduced Green function is defined as (cf. Eq. (9.73))

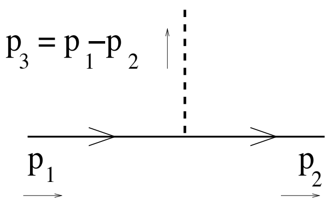
$$
\begin{aligned}
&G(p_1,s_1;p_1',s_1';...p_1^{(m)},s_1^{(m)};q_1,r_1;...q_1^{(m')},r_1^{(m')};k_1,...k_1^{(m'')} \to p_2,s_2;p_2',s_2';...p_2^{(n)},s_1^{(m)};q_2,r_2;...q_2^{(n')},r_2^{(n')};k_2,...k_2^{(n'')})\\
&= \ (-i)^{m+m'+m''+n+n'+n''-1}(2\pi)^4\delta\big(\sum p_1^{(i)}+\sum q_1^{(i)}+\sum k_1^{(i)}\big) \qquad (18.4)\\
&\times \mathcal{G}(p_1,s_1;p_1',s_1';...p_1^{(m)},s_1^{(m)};q_1,r_1;...q_1^{(m')},r_1^{(m')};k_1,...k_1^{(m'')} \to p_2,s_2;p_2',s_2';...p_2^{(n)},s_1^{(m)};q_2,r_2;...q_2^{(n')},r_2^{(n')};k_2,...k_2^{(n'')})
\end{aligned}
$$

Feynman rules for reduced Green functions for Yukawa theory

1. Scalar boson propagator  $= \dfrac{1}{M^2-p^2-i\varepsilon}$

2. Dirac fermion propagator  $= \dfrac{m+\not{p}}{m^2-p^2-i\varepsilon}$

(Arrow on the fermion line in the direction of the flow of negative charge)

3. Vertex  $= -g$

4. Integrate over boson loop momenta p_k $\quad \prod_k \int \dfrac{d^4p_k}{(2\pi)^4 i}$

5. Integrate over fermion loop momenta p_k $\quad (-1)\prod_k \int \dfrac{d^4p_k}{(2\pi)^4 i}$ (Extra factor -1 for each fermion loop)

6. Negative relative sign between two amplitudes obtained by permutation of identical external lines corresponding to fermions

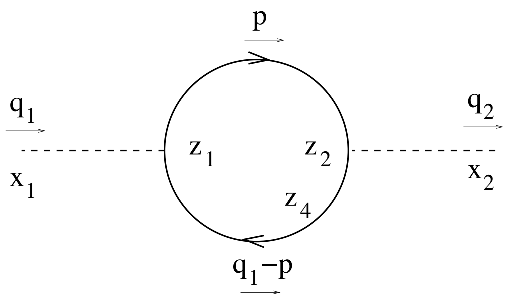
7. No symmetry factors in this theory

Matrix element of $\mathcal{M}$-matrix is a reduced amputated Green function on a mass shell multiplied by:

8. $\bar{u}(p,s)$ **for each outgoing fermion,** $u(p,s)$ **for each incoming fermion,** $v(p,s)$ **for each outgoing antifermion, and** $\bar{v}(p,s)$ **for each incoming antifermion.**

18.0.2 Extra (-1) for fermion loop

Consider an example of two-point scalar Green function in the second order in perturbation theory for example



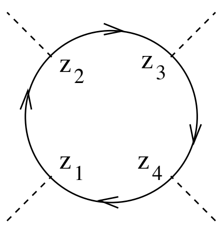
$$
\begin{aligned}
G(q_1 \to q_2) \;&=\; \int d^4x_1 d^4x_2 \; e^{-iq_1x_1+iq_2x_2} \langle \Omega | \mathrm{T}\{\hat{\phi}(x_1)\hat{\phi}(x_2)\}|\Omega\rangle \\
=\;& \int d^4x_1 d^4x_2 \; e^{-iq_1x_1+iq_2x_2} \frac{\langle 0|\mathrm{T}\{\hat{\phi}_I(x_1)\hat{\phi}_I(x_2) e^{-ig\int d^4z \hat{\phi}_I(z)\hat{\bar{\psi}}_I(z)\hat{\psi}_I(z)}\}|0\rangle}{\langle 0|\mathrm{T}\{e^{-ig\int d^4z\hat{\phi}(z)\hat{\bar{\psi}}(z)\hat{\psi}(z)}\}|0\rangle} \\
=\;& -\frac{g^2}{2}\int d^4x_1 d^4x_2 \; e^{-iq_1x_1+iq_2x_2}\int d^4z_1 d^4z_2 \langle 0|\mathrm{T}\{\hat{\phi}_I(x_1)\hat{\phi}_I(x_2)\hat{\phi}_I(z_1)\hat{\phi}_I(z_2)\hat{\bar{\psi}}_{I\xi}(z_1)\hat{\psi}_{I\xi}(z_1)\hat{\bar{\psi}}_{I\eta}(z_2)\hat{\psi}_{I\eta}(z_2)\}|0\rangle \\
=\;& -g^2\int d^4x_1 d^4x_2 \; e^{-iq_1x_1+iq_2x_2}\int d^4z_1 d^4z_2 \hat{\phi}_I(x_1)\hat{\phi}_I(z_1)\hat{\phi}_I(z_2)\hat{\phi}_I(x_2)(-1)\hat{\psi}_{I\eta}(z_2)\hat{\bar{\psi}}_{I\xi}(z_1)\hat{\psi}_{I\xi}(z_1)\hat{\bar{\psi}}_{I\eta}(z_2) \\
=\;& -g^2\int d^4x_1 d^4x_2 \; e^{-iq_1x_1+iq_2x_2}\int d^4z_1 d^4z_2 \int \frac{\not{d}^4k_2}{i}\frac{e^{-ik_2(x_2-z_2)}}{M^2-k_2^2-i\epsilon}\int\frac{\not{d}^4k_1}{i}\frac{e^{-ik_1(z_1-x_1)}}{M^2-k_1^2-i\epsilon} \\
\times\;& (-1)\int\frac{\not{d}^4p}{i}\frac{e^{-ip(z_1-z_2)}}{m^2-p^2-i\epsilon}(m+\not{p})_{\eta\xi}\int\frac{\not{d}^4p'}{i}\frac{e^{-ip'(z_2-z_1)}}{m^2-p'^2-i\epsilon}(m+\not{p}')_{\xi\eta} \\
=\;& -g^2\frac{(2\pi)^4\delta(q_1-q_2)}{(M^2-q_1^2-i\epsilon)^2}(-1)\int \not{d}^4p\frac{\mathrm{tr}(m+\not{p})(m+\not{p}-\not{q}_1)}{(m^2-p^2-i\epsilon)[m^2-(q_1-p)^2-i\epsilon]}
\end{aligned} \tag{18.5}
$$

By definition (9.73), the reduced Green function is Eq. (18.5) without the factor $(-i)(2\pi)^4\delta(q_1 - q_2)$ so we obtain

$$
\mathcal{G}(q) \;=\; \frac{g^2}{(M^2-q_1^2-i\epsilon)^2}(-1)\int\frac{\not{d}^4p}{i}\frac{\mathrm{tr}(m+\not{p})(m+\not{p}-\not{q}_1)}{(m^2-p^2-i\epsilon)[m^2-(q_1-p)^2-i\epsilon]} \tag{18.6}
$$

in accordance with Feynman rules for reduced Green functions in the previous page.

The extra (-1) is a general factor for any fermion loop coming from the fact that one always needs one extra permutation to replace all $\hat{\psi}$'s and $\hat{\bar{\psi}}$'s by contractions, for example



$$
\begin{aligned}
&\langle 0|\mathrm{T}\{\hat{\bar{\psi}}_\xi(z_1)\hat{\psi}_\xi(z_1)\hat{\bar{\psi}}_\eta(z_2)\hat{\psi}_\eta(z_2)\hat{\bar{\psi}}_\zeta(z_3)\hat{\psi}_\zeta(z_3)\hat{\bar{\psi}}_\chi(z_4)\hat{\psi}_\chi(z_4)\}|0\rangle \\
&= \; -\langle 0|\mathrm{T}\{\hat{\psi}_\chi(z_4)\hat{\bar{\psi}}_\xi(z_1)\hat{\psi}_\xi(z_1)\hat{\bar{\psi}}_\eta(z_2)\hat{\psi}_\eta(z_2)\hat{\bar{\psi}}_\zeta(z_3)\hat{\psi}_\zeta(z_3)\hat{\bar{\psi}}_\chi(z_4)\}|0\rangle \\
&= \; -\overbrace{\hat{\psi}_\chi(z_4)\hat{\bar{\psi}}_\xi(z_1)}\overbrace{\hat{\psi}_\xi(z_1)\hat{\bar{\psi}}_\eta(z_2)}\overbrace{\hat{\psi}_\eta(z_2)\hat{\bar{\psi}}_\zeta(z_3)}\overbrace{\hat{\psi}_\zeta(z_3)\hat{\bar{\psi}}_\chi(z_4)} \\
&= \; -\operatorname{tr}\big(S_F(z_4-z_1)S_F(z_1-z_2)S_F(z_2-z_3)S_F(z_3-z_4)\big)
\end{aligned}
\tag{18.7}
$$

Part XVII

19 Quantum theory of free electromagnetic field

19.1 Reminder: classical electrodynamics

Maxwell's equations in a free space

$$
\vec{\nabla}\times\vec{E} \;=\; -\dot{\vec{B}}, \qquad \vec{\nabla}\cdot\vec{B} \;=\; 0 \qquad \text{“first pair''} \tag{19.1}
$$

$$
\vec{\nabla}\cdot\vec{E} \;\overset{\text{Gauss law}}{=}\; 0, \qquad \vec{\nabla}\times\vec{B} \;=\; \dot{\vec{E}} \qquad \text{“second pair''} \tag{19.2}
$$

In terms of potentials

$$
\begin{aligned}
\vec{E} &= -\vec{\nabla}\Phi - \dot{\vec{A}}, && \Phi \;-\; \text{scalar potential} \\
\vec{B} &= \vec{\nabla}\times\vec{A} && \vec{A} \;-\; \text{vector potential}
\end{aligned}
\tag{19.3}
$$

4-vector potential is defined as $A^\mu \equiv (\Phi, \vec{A})$. It is usually (but somewhat confusing) called an “electromagnetic field”. The relativistic invariant form of Eq. (19.3) looks like

$$
F^{\mu\nu}(x) \;=\; \frac{\partial}{\partial x_\mu}A^\nu(x) - \frac{\partial}{\partial x_\nu}A^\mu(x) \tag{19.4}
$$

where

$$
F^{\mu\nu}(x) \;=\; \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{19.5}
$$

is called a “field strength tensor”. If $F_{\mu\nu}$ is obtained as (19.4) from some potential A_μ the first pair of Maxwell's equations (19.1) is satisfied automatically and the second pair (19.2) looks like

$$
\frac{\partial}{\partial x^\mu}F^{\mu\nu}(x) \;=\; 0 \tag{19.6}
$$

19.1.1 Lagrangian and Hamiltonian

The Lagrangian (density) for the free electromagnetic field is

$$
\mathcal{L}(x) \;=\; -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}(x) \;=\; \frac{1}{2}(\vec{E}^2 - \vec{B}^2) \tag{19.7}
$$

Proof: Euler-Lagrange equation for (19.7) reproduce Maxwell's equations (19.6)

$$
\frac{\partial\mathcal{L}}{\partial A^\mu}(x) = 0, \quad \frac{\partial\mathcal{L}}{\partial^\mu A^\nu}(x) = -F_{\mu\nu}(x) \quad\Rightarrow\quad \partial^\mu\frac{\partial\mathcal{L}}{\partial^\mu A^\nu}(x) \;=\; \frac{\partial\mathcal{L}}{\partial A^\nu}(x) \quad\Leftrightarrow\quad \partial^\mu F_{\mu\nu}(x) \;=\; 0 \tag{19.8}
$$

One can choose potentials $A_\mu(x)$ as canonical coordinates, then the canonical momenta are

$$\begin{aligned}\pi^0(x) &= \frac{\partial\mathcal{L}}{\partial\dot{A}^0}(x) = 0\\ \pi^i(x) &= \frac{\partial\mathcal{L}}{\partial\dot{A}_i}(x) = -\dot{A}^i(x) + \partial^i A_0(x) = E^i(x)\end{aligned} \tag{19.9}$$

and the Hamiltonian density for free electromagnetic field has the form

$$\begin{aligned}\mathcal{H}(t,\vec{x}) &= \sum \pi_i(x)\dot{A}^i(x) - \mathcal{L}(x) = -\vec{\pi}(x)\cdot\dot{\vec{A}}(x) - \mathcal{L}(x) = \vec{\pi}(x)\cdot\big(\vec{E}(x)+\vec{\nabla}\Phi(x)\big) - \mathcal{L}(x)\\ &= \vec{E}(x)\cdot\big(\vec{E}(x)+\vec{\nabla}\Phi(x)\big) - \frac{1}{2}\big(\vec{E}^2(x)-\vec{B}^2(x)\big) = \frac{1}{2}\big(\vec{E}^2(x)+\vec{B}^2(x)\big) + \vec{E}(x)\cdot\vec{\nabla}\Phi(x)\end{aligned} \tag{19.10}$$

which reproduces the Hamiltonian from E & M textbooks after integration by parts

$$\begin{aligned}H(t) &= \int d^3x\ \mathcal{H}(t,\vec{x}) = \frac{1}{2}\int d^3x\ \big[\vec{E}^2(t,\vec{x})+\vec{B}^2(t,\vec{x})\big] + \int d^3x\ \vec{E}(t,\vec{x})\cdot\vec{\nabla}\Phi(t,\vec{x})\\ &= \frac{1}{2}\int d^3x\ \big[\vec{E}^2(t,\vec{x})+\vec{B}^2(t,\vec{x})\big] - \int d^3x\ \Phi(t,\vec{x})\vec{\nabla}\cdot\vec{E}(t,\vec{x}) \stackrel{\text{Gauss}}{=} \frac{1}{2}\int d^3x\ \big[\vec{E}^2(t,\vec{x})+\vec{B}^2(t,\vec{x})\big]\end{aligned} \tag{19.11}$$

19.1.2 Gauge invariance

It is known that the potentials (19.3) are not unique: the change

$$A^\mu(x) \rightarrow A^\mu(x) + \frac{\partial}{\partial x_\mu}\Lambda(x) \tag{19.12}$$

(where $\Lambda(x)$ is an arbitrary scalar function) leaves $F^{\mu\nu}$ (19.4) intact so the electric and magnetic fields $\vec{E}$ and $\vec{B}$ will be the same for both potentials in the l.h.s. and in the r.h.s. of Eq. (19.12).

There are two popular choices for the potential: Lorenz gauge $\partial^\mu A_\mu = 0$ and Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$. Lorenz gauge has an advantage of being relativistic invariant but quantization in this gauge is somewhat complicated so we will use Coulomb gauge.

For a free electomagnetic field we can additionally have $A_0 = 0$. Let us prove by construction that for a given field $\vec{E}(t,\vec{x}), \vec{B}(t,\vec{x})$ the potential satisfying gauge conditions $\vec{\nabla}\cdot\vec{A} = 0$ and $\vec{\nabla}\cdot\vec{A} = 0$ does exist. Our guess:

$$A^0_{\text{guess}} = 0, \qquad A^i_{\text{guess}}(t,\vec{x}) = \int_{-\infty}^t dt'\ F^{0i}(t',\vec{x}) \quad\Leftrightarrow\quad \vec{A}_{\text{guess}}(t,\vec{x}) = -\int_{-\infty}^t dt'\ \vec{E}(t',\vec{x}) \tag{19.13}$$

We need to check that the potential (19.13) reproduces electric and magnetic fields $\vec{E}$ and $\vec{B}$. From Eq. (19.3) we get

$$\vec{E}_{\text{guess}}(t,\vec{x}) = -\dot{\vec{A}}_{\text{guess}}(t,\vec{x}) = \frac{d}{dt}\int_{-\infty}^t dt'\ \vec{E}(t',\vec{x}) = \vec{E}(t,\vec{x}), \tag{19.14}$$

$$\vec{B}_{\text{guess}}(t,\vec{x}) = \vec{\nabla}\times\vec{A}_{\text{guess}}(t,\vec{x}) = -\int_{-\infty}^t dt'\ \vec{\nabla}\times\vec{E}(t',\vec{x}) = \int_{-\infty}^t dt'\ \dot{\vec{B}}(t',\vec{x}) = \vec{B}(t,\vec{x})$$

The last check is

$$\vec{\nabla}\cdot\vec{A}_{\text{guess}}(t,\vec{x}) \;=\; -\int_{-\infty}^{t} dt'\;\vec{\nabla}\cdot\vec{E}(t',\vec{x}) \;\overset{\text{Gauss}}{=}\; 0 \tag{19.15}$$

so we have constructed the potential (19.13) satisfying Coulomb gauge conditions.

19.1.3 Expansion in plane waves

$$\begin{aligned}
\partial_0 F^{0i} + \partial_j F^{ji} \;&=\; \partial_0(\partial^0 A^i - \partial^i \cancel{A^0}) + \partial_j(\partial^j A^i - \partial^i A^j) \;=\; \partial_0^2 A^i + \partial_j\partial^j A^i - \partial^i(\partial_j A^j)\\
&=\; \partial_0^2 A^i + \partial_j\partial^j A^i - \partial^i(\vec{\nabla}\cdot\vec{A}) \;=\; \partial^2 A^i \;=\; 0
\end{aligned} \tag{19.16}$$

$\Rightarrow$ we have three KG equations $\partial^2 A^i \;=\; 0$ plus additional condition $\partial_i A^i = 0$. The solution of these equations is

$$A_i(t,\vec{x}) \;=\; \sum_{\lambda=1,2}\int \frac{d\!\!{}^-{}^3k}{\sqrt{2\omega_k}}\; e_i^\lambda(\vec{k})\big(a_{\vec{k}}^\lambda e^{-ikx} + a_{\vec{k}}^{\lambda *} e^{ikx}\big)\Big|_{k_0=\omega_k=|\vec{k}|} \tag{19.17}$$

where $e^{(1)}(\vec{k})$ and $e^{(2)}(\vec{k})$ are two polarization vectors orthogonal to $\vec{k}$ due to Coulomb gauge

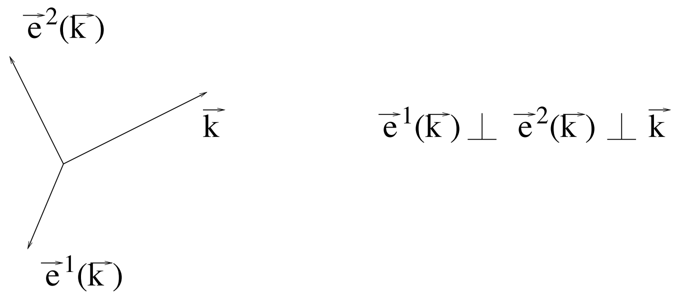


Figure 20. Polarization vectors

condition

$$\partial^i A_i(t,\vec{x}) \;=\; 0 \;\;\Leftrightarrow\;\; k^i e_i^\lambda(\vec{k}) \;=\; \vec{k}\cdot\vec{e}(\vec{k}) \;=\; 0 \tag{19.18}$$

The canonical momenta $\pi_i = -\dot{A}_i$ are given by similar formula

$$\pi_i(t,\vec{x}) \;=\; E_i(t,\vec{x}) \;=\; i\sum_{\lambda=1,2}\int \frac{d\!\!{}^-{}^3k}{\sqrt{2\omega_k}}\;\omega_k e_i^\lambda(\vec{k})\big(a_{\vec{k}}^\lambda e^{-ikx} - a_{\vec{k}}^{\lambda *} e^{ikx}\big)\Big|_{k_0=\omega_k=|\vec{k}|} \tag{19.19}$$

Sometimes it is convenient to expand the field in the circularly polarized waves rather than linearly polarized. The vectors of circular polarization are defined as

$$\begin{aligned}
\vec{e}_i^R(\vec{k}) \;&\overset{\text{def}}{=}\; \frac{1}{\sqrt{2}}\big(\vec{e}^{(1)}(\vec{k}) + i\vec{e}^{(2)}(\vec{k})\big) \qquad \text{right polarization}\\
\vec{e}_i^L(\vec{k}) \;&\overset{\text{def}}{=}\; \frac{1}{\sqrt{2}}\big(\vec{e}^{(1)}(\vec{k}) - i\vec{e}^{(2)}(\vec{k})\big) \qquad \text{left polarization}
\end{aligned} \tag{19.20}$$

and the expansion in circularly polarized plane waves has the form

$$A_i(t,\vec{x}) \;=\; \sum_{\Lambda=R,L}\int \frac{d\!\!{}^-{}^3k}{\sqrt{2\omega_k}}\,\left(e_i^\Lambda(\vec{k})a_{\vec{k}}^\Lambda e^{-ikx} + e_i^{\Lambda *}(\vec{k})a_{\vec{k}}^{\Lambda *}e^{ikx}\right)\Big|_{k_0=\omega_k=|\vec{k}|} \tag{19.21}$$

where

$$a_{\vec{k}}^R \;\stackrel{\rm def}{=}\; \frac{1}{\sqrt{2}}(a_{\vec{k}}^{(1)} - ia_{\vec{k}}^{(2)}), \qquad a_{\vec{k}}^L \;\equiv\; a_{\vec{k}}^L \;\stackrel{\rm def}{=}\; \frac{1}{\sqrt{2}}(a_{\vec{k}}^{(1)} + ia_{\vec{k}}^{(2)}) \tag{19.22}$$

19.2 Quantization of free electromagnetic field

As usual, we would like to promote the canonical coordinates and canonical momenta to operators

$$A^\mu(t,\vec{x}) \;\to\; \hat{A}^\mu(\vec{x}), \qquad \pi^i(t,\vec{x}) \;\to\; \hat{\pi}^i(\vec{x}), \qquad (\pi^0 = 0) \tag{19.23}$$

satisfying the canonical commutation relations

$$\begin{aligned}
[\hat{A}^\mu(\vec{x}),\hat{A}^\nu(\vec{y})] \;&=\; [\pi^i(\vec{x}),\pi^j(\vec{y})] \;=\; [\hat{A}^0(\vec{x}),\pi^j(\vec{y})] \;=\; 0,\\
[\hat{A}^i(\vec{x}),\pi_j(\vec{y})] \;&=\; i\delta_{ij}\delta(\vec{x}-\vec{y}) \quad\Leftrightarrow\quad [\hat{A}^i(\vec{x}),\pi^j(\vec{y})] \;=\; -i\delta_{ij}\delta(\vec{x}-\vec{y})
\end{aligned} \tag{19.24}$$

However, second line of this equation contradicts to Gauss law. Indeed, we would like to have $\vec{\nabla}\cdot\hat{\vec{E}}(\vec{x}) \;=\; 0$ just as in a classical field theory, but

$$[\hat{A}^i(\vec{x}),\pi^j(\vec{y})] \;=\; -i\delta_{ij}\delta(\vec{x}-\vec{y}) \quad\Rightarrow\quad [\hat{A}^i(\vec{x}),\partial_j\pi^j(\vec{y})] \;=\; -i\frac{\partial}{\partial y_i}\delta(\vec{x}-\vec{y}) \;=\; -i[\hat{A}^i(\vec{x}),\vec{\nabla}\cdot\hat{\vec{E}}(\vec{x})] \;\neq\; 0 \tag{19.25}$$

Way out (Bjorken & Drell textbook): impose CCR

$$\begin{aligned}
[\hat{A}^\mu(\vec{x}),\hat{A}^\nu(\vec{y})] \;&=\; [\pi^i(\vec{x}),\pi^j(\vec{y})] \;=\; [\hat{A}^0(\vec{x}),\pi^j(\vec{y})] \;=\; 0,\\
[\hat{A}^i(\vec{x}),\pi^j(\vec{y})] \;&=\; -i\delta_{ij}^{\rm tr}(\vec{x}-\vec{y}), \qquad \delta_{ij}^{\rm tr}(\vec{x}-\vec{y}) \;\stackrel{\rm def}{\equiv}\; \int d\!\!{}^-{}^3p\,\left(\delta_{ij} - \frac{\vec{p}_i\vec{p}_j}{\vec{p}^2}\right)e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}
\end{aligned} \tag{19.26}$$

The Gauss law is now satisfied since

$$[\hat{A}^i(\vec{x}),\vec{\nabla}\cdot\hat{\vec{E}}(\vec{x})] \;=\; -i\frac{\partial}{\partial y_j}\delta_{ij}^{\rm tr}(\vec{x}-\vec{y}) \;=\; -i\frac{\partial}{\partial y_j}\int d\!\!{}^-{}^3p\,\left(\delta_{ij} - \frac{\vec{p}_i\vec{p}_j}{\vec{p}^2}\right)e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \;=\; 0 \tag{19.27}$$

We see that $\vec{\nabla}\cdot\hat{\vec{E}}(\vec{x})$ commutes with all canonical coordinates and canonical momenta so it is a usual function (c-number). Similarly, $\hat{A}^0(\vec{x})$ commutes with all $\hat{A}^i$ and $\hat{\pi}^i$ (see Eq. (19.26) so $A^0(\vec{x})$ is also a c-number. Since both $\vec{\nabla}\cdot\hat{\vec{E}}(\vec{x})$ and $A^0(\vec{x})$ are c-numbers, the conditions $\vec{\nabla}\cdot\hat{\vec{E}}(\vec{x}) = 0$ and $A^0(\vec{x}) = 0$ are consistent with all commutation relations and we can put them to zero choosing the Coulomb gauge (19.13) [16].

[16] Later we will prove that Heisenberg operators $\hat{F}^{\mu\nu}(t,\vec{x})$ satisfy equations of motion $\partial_\mu\hat{F}^{\mu\nu}(x) = 0$ and then the Coulomb-gauge representation of potentials $\hat{A}^i(t,\vec{x}) \;=\int_{-\infty}^0 dt'\hat{F}^{0i}(t',\vec{x})$ can be proven exactly like Eq. (19.13).

19.2.1 Quantization in the Coulomb gauge

As usually, we

- Promote classical fields and classical canonical momenta at $t=0$ to operators
- Impose canonical commutation relations (at $t=0$)
- Define ladder operators
- Express Heisenberg operators $\hat{A}^i(t,\vec{x}) = e^{i\hat{H}t}\hat{A}^i(\vec{x})e^{-i\hat{H}t}$ in terms of ladder oerators
- Define vacuum as the lowest-energy state of quantum Hamiltonian
- Construct one-particle states and check that they are eigenstates of Hamiltonian and momentum operators.

Our canonical coordinates are $A^i(t,\vec{x}) \to \hat{A}^i(\vec{x})$ and canonical momenta $\pi^i(t,\vec{x}) = E^i(t,\vec{x}) \to \hat{\pi}^i(\vec{x})$ with CCR (19.26)

$$\begin{aligned}
[\hat{A}^\mu(\vec{x}),\hat{A}^\nu(\vec{y})] &= [\pi^i(\vec{x}),\pi^j(\vec{y})] = 0, \\
[\hat{A}^i(\vec{x}),\pi^j(\vec{y})] &= \delta^{\rm tr}_{ij}\delta(\vec{x}-\vec{y})
\end{aligned} \tag{19.28}$$

The ladder operators are defined according to Eqs. (19.17) and (19.19)

$$\begin{aligned}
\hat{A}_i(\vec{x}) &= \sum_{\lambda=1,2}\int\frac{d\!\!\bar{}^3k}{\sqrt{2|\vec{k}|}}\, e_i^\lambda(\vec{k})\big(\hat{a}^\lambda_{\vec{k}}e^{i\vec{k}\cdot\vec{x}} + \hat{a}^{\lambda\dagger}_{\vec{k}}e^{-i\vec{k}\cdot\vec{x}}\big)\Big|_{k_0=|\vec{k}|} \\
\hat{\pi}_i(\vec{x}) &= i\sum_{\lambda=1,2}\int\frac{d\!\!\bar{}^3k}{\sqrt{2|\vec{k}|}}\, |\vec{k}|e_i^\lambda(\vec{k})\big(\hat{a}^\lambda_{\vec{k}}e^{i\vec{k}\cdot\vec{x}} - \hat{a}^{\lambda\dagger}_{\vec{k}}e^{-i\vec{k}\cdot\vec{x}}\big)\Big|_{k_0=|\vec{k}|}
\end{aligned} \tag{19.29}$$

Canonical commutation relations for ladder operators are

$$\begin{aligned}
[\hat{a}^\lambda_{\vec{k}},\hat{a}^{\dagger\lambda'}_{\vec{k}'}] &= (2\pi)^3\delta_{\lambda\lambda'}\delta(\vec{k}-\vec{k}') \\
[\hat{a}^\lambda_{\vec{k}},\hat{a}^{\lambda'}_{\vec{k}'}] &= [\hat{a}^{\dagger\lambda}_{\vec{k}},\hat{a}^{\dagger\lambda'}_{\vec{k}'}] = 0
\end{aligned} \tag{19.30}$$

Proof: CCR (19.30) $\Rightarrow$ CCR (19.28). Let us check that

$$\begin{aligned}
&[\hat{\pi}_i(\vec{x}),\hat{A}_j(\vec{y})] \\
&= i\sum_{\lambda=1,2}\sum_{\lambda'=1,2}\int\frac{d\!\!\bar{}^3k\,d\!\!\bar{}^3k'}{2\sqrt{\omega_k\omega_{k'}}}e_i^\lambda(\vec{k})e_j^\lambda(\vec{k}')\omega_k\big[\hat{a}^\lambda_{\vec{k}}e^{i\vec{k}\cdot\vec{x}} - \hat{a}^{\lambda\dagger}_{\vec{k}}e^{-i\vec{k}\cdot\vec{x}},\hat{a}^{\lambda'}_{\vec{k}'}e^{i\vec{k}'\cdot\vec{y}} + \hat{a}^{\lambda'\dagger}_{\vec{k}'}e^{-i\vec{k}'\cdot\vec{y}}\big]\Big|_{\omega_k=|\vec{k}|,\omega'_k=|\vec{k}'|} \\
&= i\int\frac{d\!\!\bar{}^3k}{2}\sum_{\lambda=1,2}e_i^\lambda(\vec{k})e_j^\lambda(\vec{k})\big(e^{i\vec{k}\cdot(\vec{x}-\vec{y})} + e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}\big) = i\delta^{\rm tr}_{ij}(\vec{x}-\vec{y})
\end{aligned} \tag{19.31}$$

because

$$\sum_{\lambda=1,2}e_i^\lambda(\vec{k})e_j^\lambda(\vec{k}) = \delta_{ij} - \frac{\vec{k}_i\vec{k}_j}{\vec{k}^2} \tag{19.32}$$

Let us now promote classical Hamiltonian (19.11) to quantum operator

$$\hat{H} \;=\; \frac{1}{2}\int d^3x\ \left[\hat{\vec{E}}^2(\vec{x}) + \hat{\vec{B}}^2(\vec{x})\right] \tag{19.33}$$

and express it in terms of ladder operators. We get

$$\begin{aligned}
\hat{E}(\vec{x}) \;&=\; \hat{\pi}(\vec{x}) \;=\; i\sum_{\lambda=1,2}\int \frac{d\!\!{}^{-}{}^3k}{\sqrt{2\omega_k}}\ \omega_k \vec{e}^{\lambda}(\vec{k})\big(\hat{a}^{\lambda}_{\vec{k}}e^{i\vec{k}\cdot\vec{x}} - \hat{a}^{\lambda\dagger}_{\vec{k}}e^{-i\vec{k}\cdot\vec{x}}\big)\Big|_{\omega_k=|\vec{k}|}\\
\hat{B}(\vec{x}) \;&=\; \vec{\nabla}\times\hat{\vec{A}}(\vec{x}) \;=\; \sum_{\lambda=1,2}\vec{\nabla}\times\int\frac{d\!\!{}^{-}{}^3k}{\sqrt{2\omega_k}}\ \vec{e}^{\lambda}(\vec{k})\big(\hat{a}^{\lambda}_{\vec{k}}e^{i\vec{k}\cdot\vec{x}} + \hat{a}^{\lambda\dagger}_{\vec{k}}e^{-i\vec{k}\cdot\vec{x}}\big)\Big|_{\omega_k=|\vec{k}|}\\
&=\; i\sum_{\lambda=1,2}\int\frac{d\!\!{}^{-}{}^3k}{\sqrt{2\omega_k}}\ \vec{k}\times\vec{e}^{\lambda}(\vec{k})\big(\hat{a}^{\lambda}_{\vec{k}}e^{i\vec{k}\cdot\vec{x}} - \hat{a}^{\lambda\dagger}_{\vec{k}}e^{-i\vec{k}\cdot\vec{x}}\big)\Big|_{\omega_k=|\vec{k}|}
\end{aligned} \tag{19.34}$$

and therefore (cf. Eq. (5.18) for the KG theory)

$$\begin{aligned}
&\frac{1}{2}\int d^3x\big(\hat{\vec{E}}^2(\vec{x}) + \hat{\vec{B}}^2(\vec{x})\big) \;=\; -\int d^3x \sum_{\lambda,\lambda'=1,2}\int\frac{d\!\!{}^{-}{}^3k\, d\!\!{}^{-}{}^3k'}{4\sqrt{\omega_k\omega_{k'}}}\ \big[\omega_k\omega_{k'}\vec{e}^{\lambda}(\vec{k})\cdot\vec{e}^{\lambda'}(\vec{k}') + (\vec{k}\times\vec{e}^{\lambda}(\vec{k}))\cdot(\vec{k}'\times\vec{e}^{\lambda'}(\vec{k}'))\big]\\
&\qquad\times\ \big(\hat{a}^{\lambda}_{\vec{k}}e^{i\vec{k}\cdot\vec{x}} - \hat{a}^{\lambda\dagger}_{\vec{k}}e^{-i\vec{k}\cdot\vec{x}}\big)\big(\hat{a}^{\lambda'}_{\vec{k}'}e^{i\vec{k}'\cdot\vec{x}} - \hat{a}^{\lambda'\dagger}_{\vec{k}'}e^{-i\vec{k}\cdot\vec{x}}\big)\Big|_{\omega_k=|\vec{k}|,\omega_{k'}=|\vec{k}'|}\\
&=\; \sum_{\lambda,\lambda'=1,2}\int\frac{d\!\!{}^{-}{}^3k}{4|\vec{k}|}\ \big\{-\big(\hat{a}^{\lambda}_{\vec{k}}\hat{a}^{\lambda'}_{-\vec{k}} + \hat{a}^{\lambda\dagger}_{\vec{k}}\hat{a}^{\lambda'\dagger}_{-\vec{k}}\big)\big(|\vec{k}|^2\vec{e}^{\lambda}(\vec{k})\cdot\vec{e}^{\lambda'}(-\vec{k}) - (\vec{k}\times\vec{e}^{\lambda}(\vec{k}))\cdot(\vec{k}\times\vec{e}^{\lambda'}(-\vec{k})\big]\\
&+\ \big(\hat{a}^{\lambda}_{\vec{k}}\hat{a}^{\lambda'\dagger}_{\vec{k}} + \hat{a}^{\lambda'\dagger}_{\vec{k}}\hat{a}^{\lambda}_{\vec{k}}\big)\big[\vec{k}^2\vec{e}^{\lambda}(\vec{k})\cdot\vec{e}^{\lambda'}(\vec{k}) + (\vec{k}\times\vec{e}^{\lambda}(\vec{k}))\cdot(\vec{k}\times\vec{e}^{\lambda'}(\vec{k})\big)\\
&=\; \sum_{\lambda}\int d\!\!{}^{-}{}^3k\ \frac{|\vec{k}|}{2}\big(\hat{a}^{\lambda}_{\vec{k}}\hat{a}^{\lambda\dagger}_{\vec{k}} + \hat{a}^{\lambda\dagger}_{\vec{k}}\hat{a}^{\lambda}_{\vec{k}}\big) \;=\; \int d\!\!{}^{-}{}^3k\ \omega_k\sum_{\lambda}\hat{a}^{\lambda\dagger}_{\vec{k}}\hat{a}^{\lambda}_{\vec{k}} \;+\; \mathcal{V}\int d\!\!{}^{-}{}^3k\frac{\omega_k}{2} \;\to\; \int d\!\!{}^{-}{}^3k\ \omega_k\sum_{\lambda}\hat{a}^{\lambda\dagger}_{\vec{k}}\hat{a}^{\lambda}_{\vec{k}}
\end{aligned}$$

where we used

$$\begin{aligned}
&(\vec{k}\times\vec{e}^{\lambda}(\vec{k}))\cdot(\vec{k}\times\vec{e}^{\lambda'}(-\vec{k})) \;=\; (\vec{k}\times\vec{e}^{\lambda}(\vec{k}))_i(\vec{k}\times\vec{e}^{\lambda'}(-\vec{k}))_i \;=\; \epsilon_{ijk}\vec{k}_j\vec{e}^{\lambda}_k(\vec{k})\epsilon_{imn}\vec{k}_m\vec{e}^{\lambda'}_n(-\vec{k}) \\
&=\; \big(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\big)\vec{k}_j\vec{e}^{\lambda}_k(\vec{k})\vec{k}_m\vec{e}^{\lambda'}_n(-\vec{k}) \;=\; \vec{k}^2\vec{e}^{\lambda}(\vec{k})\cdot\vec{e}^{\lambda'}(-\vec{k}) - \big(\vec{k}\cdot\vec{e}^{\lambda}(\vec{k})\big)\big(\vec{k}\cdot\vec{e}^{\lambda'}(-\vec{k})\big) \;=\; \vec{k}^2\vec{e}^{\lambda}(\vec{k})\cdot\vec{e}^{\lambda'}(-\vec{k})
\end{aligned} \tag{19.35}$$

(recall that $\vec{k}\cdot\vec{e}^{\lambda}(\vec{k}) \;=\; 0$) and similar equation

$$(\vec{k}\times\vec{e}^{\lambda}(\vec{k}))\cdot(\vec{k}\times\vec{e}^{\lambda'}(\vec{k})) \;=\; \vec{k}^2\vec{e}^{\lambda}(\vec{k})\cdot\vec{e}^{\lambda'}(\vec{k}) \;=\; \vec{k}^2\delta^{\lambda\lambda'} \tag{19.36}$$

Since

$$\hat{H} \;=\; \int d\!\!{}^{-}{}^3k\ \omega_k\sum_{\lambda}\hat{a}^{\lambda\dagger}_{\vec{k}}\hat{a}^{\lambda}_{\vec{k}}\Big|_{\omega_k=|\vec{k}|} \tag{19.37}$$

we can define vacuum as a state annihilated by $\hat{a}^{\lambda}_{\vec{k}}$ [17]

$$\hat{a}^{\lambda}_{\vec{k}}|0\rangle \;=\; 0 \tag{19.38}$$

It is clear that such state is the lowest-energy eigenstate since for any other state $|\Psi\rangle$

$$\langle\Psi|\hat{H}|\Psi\rangle \;=\; \sum_{\text{all states}|n\rangle}\int d\!\!{}^-{}^3k\; E_k \sum_{\lambda}\langle\Psi|\hat{a}^{\lambda\dagger}_{\vec{k}}|n\rangle\langle n|\hat{a}^{\lambda}_{\vec{k}}|\Psi\rangle \;=\; \sum_{\text{all states}|n\rangle}\int d\!\!{}^-{}^3k\; E_k \sum_{\lambda}|\langle n|\hat{a}^{\lambda}_{\vec{k}}|\Psi\rangle|^2 \;>\; 0 \tag{19.39}$$

Let us define would-be one-photon state

$$|\vec{k},\lambda\rangle \;=\; \sqrt{2|\vec{k}|}\hat{a}^{\dagger\lambda}_{\vec{k}}|0\rangle, \tag{19.40}$$

(Peskin's normalization is $\langle\vec{k},\lambda|\vec{k}',\lambda'\rangle \;=\; 2|\vec{k}|\delta_{\lambda\lambda'}(2\pi)^3\delta(\vec{k}-\vec{k}')$). Using the commutators

$$[\hat{H},\hat{a}^{\lambda\dagger}_{\vec{k}}] \;=\; |\vec{k}|\hat{a}^{\lambda\dagger}_{\vec{k}}, \qquad [\hat{H},\hat{a}^{\lambda}_{\vec{k}}] \;=\; -\,|\vec{k}|\hat{a}^{\lambda\dagger}_{\vec{k}} \tag{19.41}$$

we can easily check that the state (19.40) is an eigenstate of Hamiltonian (19.37) with energy $\omega_k = |\vec{k}|$:

$$\hat{H}|\vec{k},\lambda\rangle \;=\; \sqrt{2|\vec{k}|}\hat{H}\hat{a}^{\lambda\dagger}_{\vec{k}}|0\rangle \;=\; \sqrt{2|\vec{k}|}[\hat{H},\hat{a}^{\lambda\dagger}_{\vec{k}}]|0\rangle \;=\; \sqrt{2|\vec{k}|}|\vec{k}|\hat{a}^{\lambda\dagger}_{\vec{k}}|0\rangle \;=\; \omega_k|\vec{k},\lambda\rangle \tag{19.42}$$

However, as in the KG case, before interpreting (19.40) as a one-photon state we need to construct the momentum operator for electromagnetic field and check that the state (19.40) is an eigenstate of the momentum operator with the correct relation between energy and momentum $\omega_k = \sqrt{\vec{k}^2}$.

19.3 Momentum operator for free electromagnetic field

19.3.1 Momentum of classical electromagnetic field

A general formula for stress-energy tensor is

$$T^{\mu\nu} \;=\; \sum \frac{\partial L}{\partial_\mu\Phi}\partial^\nu\Phi - g^{\mu\nu}\mathcal{L} \tag{19.43}$$

which gives

$$\begin{aligned} T^{\mu\nu} \;&=\; \sum \frac{\partial L}{\partial_\mu A^\alpha}\partial^\nu A^\alpha - g^{\mu\nu}\mathcal{L} \qquad (19.44)\\ &=\; -\,F^{\mu\alpha}\partial^\nu A^\alpha - g^{\mu\nu}\mathcal{L} \;=\; -\,F^{\mu\alpha}F^{\nu}{}_{\alpha} + \frac{g^{\mu\nu}}{4}F^{\xi\eta}F_{\xi\eta} - \partial_\alpha(F^{\mu\alpha}A^\nu) \end{aligned}$$

[17] Actually, in free electrodynamics the explicit form of the vacuum wave functional is known (cf. Eq. (4.55)):

$$\langle\{\vec{A}(\vec{x})\}|\Psi_0\rangle \;\sim\; e^{-\frac{1}{2}\int d^3x\;\vec{B}(x)\frac{1}{W}\vec{B}(x)} \;=\; e^{-\frac{1}{32\pi^2}\int d^3x d^3y\;\vec{B}(x)\frac{1}{|\vec{x}-\vec{y}|}\vec{B}(y)}$$

where

$$\frac{1}{W}\vec{B}(\vec{x}) \;=\; \int\frac{d\!\!{}^-{}^3k}{|\vec{k}|}e^{i\vec{p}\cdot\vec{x}}\vec{B}(\vec{k}) \;=\; \int d^3y\;\frac{1}{4\pi|\vec{x}-\vec{y}|}\vec{B}(y),$$

but, as usual, we do not need the explicit form of vacuum wave functional - the property (19.38) is sufficient for calculation of all amplitudes.

The last term $\sim \partial_\alpha(F^{\mu\alpha}A^\nu)$ does not contribute to either H or to P_i so one can take

$$T^{\mu\nu} \;=\; -F^{\mu\alpha}F^\nu_{\ \alpha} + \frac{g^{\mu\nu}}{4}F^{\xi\eta}F_{\xi\eta} \tag{19.45}$$

as a stress-energy tensor. It is worth noting that exactly this form of $T^{\mu\nu}$ is obtained as a variational derivative of Lagrangian with respect to metric tensor $G_{\mu\nu}$.

The momentum of a classical electromagnetic field is defined as

$$P^i(t) \;=\; \int d^3x\; T^{0i}(t,\vec{x}) \;=\; \int d^3x\big(-\dot{A}^k\partial^i A_k + \dot{A}^k\partial_k A^i\big) \overset{\text{by parts}}{=} \int d^3x\; E^k\partial^i A_k \tag{19.46}$$

so

$$\vec{P}(t) \;=\; \int d^3x\; E^k(t,\vec{x})\vec{\nabla}A_k(t,\vec{x}) \;=\; -\int d^3x\; \dot{A}^k(t,\vec{x})\vec{\nabla}A_k(t,\vec{x}) \tag{19.47}$$

Let us prove that this equation coincides with the familiar Poynting form of the momentum of an electromagnetic field

$$\vec{P}(t) \;=\; \int d^3x\; \vec{E}(t,\vec{x})\times\vec{B}(t,\vec{x}) \;=\; \int d^3x\; \vec{E}(t,\vec{x})\times(\vec{\nabla}\times\vec{A}(t,\vec{x})) \tag{19.48}$$

Proof:

$$\begin{aligned}
\vec{P}_i(t) \;&=\; \int d^3x\; (\vec{E}(t,\vec{x})\times\vec{B}(t,\vec{x}))_i \;=\; \epsilon_{ijk}\int d^3x\; \vec{E}_j(t,\vec{x})\vec{B}_k(t,\vec{x}) \;=\; \epsilon_{ijk}\epsilon_{klm}\int d^3x\; \vec{E}_j(t,\vec{x})\partial_l\vec{A}_m(t,\vec{x}) \\
&=\; \int d^3x(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})\vec{E}_j(t,\vec{x})\partial_l\vec{A}_m(t,\vec{x}) \;=\; \int d^3x\; \vec{E}_j(t,\vec{x})\partial_i\vec{A}_j(t,\vec{x}) - \int d^3x\; \vec{E}_j(t,\vec{x})\partial_j\vec{A}_i(t,\vec{x}) \overset{\text{by parts}}{=} \\
&=\; \int d^3x\; \vec{E}_j(t,\vec{x})\partial_i\vec{A}_j(t,\vec{x}) + \int d^3x\; (\vec{\nabla}\cdot\vec{E}(t,\vec{x}))\vec{A}_k(t,\vec{x}) \;=\; \int d^3x\; \vec{E}_j(t,\vec{x})\partial_i\vec{A}_j(t,\vec{x}) \;=\; -\int d^3x\; \dot{\vec{A}}_j(t,\vec{x})\partial_i\vec{A}_j(t,\vec{x})
\end{aligned} \tag{19.49}$$

which coincides with Eq. (19.50) (recall that $A_i = -A^i = -\vec{A}_i$).

19.3.2 Quantum operator of momentum

As usually, to get a quantum operator we take classical expression at $t=0$ and promote canonical coordinates (A_i) and canonical momenta ($\pi_i \equiv E_i$) to operators:

$$\hat{\vec{P}} \;=\; -\int d^3x\dot{\hat{A}}^k(\vec{x})\vec{\nabla}\hat{A}_k(\vec{x}) \;=\; \int d^3x\; \hat{\vec{E}}(\vec{x})\times\hat{\vec{B}}(\vec{x}) \tag{19.50}$$

After some algebra on can express $\hat{\vec{P}}$ in terms of ladder operators (cf. Eqs. (5.33) and (14.36))

$$\hat{P}^i \;=\; \sum_{\lambda=1,2}\int d\!\!{}^-{}^3k\; k^i\hat{a}^{\lambda\dagger}_{\vec{k}}\hat{a}^\lambda_{\vec{k}} \tag{19.51}$$

so one can define quantum operator of 4-momentum as usual:

$$P^\mu \;\equiv\; (\hat{H},\hat{\vec{P}}) \;=\; \sum_{\lambda=1,2}\int d\!\!{}^-{}^3k\; k^\mu\hat{a}^{\lambda\dagger}_{\vec{k}}\hat{a}^\lambda_{\vec{k}} \tag{19.52}$$

Using Eq. (19.52) it is easy to check that

$$[\hat{P}^\mu,\hat{a}^{\lambda\dagger}_{\vec{k}}] \;=\; k^\mu\hat{a}^{\lambda\dagger}_{\vec{k}}, \qquad [\hat{P}^\mu,\hat{a}^\lambda_{\vec{k}}] \;=\; -k^\mu\hat{a}^\lambda_{\vec{k}} \tag{19.53}$$

so

$$\hat{\vec{P}}|\vec{k},\lambda\rangle \;=\; \sqrt{2|\vec{k}|}\hat{\vec{P}}\hat{a}^{\lambda\dagger}_{\vec{k}}|0\rangle \;=\; \sqrt{2|\vec{k}|}[\hat{\vec{P}},\hat{a}^{\lambda\dagger}_{\vec{k}}]|0\rangle \;=\; \sqrt{2|\vec{k}|}\vec{k}\hat{a}^{\lambda\dagger}_{\vec{k}}|0\rangle \;=\; \vec{k}|\vec{k},\lambda\rangle \tag{19.54}$$

which means that the state $|\vec{k},\lambda\rangle$ can be interpreted as one-photon state with momentum $\vec{k}$ and energy $E_k = |\vec{k}|$. The states corresponding to right- and left- circularly polarized photons are defined as

$$\begin{aligned}
|\vec{k},R\rangle \;&=\; \sqrt{2|\vec{k}|}\hat{a}^{R\dagger}_{\vec{k}} \;\equiv\; \sqrt{|\vec{k}|}(\hat{a}^{1\dagger}_{\vec{k}} + i\hat{a}^{2\dagger}_{\vec{k}})|0\rangle, \\
|\vec{k},L\rangle \;&=\; \sqrt{2|\vec{k}|}\hat{a}^{L\dagger}_{\vec{k}} \;\equiv\; \sqrt{|\vec{k}|}(\hat{a}^{1\dagger}_{\vec{k}} - i\hat{a}^{2\dagger}_{\vec{k}})|0\rangle
\end{aligned} \tag{19.55}$$

As we shall see later, the circular polarization is related to the helicity of the photon.

Using Eq. (19.53) it is easy to prove that

$$e^{i\hat{P}x}\hat{a}^{\lambda}_{\vec{k}}e^{-i\hat{P}x} \;=\; \hat{a}^{\lambda}_{\vec{k}}e^{-ikx}, \qquad e^{i\hat{P}x}\hat{a}^{\lambda\dagger}_{\vec{k}}e^{-i\hat{P}x} \;=\; \hat{a}^{\lambda\dagger}_{\vec{k}}e^{ikx} \tag{19.56}$$

and therefore Heisenberg operators for electromagnetic field take the familiar form (cf. Eq. (5.13)

$$\hat{A}_i(x) \;\equiv\; e^{i\hat{H}x_0}\hat{A}_i(\vec{x})e^{-i\hat{H}x_0} \;=\; \sum_{\lambda=1,2}\int\frac{d\!\!\!{}^-{}^3k}{\sqrt{2\omega_k}}\,e^{\lambda}_i(\vec{k})\big(\hat{a}^{\lambda}_{\vec{k}}e^{-ikx} + \hat{a}^{\lambda\dagger}_{\vec{k}}e^{ikx}\big)\Big|_{\omega_k=|\vec{k}|} \tag{19.57}$$

$$\hat{\pi}_i(x) = \hat{E}_i(x) \;\equiv\; e^{i\hat{H}x_0}\hat{\pi}_i(\vec{x})e^{-i\hat{H}x_0} \;=\; i\sum_{\lambda=1,2}\int\frac{d\!\!\!{}^-{}^3k}{\sqrt{2\omega_k}}\,\omega_k e^{\lambda}_i(\vec{k})\big(\hat{a}^{\lambda}_{\vec{k}}e^{-ikx} - \hat{a}^{\lambda\dagger}_{\vec{k}}e^{ikx}\big)\Big|_{\omega_k=|\vec{k}|}$$

In addition, from

$$[P^\mu,\hat{A}^i(x)] \;=\; -\,i\partial^\mu\hat{A}^i(x) \tag{19.58}$$

we can check (similarly to Eq. (6.16) [18]) that $\hat{P}$ induces shifts in the arguments of field operators:

$$e^{i\hat{P}a}\hat{A}^i(x)e^{-i\hat{P}a} \;=\; \hat{A}^i(x+a) \tag{19.59}$$

so it is indeed a quantum operator of momentum since it describes the response of the system to shifts of coordinates. In the next Section we will construct quantum operator of angular momentum and check the spin of of the one-photon states (19.54)

19.4 Angular momentum and spin of the electromagnetic field

19.4.1 Angular momentum of classical electromagnetic field

The angular momentum of classical electromagnetic field is given by (see *Jackson*)

$$\vec{J}(t) \;=\; \int d^3x\;\vec{x}\times\big(\vec{E}(t,\vec{x})\times\vec{B}(t,\vec{x})\big) \tag{19.60}$$

[18] A more simple way to see this is to use Eqs. (19.57) and (19.56).

Let us prove that it is conserved. From Maxwell's equations (19.1) and (19.2) we see that

$$\begin{aligned}
\frac{d}{dt}\vec{J}(t) &= \int d^3x\ \vec{x}\times\big(\vec{E}(t,\vec{x})\times\dot{\vec{B}}(t,\vec{x}) - \vec{B}(t,\vec{x})\times\dot{\vec{E}}(t,\vec{x})\big) \qquad (19.61)\\
&= -\int d^3x\ \vec{x}\times\big[\vec{E}(t,\vec{x})\times(\vec{\nabla}\times\vec{E}(t,\vec{x})) + \vec{B}(t,\vec{x})\times(\vec{\nabla}\times\vec{B}(t,\vec{x}))\big]\\
&= -\int d^3x\ \vec{x}\times\big[\frac{1}{2}\vec{\nabla}|\vec{E}(t,\vec{x})|^2 - (\vec{E}(t,\vec{x})\cdot\vec{\nabla})\vec{E}(t,\vec{x})) + \frac{1}{2}\vec{\nabla}|\vec{B}(t,\vec{x})|^2(\vec{B}(t,\vec{x})\cdot\vec{\nabla})\vec{B}(t,\vec{x}))\big]
\end{aligned}$$

where we used formula

$$\vec{a}\times(\vec{\nabla}\times\vec{a}) = \frac{1}{2}\vec{\nabla}(\vec{a}^2) - (\vec{a}\cdot\vec{\nabla})\vec{a} \qquad (19.62)$$

Now we need to integrate by parts which is easily done in components

$$\begin{aligned}
\frac{d}{dt}\vec{J}_i(t) &= -\int d^3x\ \epsilon_{ijk}\vec{x}_j\big[\frac{1}{2}\frac{\partial}{\partial\vec{x}_k}|\vec{E}(t,\vec{x})|^2 - \vec{E}_l(t,\vec{x})\frac{\partial}{\partial\vec{x}_l}\vec{E}_k(t,\vec{x}) + \frac{1}{2}\frac{\partial}{\partial\vec{x}_k}|\vec{B}(t,\vec{x})|^2 - \vec{B}_l(t,\vec{x})\frac{\partial}{\partial\vec{x}_l}\vec{B}_k(t,\vec{x})\big]\\
&\overset{\partial_l\vec{E}_l=\partial_l\vec{B}_l=0}{=} -\int d^3x\ \epsilon_{ijk}\vec{x}_j\big[\frac{1}{2}\frac{\partial}{\partial\vec{x}_k}|\vec{E}(t,\vec{x})|^2 - \frac{\partial}{\partial\vec{x}_l}(\vec{E}_l(t,\vec{x})\vec{E}_k(t,\vec{x})) + \frac{1}{2}\frac{\partial}{\partial\vec{x}_k}|\vec{B}(t,\vec{x})|^2 - \frac{\partial}{\partial\vec{x}_l}((\vec{B}_l(t,\vec{x})\vec{B}_k(t,\vec{x}))\big]\\
&\overset{\text{by parts}}{=} -\int d^3x\ \epsilon_{ilk}[\vec{E}_l(t,\vec{x})\vec{E}_k(t,\vec{x}) + \vec{B}_l(t,\vec{x})\vec{B}_k(t,\vec{x})] = 0 \qquad (19.63)
\end{aligned}$$

19.4.2 Quantum operator of angular momentum

As usual, to get quantum operator we take the corresponding classical quantity (19.64) and promote canonical coordinates A^i and canonical momenta E^i to operators: $A^i(t,\vec{x}) \to \hat{A}^i(\vec{x})$ and $E^i(t,\vec{x}) \to \hat{E}^i(\vec{x})$

$$\hat{\vec{J}} = \int d^3x\ \vec{x}\times\big(\hat{\vec{E}}(\vec{x})\times\hat{\vec{B}}(t,\vec{x})\big) \qquad (19.64)$$

It is instructive to rewrite this operator as

$$\begin{aligned}
\hat{\vec{J}}_i &= \epsilon_{ijk}\int d^3x\ \vec{x}_j\big(\hat{\vec{E}}(\vec{x})\times\hat{\vec{B}}(t,\vec{x})\big)_k = \epsilon_{ijk}\epsilon_{klm}\int d^3x\ \vec{x}_j\hat{\vec{E}}_l(\vec{x})\hat{\vec{B}}_m(t,\vec{x})\\
&= \epsilon_{ijk}\epsilon_{klm}\epsilon_{mnr}\int d^3x\ \vec{x}_j\hat{\vec{E}}_l(\vec{x})\frac{\partial}{\partial\vec{x}_n}\hat{\vec{A}}_r(t,\vec{x}) = \epsilon_{ijk}\int d^3x\ \vec{x}_j\big(\hat{\vec{E}}_l(\vec{x})\frac{\partial}{\partial\vec{x}_k}\hat{\vec{A}}_l(\vec{x}) - \hat{\vec{E}}_l(\vec{x})\frac{\partial}{\partial\vec{x}_l}\hat{\vec{A}}_k(\vec{x})\big)\\
&\overset{\text{by parts}}{=} \epsilon_{ijk}\int d^3x\ \vec{x}_j\hat{\vec{E}}_l(\vec{x})\frac{\partial}{\partial\vec{x}_k}\hat{\vec{A}}_l(\vec{x}) + \epsilon_{ijk}\int d^3x\ \hat{\vec{E}}_j(\vec{x})\hat{\vec{A}}_k(\vec{x}) \qquad (19.65)
\end{aligned}$$

which in vector form looks like

$$\hat{\vec{J}} = \hat{\vec{O}} + \hat{\vec{S}}, \qquad \hat{\vec{O}} = \int d^3x\ \vec{x}\times\hat{\vec{E}}_k\vec{\nabla}\hat{\vec{A}}_k, \qquad \hat{\vec{S}} = \int d^3x\ \hat{\vec{E}}(\vec{x})\times\vec{A}(\vec{x}) \quad (19.66)$$

Looking at the momentum density (19.46) it is easy to interpret the operator $\hat{\vec{O}}$ as an orbital angular momentum. We will now demonstrate that the second term can be interpreted as an operator of spin. [19]

[19] Strictly speaking, we define the operators $\hat{\vec{O}}$ and $\hat{\vec{S}}$ as normal products of corresponding ladder operators $:\hat{\vec{O}}:$ and $:\hat{\vec{S}}:$ so that $\hat{\vec{O}}|0\rangle = \hat{\vec{S}}|0\rangle = 0$, see the footnote at page 106

First, let us check the self-consistency of our definition of orbital momentum. Let us take take photon with momentum $\vec{k} \parallel \vec{z}$ described by $\hat{a}^{\lambda}_{\vec{k}}$ and check that the z-component of the orbital angular momentum of such photon vanishes. We get

$$
\begin{aligned}
\hat{\vec{O}}_3 \hat{a}^{\lambda}_{\vec{k}}|0\rangle &\overset{\text{see footnote}}{=} [\hat{\vec{O}}_3, \hat{a}^{\lambda}_{\vec{k}}]|0\rangle \;=\; \int d^3x\; [(\vec{x}\times \hat{\vec{E}}_i \vec{\nabla}\hat{\vec{A}}_i)_3, \hat{a}^{\lambda\dagger}_{\vec{k}}]|0\rangle \\
&= \int d^3x\; [\big(\vec{x}_1 \hat{E}_i(\vec{x}) \frac{\partial}{\partial \vec{x}_2}\hat{A}_i(\vec{x}) - \vec{x}_2 \hat{E}_i(\vec{x}) \frac{\partial}{\partial \vec{x}_1}\hat{A}_i(\vec{x})\big), \hat{a}^{\lambda\dagger}_{\vec{k}}]|0\rangle \\
&= \int d^3x\; \Big(\vec{x}_1 \hat{E}_i(\vec{x}) \frac{\partial}{\partial \vec{x}_2}[\hat{A}_i(\vec{x}), \hat{a}^{\lambda\dagger}_{\vec{k}}] - \vec{x}_2 \hat{E}_i(\vec{x}) \frac{\partial}{\partial \vec{x}_1}[\hat{A}_i(\vec{x}), \hat{a}^{\lambda\dagger}_{\vec{k}}]\Big)|0\rangle \\
&+ \int d^3x\; \Big(\vec{x}_1 [\hat{E}_i(\vec{x}), \hat{a}^{\lambda\dagger}_{\vec{k}}] \frac{\partial}{\partial \vec{x}_2}\hat{A}_i(\vec{x}) - \vec{x}_2 [\hat{E}_i(\vec{x}), \hat{a}^{\lambda\dagger}_{\vec{k}}] \frac{\partial}{\partial \vec{x}_1}\hat{A}_i(\vec{x})\Big)|0\rangle \\
&\overset{\text{by parts}}{=} \int d^3x\; \Big(\vec{x}_1 \hat{E}_i(\vec{x}) \frac{\partial}{\partial \vec{x}_2}[\hat{A}_i(\vec{x}), \hat{a}^{\lambda\dagger}_{\vec{k}}] - \vec{x}_2 \hat{E}_i(\vec{x}) \frac{\partial}{\partial \vec{x}_1}[\hat{A}_i(\vec{x}), \hat{a}^{\lambda\dagger}_{\vec{k}}]\Big)|0\rangle \\
&+ \int d^3x\; \Big(\vec{x}_1 \hat{A}_i(\vec{x}) \frac{\partial}{\partial \vec{x}_2}[\hat{E}_i(\vec{x}), \hat{a}^{\lambda\dagger}_{\vec{k}}] - \vec{x}_2 \hat{A}_i(\vec{x}) \frac{\partial}{\partial \vec{x}_1}[\hat{E}_i(\vec{x}), \hat{a}^{\lambda\dagger}_{\vec{k}}]\Big)|0\rangle \qquad (19.67)
\end{aligned}
$$

Using the commutators

$$
[\hat{A}_j(\vec{x}), \hat{a}^{\lambda\dagger}_{\vec{k}}] \;=\; \frac{e^{\lambda}_j(\vec{k})}{\sqrt{2|\vec{k}|}} e^{i\vec{k}\vec{x}}, \qquad [\hat{E}_j(\vec{x}), \hat{a}^{\lambda\dagger}_{\vec{k}}] \;=\; i|\vec{k}| \frac{e^{\lambda}_j(\vec{k})}{\sqrt{2|\vec{k}|}} e^{i\vec{k}\vec{x}} \qquad (19.68)
$$

we see that in our $\vec{k} \parallel z$ case $\frac{\partial}{\partial \vec{x}_1} e^{i|k|x_3}$ and $\frac{\partial}{\partial \vec{x}_2} e^{i|k|x_3}$ vanish so $\hat{O}_3|\vec{k}, \lambda\rangle \;=\; 0$ if $\vec{k} \parallel \vec{z}$. [20]

We will now check the interpretation of $\hat{S}$ given by Eq. (19.66) as a photon spin operator. Let us apply the operator $\hat{S}_3$ to state corresponding to photon with right circular polarization (see Eq. (19.55))

$$
|\vec{k}, R\rangle \;\equiv\; \sqrt{2|\vec{k}|}\hat{a}^{R\dagger}_{\vec{k}}|0\rangle \qquad (19.69)
$$

We will need commutators

$$
[\hat{A}_j(\vec{x}), \hat{a}^{R\dagger}_{\vec{k}}] \;=\; \frac{e^{R}_j(\vec{k})}{\sqrt{2|\vec{k}|}} e^{i\vec{k}\vec{x}}, \qquad [\hat{E}_j(\vec{x}), \hat{a}^{R\dagger}_{\vec{k}}] \;=\; i|\vec{k}| \frac{e^{R}_j(\vec{k})}{\sqrt{2|\vec{k}|}} e^{i\vec{k}\vec{x}} \qquad (19.70)
$$

where $\vec{e}^R = \frac{1}{\sqrt{2}}(\vec{e}^1 + i\vec{e}^2)$, see Eq. (19.55).

[20] We could consider the Poynting vector itself rather than $\vec{\hat{E}}_k \vec{\nabla}\vec{\hat{A}}_k$ as a density of momentum but then we would get a wrong result that the z-component of the orbital angular momentum of the photon moving in z direction does not vanish.

We get

$$
\begin{aligned}
\hat{S}_3|\vec{k},+\rangle &= \sqrt{2|\vec{k}|}[\hat{S}_3,\hat{a}^{R\dagger}_{\vec{k}}]|0\rangle = \sqrt{2|\vec{k}|}\int d^3x[\vec{E}_1(\vec{x})\vec{A}_2(\vec{x})-\vec{E}_2(\vec{x})\vec{A}_1(\vec{x}),\hat{a}^{R\dagger}_{\vec{k}}]|0\rangle \\
&= \sqrt{2|\vec{k}|}\int d^3x[\vec{E}_1(\vec{x}),\hat{a}^{R\dagger}_{\vec{k}}]\vec{A}_2(\vec{x})-[\vec{E}_2(\vec{x}),\hat{a}^{+\dagger}_{\vec{k}}]\vec{A}_1(\vec{x}),\hat{a}^{R\dagger}_{\vec{k}}]+\vec{E}_1(\vec{x})[\vec{A}_2(\vec{x}),\hat{a}^{R\dagger}_{\vec{k}}]-\vec{E}_2(\vec{x})[\vec{A}_1(\vec{x}),\hat{a}^{R\dagger}_{\vec{k}}]|0\rangle \\
&= \int d^3x\; e^{i\vec{k}\cdot\vec{x}}\Big\{i|\vec{k}|\vec{e}^{\,R}_1(\vec{k})\vec{A}_2(\vec{x})-i|\vec{k}|\vec{e}^{\,R}_2(\vec{k})\vec{A}_1(\vec{x})+\vec{E}_1(\vec{x})\vec{e}^{\,R}_2(\vec{k})-\vec{E}_2(\vec{x})\vec{e}^{\,R}_1(\vec{k})\Big\}|0\rangle \\
&= \int d^3x\; e^{i\vec{k}\cdot\vec{x}}\Big\{i|\vec{k}|\vec{A}_2(\vec{x})-\vec{E}_2(\vec{x})+|\vec{k}|\vec{A}_1(\vec{x})+i\vec{E}_1(\vec{x})\Big\}|0\rangle \\
&= \int d^3x\; e^{i\vec{k}\cdot\vec{x}}\Big\{|\vec{k}|\big(\vec{A}_1(\vec{x})+i\vec{A}_2(\vec{x})\big)+\big(i\vec{E}_1(\vec{x})-\vec{E}_2(\vec{x})\big)\Big\}|0\rangle \\
&= \frac{\sqrt{|\vec{k}|}}{2}\sum_{\lambda=1,2}\Big\{(\vec{e}^{\,\lambda}_1(\vec{k})+i\vec{e}^{\,\lambda}_2(\vec{k}))(\hat{a}^{\lambda}_{-\vec{k}}+\hat{a}^{\lambda\dagger}_{\vec{k}})-(\vec{e}^{\,\lambda}_1(\vec{k})+i\vec{e}^{\,\lambda}_2(\vec{k}))(\hat{a}^{\lambda}_{-\vec{k}}-\hat{a}^{\lambda\dagger}_{\vec{k}})\Big\}|0\rangle \\
&= \sqrt{|\vec{k}|}\sum_{\lambda=1,2}(\vec{e}^{\,\lambda}_1(\vec{k})+i\vec{e}^{\,\lambda}_2(\vec{k}))\hat{a}^{\lambda\dagger}_{\vec{k}}|0\rangle = \sqrt{|\vec{k}|}(\hat{a}^{1\dagger}_{\vec{k}}+i\hat{a}^{2\dagger}_{\vec{k}})|0\rangle = \sqrt{2|\vec{k}|}\hat{a}^{R\dagger}_{\vec{k}}|0\rangle = |\vec{k},R\rangle
\end{aligned}
\tag{19.71}
$$

where we used Eq. (19.31):

$$
\begin{aligned}
\hat{A}_i(\vec{x}) &= \sum_{\lambda=1,2}\int\frac{d\!\!{}^-{}^3k}{\sqrt{2|\vec{k}|}}\; e^{\lambda}_i(\vec{k})\big(\hat{a}^{\lambda}_{\vec{k}}e^{i\vec{k}\cdot\vec{x}}+\hat{a}^{\lambda\dagger}_{\vec{k}}e^{-i\vec{k}\cdot\vec{x}}\big)\Big|_{\omega_k=|\vec{k}|} \\
\hat{\pi}_i(\vec{x}) &= i\sum_{\lambda=1,2}\int\frac{d\!\!{}^-{}^3k}{\sqrt{2|\vec{k}|}}\;|\vec{k}|e^{\lambda}_i(\vec{k})\big(\hat{a}^{\lambda}_{\vec{k}}e^{i\vec{k}\cdot\vec{x}}-\hat{a}^{\lambda\dagger}_{\vec{k}}e^{-i\vec{k}\cdot\vec{x}}\big)\Big|_{\omega_k=|\vec{k}|}
\end{aligned}
\tag{19.72}
$$

Similarly one can check that $\hat{S}_3|\vec{k},L\rangle = -|\vec{k},L\rangle$ so the operator $\hat{S}_3$ measures the helicity of the photon with momentum $\vec{k}\parallel z$.

Part XVIII

20 QED

20.1 Classical theory of interacting Dirac and electromagnetic fields

The Lagrangian (density) for QED has the form

$$
\mathcal{L}(x) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\bar{\psi}(i\not{D}-m)\psi, \qquad D_\mu\psi(x) \equiv \big(\partial_\mu - ieA_\mu\big)(x) \tag{20.1}
$$

where $e = e_{positron} = -e_{\text{electron}}$. It is a sum of the free electromagnetic Lagrangian (19.7), free Dirac Lagrangian (13.37) and the interaction Lagrangian

$$
\mathcal{L}_{\text{int}}(x) = e\bar{\psi}(x)\not{A}(x)\psi(x) \tag{20.2}
$$

describing interaction of electromagnetic field with the Dirac field.

The corresponding Euler-Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial \bar\psi}(x) \;=\; \frac{d}{dx_\mu}\frac{\partial \mathcal{L}}{\partial \partial^\mu \bar\psi} \qquad \Rightarrow \qquad (i\,\not{D} - m)\psi(x) \;=\; 0 \tag{20.3}$$

$$\frac{\partial \mathcal{L}}{\partial \psi}(x) \;=\; \frac{d}{dx_\mu}\frac{\partial \mathcal{L}}{\partial \partial^\mu \psi}(x) \qquad \Rightarrow \qquad e\bar\psi(x)\not{A}(x) - m\bar\psi(x) \;=\; i\partial^\mu \bar\psi(x)\gamma_\mu \qquad \Leftrightarrow \qquad \bar\psi(x)\big(i\overleftarrow{\not{D}} + m\big) \;=\; 0$$

where $\bar\psi(x)\overleftarrow{D}_\mu \;\equiv\; \partial^\mu \bar\psi(x) + ie\bar\psi(x)A^\mu(x)$, and

$$\frac{\partial \mathcal{L}}{\partial A^\nu}(x) \;=\; \frac{d}{dx_\mu}\frac{\partial \mathcal{L}}{\partial \partial^\mu A^\nu}(x) \qquad \Rightarrow \qquad e\bar\psi\gamma_\nu\psi(x) \;=\; -\,\partial^\mu F_{\mu\nu}(x) \qquad \Leftrightarrow \qquad \partial^\mu F_{\mu\nu}(x) \;=\; -\,ej_\nu(x) \tag{20.4}$$

The first two equations (20.3) are Dirac equations in an "external field" A_μ and the last (20.4) is the Maxwell equation with the source.

20.1.1 Gauge invariance

This theory of interacting fields is invariant under *gauge transformations*

$$\begin{aligned} \psi(x) \;&\rightarrow\; e^{i\alpha(x)}\psi(x) \\ \bar\psi(x) \;&\rightarrow\; e^{-i\alpha(x)}\bar\psi(x) \\ A_\mu(x) \;&\rightarrow\; A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x) \end{aligned} \tag{20.5}$$

where $\alpha(x)$ is an arbitrary (scalar) function of coordinates. Let us check that the Lagrangian (20.1) is gauge invariant. For $F_{\mu\nu}F^{\mu\nu}$ it is obvious since $F_{\mu\nu} \;\rightarrow\; F_{\mu\nu}$ (see previous Section) and for the mass term we get $m\bar\psi(x)\psi(x) \rightarrow m\bar\psi(x)e^{-i\alpha(x)}e^{i\alpha(x)}\psi(x) = m\bar\psi(x)\psi(x)$. Let us now prove that the "covariant derivative" D_μ is gauge invariant (the better word is "gauge covariant")

$$\begin{aligned} D_\mu\psi(x) \;&=\; \big(\frac{\partial}{\partial x^\mu} - ieA_\mu(x)\big)\psi(x) \;\rightarrow\; \big(i\frac{\partial}{\partial x^\mu} - ieA'_\mu(x)\big)\psi'(x) \\ &=\; i\frac{\partial}{\partial x^\mu}\big(e^{i\alpha(x)}\psi(x)\big) - ie\big(A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x)\big)e^{i\alpha(x)}\psi(x) \\ &=\; e^{i\alpha(x)}\big[i\frac{\partial}{\partial x^\mu}\psi(x) - ieA_\mu(x)\psi(x)\big] \;=\; e^{i\alpha(x)}D_\mu\psi(x) \end{aligned} \tag{20.6}$$

Similarly one can prove that $\bar\psi(x)\overleftarrow{D}_\mu \;\rightarrow\; \bar\psi(x)\overleftarrow{D}_\mu\; e^{-i\alpha(x)}$ under the transformation (20.5).

Now we see that the term $\bar\psi(x)\gamma^\mu D_\mu\psi(x) \rightarrow \bar\psi(x)e^{-i\alpha(x)}e^{i\alpha(x)}\gamma^\mu D_\mu\psi(x)$ is gauge invariant and so is the Lagrangian (20.1).

The Dirac current is defined as

$$j^\mu(x) \;\stackrel{\text{def}}{\equiv}\; \bar\psi(x)\gamma^\mu\psi(x) \tag{20.7}$$

This current is conserved (even in the interacting theory) since

$$\begin{aligned} \partial_\mu j^\mu(x) \;&=\; \big(\frac{\partial}{\partial x^\mu}\bar\psi(x)\big)\gamma^\mu\psi(x) + \bar\psi(x)\frac{\partial}{\partial x^\mu}\gamma^\mu\psi(x) \;=\; \bar\psi(x)\big(\overleftarrow{D}_\mu - ieA_\mu(x)\big)\gamma^\mu\psi(x) + \bar\psi(x)\gamma^\mu\big(D_\mu + ieA_\mu(x)\big)\psi(x) \\ &=\; \bar\psi(x)\overleftarrow{\not{D}}\,\psi(x) + \bar\psi(x)\not{D}\psi(x) \;=\; \bar\psi(x)\big(\overleftarrow{\not{D}} - im\big)\psi(x) + \bar\psi(x)\big(\not{D} + im\big)\psi(x) \;=\; 0 \end{aligned} \tag{20.8}$$

where we used Dirac equations (20.3). The charge Q defined as [21]

$$Q(t) \stackrel{\rm def}{\equiv} -e\int d^3x\ j_0(t,\vec{x}) = -e\int d^3x\ \bar{\psi}(t,\vec{x})\gamma^0\psi(t,\vec{x}) \tag{20.9}$$

is conserved:

$$\frac{d}{dt}Q(t) = -e\int d^3x\ [\frac{\partial}{\partial t}j_0 + \vec{\nabla}\cdot\vec{j}(x,t) - \vec{\nabla}\cdot\vec{j}(x,t)] \stackrel{\rm by\ parts}{=} -e\int d^3x\ \partial_\mu j^\mu(t,\vec{x}) = 0 \tag{20.10}$$

In a free electromagnetic theory we used the Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$, $A_0 = 0$. In the theory with the interaction it is impossible to satisfy both of these conditions. Indeed, suppose $\vec{\nabla}\cdot\vec{A} = 0$ then from Maxwell equation (20.4) we get ($\Phi \equiv A_0$)

$$\partial_\mu F^{\mu 0} = \partial_i F^{i0} = \partial_i(\partial^i A_0 - \partial_0 A^i) = -\vec{\nabla}^2 A_0 - \partial_0(\vec{\nabla}\cdot\vec{A}) = -\vec{\nabla}^2 A_0 = -ej^0 \tag{20.11}$$

so one cannot put $A_0 = 0$ if there are sources of the electromagnetic field, so we will drop $A_0 = 0$ condition [22] and use Coulomb gauge

$$\vec{\nabla}\cdot\vec{A} = 0 \tag{20.12}$$

Actually, the field $A_0(x) \equiv \Phi(x)$ is completely determined by those sources since from classical electrodynamics we know that

$$\vec{\nabla}^2\Phi(t,\vec{x}) = -\rho(t,\vec{x}) \quad\Rightarrow\quad \Phi(t,\vec{x}) = \int d^3x'\,\frac{\rho(t,\vec{x}')}{4\pi|\vec{x}-\vec{x}'|} \tag{20.13}$$

where $\rho(t,\vec{x})$ is a density of continuous distribution of charge (this formula is easy to check using $\vec{\nabla}^2\frac{1}{|\vec{x}-\vec{x}'|} = -4\pi\delta(\vec{x}-\vec{x}')$). In our case $\rho(t,\vec{x}) = -ej^0(t,\vec{x}) = -e\psi^\dagger(t,\vec{x})\psi(t,\vec{x})$ so the field

$$A_0(t,\vec{x}) = -e\int d^3x'\,\frac{\psi^\dagger(t,\vec{x}')\psi(t,\vec{x}')}{4\pi|\vec{x}-\vec{x}'|} \tag{20.14}$$

is not an independent dynamical variable.

Thus, the dynamical variables (= canonical coordinates) in our theory are $A^i(x)$ and $\psi(x)$. The corresponding canonical momenta have the form

$$\pi(t,\vec{x}) \equiv \frac{\partial\mathcal{L}}{\partial\dot{\psi}}(t,\vec{x}) = i\psi^\dagger(t,\vec{x}) \tag{20.15}$$

$$\pi^k(t,\vec{x}) \equiv \frac{\partial\mathcal{L}}{\partial\dot{A}_k}(t,\vec{x}) = -\dot{A}^k(t,\vec{x}) + \partial^k A_0(t,\vec{x}) = E^k(t,\vec{x})$$

(self-consistency check is $\pi^0(t,\vec{x}) \equiv \frac{\partial\mathcal{L}}{\partial\dot{A}_0}(t,\vec{x}) = 0$ which agrees with the fact tat A^0 is not a dynamical variable).

The classical Hamiltonian has the form

$$H(t) = \int d^3x\big\{\pi(t,\vec{x})\dot{\psi}(t,\vec{x}) + \pi_k(t,\vec{x})\dot{A}^k(t,\vec{x}) - \mathcal{L}(t,\vec{x})\big\} \tag{20.16}$$

$$= \int d^3x\big\{i\psi^\dagger(t,\vec{x})\dot{\psi}(t,\vec{x}) + E_k(t,\vec{x})[-E^k(t,\vec{x}) + \partial^k\Phi(t,\vec{x})] - \mathcal{L}(t,\vec{x})\big\}$$

$$= \int d^3x\ \Big[\psi^\dagger(m - i\vec{\gamma}\cdot\vec{\nabla})\psi(t,\vec{x}) + \frac{1}{2}[\vec{E}^2(t,\vec{x}) + \vec{B}^2(t,\vec{x})] + \vec{E}\cdot\vec{\nabla}\Phi(t,\vec{x}) - e\bar{\psi}A\psi(t,\vec{x})\Big]$$

[21] recall that $e_{\rm electron} = -e$, see the sign in Eq. (15.36) for quantum charge operator

[22] Alternatively, one can drop $\vec{\nabla}\cdot\vec{A} = 0$ condition and use the "temporal gauge" $A_0 = 0$

where we used $\mathcal{L} = -\frac{1}{2}(\vec{E}^2 - \vec{B}^2) + \bar{\psi}(i\partial\!\!\!/ - m)\psi + e\bar{\psi}A\!\!\!/\psi$. Now, integrating by parts term $\sim \int d^3x\ \vec{E}(t,\vec{x})\cdot\vec{\nabla}\Phi(t,\vec{x}) = -\int d^3x\ \rho(t,\vec{x})\Phi(t,\vec{x})$, we can rewrite the Hamiltonian (20.16) in the form

$$H(t) = \int d^3x\ \Big(\psi^\dagger(m - i\vec{\gamma}\cdot\vec{\nabla} - e\vec{\gamma}\cdot\vec{A})\psi(t,\vec{x}) + \frac{1}{2}[\vec{E}^2(t,\vec{x}) + \vec{B}^2(t,\vec{x})]\Big) \qquad (20.17)$$

For future use let us divide $\vec{E}(x)$ in the transverse and longitudinal parts

$$\begin{aligned}\vec{E} &= -\vec{\nabla}\cdot\Phi(x) - \dot{\vec{A}}(x) = \vec{E}^{\rm ln} + \vec{E}^{\rm tr} \qquad (20.18)\\ \vec{E}^{\rm ln}(x) &\equiv -\vec{\nabla}\cdot\Phi(x)\\ \vec{E}^{\rm tr}(x) &\equiv -\dot{\vec{A}}(x) \qquad \text{“transverse'' because } \vec{\nabla}\cdot\vec{E}^{\rm tr} = \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = 0\end{aligned}$$

Note that $\vec{E}^{\rm ln}(x) = \vec{\nabla}\cdot\Phi(x)$ is not an independent dynamical variable since it is determined by Dirac fields just like $\Phi(x)$ itself.

We get

$$\frac{1}{2}[\vec{E}^2 + \vec{B}^2] = \frac{1}{2}(\vec{E} + \vec{\nabla}\Phi)^2 + \frac{1}{2}\vec{B}^2 + \frac{1}{2}(\vec{\nabla}\Phi)^2 = \frac{1}{2}(\dot{\vec{A}}^2 + \vec{B}^2) + \frac{1}{2}(\vec{\nabla}\Phi)^2 \qquad (20.19)$$

so the Hamiltonian takes the form

$$\begin{aligned}H(t) &= \int d^3x\ \mathcal{H}(t,\vec{x}) \qquad (20.20)\\ \mathcal{H}(t,\vec{x}) &= \psi^\dagger(t,\vec{x})(m - i\vec{\gamma}\cdot\vec{\nabla} - e\vec{\gamma}\cdot\vec{A})\psi(t,\vec{x}) + \frac{1}{2}[\dot{\vec{A}}^2(t,\vec{x}) + \vec{B}^2(t,\vec{x})] + \frac{1}{2}(\vec{\nabla}\Phi(t,\vec{x}))^2\end{aligned}$$

20.2 Quantization

As usually, we take classical canonical coordinates and momenta (20.15) at $t = 0$ and promote them to operators

$$\begin{aligned}A^i(0,\vec{x}) &\to \hat{A}(\vec{x}), \qquad \pi^i(0,\vec{x}) \to \hat{\pi}^i(\vec{x}) = \hat{E}^i(\vec{x})\\ \psi(0,\vec{x}) &\to \hat{\psi}(\vec{x}), \qquad \pi(0,\vec{x}) = i\psi^\dagger(0,\vec{x}) \to i\hat{\psi}^\dagger(\vec{x})\end{aligned}$$

In addition, we define the operator $\hat{A}^0 \equiv \hat{\Phi}$ by Eq. (20.21)

$$\hat{A}_0(\vec{x}) = \hat{\Phi}(\vec{x}) \stackrel{\rm def}{\equiv} -e\int d^3x'\,\frac{\hat{\psi}^\dagger(\vec{x}')\hat{\psi}(\vec{x}')}{4\pi|\vec{x} - \vec{x}'|} \qquad (20.21)$$

We impose the canonical (anti)commutation relations of the usual form ($\pi^i \equiv E^i$)

$$\begin{aligned}&\{\hat{\psi}_\xi(\vec{x}), \hat{\psi}^\dagger_\eta(\vec{y})\} = \delta_{\xi\eta}(\vec{x} - \vec{y}), \qquad \{\hat{\psi}_\xi(\vec{x}), \hat{\psi}_\eta(\vec{y})\} = \{\hat{\psi}^\dagger_\xi(\vec{x}), \hat{\psi}^\dagger_\eta(\vec{y})\} = 0\\ &[\hat{A}^i(\vec{x}), \hat{E}^j_{\rm tr}(\vec{y})] = \delta^{\rm tr}_{ij}\delta(\vec{x} - \vec{y}), \qquad [\hat{A}^i(\vec{x}), \hat{A}^j(\vec{y})] = [\hat{E}^i_{\rm tr}(\vec{x}), \hat{E}^j_{\rm tr}(\vec{y})] = 0,\\ &[\hat{A}^i(\vec{x}), \hat{\psi}_\xi(\vec{y})] = [\hat{A}^i(\vec{x}), \hat{\psi}^\dagger_\xi(\vec{y})] = [\hat{E}^{\rm tr}_i(\vec{x}), \hat{\psi}(\vec{x}')] = [\hat{E}^{\rm tr}_i(\vec{x}), \hat{\psi}(\vec{x}')] = 0 \qquad (20.22)\end{aligned}$$

Also, from equations (20.25) we get

$$[\hat{A}^0(\vec{x}), \hat{A}^i(\vec{y})] = 0 \qquad (20.23)$$

(because $\hat{A}^i$ commute with $\hat{\psi}$ and $\hat{\psi}^\dagger$) and

$$
\begin{aligned}
[\hat{A}^0(\vec{x}),\hat{A}^0(\vec{y})] &= \int d^3x'd^3y'\,\frac{[\hat{\psi}^\dagger(\vec{x}')\hat{\psi}(\vec{x}'),\hat{\psi}^\dagger(\vec{y}')\hat{\psi}(\vec{y}')]}{16\pi^2|\vec{x}-\vec{x}'||\vec{y}-\vec{y}'|}\\
&= \int d^3x'd^3y'\,\frac{\hat{\psi}^\dagger(\vec{x}')\{\hat{\psi}(\vec{x}'),\hat{\psi}^\dagger(\vec{y}')\}\hat{\psi}(\vec{y}')-\hat{\psi}^\dagger(\vec{y}')\{\hat{\psi}(\vec{y}'),\hat{\psi}^\dagger(\vec{x}')\}\hat{\psi}(\vec{x}')}{16\pi^2|\vec{x}-\vec{x}'||\vec{y}-\vec{y}'|}\\
&= \int d^3x'd^3y'\;\delta(\vec{x}'-\vec{y}')\frac{\hat{\psi}^\dagger(\vec{x}')\hat{\psi}(\vec{y}')-\hat{\psi}^\dagger(\vec{y}')\hat{\psi}(\vec{x}')}{16\pi^2|\vec{x}-\vec{x}'||\vec{y}-\vec{y}'|} \;=\; 0 \qquad (20.24)
\end{aligned}
$$

Because $\vec{\hat{E}}^{\rm l}(\vec{x}) = \vec{\nabla}\hat{\Phi}(\vec{x}) = \vec{\nabla}A^0(\vec{x})$ commutes with $\hat{A}^i(\vec{y})$, with $\vec{\hat{E}}^{\rm tr}(\vec{y})$ and with itself (see the two above equations) one can replace $\hat{E}^i_{\rm tr}$ in the second line in Eq. (20.22) by full $\hat{E}^i$:

$$
\begin{aligned}
&\{\hat{\psi}_\xi(\vec{x}),\hat{\psi}^\dagger_\eta(\vec{y})\} = \delta_{\xi\eta}(\vec{x}-\vec{y}), \qquad \{\hat{\psi}_\xi(\vec{x}),\hat{\psi}_\eta(\vec{y})\} = \{\hat{\psi}^\dagger_\xi(\vec{x}),\hat{\psi}^\dagger_\eta(\vec{y})\} = 0\\
&[\hat{A}^i(\vec{x}),\hat{E}^j(\vec{y})] = \delta^{\rm tr}_{ij}\delta(\vec{x}-\vec{y}), \qquad [\hat{A}^\mu(\vec{x}),\hat{A}^\nu(\vec{y})] = [\hat{E}^i(\vec{x}),\hat{E}^j(\vec{y})] = 0,\\
&[\hat{A}^i(\vec{x}),\hat{\psi}_\xi(\vec{y})] = [\hat{A}^i(\vec{x}),\hat{\psi}^\dagger_\xi(\vec{y})] = [\hat{E}^{\rm tr}_i(\vec{x}),\hat{\psi}(\vec{x}')] = [\hat{E}^{\rm tr}_i(\vec{x}),\hat{\psi}(\vec{x}')] = 0 \qquad (20.25)
\end{aligned}
$$

However, $\hat{A}^0 = \hat{\Phi}$ does not commute with Dirac operators:

$$
[\hat{\Phi}(\vec{x}),\hat{\psi}(\vec{x}')] = \frac{e\hat{\psi}(\vec{x}')}{4\pi|\vec{x}-\vec{x}'|}, \qquad [\hat{\Phi}(\vec{x}),\hat{\psi}^\dagger(\vec{x}')] = -\frac{e\hat{\psi}^\dagger(\vec{x}')}{4\pi|\vec{x}-\vec{x}'|} \qquad (20.26)
$$

and therefore the in last line in Eq. (20.25) one cannot replace $\hat{E}^{\rm tr}_i$ by $\hat{E}_i$

Since $\partial_i\hat{A}^i(\vec{x}) = -\vec{\nabla}\cdot\vec{A}(\vec{x})$ commutes with all canonical coordinates $\hat{A}^i$ (evident) and all canonical momenta $\hat{\pi}^i$ (because $\partial_i\delta^{\rm tr}_{ij}(\vec{x}-\vec{x}') = 0$ it is a c-number so we can safely set the Coulomb gauge condition in the operator form

$$
\vec{\nabla}\cdot\vec{\hat{A}}(\vec{x}) = 0 \qquad (20.27)
$$

- it will not contradict any commutation relations.

The Hamiltonian is obtained by promotion of classical fields in Eq. (20.20) to operators:

$$
\hat{H} = \int d^3x\Big[\psi^\dagger(\vec{x})[m-i\vec{\gamma}\cdot\vec{\nabla}-e\vec{\gamma}\cdot\vec{A}(\vec{x})]\psi(\vec{x})+\frac{1}{2}[\vec{\hat{E}}^2_{\rm tr}(\vec{x})+\vec{B}^2(\vec{x})]+\frac{1}{2}(\vec{\nabla}\Phi(\vec{x}))^2\Big] \qquad (20.28)
$$

The Heisenberg operators are defined as usual

$$
\begin{aligned}
&\hat{A}^\mu(t,\vec{x}) \equiv e^{i\hat{H}t}\hat{A}^\mu(\vec{x})e^{-i\hat{H}t}, \qquad \hat{\pi}^i(t,\vec{x}) = e^{i\hat{H}t}\hat{\pi}^i(\vec{x})e^{-i\hat{H}t} = e^{i\hat{H}t}\hat{E}^i(\vec{x})e^{-i\hat{H}t} \equiv \hat{E}^i(t,\vec{x})\\
&\hat{\psi}(t,\vec{x}) \equiv e^{i\hat{H}t}\hat{\psi}(\vec{x})e^{-i\hat{H}t}, \qquad \hat{\bar{\psi}}(t,\vec{x}) \equiv e^{i\hat{H}t}\hat{\bar{\psi}}(\vec{x})e^{-i\hat{H}t} \qquad (20.29)
\end{aligned}
$$

It is easy to see that they satisfy the equal-time commutation relations (cf. Eq. (7.14))

$$
\begin{aligned}
&\{\hat{\psi}_\xi(t,\vec{x}),\hat{\psi}^\dagger_\eta(t,\vec{y})\} = \delta_{\xi\eta}(\vec{x}-\vec{y}), \qquad \{\hat{\psi}_\xi(t,\vec{x}),\hat{\psi}_\eta(t,\vec{y})\} = \{\hat{\psi}^\dagger_\xi(t,\vec{x}),\hat{\psi}^\dagger_\eta(t,\vec{y})\} = 0\\
&[\hat{A}^i(t,\vec{x}),\hat{E}^j(t,\vec{y})] = -i\delta^{\rm tr}_{ij}\delta(\vec{x}-\vec{y}), \qquad [\hat{A}^\mu(t,\vec{x}),\hat{A}^\nu(t,\vec{y})] = [\hat{E}^i(t,\vec{x}),\hat{E}^j(t,\vec{y})] = 0,\\
&[\hat{A}^i(t,\vec{x}),\hat{\psi}_\xi(t,\vec{y})] = [\hat{A}^i(t,\vec{x}),\hat{\psi}^\dagger_\xi(t,\vec{y})] = [\dot{\hat{A}}^i(t,\vec{x}),\hat{\psi}_\xi(t,\vec{y})] = [\dot{\hat{A}}^i(t,\vec{x}),\hat{\psi}^\dagger_\xi(t,\vec{y})] = 0\\
&[\hat{A}^0(t,\vec{x}),\hat{A}^i(t,\vec{y})] = [\hat{A}^0(t,\vec{x}),\hat{E}^i(t,\vec{y})] = [\hat{A}^0(t,\vec{x}),\hat{A}^0(t,\vec{y})] = 0 \qquad (20.30)
\end{aligned}
$$

The two last formulas in the third line require explanation. From Eq. (20.25) and (20.29) we get

$$e^{i\hat{H}t}\hat{E}_i^{\rm tr}(\vec{x})e^{-i\hat{H}t} ~=~ e^{i\hat{H}t}(\hat{E}_i(\vec{x})-\partial_i\hat{\Phi}(\vec{x}))e^{-i\hat{H}t} ~=~ \hat{E}_i(t,\vec{x})-\partial_i\hat{\Phi}(t,\vec{x}) ~=~ -\dot{\hat{A}}_i(t,\vec{x}) \tag{20.31}$$

because

$$\begin{aligned}
-\partial^0\hat{A}^i(t,\vec{x})+\partial^i\hat{\Phi}(t,\vec{x}) ~&=~ e^{i\hat{H}t}\big(-i[\hat{H},\vec{A}_i(\vec{x})]+\vec{\partial}_i\hat{\Phi}(\vec{x})\big)e^{-i\hat{H}t} \qquad (20.32)\\
=~ e^{i\hat{H}t}\big\{&-i\int d^3x'[\hat{\vec{E}}_j(\vec{x}'),\vec{A}_i(\vec{x})]\big(\hat{\vec{E}}_j(\vec{x}')+\vec{\partial}_j\hat{\Phi}(\vec{x})\big)+\vec{\partial}_i\hat{\Phi}(\vec{x})\big\}e^{-i\hat{H}t}\\
=~ e^{i\hat{H}t}\big\{&\int d^3x'\delta^{\rm tr}_{ij}(\vec{x}-\vec{x}')\big(\hat{\vec{E}}_j(\vec{x}')+\vec{\partial}_j\hat{\Phi}(\vec{x})\big)+\vec{\partial}_i\hat{\Phi}(\vec{x})\big\}e^{-i\hat{H}t}\\
=~ e^{i\hat{H}t}\big\{&\hat{\vec{E}}_i^{\rm tr}(\vec{x})+\vec{\partial}_i\hat{\Phi}(\vec{x})\big\}e^{-i\hat{H}t} ~=~ e^{i\hat{H}t}\hat{\vec{E}}_i(\vec{x})e^{-i\hat{H}t} ~=~ \hat{\vec{E}}_i(t,\vec{x}) ~=~ \hat{E}^i(t,\vec{x})
\end{aligned}$$

Also, repeating the derivation of Eq. (7.15) we see that

$$\hat{H}(t) ~=\int d^3x~\Big[\{\hat{\psi}^\dagger(t,\vec{x})[m-i\vec{\gamma}\cdot\vec{\nabla}-e\vec{\gamma}\cdot\vec{A}(\vec{x})]\hat{\psi}(t,\vec{x})+~\frac{1}{2}[\dot{\hat{\vec{A}}}{}^2(t,\vec{x})+\hat{\vec{B}}^2(t,\vec{x})]+\frac{1}{2}\big(\vec{\nabla}\hat{\Phi}(t,\vec{x})\big)^2\Big] ~=~ \hat{H} \tag{20.33}$$

actually does not depend on t (but different parts like $\hat{H}_{\rm D}(t) ~=~ -i\int d^3x\hat{\psi}^\dagger(t,\vec{x})\vec{\gamma}\cdot\vec{\nabla}\hat{\psi}(t,\vec{x})$ may depend on t!)

The operators (20.29) satisfy the same equations (20.3) and (20.4) as their classical counterparts. Let us prove this for Gauss law $\vec{\nabla}\cdot\vec{E}(t,\vec{x}) ~=~ \rho(t,\vec{x}) ~\equiv~ -e\psi^\dagger\psi(t,\vec{x})$. First, from Eq. (20.32) we see that $\hat{E}^i(t,\vec{x}) ~=~ -\partial^0\hat{A}^i(t,\vec{x})+\partial^i\hat{\Phi}(t,\vec{x})$, same as for classical fields. Now it is easy to obtain Gauss law in the operator form reads

$$\begin{aligned}
\vec{\nabla}\cdot\hat{\vec{E}}(t,\vec{x}) ~&=~ -\partial_i\hat{E}^i(t,\vec{x}) ~=~ -\partial_i\partial^0\hat{A}^i(t,\vec{x})+\partial_i\partial^i\hat{\Phi}(t,\vec{x}) ~=~ -\partial^0(e^{i\hat{H}t}\partial_i\hat{A}^i(\vec{x})e^{-i\hat{H}t})+\partial_i\partial^i\hat{\Phi}(t,\vec{x})\\
&=~ -\vec{\nabla}^2\hat{\Phi}(t,\vec{x}) ~=~ -e\hat{\bar{\psi}}^\dagger(t,\vec{x})\hat{\psi}(t,\vec{x}) ~=~ -e\hat{j}^0(t,\vec{x}) ~=~ \hat{\rho}(t,\vec{x}) \qquad (20.34)
\end{aligned}$$

Similarly one can check that the operators (20.29) satisfy the same equations (20.3) and (20.4) as their classical counterparts

$$\begin{aligned}
\partial_\mu\hat{F}^{\mu\nu}(x) ~&=~ -e\hat{j}^\nu(x)\\
(i\hat{\not{D}}-m)\hat{\psi}(x) ~=~ 0,\qquad \hat{\bar{\psi}}(x)\big(i\overleftarrow{\hat{\not{D}}}+m\big) ~&=~ 0 \qquad (20.35)
\end{aligned}$$

where $\hat{j}^\mu(x) ~=~ \hat{\bar{\psi}}(x)\gamma^\mu\hat{\psi}(x)$ and $\hat{D}_\mu ~\equiv~ \partial_\mu-ie\hat{A}_\mu$

Part XIX

20.3 Interaction picture

To quantize electrodynamics in the interaction picture we separate the Hamiltonian (20.33) in four parts:

$$
\begin{aligned}
\hat{H} &= \hat{H}_{\rm D} + \hat{H}_{\rm em} + \hat{H}_{\rm int} + \hat{H}_{\rm Coul}, \qquad (20.36)\\
\hat{H}_{\rm D} &= \int d^3x\ \hat{\psi}^\dagger(\vec{x})(m - i\vec{\gamma}\cdot\vec{\nabla})\hat{\psi}(\vec{x})\\
\hat{H}_{\rm em} &= \frac{1}{2}\int d^3x\ [\dot{\hat{\vec{A}}}{}^2(\vec{x}) + \hat{\vec{B}}^2(\vec{x})]\\
\hat{H}_{\rm int} &= -e\int d^3x\ \hat{\bar{\psi}}(\vec{x})\vec{\gamma}\cdot\hat{\vec{A}}(\vec{x})\hat{\psi}(\vec{x})\\
\hat{H}_{\rm Coul} &= \frac{1}{2}\int d^3x\ (\vec{\nabla}\hat{\Phi}(\vec{x}))^2 = -\frac{1}{2}\int d^3x\ \hat{\Phi}(\vec{x})\vec{\nabla}^2\hat{\Phi}(\vec{x}) = \int d^3x d^3y\ \hat{\psi}^\dagger(\vec{x})\hat{\psi}(\vec{x})\frac{e^2}{8\pi|\vec{x}-\vec{y}|}\hat{\psi}^\dagger(\vec{y})\hat{\psi}(\vec{y})
\end{aligned}
$$

where in the last line we used equation $\vec{\nabla}^2\hat{\Phi}(\vec{x}) = e\psi^\dagger(\vec{x})\psi(\vec{x})$ following from the definition (20.21).

Now let us define "perturbative Hamiltonian" as

$$\hat{H}_0 \stackrel{\rm def}{=} \hat{H}_{\rm D} + \hat{H}_{\rm em} \qquad (20.37)$$

then $\hat{H} = \hat{H}_0 + \hat{H}_{\rm int} + \hat{H}_{\rm Coul}$.

Operators in the interaction representation are defined as usual (note that Eq. (20.30) $\Rightarrow$ $[\hat{H}_{\rm em}, \hat{H}_{\rm D}] = 0$)

$$
\begin{aligned}
\hat{A}_I(z) &= e^{i\hat{H}_0 z_0}\hat{A}(\vec{z})e^{-i\hat{H}_0 z_0} = e^{i\hat{H}_{\rm em} z_0}\hat{A}^i(\vec{z})e^{-i\hat{H}_{\rm em} z_0}\\
\hat{\psi}_I(z) &= e^{i\hat{H}_0 z_0}\hat{\psi}(\vec{z})e^{-i\hat{H}_0 z_0} = e^{i\hat{H}_{\rm D} z_0}\hat{\psi}(\vec{z})e^{-i\hat{H}_{\rm D} z_0}\\
\hat{\bar{\psi}}_I(z) &= e^{i\hat{H}_0 z_0}\hat{\bar{\psi}}(\vec{z})e^{-i\hat{H}_0 z_0} = e^{i\hat{H}_{\rm D} z_0}\hat{\bar{\psi}}(\vec{z})e^{-i\hat{H}_{\rm D} z_0} \qquad (20.38)
\end{aligned}
$$

$\Rightarrow$ the expansion in ladder operators is a combination of Eqs. (19.57) and (14.21)

$$
\begin{aligned}
\hat{\psi}_I(x) &= e^{i\hat{H}_{\rm D}t}\hat{\psi}(\vec{x})e^{-i\hat{H}_{\rm D}t} = \sum_s\int\frac{d\!\!{}^-{}^3p}{\sqrt{2E_p}}[u(\vec{p},s)e^{-ipx}\hat{a}^s_{\vec{p}} + v(\vec{p},s)e^{ipx}\hat{b}^{s\dagger}_{\vec{p}}]\Big|_{p_0=E_p},\\
\hat{\bar{\psi}}_I(\vec{x}) &= e^{i\hat{H}_{\rm D}t}\hat{\bar{\psi}}(\vec{x})e^{-i\hat{H}_{\rm D}t} = \sum_s\int\frac{d\!\!{}^-{}^3p}{\sqrt{2E_p}}[\bar{v}(\vec{p},s)e^{-ipx}\hat{b}^s_{\vec{p}} + \bar{u}(\vec{p},s)e^{ipx}\hat{a}^{s\dagger}_{\vec{p}}]\Big|_{p_0=E_p},\\
\hat{A}^i_I(x) &= \sum_{\lambda=1,2}\int\frac{d\!\!{}^-{}^3k}{\sqrt{2\omega_k}}\ e^{\lambda i}(\vec{k})\big(\hat{a}^\lambda_{\vec{k}}e^{-ikx} + \hat{a}^{\lambda\dagger}_{\vec{k}}e^{ikx}\big)\Big|_{k_0=\omega_k=|\vec{k}|} \qquad (20.39)
\end{aligned}
$$

20.4 Perturbative series for Green functions

The (exact) Green functions are defined in a usual way (cf Eq. (7.20):

$$G(x_1,...x_l,y_1,...y_m,z_1,...z_n) \equiv \langle\Omega|\mathrm{T}\{\hat{\psi}(x_1),...\hat{\psi}(x_l),\hat{\bar{\psi}}(y_1),....\hat{\bar{\psi}}(y_m),\hat{A}^{\mu_1}(z_1),...\hat{A}^{\mu_n}(z_n)|\Omega\rangle \qquad (20.40)$$

where $|\Omega\rangle$ is a "true vacuum" of QED (lowest-energy eigenstate of the Hamiltonian (20.36) and $\hat{\psi}, \hat{A}$ are Heisenberg operators (20.29).

As usual, we define the "perturbative vacuum" as a direct product of vacuum of free Dirac theory and vacuum of free electromagnetic theory:

$$|0\rangle \;=\; |0_{\rm D}\rangle|0_{\rm em}\rangle \tag{20.41}$$

This vacuum is annihilated by both fermion and photon annihilation operators

$$\hat{a}^{\lambda}_{\vec{p}}|0\rangle \;=\; \hat{b}^{\lambda}_{\vec{p}}|0\rangle \;=\; \hat{a}^{\lambda}_{\vec{k}}|0\rangle \;=\; 0 \tag{20.42}$$

Next, we use the property (9.26)

$$e^{-i\hat{H}T}|0\rangle \overset{T\to\infty(1-i\epsilon)}{=} e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle, \tag{20.43}$$

repeat the steps from Eq. (9.31) to Eq. (9.41) and get the Green function (20.40) in the interaction representation:

$$\begin{aligned}&\langle\Omega|\mathrm{T}\{\hat{\psi}(x_1),...\hat{\psi}(x_l),\hat{\bar{\psi}}(y_1),....\hat{\bar{\psi}}(y_m),\hat{A}^{\mu_1}(z_1),...\hat{A}^{\mu_n}(z_n)|\Omega\rangle \\ &= \frac{\langle 0|\mathrm{T}\{\hat{\psi}_I(x_1),...\hat{\psi}_I(x_l),\hat{\bar{\psi}}_I(y_1),....\hat{\bar{\psi}}_I(y_m),\hat{A}^{\mu_1}_I(z_1),...\hat{A}^{\mu_n}_I(z_n)e^{-i\int dt\ (\hat{H}_{\rm int}(t)+\hat{H}_{\rm C}(t))}\}|0\rangle}{\langle 0|\mathrm{T}\{e^{-i\int dt\ (\hat{H}_{\rm int}(t)+\hat{H}_{\rm C}(t))}\}|0\rangle}\end{aligned} \tag{20.44}$$

where

$$\hat{H}_{\rm int}(t) \;=\; e^{i\hat{H}_0t}\hat{H}_{\rm int}e^{-i\hat{H}_0t} \;=\; -\,e\int d^3x\ \hat{\bar{\psi}}_I(t,\vec{x})\vec{\gamma}\cdot\hat{\vec{A}}_I(t,\vec{x})\hat{\psi}_I(t,\vec{x}) \tag{20.45}$$

$$\hat{H}_C(t) \;=\; e^{i\hat{H}_0t}\hat{H}_{\rm Coul}e^{-i\hat{H}_0t} \;=\; \int d^3xd^3y\ \hat{\psi}^\dagger_I(t,\vec{x})\hat{\psi}_I(t,\vec{x})\frac{e^2}{8\pi|\vec{x}-\vec{y}|}\hat{\psi}^\dagger_I(t,\vec{y})\hat{\psi}_I(t,\vec{y}) \tag{20.46}$$

Now we can expand the r.h.s. of Eq. (20.44) in powers of e ($\Leftrightarrow$ in powers of $\hat{H}_I$ and $\hat{H}_C$) and use Wick's theorem to get all possible Feynman diagrams. The contractions are:
1. Feynman propagator of Dirac particle

$$\overbrace{\hat{\psi}_I(x)\hat{\bar{\psi}}_I}(y) \;=\; \langle 0|\mathrm{T}\{\hat{\psi}_I(x)\hat{\bar{\psi}}_I(y)\}|0\rangle \;=\; \int\frac{d\!\!{}^-{}^4p}{i}e^{-ip(x-y)}\frac{m+\not{p}}{m^2-p^2-i\epsilon} \;=\; S_F(x-y) \tag{20.47}$$

and
2. Propagator of transverse photon

$$\begin{aligned}&\overbrace{\hat{A}^i_I(x)\hat{A}^j_I}(y) \;=\; \langle 0|\mathrm{T}\{\hat{A}^i_I(x)\hat{A}^j_I(y)\}|0\rangle \;=\; \langle 0|\mathrm{T}\{\hat{\vec{A}}_{Ii}(x)\hat{\vec{A}}_{Ij}(y)\}|0\rangle \\ &= \int\frac{d\!\!{}^-{}^3k}{2\omega_k}\sum_{\lambda=1,2}\vec{e}^{\,\lambda}_i(\vec{k})\vec{e}^{\,\lambda}_j(\vec{k})\big(\theta(x_0-y_0)e^{-ik(x-y)}+\theta(y_0-x_0)e^{ik(x-y)}\big) \\ &= \int\frac{d\!\!{}^-{}^4k}{i}\frac{1}{-k^2-i\epsilon}e^{-ik(x-y)}\Big(\delta_{ij}-\frac{\vec{k}_i\vec{k}_j}{\vec{k}^2}\Big) \;=\; \int\frac{d\!\!{}^-{}^4k}{i}\frac{1}{k^2+i\epsilon}e^{-ik(x-y)}\Big(g^{ij}+\frac{k^ik^j}{\vec{k}^2}\Big) \;=\; D^{ij}_{\rm tr}(x-y)\end{aligned} \tag{20.48}$$

where we used Eq. (19.32).

20.5 Feynman photon propagator and Lorentz invariance of Feynman diagrams

Let us introduce the unit 4-vector in time direction $\eta \equiv (1,0,0,0)$ and rewrite the transverse photon propagator (20.48) as follows

$$D^{\mu\nu}_{\rm tr}(x-y) \;=\; \int \frac{d\!\!{}^-{}^4k}{i}\frac{1}{k^2+i\epsilon}e^{-ik(x-y)}\Big(g^{\mu\nu}+\frac{k^\mu k^\nu}{\vec{k}^2}-\frac{k^0}{\vec{k}^2}(k^\mu\eta^\nu+k^\nu\eta^\mu)+k^2\frac{\eta^\mu\eta^\nu}{\vec{k}^2}\Big) \tag{20.49}$$

which can be rewritten as a Feynman propagator plus two additional terms

$$\begin{aligned}
D^{\mu\nu}_{\rm tr}(x-y) &= D^{\mu\nu}_F(x-y)+D^{\mu\nu}_{\rm Ward}(x-y)+D^{\mu\nu}_{\rm inst}(x-y) \qquad (20.50)\\
D^{\mu\nu}_F(x-y) &= \int \frac{d\!\!{}^-{}^4k}{i}\frac{g^{\mu\nu}}{k^2+i\epsilon}e^{-ik(x-y)}\\
D^{\mu\nu}_{\rm W}(x-y) &= \int \frac{d\!\!{}^-{}^4k}{i}\frac{g^{\mu\nu}}{k^2+i\epsilon}e^{-ik(x-y)}\Big(\frac{k^\mu k^\nu}{\vec{k}^2}-\frac{k^0}{\vec{k}^2}(k^\mu\eta^\nu+k^\nu\eta^\mu)\Big)\\
D^{\mu\nu}_{\rm inst}(x-y) &= \int \frac{d\!\!{}^-{}^4k}{i}\frac{\eta^\mu\eta^\nu}{\vec{k}^2}e^{-ik(x-y)} \;=\; -\,i\delta(x_0-y_0)\int d\!\!{}^-{}^3k\frac{\eta^\mu\eta^\nu}{\vec{k}^2}e^{i\vec{k}(\vec{x}-\vec{y})} \;=\; -\,i\delta(x_0-y_0)\frac{\eta^\mu\eta^\nu}{4\pi|\vec{x}-\vec{y}|}
\end{aligned}$$

The contribution of the term $D^{\mu\nu}_{\rm W}(x-y)$ vanishes for any S-matrix element due to Ward identity (see Peskin's textbook). In addition, the term $D^{\mu\nu}_{\rm inst}(x-y)$ cancels the contribution of the instantaneous term H_c in the Hamiltonian so one can use Feynman propagator

$$D^{\mu\nu}_F(x-y) \;=\; \int \frac{d\!\!{}^-{}^4k}{i}\frac{g^{\mu\nu}}{k^2+i\epsilon}e^{-ik(x-y)} \tag{20.51}$$

and the interaction Hamiltonian

$$\hat{H}_I(t) \;=\; e\int d^3x\;\hat{\bar{\psi}}_I(t,\vec{x})\,\hat{A}_I(t,\vec{x})\hat{\psi}_I(t,\vec{x}) \tag{20.52}$$

for the calculation of S-matrix elements (= physical cross sections).

20.5.1 Electron-positron scattering in the lowest order in perturbation theory

Let us illustrate this using simple example of elastic electron-positron scattering in the lowest order in perturbation theory. Due to LSZ theorem (we can use Eq. (17.8) without scalar bosons) we get [23]

$$\begin{aligned}
{}_{\rm out}\langle p_2,s_2;q_2,r_2|p_1,s_1;q_1,r_1\rangle_{\rm in} \;&=\; \lim_{p_i^2,q_i^2\to m^2}\int d^4x_1d^4y_1d^4x_2d^4y_2\;e^{-ip_1x_1-iq_1y_1+ip_2x_2+iq_2y_2}\\
&\times\langle\Omega|{\rm T}\{\bar{u}_\xi(p_2,s_2)(m-\not{p}_2)_{\xi\eta}\hat{\psi}_\eta(x_2)\hat{\bar{\psi}}_\omega(y_2)(m+\not{q}_2)_{\omega\chi}v_\chi(q_2,r_2)\\
&\quad\times\hat{\bar{\psi}}_\lambda(x_1)(m-\not{p}_1)_{\lambda\rho}u_\rho(p_1,s_1)\bar{v}_\zeta(q_1,r_1)(m+\not{q}_1)_{\zeta\sigma}\hat{\psi}_\sigma(y_1)\}|\Omega\rangle \qquad (20.53)\\
=\;\lim_{p_i^2,q_i^2\to m^2}&\int d^4x_1d^4y_1d^4x_2d^4y_2\;e^{-ip_1x_1-iq_1y_1+ip_2x_2+iq_2y_2}\bar{u}_\xi(p_2,s_2)(m-\not{p}_2)_{\xi\eta}(m+\not{q}_2)_{\omega\chi}v_\chi(q_2,r_2)\\
\times\,(m-\not{p}_1)_{\lambda\rho}&u_\rho(p_1,r_1)\bar{v}_\zeta(q_1,r_1)(m+\not{q}_1)_{\zeta\sigma}\frac{\langle 0|{\rm T}\{\hat{\psi}_\eta(x_2)\hat{\bar{\psi}}_\omega(y_2)\hat{\bar{\psi}}_\lambda(x_1)\hat{\psi}_\sigma(y_1)e^{-i\int dt\;(\hat{H}_{\rm int}(t)+\hat{H}_{\rm C}(t))}\}|0\rangle}{\langle 0|{\rm T}\{e^{-i\int dt\;(\hat{H}_{\rm int}(t)+\hat{H}_{\rm C}(t))}\}|0\rangle}
\end{aligned}$$

[23]To save space, we omit the interaction-representation index "I" from the operators in what follows

First, we consider the ratio in the last line in the second order in e and prove that the contribution of $\hat{H}_{\rm C}(t))$ cancels with the contribution of the last term in transverse photon propagator

$$\frac{\langle 0|{\rm T}\{\hat{\psi}_\eta(x_2)\hat{\bar{\psi}}_\omega(y_2)\hat{\bar{\psi}}_\lambda(x_1)\hat{\psi}_\sigma(y_1)e^{-i\int dt\ (\hat{H}_{\rm int}(t)+\hat{H}_{\rm C}(t))}\}|0\rangle}{\langle 0|{\rm T}\{e^{-i\int dt\ (\hat{H}_{\rm int}(t)+\hat{H}_{\rm C}(t))}\}|0\rangle} \tag{20.54}$$

$$= \langle 0|{\rm T}\{\hat{\psi}_\eta(x_2)\hat{\bar{\psi}}_\omega(y_2)\hat{\bar{\psi}}_\lambda(x_1)\hat{\psi}_\sigma(y_1)\Big(-\frac{1}{2}\int dt\ \hat{H}_{\rm int}(t)\int dt'\ \hat{H}_{\rm int}(t') - i\int dt\hat{H}_{\rm C}(t))\Big)\}|0\rangle$$

$$= e^2\langle 0|{\rm T}\{\hat{\psi}_\eta(x_2)\hat{\bar{\psi}}_\omega(y_2)\hat{\bar{\psi}}_\lambda(x_1)\hat{\psi}_\sigma(y_1)\Big(-\int\frac{d^4zd^4z'}{2}\hat{\bar{\psi}}(z)\vec{\gamma}\cdot\hat{\vec{A}}(z)\hat{\psi}(z)\hat{\bar{\psi}}(z')\vec{\gamma}\cdot\hat{\vec{A}}(z')\hat{\psi}(z') - i\int dt\hat{H}_{\rm C}(t))\Big)\}|0\rangle$$

By Wick's theorem we can replace $...\hat{A}_i(z)...\hat{A}_j(z')...$ by contraction $\widehat{\hat{A}_i(z)...\hat{A}_j}(z') = D^{\rm tr}_{ij}(z-z')$ and get

$$= -\frac{e^2}{2}\int d^4zd^4z'D^{\mu\nu}_{\rm tr}(z-z')\langle 0|{\rm T}\{\hat{\psi}_\eta(x_2)\hat{\bar{\psi}}_\omega(y_2)\hat{\bar{\psi}}_\lambda(x_1)\hat{\psi}_\sigma(y_1)\big(\hat{\bar{\psi}}(z)\gamma_\mu\hat{\psi}(z)\big)\big(\hat{\bar{\psi}}(z')\gamma_\nu\hat{\psi}(z')\big)\}|0\rangle$$

$$-ie^2\int dtd^3zd^3z'\ \frac{\eta^\mu\eta^\nu}{8\pi|\vec{x}-\vec{y}|}\langle 0|{\rm T}\{\hat{\psi}_\eta(x_2)\hat{\bar{\psi}}_\omega(y_2)\hat{\bar{\psi}}_\lambda(x_1)\hat{\psi}_\sigma(y_1)\big(\hat{\bar{\psi}}(t,\vec{z})\gamma_\mu\hat{\psi}(t,\vec{z})\big)\big(\hat{\bar{\psi}}\gamma_\nu(t,\vec{z})\hat{\psi}(t,\vec{z}')\big)\}|0\rangle$$

$$= -\frac{e^2}{2}\int d^4zd^4z'\Big[D^{\mu\nu}_{\rm tr}(z-z') + i\delta(z_0-z'_0)\frac{\eta^\mu\eta^\nu}{4\pi|\vec{x}-\vec{y}|}\Big]$$
$$\times\ \langle 0|{\rm T}\{\hat{\psi}_\eta(x_2)\hat{\bar{\psi}}_\omega(y_2)\hat{\bar{\psi}}_\lambda(x_1)\hat{\psi}_\sigma(y_1)\big(\hat{\bar{\psi}}(z)\gamma_\mu\hat{\psi}(z)\big)\big(\hat{\bar{\psi}}(z')\gamma_\nu\hat{\psi}(z')\big)\}|0\rangle$$

$$= -\frac{e^2}{2}\int d^4zd^4z'\Big[D^{\mu\nu}_F(z-z') + D^{\mu\nu}_{\rm W}(z-z')\Big]$$
$$\times\ \langle 0|{\rm T}\{\hat{\psi}_\eta(x_2)\hat{\bar{\psi}}_\omega(y_2)\hat{\bar{\psi}}_\lambda(x_1)\hat{\psi}_\sigma(y_1)\big(\hat{\bar{\psi}}(z)\gamma^\mu\hat{\psi}(z)\big)\big(\hat{\bar{\psi}}(z')\gamma^\nu\hat{\psi}(z')\big)\}|0\rangle \tag{20.55}$$

It is an exercise in combinatorics (see *Bjorken & Drell*) to prove that in any order in perturbation theory the instantaneous term $D^{\mu\nu}_{\rm inst}$ in the transverse photon propagator (20.50) cancels with the Coulomb instantaneous term (20.46) in the interaction Hamiltonian so we can build perturbation theory with the interaction Hamiltonian (20.52) and the photon propagator $D^{\mu\nu}_F(z-z') + D^{\mu\nu}_{\rm W}(z-z')$.

A the second step we will demonstrate that one can drop the term $D^{\mu\nu}_{\rm W}(z-z')$ when calculating physical amplitude (20.53). Substituting our result

$$\frac{\langle 0|{\rm T}\{\hat{\psi}_\eta(x_2)\hat{\bar{\psi}}_\omega(y_2)\hat{\bar{\psi}}_\lambda(x_1)\hat{\psi}_\sigma(y_1)e^{-i\int dt\ (\hat{H}_{\rm int}(t)+\hat{H}_{\rm C}(t))}\}|0\rangle}{\langle 0|{\rm T}\{e^{-i\int dt\ (\hat{H}_{\rm int}(t)+\hat{H}_{\rm C}(t))}\}|0\rangle} \tag{20.56}$$

$$= -e^2\int d^4zd^4z'\Big[D^{\mu\nu}_F(z-z') + D^{\mu\nu}_{\rm W}(z-z')\Big]$$
$$\times\ \frac{1}{2}\langle 0|{\rm T}\{\hat{\psi}_\eta(x_2)\hat{\bar{\psi}}_\omega(y_2)\hat{\bar{\psi}}_\lambda(x_1)\hat{\psi}_\sigma(y_1)\big(\hat{\bar{\psi}}(z)\gamma_\mu\hat{\psi}(z)\big)\big(\hat{\bar{\psi}}(z')\gamma_\nu\hat{\psi}(z')\big)\}|0\rangle$$

in the r.h.s. of Eq. (20.53) we obtain

$$
\begin{aligned}
&{}_{\rm out}\langle p_2,s_2;q_2,r_2|p_1,s_1;q_1,r_1\rangle_{\rm in} \qquad\qquad (20.57)\\
&= \; -e^2 \lim_{p_i^2,q_i^2\to m^2}\int d^4x_1 d^4y_1 d^4x_2 d^4y_2\; e^{-ip_1x_1-iq_1y_1+ip_2x_2+iq_2y_2}\bar u_\xi(p_2,s_2)(m-\not p_2)_{\xi\eta}(m+\not q_2)_{\omega\chi}v_\chi(q_2,r_2)\\
&\times\,(m-\not p_1)_{\lambda\rho}u_\rho(p_1,r_1)\bar v_\zeta(q_1,r_1)(m+\not q_1)_{\zeta\sigma}\int d^4z d^4z'\Big[D_F^{\mu\nu}(z-z')+D_{\rm W}^{\mu\nu}(z-z')\Big]\\
&\qquad\qquad\times\,\frac{1}{2}\langle 0|{\rm T}\{\hat\psi_\eta(x_2)\hat{\bar\psi}_\omega(y_2)\hat{\bar\psi}_\lambda(x_1)\hat\psi_\sigma(y_1)\big(\hat{\bar\psi}(z)\gamma_\mu\hat\psi(z)\big)\big(\hat{\bar\psi}(z')\gamma_\nu\hat\psi(z')\big)\}|0\rangle
\end{aligned}
$$

By Wick's theorem, the T-product in the r.h.s. of this formula reduces to the sum of two contractions

$$
\begin{aligned}
&\frac{1}{2}\langle 0|{\rm T}\{\hat\psi_\eta(x_2)\hat{\bar\psi}_\omega(y_2)\hat{\bar\psi}_\lambda(x_1)\hat\psi_\sigma(y_1)\big(\hat{\bar\psi}(z)\gamma^\mu\hat\psi(z)\big)\big(\hat{\bar\psi}(z')\gamma^\nu\hat\psi(z')\big)\}|0\rangle\\
&= \; \hat\psi_\eta(x_2)\big(\hat{\bar\psi}(z')\gamma^\nu\hat\psi(z')\big)\hat{\bar\psi}_\lambda(x_1)\hat\psi_\sigma(y_1)\big(\hat{\bar\psi}(z)\gamma^\mu\hat\psi(z)\big)\hat{\bar\psi}_\omega(y_2)\\
&- \; \hat\psi_\eta(x_2)\big(\hat{\bar\psi}(z')\gamma^\nu\hat\psi(z')\big)\hat{\bar\psi}_\omega(y_2)\hat\psi_\sigma(y_1)\big(\hat{\bar\psi}(z)\gamma^\mu\hat\psi(z)\big)\hat{\bar\psi}_\lambda(x_1)\\
&= \; \big(S_F(x_2-z')\gamma_\nu S_F(z'-x_1)\big)_{\eta\lambda}\big(S_F(y_1-z)\gamma_\mu S_F(z-y_2)\big)_{\sigma\omega}\\
&- \; \big(S_F(x_2-z')\gamma_\nu S_F(z'-y_2)\big)_{\eta\omega}\big(S_F(y_1-z)\gamma_\mu S_F(z-x_1)\big)_{\sigma\lambda} \qquad (20.58)
\end{aligned}
$$

Performing Fourier integrations, one obtains ($k\equiv p_2-p_1=q_1-q_2$, $k'\equiv p_1+q_1=p_2+q_2$)

$$
\begin{aligned}
&{}_{\rm out}\langle p_2,s_2;q_2,r_2|p_1,s_1;q_1,r_1\rangle_{\rm in} \;=\; -e^2(2\pi)^4\delta(p_2+q_2-p_1-q_1)\\
&\times \lim_{p_i^2,q_i^2\to m^2}\bar u_\xi(p_2,s_2)(m-\not p_2)_{\xi\eta}(m+\not q_2)_{\omega\chi}v_\chi(q_2,r_2)(m-\not p_1)_{\lambda\rho}u_\rho(p_1,s_1)\bar v_\zeta(q_1,r_1)(m+\not q_1)_{\zeta\sigma}\\
&\times\Big[\frac{1}{ik^2}\Big(g^{\mu\nu}+\frac{k^\mu k^\nu}{\vec k^2}-\frac{k^0}{\vec k^2}(k^\mu\eta^\nu+k^\nu\eta^\mu)\Big)\Big(\frac{m+\not p_2}{i(m^2-p_2^2)}\gamma_\nu\frac{m+\not p_1}{i(m^2-p_1^2)}\Big)_{\eta\lambda}\Big(\frac{m-\not q_1}{i(m^2-q_1^2)}\gamma_\mu\frac{m-\not q_2}{i(m^2-q_2^2)}\Big)_{\sigma\omega}\\
&-\frac{1}{ik'^2}\Big(g^{\mu\nu}+\frac{k'^\mu k'^\nu}{\vec k'^2}-\frac{k'^0}{\vec k'^2}(k'^\mu\eta^\nu+k'^\nu\eta^\mu)\Big)\Big(\frac{m+\not p_2}{i(m^2-p_2^2)}\gamma_\nu\frac{m-\not q_2}{i(m^2-q_2^2)}\Big)_{\eta\omega}\Big(\frac{m-\not q_1}{i(m^2-q_1^2)}\gamma_\mu\frac{m+\not p_1}{i(m^2-p_1^2)}\Big)_{\sigma\lambda}\Big]\\
&= \; i(2\pi)^4\delta(p_1+q_1-p_2-q_2)\\
&\times\frac{e^2}{t}\big[\bar u(p_2,s_2)\gamma_\nu u(p_1,s_1)\big]\big[\bar v(q_1,r_1)\gamma_\mu v(q_2,r_2)\big]\Big(g^{\mu\nu}+\frac{k^\mu k^\nu}{\vec k^2}-\frac{k^0}{\vec k^2}(k^\mu\eta^\nu+k^\nu\eta^\mu)\Big)\\
&-\frac{e^2}{s}\big[\bar u(p_2,s_2)\gamma_\nu v(q_2,r_2)\big]\big[\bar v(q_1,r_1)\gamma_\mu u(p_1,s_1)\big]\Big(g^{\mu\nu}+\frac{k'^\mu k'^\nu}{\vec k'^2}-\frac{k'^0}{\vec k'^2}(k'^\mu\eta^\nu+k'^\nu\eta^\mu)\Big) \qquad (20.59)
\end{aligned}
$$

where $k\equiv p_2-p_1=q_1-q_2$, $k'\equiv p_1+q_1=p_2+q_2$ and we used Mandelstam notations $t=k^2$, $s=k'^2$.

Now, it is easy to see that due to the properties

$$
\begin{aligned}
\bar u(p_2,s_2)\not k u(p_1,s_1) &= \bar u(p_2,s_2)[(\not p_2-m)-(\not p_1-m)]u(p_1,s_1) = 0\\
\bar v(q_1,r_1)\not k v(q_2,r_2) &= \bar v(q_1,r_1)[(\not q_1+m)-(\not q_2+m)]v(q_2,r_2) = 0\\
\bar u(p_2,s_2)\not k' v(q_2,r_2) &= \bar u(p_2,s_2)[(\not p_2-m)+(\not q_2+m)]v(q_2,r_2) = 0\\
\bar v(q_1,r_1)\not k' u(p_1,s_1) &= \bar v(q_1,r_1)[(\not q_1+m)+(\not p_1-m)]u(p_1,s_1) = 0 \qquad (20.60)
\end{aligned}
$$

the contributions of $D_W^{\mu\nu}(k)$ and $D_W^{\mu\nu}(k')$ to the r.h.s. of Eq. (20.58) cancel and we get the result

$$
\begin{aligned}
{}_{\rm out}\langle p_2,s_2;q_2,r_2|p_1,s_1;q_1,r_1\rangle_{\rm in} \;&=\; ie^2(2\pi)^4\delta(p_2+q_2-p_1-q_1) \qquad (20.61)\\
\times \Big(\big[\bar u(p_2,s_2)\gamma_\nu u(p_1,s_1)\big]\frac{g^{\mu\nu}}{t}\big[\bar v(q_1,r_1)\gamma_\mu v(q_2,r_2)\big] &- \big[\bar u(p_2,s_2)\gamma_\nu v(q_2,r_2)\big]\frac{g^{\mu\nu}}{s}\big[\bar v(q_1,r_1)\gamma_\mu u(p_1,s_1)\big]\Big)
\end{aligned}
$$

Note that for the amputated Green function at arbitrary momenta

$$
\begin{aligned}
G^{\rm amp}_{\eta\omega\lambda\sigma}(p_1,q_1\to p_2,q_2) \;&=\; \Pi(p_i^2-m^2)(q_i^2-m^2)\int d^4x_1d^4y_1d^4x_2d^4y_2 \qquad (20.62)\\
\times\; e^{-ip_1x_1-iq_1y_1+ip_2x_2+iq_2y_2}&\frac{\langle 0|{\rm T}\{\hat\psi_\eta(x_2)\hat{\bar\psi}_\omega(y_2)\hat{\bar\psi}_\lambda(x_1)\hat\psi_\sigma(y_1)e^{-i\int dt\ (\hat H_{\rm int}(t)+\hat H_{\rm C}(t))}\}|0\rangle}{\langle 0|{\rm T}\{e^{-i\int dt\ (\hat H_{\rm int}(t)+\hat H_{\rm C}(t))}\}|0\rangle}
\end{aligned}
$$

we can cancel the contribution of the instantaneous term $D^{\mu\nu}_{\rm inst}$ in the transverse photon propagator (20.50) with the Coulomb instantaneous term (20.46) in the interaction Hamiltonian as demonstrated in Eq. (20.56), but in general we cannot remove $D^{\mu\nu}_{\rm inst}$ from the photon propagator. Indeed, from Eq. (20.58) we get

$$
\begin{aligned}
G_{\eta\omega\lambda\sigma}(p_1,q_1\to p_2,q_2) \;&=\; -\,e^2(2\pi)^4 i\delta(p_2+q_2-p_1-q_1) \qquad (20.63)\\
\times\Big[\frac{1}{k^2}\Big(g^{\mu\nu}+\frac{k^\mu k^\nu}{\vec k^2}-\frac{k^0}{\vec k^2}(k^\mu\eta^\nu+k^\nu\eta^\mu)\Big)&[(m+\not p_2)\gamma_\nu(m+\not p_1)]_{\eta\lambda}[(m-\not q_1)\gamma_\mu(m-\not q_2)]_{\sigma\omega}\\
-\frac{1}{k'^2}\Big(g^{\mu\nu}+\frac{k'^\mu k'^\nu}{\vec k'^2}-\frac{k'^0}{\vec k'^2}(k'^\mu\eta^\nu+k'^\nu\eta^\mu)\Big)&[(m+\not p_2)\gamma_\nu(m-\not q_2)]_{\eta\omega}[(m-\not q_1)\gamma_\mu(m+\not p_1)]_{\sigma\lambda}\Big]
\end{aligned}
$$

and, for example,

$$
\begin{aligned}
k^\nu(m+\not p_2)\gamma_\nu(m+\not p_1) \;&=\; (m+\not p_2)[m-\not p_1-(m-\not p_2)](m+\not p_1) \;=\; (m+\not p_1)(m^2-p_2^2)-(m+\not p_2)(m^2-p_1^2)\\
k'^\nu(m+\not p_2)\gamma_\nu(m-\not q_2) \;&=\; (m+\not p_2)[m+\not q_2-(m-\not p_2)](m-\not q_2) \;=\; (m+\not p_2)(m^2-q_2^2)-(m-\not q_2)(m^2-p_1^2)
\end{aligned}
\qquad (20.64)
$$

so we see that the contribution of the term $D_W^{\mu\nu}$ drops out only if all fermions are on the mass shell ($p_i^2=q_i^2=m^2$). This is a manifestation of the Ward identity which in general reads as follows

Ward identity:

Suppose we have a general amputated Green function [24] $G^{\rm amp}_{\mu_1,\ldots,\mu_m}(k_1,\ldots k_m,p_1,\ldots p_l)$ with all electron and positron momenta $p_1,\ldots p_l$ on the mass shell ($p_i^2=m^2$), then [25]

$$
k_i^{\mu_i}G^{\rm amp}_{\mu_1,\ldots,\mu_m}(k_1,\ldots k_m,p_1,\ldots p_l)=0 \qquad (20.65)
$$

[24] Note that sometimes the name "Ward identity" (or "Ward-Takahshi identity" is reserved for a more general formula relating different off-shell Green functions of which our property (20.65) is a consequence.

[25] This identity is generally not true for individual Feynman diagrams but it restores when we sum over the diagrams for G at any given order.

Using Ward identity it is easy to prove (by induction, See *Peskin & Schroeder*) that the terms $\sim a_\mu k_\nu$ (or $k_\mu k_\nu$) in the photon propagator do not contribute to any physical S-matrix element.

Summarising, we have proved (albeit on a simplest example) that one can use the Feynman photon propagator (20.51) and the interaction Hamiltonian (20.52) for the calculation of S-matrix elements ($\equiv$ scattering cross sections). The result for an arbitrary Green function reads

$$\langle\Omega|\mathrm{T}\{\hat\psi(x_1)\hat{\bar\psi}(x_2)\hat{\bar\psi}(x_3)....\hat\psi(x_m)\hat A(y_1)...\hat A(y_n)\}|\Omega\rangle \tag{20.66}$$
$$= \frac{\langle 0|\mathrm{T}\{\hat\psi_I(x_1)\hat{\bar\psi}_I(x_2)\hat{\bar\psi}_I(x_3)....\hat\psi_I(x_m)\hat A_I(y_1)...\hat A_I(y_n)e^{ie\int d^4z\hat{\bar\psi}_I(z)\hat{\not A}_I(z)\hat\psi_I(z)}\}|0\rangle}{\langle 0|\mathrm{T}\{e^{ie\int d^4z\hat{\bar\psi}_I(z)\hat{\not A}_I(z)\hat\psi_I(z)}\}|0\rangle}$$

where fermion contractions are given by Dirac propagator (16.3) and photon contractions are given by Feynman photon propagator (20.51).

From Eq. (20.61) we see that the matrix element of the transition matrix is

$$\mathcal{M}(p_1,s_1;q_1,r_1\to p_2,s_2;q_2,r_2) \tag{20.67}$$
$$= e^2\Big(\big[\bar u(p_2,s_2)\gamma_\nu u(p_1,s_1)\big]\frac{g^{\mu\nu}}{t}\big[\bar v(q_1,r_1)\gamma_\mu v(q_2,r_2)\big] - \big[\bar u(p_2,s_2)\gamma_\nu v(q_2,r_2)\big]\frac{g^{\mu\nu}}{s}\big[\bar v(q_1,r_1)\gamma_\mu u(p_1,s_1)\big]\Big)$$

which is depicted by two diagrams in Fig. The relative (-) sign is obvious if one redraws

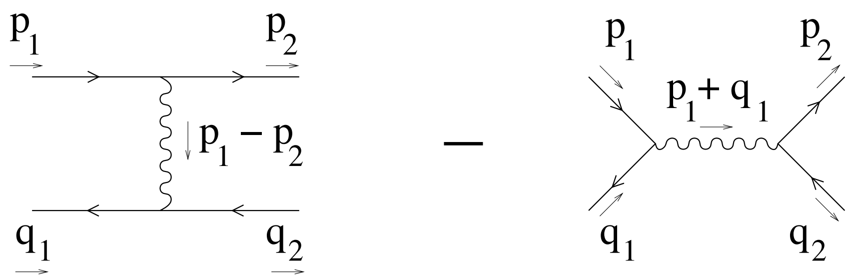
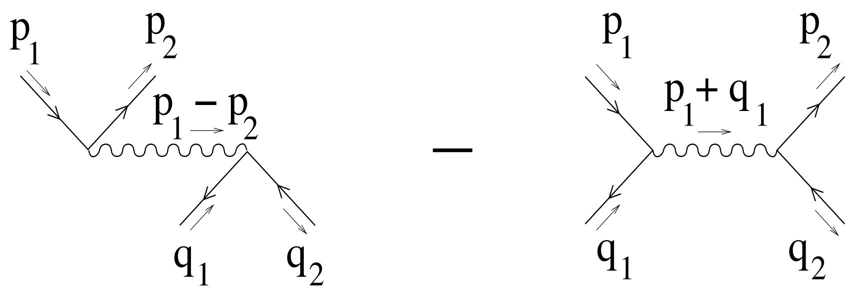


Figure 21. Diagrams for the elastic e^+e^- scattering

these diagrams as



It is easy to see now that the two diagrams differ by exchange of the fermion lines with momenta p_2 and $-q_1$ going out from left and right vertex.

20.5.2 QED interaction vertex

The formula (20.67) tells us that the fermion-fermion-photon interaction vertex is $\pm e$ in the set of Feynman rules for reduced Green functions. To fix the sign, let us consider the three-point fermion-fermion-photon Green function in the lowest order in perturbation theory.

$$G(k \to p, q)_{\xi\eta} \equiv G(k, -p, -q)_{\xi\eta} = \int d^4x d^4y d^4z \; e^{-ikz+ipx+iqy} \langle \Omega | \mathrm{T}\{\hat{A}^\mu(z)\hat{\psi}_\xi(x)\hat{\bar{\psi}}_\eta(y)\}|\Omega\rangle \quad (20.68)$$

$$= \int d^4x d^4y d^4z \; e^{-ikz+ipx+iqy} \frac{\langle 0|\mathrm{T}\{\hat{A}^\mu(z)\hat{\psi}_\xi(x)\hat{\bar{\psi}}_\eta(y) e^{ie\int d^4w \hat{\bar{\psi}}(w)\hat{\not{A}}(w)\hat{\psi}(w)}\}|0\rangle}{\langle 0|\mathrm{T}\{e^{ie\int d^4z \hat{\bar{\psi}}(w)\hat{\not{A}}(w)\hat{\psi}(w)}\}|0\rangle}$$

$$= ie\int d^4x d^4y d^4z d^4w \; e^{-ikz+ipx+iqy} \langle 0|\mathrm{T}\{\hat{A}^\mu(z)\hat{\psi}_\xi(x)\big(\hat{\bar{\psi}}(w)\,\hat{\not{A}}(w)\hat{\psi}(w)\big)\hat{\bar{\psi}}_\eta(y)\}|0\rangle + O(e^3)$$

$$= ie\int d^4x d^4y d^4z d^4w \; e^{-ikz+ipx+iqy} D_F^{\mu\nu}(z-w)\big[S_F(x-w)\gamma_\nu S_F(w-y)\big]_{\xi\eta} + O(e^3)$$

$$= ie(2\pi)^4\delta(p+q-k)\frac{g^{\mu\nu}}{i(k^2+i\epsilon)}\Big(\frac{m+\not{p})}{i(m^2-p^2-i\epsilon)}\gamma_\nu\big(\frac{m-\not{q})}{i(m^2-q^2-i\epsilon)}\Big)_{\xi\eta} \quad (20.69)$$

so we see that the interaction vertex for the set of Feynman rules in the momentum space is $ie(2\pi)^4\delta(\sum p_i)$.

Let us figure out the sign of the vertex for the set of Feynman rules for reduced Green functions. The definition of a reduced Green function (9.73) reads

$$G(p_1, ..., p_N) = (-i)^{N-1}(2\pi)^4\delta\Big(\sum_{i=1}^N p_i\Big)\mathcal{G}(p_1, ..., p_N) \quad (20.70)$$

First, we see that the photon propagator in this set is $\frac{g^{\mu\nu}}{k^2+i\epsilon}$:

$$-i\mathcal{D}_F^{\mu\nu}(k_1)(2\pi)^4\delta(k_1-k_2) = G^{\mu\nu}(k_1, -k_2)_{\xi\eta} = \int d^4z_1 d^4z_2 e^{-ik_1z_1+ik_2z_2} \; \widehat{\hat{A}^\mu(z_1)\hat{A}^\nu}(z_2) \quad (20.71)$$

$$= \int d^4z_1 d^4z_2 e^{-ik_1z_1+ik_2z_2} \; D_F^{\mu\nu}(z_1-z_2) = (2\pi)^4\delta(k_1-k_2)\frac{g^{\mu\nu}}{k_1^2+i\epsilon} \quad \Rightarrow \quad \mathcal{D}_F^{\mu\nu}(k) = \frac{g^{\mu\nu}}{k^2+i\epsilon}$$

Note that the sign is different from the propagator of a massless scalar particle (Eq. (6.43) with $m=0$) because physical photons correspond to $g^{ij} = -\delta_{ij}$.

Second, from Eqs. (20.69) and (20.70) we get

$$\mathcal{G}(k \to p, q)_{\xi\eta} = e\frac{g^{\mu\nu}}{(p+q)^2+i\epsilon}\Big(\frac{m+\not{p}}{m^2-p^2-i\epsilon}\gamma_\nu\big(\frac{m-\not{q})}{m^2-q^2-i\epsilon}\Big)_{\xi\eta} \quad (20.72)$$

Recalling that the Dirac propagator in this set of rules is $\frac{m+\not{p}}{m^2-p^2-i\epsilon}$ we see that the fermion-fermion-photon vertex in the set of Feynman rules for reduced Green functions is e (= charge of the positron).

20.6 LSZ theorem for QED

To finalize the set of Feynman rules for QED we need the LSZ theorem for matrix elements of S-matrix. It has the form (up to renormalization Z-factors to be discussed later)

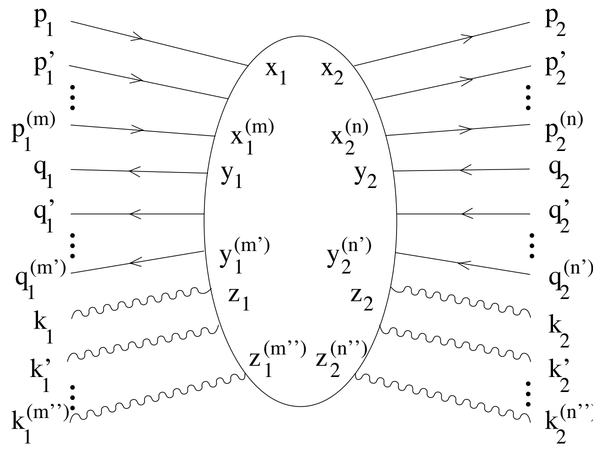


Figure 22. LSZ theorem for QED

$$
{}_{\rm out}\langle p_2,s_2;p_2',s_2';...p_2^{(n)},s_2^{(n)};q_2,r_2;...q_2^{(n')},r_2^{(n')};k_2,\lambda_2;...k_2^{(n'')},\lambda_2^{(n'')}|p_1,s_1;p_1',s_1';...p_1^{(m)},s_1^{(m)};q_1,r_1;...q_1^{(m')},r_1^{(m')};k_1,\lambda_1;...k_1^{(m'')},\lambda_1^{(m'')}\rangle_{\rm in}
$$

$$
\begin{aligned}
&= \lim_{k_1^{(k)2}\to 0}\lim_{k_2^{(n)2}\to 0}\lim_{p_1^{(i)2}\to m^2}\lim_{p_2^{(l)2}\to m^2}\lim_{q_1^{(j)2}\to m^2}\lim_{q_2^{(m)2}\to m^2}\int\prod_{i=1}^{m}dx_1^{(i)}\prod_{j=1}^{m'}dy_1^{(j)}\prod_{k=1}^{m''}dz_1^{(k)}\prod_{i=1}^{l}dx_2^{(l)}\prod_{m=1}^{n'}dy_2^{(m)}\prod_{n=1}^{n''}dz_1^{(n)}\\
&\times\, i^{m+n-m'-n'+m''+n''}\exp\Big\{-i\sum_{i=1}^{m}p_1^{(i)}x_1^{(i)}-i\sum_{j=1}^{m'}q_1^{(j)}y_1^{(j)}-i\sum_{k=1}^{m''}k_1^{(k)}z_1^{(k)}+i\sum_{l=1}^{n}p_2^{(l)}x_2^{(l)}+i\sum_{m=1}^{n'}q_2^{(m)}y_2^{(m)}+i\sum_{n=1}^{n''}k_2^{(n)}z_2^{(n)}\Big\}\\
&\times\,\langle\Omega|\mathrm{T}\{\prod_{l=1}^{n}\bar u_\xi(p_2^{(l)},s_2^{(l)})(m-\not p_2^{(l)})_{\xi\eta}\hat\psi_\eta(x_2^{(l)})\prod_{m=1}^{n'}\hat{\bar\psi}_\omega(y_2^{(m)})(m+\not q_2^{(m)})_{\omega\chi}v_\chi(q_2^{(m)},r_2^{(m)})\prod_{n=1}^{n''}e^{\lambda_2^{(n)}}_{\mu_2^{(n)}}(k_2^{(n)})\;k_2^{(n)2}\hat A^{\mu_2^{(n)}}(z_2^{(n)})\\
&\times\prod_{l=1}^{m}\hat{\bar\psi}_\lambda(x_1^{(i)})(m-\not p_1^{(l)})_{\lambda\rho}u_\rho(p_1^{(i)},r_1^{(i)})\prod_{j=1}^{m'}\bar v_\zeta(q_1^{(j)},r_1^{(j)})(m+\not q_1^{(j)})_{\zeta\sigma}\hat\psi_\sigma(y_1^{(j)})\prod_{k=1}^{m''}e^{\lambda_1^{(k)}}_{\mu_1^{(k_1)}}(k_1^{(k)})\;k_1^{(k)2}\hat A^{\mu_1^{(k)}}(z_1^{(k)})\}|\Omega\rangle
\end{aligned}
\tag{20.73}
$$

The proof is similar to Sect. 8.2 (see textbook by Bjorken & Drell or Peskin). As in Yukawa theory, we have $\bar u(p,s)$ for each outgoing electron, $u(p,s)$ for each incoming electron, $v(p,s)$ for each outgoing positron, and $\bar v(p,s)$ for each incoming positron. In addition, each incoming and outgoing photon with polarization λ brings in the factor $e^\lambda_\mu(k)$ (convoluted with corresponding Lorentz index from the amplitude $\langle ...A^\mu(k)...\rangle$).

This LSZ theorem (without Z-factors) is equivalent to Peskin's mnemonic rule

$$
{}_{\rm out}\langle p_2,s_2;p_2',s_2';...p_2^{(n)},s_2^{(n)};q_2,r_2;...q_2^{(n')},r_2^{(n')};k_2,\lambda_2;...k_2^{(n'')},\lambda_2^{(n'')}|p_1,s_1;p_1',s_1';...p_1^{(m)},s_1^{(m)};q_1,r_1;...q_1^{(m')},r_1^{(m')};k_1,\lambda_1;...k_1^{(m'')},\lambda_1^{(m'')}\rangle_{\rm in}
$$

$$
\begin{aligned}
=\;&\langle p_2,s_2;p_2',s_2';...p_2^{(n)},s_2^{(n)};q_2,r_2;...q_2^{(n')},r_2^{(n')};k_2,\lambda_2;...k_2^{(n'')},\lambda_2^{(n'')}|\exp\{ie\int d^4z\;\hat{\bar\psi}(z)\;\hat A(z)\hat\psi(z)\}\\
&\times|p_1,s_1;p_1',s_1';...p_1^{(m)},s_1^{(m)};q_1,r_1;...q_1^{(m')},r_1^{(m')};k_1,\lambda_1;...k_1^{(m'')},\lambda_1^{(m'')}\rangle
\end{aligned}
$$

where all the states and operators in the r.h.s. of this formula are in the interaction representation. The contractions of Dirac ladder operators with fermion fields are presented in Eq. (17.14)

$$\widehat{\hat{a}^{s_2}_{\vec{p}_2}\ \hat{\bar{\psi}}_\xi}(z) \ \stackrel{\text{def}}{\equiv}\ \{\hat{a}^{s_2}_{\vec{p}_2}, \hat{\bar{\psi}}_\xi(z)\} \ =\ \sum_s \int \frac{d\!\!\!^{-}{}^3p}{\sqrt{2E_p}}\{\hat{a}^{s_2}_{\vec{p}_2}, \big[\bar{v}_\xi(\vec{p},s)e^{-ipz}\hat{b}^s_{\vec{p}} + \bar{u}_\xi(\vec{p},s)e^{ipz}\hat{a}^{s\dagger}_{\vec{p}}\big]\} \ =\ \frac{e^{ip_2z}}{\sqrt{2E_{p_2}}}\ \bar{u}_\xi(\vec{p}_2, s_2)$$

$$\widehat{\hat{\psi}_\eta(z)\ \hat{a}^{s_1\dagger}_{\vec{p}_1}} \ \stackrel{\text{def}}{\equiv}\ \{\hat{\psi}_\eta(z), \hat{a}^{s_1\dagger}_{\vec{p}_1}\} \ =\ \sum_s \int \frac{d\!\!\!^{-}{}^3p}{\sqrt{2E_p}}\{\big[u_\eta(\vec{p},s)e^{-ipz}\hat{a}^s_{\vec{p}} + v_\eta(\vec{p},s)e^{ipz}\hat{b}^{s\dagger}_{\vec{p}}\big], \hat{a}^{s_1\dagger}_{\vec{p}_1}\} \ =\ \frac{e^{-ip_1z}}{\sqrt{2E_{p_1}}}\ u_\eta(\vec{p}_1, s_1),$$

$$\widehat{\hat{\bar{\psi}}_\xi(z)\ \hat{b}^{r_1\dagger}_{\vec{q}_1}} \ \stackrel{\text{def}}{\equiv}\ \{\hat{\bar{\psi}}_\xi(z), \hat{b}^{r_1\dagger}_{\vec{q}_1}\} \ =\ \sum_s \int \frac{d\!\!\!^{-}{}^3p}{\sqrt{2E_p}}\{\big[\bar{v}_\xi(\vec{p},s)e^{-ipz}\hat{b}^s_{\vec{p}} + \bar{u}_\xi(\vec{p},s)e^{ipz}\hat{a}^{s\dagger}_{\vec{p}}\big], \hat{b}^{r_1\dagger}_{\vec{q}_1}\} \ =\ \frac{e^{-iq_1z}}{\sqrt{2E_{q_1}}}\ \bar{v}_\xi(\vec{q}_1, r_1)$$

$$\widehat{\hat{b}^{r_2}_{\vec{q}_2}\ \hat{\psi}_\eta}(z) \ \stackrel{\text{def}}{\equiv}\ \{\hat{b}^{r_2}_{\vec{q}_2}, \hat{\psi}_\eta(z)\} \ =\ \sum_s \int \frac{d\!\!\!^{-}{}^3p}{\sqrt{2E_p}}\{\hat{b}^{r_2}_{\vec{q}_2}, \big[u_\eta(\vec{p},s)e^{-ipz}\hat{a}^s_{\vec{p}} + v_\eta(\vec{p},s)e^{ipz}\hat{b}^{s\dagger}_{\vec{p}}\big]\} \ =\ \frac{e^{iq_2z}}{\sqrt{2E_{q_2}}}\ v_\eta(\vec{q}_2, r_2), \tag{20.74}$$

and the contractions of photon ladder operators with $\hat{A}^\mu$ are given by

$$\widehat{\hat{a}^{\lambda_2}_{\vec{k}_2}\ \hat{A}_\mu}(z) \ \stackrel{\text{def}}{\equiv}\ [\hat{a}^{\lambda_2}_{\vec{k}_2}, \hat{A}_\mu(z)] \ =\ \sum_\lambda \int \frac{d\!\!\!^{-}{}^3k}{\sqrt{2|\vec{k}|}} e^\lambda_\mu(\vec{k})\ [\hat{a}^{\lambda_2}_{\vec{k}_2}, (e^{-ikz}\hat{a}^\lambda_{\vec{k}} + e^{ikz}\hat{a}^{\lambda\dagger}_{\vec{k}})] \ =\ \frac{e^{ik_2z}}{\sqrt{2|\vec{k}_2|}}\ e^{\lambda_2}_\mu(\vec{k}_2)$$

$$\widehat{\hat{A}_\mu(z)\ \hat{a}^{\lambda_1\dagger}_{\vec{k}_1}} \ \stackrel{\text{def}}{\equiv}\ [\hat{A}_\mu(z), \hat{a}^{\lambda_1\dagger}_{\vec{k}_1}] \ =\ \sum_\lambda \int \frac{d\!\!\!^{-}{}^3k}{\sqrt{2|\vec{k}|}} e^\lambda_\mu(\vec{k})\big[(e^{-ikz}\hat{a}^\lambda_{\vec{k}} + e^{ikz}\hat{a}^{\lambda\dagger}_{\vec{k}}), \hat{a}^{\lambda_1\dagger}_{\vec{k}_1}\big] \ =\ \frac{e^{-ik_1z}}{\sqrt{2|\vec{k}_1|}}\ e^{\lambda_1}_\mu(\vec{k}_1) \tag{20.75}$$

20.7 Set of Feynman rules for QED

Reminder: QED Lagrangian (density)

$$\mathcal{L}(x) \ =\ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m + eA\!\!\!/)\psi, \tag{20.76}$$

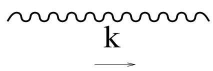
Green functions in the momentum space are defined as (cf. Eq. (18.2))

$$G\big(p_1, s_1; ...p_1^{(m)}, s_1^{(m)}; q_1, r_1; ...q_1^{(m')}, r_1^{(m')}; k_1, \lambda_1; ...k_1^{(m'')}, \lambda_1^{(m'')} \to p_2, s_2; ...p_2^{(n)}, s_1^{(m)}; q_2, r_2; ...q_2^{(n')}, r_2^{(n')}; k_2, \lambda_2; ...k_2^{(n'')}, \lambda_2^{(n'')}\big)$$
$$= \int \prod_{i=1}^m dx_1^{(i)} \prod_{j=1}^{m'} dy_1^{(j)} \prod_{k=1}^{m''} dz_1^{(k)} \prod_{i=1}^{l} dx_2^{(l)} \prod_{m=1}^{n'} dy_2^{(m)} \prod_{n=1}^{n''} dz_1^{(n)} G\big(x_1, ...x_1^{(m)}, y_1, ...y_1^{(m')}, z_1, ...z_1^{(m'')}, x_2, ...x_2^{(n)}, y_2, ...y_2^{(n')}, z_2, ...z_2^{(n'')}\big)$$
$$\times\ \exp\Big\{ -i\sum_{i=1}^m p_1^{(i)}x_1^{(i)} - i\sum_{j=1}^{m'} q_1^{(j)}y_1^{(j)} - i\sum_{k=1}^{m''} k_1^{(k)}z_1^{(k)} + i\sum_{l=1}^{n} p_2^{(l)}x_2^{(l)} + i\sum_{m=1}^{n'} q_2^{(m)}y_2^{(m)} + i\sum_{n=1}^{n''} k_2^{(n)}z_2^{(n)}\Big\} \tag{20.77}$$

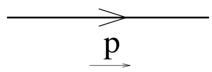
where $G\big(x_1, ...x_1^{(m)}, y_1, ...y_1^{(m')}, z_1, ...z_1^{(m'')}, x_2, ...x_2^{(n)}, y_2, ...y_2^{(n')}, z_2, ...z_2^{(n'')}\big)$ is a Green function in the coordinate space given by Eq. (20.66)

$$G\big(x_1, ...x_1^{(m)}, y_1, ...y_1^{(m')}, z_1, ...z_1^{(m'')}, x_2, ...x_2^{(n)}, y_2, ...y_2^{(n')}, z_2, ...z_2^{(n'')}\big) \tag{20.78}$$
$$= \langle\Omega|\mathrm{T}\{\prod_{l=1}^{n} \hat{\psi}(x_2^{(l)}) \prod_{m=1}^{n'} \hat{\bar{\psi}}(y_2^{(m)}) \prod_{n=1}^{n''} \hat{A}(z_2^{(n)}) \prod_{l=1}^{m} \hat{\bar{\psi}}(x_1^{(i)}) \prod_{j=1}^{m'} \hat{\psi}(y_1^{(j)}) \prod_{k=1}^{m''} \hat{A}(z_1^{(k)})\}|\Omega\rangle$$
$$= \frac{\langle 0|\mathrm{T}\{\prod_{l=1}^n \hat{\psi}_I(x_2^{(l)}) \prod_{m=1}^{n'} \hat{\bar{\psi}}_I(y_2^{(m)}) \prod_{n=1}^{n''} \hat{A}_I(z_2^{(n)}) \prod_{l=1}^{m} \hat{\bar{\psi}}_I(x_1^{(i)}) \prod_{j=1}^{m'} \hat{\psi}_I(y_1^{(j)}) \prod_{k=1}^{m''} \hat{A}_I(z_1^{(k)})\} e^{ie\int d^4z \hat{\bar{\psi}}_I(z)\hat{A\!\!\!/}(z)\hat{\psi}_I(z)}\}|0\rangle}{\langle 0|\mathrm{T}\{e^{ie\int d^4z \hat{\bar{\psi}}_I(z)\hat{A\!\!\!/}(z)\hat{\psi}_I(z)}\}|0\rangle}$$

Feynman rules for Green functions in QED

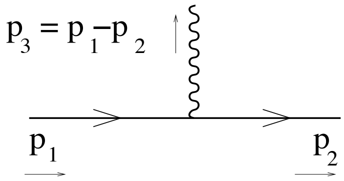
1. Photon propagator in Feynman gauge 

$= \dfrac{-ig^{\mu\nu}}{k^2+i\varepsilon}$

2. Dirac fermion propagator 

$= \dfrac{-i(m+\not p)}{m^2-p^2-i\varepsilon}$

(Arrow on the fermion line in the direction of the flow of negative charge)

3. Vertex 

$= ie\gamma^{\mu}(2\pi)^4\delta(p_1-p_2-p_3)$

4. Integrate over all momenta k_i of internal lines $\prod_i \int \dfrac{d^4k_i}{(2\pi)^4}$

5. Extra factor (-1) for each fermion loop

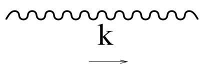
6. Negative relative sign between two amplitudes obtained by permutation of identical external lines corresponding to fermions

7. No symmetry factors in QED

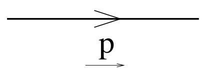
The reduced Green function is defined as usual

$$G(p_1,s_1;...p_1^{(m)},s_1^{(m)};q_1,r_1;...q_1^{(m')},r_1^{(m')};k_1,\lambda_1;...k_1^{(m'')},\lambda_1^{(m'')}\to p_2,s_2;...p_2^{(n)},s_1^{(m)};q_2,r_2;...q_2^{(n')},r_2^{(n')};k_2,\lambda_2;...k_2^{(n'')},\lambda_2^{(n'')})$$
$$= (-i)^{m+m'+m''+n+n'+n''-1}(2\pi)^4\delta\big(\sum p_1^{(i)}+\sum q_1^{(i)}+\sum k_1^{(i)}-\sum p_2^{(i)}+\sum q_2^{(i)}+\sum k_2^{(i)}\big) \tag{20.79}$$
$$\times\ \mathcal{G}(p_1,s_1;...p_1^{(m)},s_1^{(m)};q_1,r_1;...q_1^{(m')},r_1^{(m')};k_1,\lambda_1;...k_1^{(m'')}\lambda_1^{(m'')}\to p_2,s_2;...p_2^{(n)},s_1^{(m)};q_2,r_2;...q_2^{(n')},r_2^{(n')};k_2,\lambda_2;...k_2^{(n'')}\lambda_2^{(n'')})$$

Feynman rules for reduced Green functions in QED

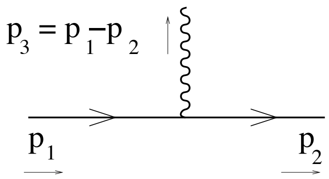
1. Photon propagator in Feynman gauge 

$= \dfrac{g^{\mu\nu}}{k^2+i\varepsilon}$

2. Dirac fermion propagator 

$= \dfrac{m+\not{p}}{m^2-p^2-i\varepsilon}$

   (Arrow on the fermion line in the direction of the flow of negative charge)

3. Vertex 

$= e\gamma^{\mu}$

4. Integrate over photon loop momenta k_j $\quad \prod_j \int \dfrac{d^4k_j}{(2\pi)^4 i}$

5. Integrate over fermion loop momenta p_j $\quad (-1)\prod_j \int \dfrac{d^4p_j}{(2\pi)^4 i}$ (Extra factor -1 for each fermion loop)

6. Negative relative sign between two amplitudes obtained by permutation of identical external lines corresponding to fermions

7. No symmetry factors in QED

Matrix element of $\mathcal{M}$-matrix in QED

Matrix element of $\mathcal{M}$-matrix is a reduced amputated Green function on a mass shell multiplied by:

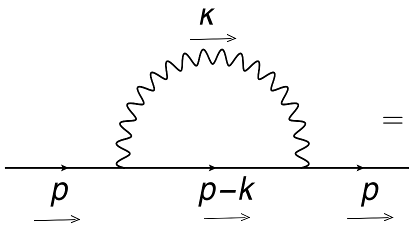
$\bar{u}(p,s)$ for each outgoing electron, $u(p,s)$ for each incoming electron, $v(p,s)$ for each outgoing positron, $\bar{v}(p,s)$ for each incoming positron, and $e_\mu^\lambda(k)$ for each incoming or outgoing photon.

Part XX

21 Renormalization in QED

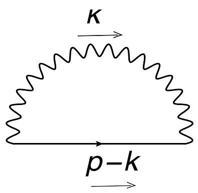
21.1 A problem with UV divergence of Feynman diagrams

A problem: some Feynman diagrams in QED are divergent at large momenta ("UV-devergent"). Example:



$$= \frac{m+\not{p}}{m^2-p^2-i\epsilon}\Big\{e^2\int\frac{đ^4k}{i}\,\frac{\gamma_\mu(m+\not{p}-\not{k})\gamma_\nu}{m^2-(p-k)^2-i\epsilon}\,\frac{g^{\mu\nu}}{k^2+i\epsilon}\Big\}\frac{m+\not{p}}{m^2-p^2-i\epsilon} \tag{21.1}$$

The expression in braces is a part of so-called "self-energy"



$$= \;-\,\Sigma(p) \;=\; e^2\int\frac{đ^4k}{i}\,\frac{4m-2(\not{p}-\not{k})}{[m^2-(p-k)^2-i\epsilon](k^2+i\epsilon)} \;=\; ? \tag{21.2}$$

A simple way to calculate (simple) Feynman integrals: Feynman formula

$$\frac{1}{AB} \;=\; \int_0^1 d\alpha\,\frac{1}{(A\alpha+B\bar{\alpha})^2} \tag{21.3}$$

where we used convenient notation $\bar{\alpha}\equiv 1-\alpha$ (nothing to do with Dirac conjugation!). Later we will need a more general formula

$$\frac{\Gamma(a)}{A^a}\frac{\Gamma(b)}{B^b} \;=\; \int_0^1 d\alpha\;\alpha^{a-1}\bar{\alpha}^{b-1}\frac{\Gamma(a+b)}{(A\alpha+B\bar{\alpha})^{a+b}} \tag{21.4}$$

Using Feynman formula (21.3) we obtain

$$\begin{aligned}\Sigma(p) \;&=\; e^2\int\frac{đ^4k}{i}\,\frac{4m+2(\not{k}-\not{p})}{[m^2-(p-k)^2-i\epsilon](-k^2-i\epsilon)} \;=\; e^2\int\frac{đ^4k}{i}\,\frac{4m+2(\not{k}-\not{p})}{[m^2\alpha-(p-k)^2\alpha-k^2\bar{\alpha}-i\epsilon]^2}\\ &=\; e^2\int\frac{đ^4k}{i}\,\frac{4m+2(\not{k}-\not{p})}{[m^2\alpha-(k-p\alpha)^2-p^2\bar{\alpha}\alpha-i\epsilon]^2} \;\overset{\text{shift }k\to k+p\alpha}{=}\; e^2\int\frac{đ^4k}{i}\int_0^1 d\alpha\frac{4m+2(\not{k}-\not{p}\bar{\alpha})}{[m^2\alpha-p^2\bar{\alpha}\alpha-k^2-i\epsilon]^2}\\ &=\; e^2\int_0^1 d\alpha(4m-2\not{p}\bar{\alpha})\int\frac{đ^4k}{i}\,\frac{1}{[m^2\alpha-p^2\bar{\alpha}\alpha-k^2-i\epsilon]^2}\end{aligned} \tag{21.5}$$

How to calculate $\int\frac{đ^4k}{i}\frac{1}{[M^2-k^2-i\epsilon]^2}$? Suppose $p^2<0$, then $M^2>0$ and

$$\int\frac{đ^4k}{i}\,\frac{1}{[M^2-k^2-i\epsilon]^2} \;=\; \int\frac{đk_0}{i}\int đ^2k\;\frac{1}{[M^2+\vec{k}^2-k_0^2-i\epsilon]^2} \tag{21.6}$$

For Euclidean integrals

$$\int d^d p\ f(p^2) \ = \ \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}\int_0^\infty dp^2\ (p^2)^{\frac{d}{2}-1}f(p^2) \tag{21.7}$$

so

$$\overset{k_0=ik_4}{=} \int\frac{d\!\!{}^-{}^4k}{i}\frac{1}{[M^2+\vec{k}^2+k_4^2]^2} \ = \ \int d\!\!{}^-{}^4k_e\ \frac{1}{[M^2+k_e^2]^2} \ = \ \frac{1}{16\pi^2}\int_0^\infty dk_e^2\ \frac{k_e^2}{(M^2+k_e^2)^2} \ \simeq \ \frac{1}{16\pi^2}\int_{M^2}^\infty\frac{dk_e^2}{k_e^2} \ = \ \ln\infty? \tag{21.8}$$

where $k_e = (k_1, k_2, k_3, k_4)$ is a 4-dimensional vector in the Euclidean space with metric $g_{\mu\mu} = \delta_{\mu\nu}$.

If we cut the divergence at some large $\mu_{\rm UV}$ ("ultraviolet cutoff"), we get

$$\Sigma(p) \ = \ \frac{e^2}{16\pi^2}\int_0^1 d\alpha\ (4m-2\not{p}\bar\alpha)\int_0^\infty dk_e^2\ \frac{k_e^2}{(m^2-p^2\bar\alpha\alpha+k_e^2)^2} \ = \ \frac{e^2}{16\pi^2}\int_0^1 d\alpha\ (4m-2\not{p}\bar\alpha)\ln\frac{\mu_{UV}^2}{m^2-p^2\bar\alpha\alpha} \tag{21.9}$$

The necessity of the cutoff may indicate the existence of some new physics at large momenta. The best known example is transition from four-fermion weak interaction at low energies to interaction mediated by W-boson at high energies ($m_W \sim 80$GeV).

21.1.1 Example of non-renormalizable theory: four-fermion "$V-A$" weak interaction

In 50's weak interactions were described by so-called four-fermion $V-A$ (= "vector minus axial-vector") Lagrangian, a part of which reads

$$\mathcal{L}_{\rm int}(x) \ = \ \frac{G_F}{\sqrt{2}}[\bar\psi(x)\gamma^\mu(1-\gamma_5)\nu(x)][\bar\nu(x)\gamma^\mu(1-\gamma_5)\psi(x)] \tag{21.10}$$

where $\bar\nu(x)$ is the neutrino field and $G_F = 1.166\times10^{-5}{\rm GeV}^{-2}$ is a Fermi constant (extra $\frac{1}{\sqrt2}$ is for historical reasons).

The free neutrino-antineutrino field is described by the Dirac Lagrangian with $m = 0$

$$\mathcal{L}(x) \ = \ \bar\nu(x)i\not{\partial}\nu(x) \tag{21.11}$$

which leads to Weyl equations (13.11) and the corresponding decomposition into ladder operators reads

$$\begin{aligned}\hat\nu(x) \ &= \ \int\frac{d\!\!{}^-{}^3p}{\sqrt{2|p|}}\left[\upsilon(\vec p)e^{-ipx}\hat\alpha_{\vec p}+\varsigma(\vec p)e^{ipx}\hat\beta^\dagger_{\vec p}\right]\Big|_{p_0=|p|}\\ \hat{\bar\nu}(\vec x) \ &= \ \int\frac{d\!\!{}^-{}^3p}{\sqrt{2|p|}}\left[\bar\varsigma(\vec p)e^{-ipx}\hat\beta_{\vec p}\ +\bar\upsilon(\vec p)e^{ipx}\hat\alpha^\dagger_{\vec p}\right]\Big|_{p_0=|p|}\end{aligned} \tag{21.12}$$

There is no sum over spins since the neutrino has negative helicity so the spinor $\upsilon(\vec p)$ is always left $\upsilon(\vec p) = \frac{1-\gamma_5}{2}\upsilon(\vec p)$ and positive-helicity antineutrino spinor is always right

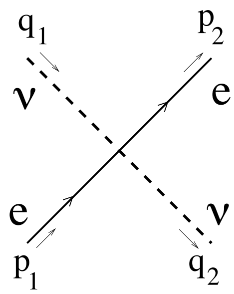


Figure 23. Four-femion scattering. Dashed line denotes neutrino

$\bar{\varsigma}(\vec{p}) = \bar{\varsigma}(\vec{p})\frac{1+\gamma_5}{2}$. The propagator is a massless Dirac propagator and we have $\bar{\upsilon}(p)$ for outgoing neutrino, $\upsilon(p)$ for incoming neutrino, $\varsigma(p)$ for outgoing antineutrino, and $\bar{\varsigma}(p)$ for incoming antineutrino.

Let us consider a neutrino-electron scattering in this model. In the leading order in G_F the amplitude of ν-e elastic scattering in this theory reads

$$\mathcal{M}(p_1, s_1; q_1 \to p_1, s_2; q_2) = \frac{G_F}{\sqrt{2}}[\bar{u}(p_2, s_2)\gamma_\mu(1-\gamma_5)\upsilon(q_1)][\bar{\upsilon}(q_2)\gamma^\mu(1-\gamma_5)u(p_1, s_1)] \quad (21.13)$$

where $\bar{\upsilon}(q_2)(1 - \gamma_5)$ denotes the neutrino spinor and $(1 - \gamma_5)\upsilon(q_1)$ the antineutrino one. The corresponding cross section is in a good agreement with experiment at low energies of electrons.

However, let us try to calculate the same cross section in the next order in G_F. The corresponding "fish" diagram is shown in Fig. 24

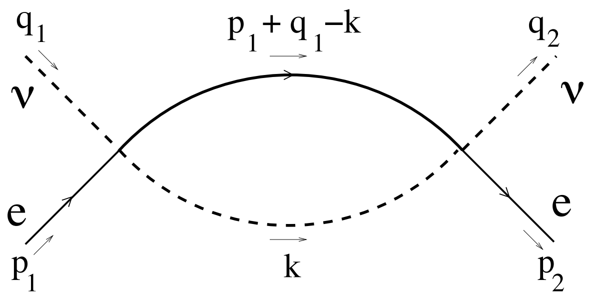


Figure 24. Four-fermion scattering in the second order in G_F.

and the result is

$$\begin{aligned}
&\mathcal{M}(p_1, s_1; q_1 \to p_1, s_2; q_2) \\
&= \frac{G_F^2}{2}\int \frac{d\!\!{}^-{}^4k}{i}[\bar{u}(p_2, s_2)\gamma_\mu(1-\gamma_5)\frac{\not{k}}{-k^2 - i\epsilon}\gamma_\nu(1-\gamma_5)u(p_1, s_1)]\bar{\upsilon}(q_2)\gamma^\mu(1-\gamma_5)\frac{m+\not{p}_1+\not{q}_1-\not{k}}{m^2-(p_1+q_1-k)^2}\gamma^\nu(1-\gamma_5)\upsilon(q_1) \\
&= [\bar{u}(p_2, s_2)\gamma_\mu(1-\gamma_5)]_\xi[\gamma_\nu(1-\gamma_5)u(p_1, s_1)]_\eta[\bar{\upsilon}(q_2)\gamma^\mu(1-\gamma_5)]_\zeta[\gamma^\nu(1-\gamma_5)\upsilon(q_1)]_\sigma \\
&\qquad \times \frac{G_F^2}{2}\int \frac{d\!\!{}^-{}^4k}{i}\frac{\not{k}_{\xi\eta}}{-k^2 - i\epsilon}\frac{(m+\not{p}_1+\not{q}_1-\not{k})_{\zeta\sigma}}{m^2-(p_1+q_1-k)^2 - i\epsilon} \qquad (21.14)
\end{aligned}$$

The momentum integral can be calculated gy Feynman's formula

$$\int\frac{d\!\!{}^-{}^4k}{i}\frac{\not{k}_{\xi\eta}}{-k^2-i\epsilon}\frac{(m+\not{p}_1+\not{q}_1-\not{k})_{\zeta\sigma}}{m^2-(p_1+q_1-k)^2-i\epsilon} = \int_0^1 d\!\!{}^-\alpha\int\frac{d\!\!{}^-{}^4k}{i}\frac{\not{k}_{\xi\eta}(m+\not{p}_1+\not{q}_1-\not{k})_{\zeta\sigma}}{\left(m^2\alpha-[k-(p_1+q_1)\alpha]^2-s\bar{\alpha}\alpha-i\epsilon\right)^2} \tag{21.15}$$

$$= \int_0^1 d\!\!{}^-\alpha\int\frac{d\!\!{}^-{}^4k}{i}\frac{(\not{k}+\alpha[\not{p}_1+\not{q}_1])_{\xi\eta}(m+[\not{p}_1+\not{q}_1]\bar{\alpha}-\not{k})_{\zeta\sigma}}{\left(m^2\alpha-k^2-s\bar{\alpha}\alpha-i\epsilon\right)^2} \simeq -\int_0^1 d\!\!{}^-\alpha\int\frac{d\!\!{}^-{}^4k}{i}\frac{\not{k}_{\xi\eta}\not{k}_{\zeta\sigma}}{\left(m^2\alpha-k^2-s\bar{\alpha}\alpha-i\epsilon\right)^2}$$

$$= -\frac{1}{4}\gamma^\rho_{\xi\eta}(\gamma_\rho)_{\zeta\sigma}\int_0^1 d\!\!{}^-\alpha\int\frac{d\!\!{}^-{}^4k}{i}\frac{k^2}{\left(m^2\alpha-k^2-s\bar{\alpha}\alpha-i\epsilon\right)^2} = \frac{\gamma^\rho_{\xi\eta}(\gamma_\rho)_{\zeta\sigma}}{4}\int_0^1 d\!\!{}^-\alpha\int d\!\!{}^-{}^4k_e\frac{k_e^2}{\left(m^2\alpha+k_e^2-s\bar{\alpha}\alpha-i\epsilon\right)^2}$$

$$= -\frac{\gamma^\rho_{\xi\eta}(\gamma_\rho)_{\zeta\sigma}}{64\pi^2}\int_0^1 d\!\!{}^-\alpha\int_0^\infty dk_e^2\frac{k_e^4}{\left(m^2\alpha+k_e^2-s\bar{\alpha}\alpha-i\epsilon\right)^2} \simeq -\frac{\gamma^\rho_{\xi\eta}(\gamma_\rho)_{\zeta\sigma}}{64\pi^2}\int_{m^2}^{\mu^2_{\rm UV}}dk_e^2 = -\frac{\gamma^\rho_{\xi\eta}(\gamma_\rho)_{\zeta\sigma}}{64\pi^2}\mu^2_{\rm UV}$$

Nowadays we know that weak interctions are mediated by W-bosons (and Z-boson) with $m_W\simeq 80$GeV so instead of "fish" diagram of Fig. 24 we have adiagram shown in Fig. 25 where the coupling constant of νeW interaction g_W is called a "weak coupling constant".

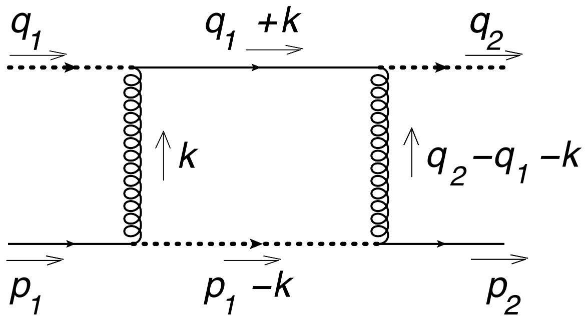


Figure 25. Four-fermion scattering mediated by W-boson (denoted by curvy line.

The corresponding $\mathcal{M}$-matrix element is

$$\begin{aligned}&\mathcal{M}(p_1,s_1;q_1\to p_1,s_2;q_2) \\ &= [\bar{u}(p_2,s_2)\gamma_\mu(1-\gamma_5)]_\xi[\gamma_\nu(1-\gamma_5)u(p_1,s_1)]_\eta[\bar{v}(q_2)\gamma^\mu(1-\gamma_5)]_\zeta[\gamma^\nu(1-\gamma_5)v(q_1)]_\sigma \\ &\quad\times\frac{g_W^4}{4}\int\frac{d\!\!{}^-{}^4k}{i}\frac{\not{k}_{\xi\eta}}{-k^2-i\epsilon}\frac{(m+\not{p}_1+\not{q}_1-\not{k})_{\zeta\sigma}}{m^2-(p_1+q_1-k)^2-i\epsilon}\frac{1}{(m_W^2-k^2-i\epsilon)(m_W^2-(q_1-q_2+k)^2-i\epsilon)}\end{aligned} \tag{21.16}$$

where the integral over loop momenta is now convergent at momenta $k^2\sim m_W^2$. If we consider the region of $k^2\ll m_W^2$ we get the old result

$$\frac{g_W^2}{4}\int\frac{d\!\!{}^-{}^4k}{i}\frac{\not{k}_{\xi\eta}}{-k^2-i\epsilon}\frac{(m+\not{p}_1+\not{q}_1-\not{k})_{\zeta\sigma}}{m^2-(p_1+q_1-k)^2-i\epsilon}\frac{1}{(m_W^2-k^2-i\epsilon)(m_W^2-(q_1-q_2+k)^2-i\epsilon)} \overset{k^2\ll m_W^2}{\simeq}$$

$$\simeq \frac{g_W^2}{m_W^4}\int\frac{d\!\!{}^-{}^4k}{i}\frac{\not{k}_{\xi\eta}}{-k^2-i\epsilon}\frac{(m+\not{p}_1+\not{q}_1-\not{k})_{\zeta\sigma}}{m^2-(p_1+q_1-k)^2-i\epsilon} \tag{21.17}$$

multiplied by $\frac{g_W^2}{4m_W^4}$ instead of $\frac{G_W^2}{2}$ so we can estimate that $G_W^2\sim\frac{g_W^2}{m_W^4}$ (the correct formula is $G_F^2=\frac{g_W^2}{8m_W^4}$).

The four-fermion model with Lagrangian (21.10) is an example of so-called "non-renormalizable" theories which are incomplete at large momenta. The indication of this incompleteness is the explicit dependence of physical cross sections on the UV cutoff μ like in the above example.

The situation in QED (and other so-called "renormalizable" theories) is more subtle: one still needs a UV cutoff $\mu_{\rm UV}$ for calculation of the individual Feynman diagrams but the cross sections do not depend explicitly on $\mu \equiv \mu_{\rm UV}$.

How can it be?

21.1.2 Renormalization program in QED

QCD Lagrangian has the form

$$\mathcal{L}_{\rm QED}(x) \;=\; \bar\psi(x)\big[i\not{\partial} - m_0 + e_0 \not{A}(x)\big]\psi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{21.18}$$

The parameters in the Lagrangian are called the "bare mass" m_0 and "bare charge" e_0. In principle, there is no reason that they should be equal to "physical mass" of the electron m and 'physical charge" e (defined as a coefficient in front of Coulomb potential between charges $V(r) = \frac{e^2}{4\pi r}$).

A priori we do not know the relation between m_0 and m and between e_0 and e. We've demonstrated that in the leading order in perturbation theory $m = m_0$ and $e = e_0$, but starting from the next-to-leading order this is no longer true (we have see it for the physical mass in the KG model, see Eq. (10.18)). In general, m and e can be expressed as an (infinite) series in coupling constant (= charge e_0)

$$\begin{aligned} m &= m_0(1 + a_1 e_0^2 + a_2 e_0^4 + \ldots) \\ e &= e_0(1 + b_1 e_0^2 + b_2 e_0^4 + \ldots) \end{aligned} \tag{21.19}$$

(It is easy to see that the parameter of the expansion is e_0^2 rathe than e_0). Now, suppose we calculated a certain cross section using Feynman diagrams following from the Lagrangian (21.18). Naturally, we will get a cross section as a perturbative series in e_0 (and m_0 will be the mass in the corresponding propagators) [26]:

$$\sigma \;=\; e_0^4(\sigma_0^{(0)} + e_0^2\sigma_1^{(0)} + e_0^4\sigma_2^{(0)} + \ldots) \tag{21.20}$$

The coefficients $\sigma_i^{(0)}$ in the expansion (21.21) are the functions of scattering momenta as well as e_0, m_0, and the ultraviolet cutoff $\mu_{\rm UV}$ [27]. (It may be demonstrated that in QED the dependence of diagrams on the UV cutoff $\mu_{\rm UV}$ is no stronger than logarithmical with a typical term being $\sim e_0^m(\ln\mu_{\rm UV}^2)^n$ with $m \geq n$.)

Renormalizability in QED:

If one expresses the cross section in terms of e and m rather than e_0 and m_0

$$\sigma \;=\; e^4(\sigma_0 + e^2\sigma_1 + e^4\sigma_2 + \ldots), \tag{21.21}$$

[26] Typically, a cross section starts from e_0^4.

[27] The renormalization is not specific to the theories with UV divergencies. Even for a theory where all Feynman diagrams are finite there may be (finite) difference between parameters of the Lagrangian and physical masses and charges.

the coefficients σ_i are finite functions of scattering momenta and physical mass m.

Summary: renormalization program in QED

- Write don Lagrangian in terms of bare mass and bare coupling constant

$$\mathcal{L} \;=\; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\,\partial\!\!\!/ - m_0 + e_0\,A\!\!\!/)\psi$$

- Calculate Feynman diagrams imposing a UV cutofff $\mu_{\rm UV}$ for (logarithmically) divergent loop momentum integrals.

- Calculate physical mass m in terms of bare Lagrangian parameters m_0, e_0

$$m \;=\; m_0(1 + a_1 e_0^2 + a_2 e_0^2 + ...) \tag{21.22}$$

- To relate e_0 to the physical charge e, calculate the non-relativistic limit of Coulomb exchange. In this limit the result for Coulomb potential between two electrons will be

$$V(r) \;=\; -\frac{e_0^2}{4\pi r}(1 + b_1 e_0^2 + b_2 e_0^2 + ...)$$

  By definition, the coefficient in front of $-\frac{1}{4\pi r}$ is (the square of) the physical charge of the electron

$$e^2 \;=\; -\frac{e_0^2}{4\pi r}(1 + b_1 e_0^2 + b_2 e_0^2 + ...) \tag{21.23}$$

  The coefficients a_i and b_i are functions of $\ln\frac{\mu^2_{\rm UV}}{m_0^2}$.

- Inverse equations (21.22) and (21.23):

$$\begin{aligned} m_0 \;&=\; m(1 + c_1 e^2 + c_2 e^4 + ...) \\ e_0 \;&=\; e(1 + d_1 e^2 + d_2 e^4 + ...) \end{aligned} \tag{21.24}$$

  where the coefficients c_i, d_i are functions of $\ln\frac{\mu^2_{\rm UV}}{m^2}$.

- Get rid of m_0 in favor of m and rewrite Feynman diagrams as a series in physical charge e. The resulting expressions for cross sections will not depend on $\mu_{\rm UV}$ $\Rightarrow$ perturbative series for cross sections will be finite.

The necessity of the cutoff may indicate our lack of understanding of physics at large momenta, but the information about this physics at large momenta (like masses of interaction mediators) is screened by the property of renormalizability (unlike the non-renormalizable theories such as four-fermion model where this information was as explicit as $G_F = \frac{g_W^2\sqrt{2}}{8m_W^2}$.

22 Renormalization program in QED at the one-loop level

22.1 LSZ theorem and physical mass

As we saw in Sect. 10.3, technically it is more convenient to calculate Feynman diagrams in terms of e_0 an physical mass m (rather than in terms of e_0 and m_0). Similarly to the scalar theory , we rewrite QED Lagrangian as follows

$$\mathcal{L}_{\rm QED} \;=\; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\psi(i\not\partial - m_0 + e_0\not A)\psi \;=\; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\psi(i\not\partial - m)\psi + \delta m\bar\psi\psi + e_0\bar\psi\not A)\psi$$

(where $\delta m \equiv m - m_0$) so the first two terms will form "new" $\mathcal{L}_0$ and the last two terms "new" $\mathcal{L}_{\rm int}$:

$$\mathcal{L}_{\rm QED} \;=\; \mathcal{L}_0 + \mathcal{L}_{\rm int} \tag{22.1}$$

$$\mathcal{L}_0 \;=\; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\psi(i\not\partial - m)\psi, \qquad \mathcal{L}_{\rm int} \;=\; \delta m\bar\psi\psi + e_0\bar\psi\not A\psi$$

The Dirac propagator for a new $\mathcal{L}_0$ is

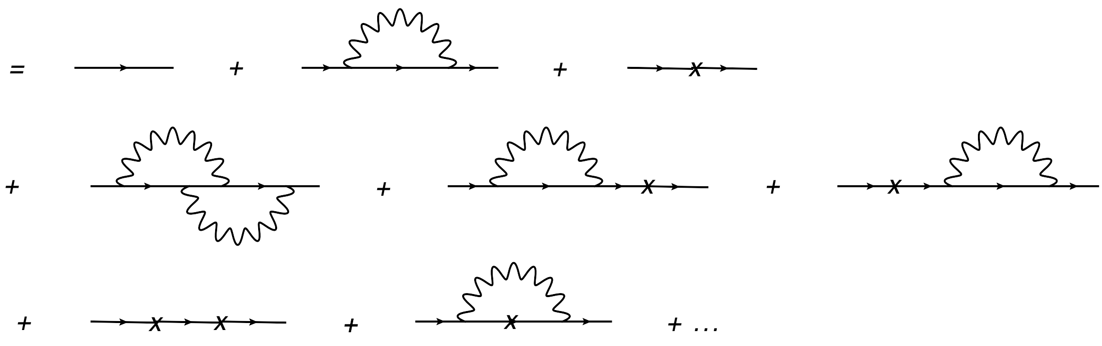
$$S_F(x-y) \;=\; \int\frac{\not d^4}{i}\frac{m+\not p}{m^2-p^2-i\epsilon}e^{-ip(x-y} \qquad \text{where } m \text{ is a } \underline{\text{physical}} \text{ mass} \tag{22.2}$$

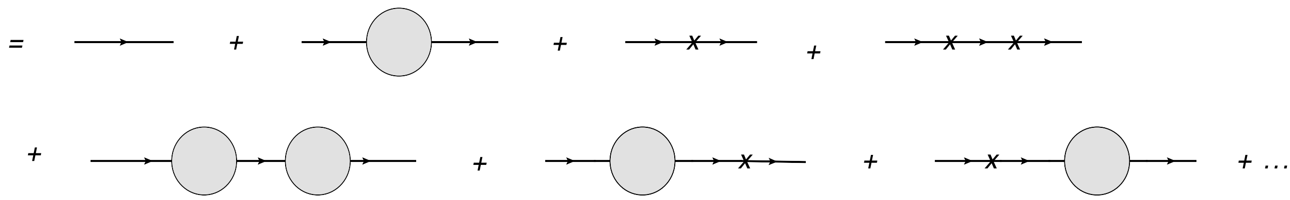
and we have "old" Dirac-Dirac-photon vertex $e_0\gamma_\mu$ and a new Dirac-Dirac vertex with constant δm. The "mass counterterm" δm is can be represented as a series in coupling constant e_0^2

$$\delta m \;=\; m(c_1e_0^2 + c_2e_0^4 + ...) \tag{22.3}$$

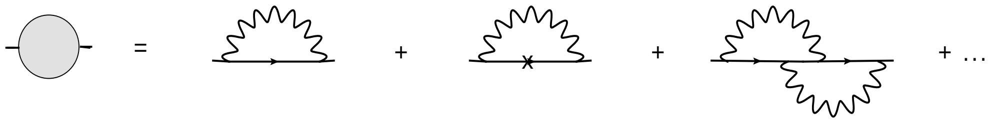
where the coefficients c_1, c_2 etc. are fixed by the requirement that the pole of the exact Dirac propagator $\mathcal{G}(p)$ remains at $p^2 = m^2$.

$$\mathcal{G}(p) \;=\; i\int d^4x\, e^{ipx}\langle\Omega|\mathrm{T}\{\hat\psi(x)\hat{\bar\psi}(y)\}|\Omega\rangle \;=\; i\int d^4x\, e^{ipx}\langle 0|\mathrm{T}\{\hat\psi_I(x)\hat{\bar\psi}_I(y)e^{i\int d^4z\ \hat L_I(z)}\}|0\rangle_{\rm connected} \tag{22.4}$$





where the sum of the one-particle irreducible (1PI) diagrams



is called (minus) self-energy $-\Sigma(p)$. Let us calculate it in the leading order in e_0^2

$$\Sigma(p) \;=\; -\,e_0^2\int\frac{đ^4k}{i}\,\frac{4m+2(\not{k}-\not{p})}{[m^2-(p-k)^2-i\epsilon](k^2+i\epsilon)} \;+\; O(e_0^4) \tag{22.5}$$

From Eq. (21.9) we know that this integral is logarithmically divergent at large k. However, formula (21.9) was approximate ($\ln\mu_{\rm UV}^2$ + const?) and we need the rigorous way to cut off the integrals over large momenta.

Rigorous definition of regularized Feynman diagrams: dimensional regularization and MS ("minimal subtraction") scheme

Part XXI

22.2 Dimensional regularization of loop integrals in Feynman diagrams

22.2.1 Step 1: calculation of Feynman integrals in arbitrary dimension of space-time

Let us start with the discussion of the integration over n-dimensional Euclidean space and consider the integral

$$\int d^np\; f(p^2) \;=\; C\int_0^\infty dp\; p^{n-1}\; f(p^2) \tag{22.6}$$

- up to some constant C it is evident from dimensional considerations (this constant defines the surface "area" of unit sphere in n dimensions). To find this constant, we should go to spherical polar coordinates in d dimensions. We will need only one mathematical formula for the element of volume in n-dimensional space

$$\int d^np \;\equiv\; \int_0^\infty p^{n-1}dp\int_0^\pi \sin^{n-2}\theta_{n-2}d\theta_{n-2}\int_0^\pi \sin^{n-3}\theta_{n-3}d\theta_{n-3}....\int_0^\pi \sin\theta_1 d\theta_1\int_0^{2\pi}d\phi \tag{22.7}$$

where $\theta_1,...\theta_{n-2}$ are (n-2) polar angles and ϕ is an azimuthal angle. If the integrand depends only on p^2 the integrals over angles can be easily performed so we obtain

$$\begin{aligned}\int d^np\; f(p^2) \;&=\; \int_0^\infty dp\; p^{n-1}f(p^2)\int_0^\pi \sin^{n-2}\theta_{n-2}d\theta_{n-2}\int_0^\pi \sin^{n-3}\theta_{n-3}d\theta_{n-3}....\int_0^\pi \sin\theta_1 d\theta_1\int_0^{2\pi}d\phi\; f(p^2)\\ &=\; \frac{2\pi^{n/2}}{\Gamma(n/2)}\int_0^\infty dp\; p^{n-1}f(p^2)\end{aligned} \tag{22.8}$$

where $\frac{2\pi^{n/2}}{\Gamma(n/2)}$ is a surface area of unit sphere in the space of n dimensions. (The quick check gives 2π for n=2 and 4π for n=3).

The integral $\int đ^d p$ in Minkowski space is defined as a product of integration over p_0 and integration over $(d-1)$-dimensional Euclidean space of vectors $\vec{p}$:

$$\int đ^d p \;\equiv\; \int \frac{dp_0}{2\pi} \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \tag{22.9}$$

Let us calculate now the Feynman integral for $\Sigma(p)$ in arbitrary number of dimensions d:

$$-\tilde{\Sigma}(p) \;\equiv\; e_0^2 \int \frac{đ^d k}{i} \frac{\gamma_\alpha[m + (\not{p} - \not{k})]\gamma^\alpha}{[m^2 - (p-k)^2 - i\epsilon](k^2 + i\epsilon)} \;+\; O(e_0^4) \tag{22.10}$$

As a first step, we re-derive the formula (21.5)

$$\begin{aligned}
\tilde{\Sigma}(p) \;&=\; e_0^2 \int \frac{đ^4 d}{i} \frac{dm + (d-2)(\not{k} - \not{p})}{[m^2 - (p-k)^2 - i\epsilon](-k^2 - i\epsilon)} \;=\; e_0^2 \int_0^1 d\alpha \int \frac{đ^4 k}{i} \frac{dm + (d-2)(\not{k} - \not{p})}{[m^2\alpha - (p-k)^2\alpha - k^2\bar{\alpha} - i\epsilon]^2} \\
&=\; e_0^2 \int_0^1 d\alpha \int \frac{đ^d k}{i} \frac{dm + (d-2)(\not{k} - \not{p})}{[m^2\alpha - (k - p\alpha)^2 - p^2\bar{\alpha}\alpha - i\epsilon]^2} \overset{\text{shift } k \to k + p\alpha}{=} e_0^2 \int_0^1 d\alpha \int \frac{đ^4 k}{i} \frac{dm + (d-2)(\not{k} - \not{p}\bar{\alpha})}{[m^2\alpha - p^2\bar{\alpha}\alpha - k^2 - i\epsilon]^2} \\
&=\; e_0^2 \int_0^1 d\alpha (dm - (d-2)\not{p}\bar{\alpha}) \int đ^{d-1}\vec{k} \int \frac{dk_0}{2\pi i} \frac{1}{[m^2\alpha - p^2\bar{\alpha}\alpha + \vec{k}^2 - k_0^2 - i\epsilon]^2}
\end{aligned} \tag{22.11}$$

where we used frmulas $\gamma^\mu\gamma_\mu = \delta^\mu_\mu = d$ and $\gamma_\mu\gamma_\alpha\gamma^\mu \;=\; (2-d)\gamma_\alpha$.

Let us assume first that $p^2 < 0$, then the poles in p_0 complex plane are located as shown in Fig. 26 and we can turn the contour of integration on 90° counterclockwise so

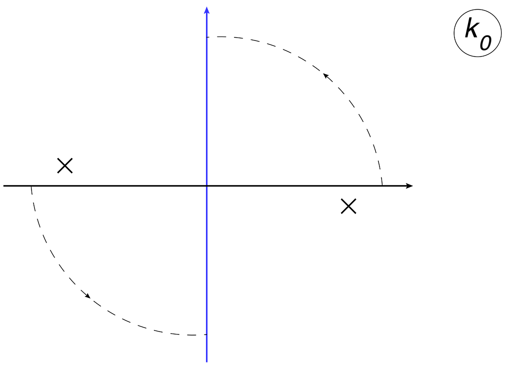


Figure 26. Wick's rotation: $\int dk_0 \to i \int dk_d$

it will run along the imaginary axis (this is sometimes called Wick's rotation to Euclidean space). Introducing new variable $k_0 = ik_d$ we get

$$\tilde{\Sigma}(p) \;=\; e_0^2 \int_0^1 d\alpha \int đ^{d-1}\vec{k} \int \frac{dk_d}{2\pi} \frac{dm - (d-2)\not{p}\bar{\alpha}}{[m^2\alpha - p^2\bar{\alpha}\alpha + \vec{k}^2 + k_d^2 - i\epsilon]^2} \tag{22.12}$$

Now the $d-1$-dimensional Euclidean space of $\vec{k}$ and one additional Euclidean coordinate k_d form the Euclidean space of d-dimensional vectors $k = (\vec{k}, k_d)$ so the integral (22.12) can be rewritten as

$$\tilde\Sigma(p) \;=\; e_0^2\int_0^1 d\alpha(dm-(d-2)\not p\bar\alpha)\int \not d^d k\frac{1}{[m^2\alpha-p^2\bar\alpha\alpha+k^2]^2} \tag{22.13}$$

$$= e_0^2\int_0^1 d\alpha(dm-(d-2)\not p\bar\alpha)\frac{\Gamma(d/2)}{(4\pi)^{d/2}}\int_0^\infty dk^2\;\frac{(k^2)^{\frac{d}{2}-1}}{[m^2\alpha-p^2\bar\alpha\alpha+k^2]^2} \;=\; e_0^2\int_0^1 d\alpha(dm-(d-2)\not p\bar\alpha)\frac{\Gamma(2-\frac{d}{2})}{(m^2\alpha-p^2\bar\alpha\alpha)^{2-\frac{d}{2}}}$$

where we've used Eq. (22.12) and the integral

$$\int_0^\infty dt\; t^{a-1}\frac{\Gamma(b)}{(t+B)^b} \;=\; \frac{\Gamma(a)\Gamma(b-a)}{B^{b-a}} \tag{22.14}$$

It can be demonstrated (by analytic continuation) that at arbitrary p^2 the result for the integral (22.13) has the form

$$\begin{aligned}\tilde\Sigma(p) \;&=\; e_0^2\int_0^1 d\alpha(dm-(d-2)\not p\bar\alpha)\int\frac{\not d^d k}{i}\frac{1}{[m^2\alpha-p^2\bar\alpha\alpha+k^2-i\epsilon]^2}\\ &=\; e_0^2\int_0^1 d\alpha(dm-(d-2)\not p\bar\alpha)\frac{\Gamma(2-\frac{d}{2})}{(m^2\alpha-p^2\bar\alpha\alpha-i\epsilon)^{2-\frac{d}{2}}}\end{aligned} \tag{22.15}$$

It should be noted that formally the integral (22.12) id defined only for $d=1,2$ and 3, but the r.h.s of Eq. (22.15) gives us an opportunity to define this integral at arbitrary (ral or complex) d by analytic continuation. It is easy to see that at $d=4$ the r.h.s. of Eq. (22.15) has a simple pole $\frac{1}{4-d}$ (recall that $\Gamma(\epsilon) \;=\; \frac{1}{\epsilon}-C+O(\epsilon)$ where $C\simeq 0.577$ is an Euler constant).

22.2.2 Step 2: definition of UV cutoff $\mu_{\rm UV}$

We define

$$\Sigma^d(p) \;\equiv\; \tilde\mu^{4-d}\tilde\Sigma^d(p) \tag{22.16}$$

and expand $\Sigma^d(p)$ as a function of d around the pole at $d=4$.

$$\Gamma(2-\frac{d}{2}) \;=\; \frac{\Gamma(3-\frac{d}{2})}{2-\frac{d}{2}} \;=\; \frac{1}{2-\frac{d}{2}}-C+O(2-\frac{d}{2}),\qquad C=-\psi(1)=0.577... \tag{22.17}$$

so

$$\frac{\tilde\mu^{4-d}(4\pi)^{2-\frac{d}{2}}}{(m^2\alpha-p^2\bar\alpha\alpha-i\epsilon)^{2-\frac{d}{2}}} \;=\; 1\;+\;(2-\frac{d}{2})\ln\frac{4\pi\tilde\mu^2}{m^2\alpha-p^2\alpha\bar\alpha-i\epsilon}\;+\;O(2-\frac{d}{2})^2 \tag{22.18}$$

and we get

$$\Sigma^d(p) \;=\; \frac{e_0^2}{(4\pi)^{d/2}}\tilde\mu^{4-d}\int_0^1 d\alpha(dm-(d-2)\not p\bar\alpha)\frac{\Gamma(2-\frac{d}{2})}{(m^2\alpha-p^2\bar\alpha\alpha)^{2-\frac{d}{2}}} \tag{22.19}$$

$$= \frac{e_0^2}{16\pi^2}\Big[\frac{4m-\not p}{2-\frac{d}{2}}-2m+\not p\;+\;\int_0^1 d\alpha(4m-2\not p\bar\alpha)\Big(\ln\frac{\tilde\mu^2/4\pi}{m^2\alpha-p^2\alpha\bar\alpha-i\epsilon}-C\;+\;O(2-\frac{d}{2})\Big)\Big]$$

$$= \frac{e_0^2}{16\pi^2}\Big[\frac{4m-\not p}{2-\frac{d}{2}}-2m+\not p\;+\;\int_0^1 d\alpha(4m-2\not p\bar\alpha)\Big(\ln\frac{\mu^2}{m^2\alpha-p^2\alpha\bar\alpha-i\epsilon}\;+\;O(2-\frac{d}{2})\Big)\Big]$$

where $\mu^2\equiv\frac{\tilde\mu^2}{4\pi}e^{-C}\equiv$ UV cutoff in the "$\overline{\rm MS}$ scheme".

22.2.3 Step 3: minimal subtraction scheme

We define the "regularized self-energy" in the leading order as

$$\Sigma^{\rm reg}(p) \;\equiv\; \lim_{d\to 4}\big(\Sigma^d(p) - \text{pole at d} = 4\big) \;=\; \frac{e_0^2}{16\pi^2}\Big[\int_0^1 d\alpha\,(4m - 2\not p\bar\alpha)\ln\frac{\mu^2}{m^2 - p^2\bar\alpha\alpha - i\epsilon} - 2m + \not p\Big] \tag{22.20}$$

In general,

$$\Sigma^{\rm reg}(p) \;= m\Sigma_1(p^2) - \not p\Sigma_2(p^2) \tag{22.21}$$

where $\Sigma^{\rm reg}_{1,2}(p) \;=\; \sum_{n=2}^{\infty} e_0^n f_n\big(\frac{p^2}{\mu^2}, \frac{m^2}{\mu^2}\big)$ and f_n are scalar logarithmical functions.

Let us return now to the calculation of exact Dirac propagator (22.4)

$$\begin{aligned}
\mathcal{G}^d(p) \;=&\; \frac{1}{m-\not p} - \frac{1}{m-\not p}\Sigma^d(p)\frac{1}{m-\not p} + \frac{1}{m-\not p}\delta m\frac{1}{m-\not p} - \frac{1}{m-\not p}\delta m\frac{1}{m-\not p}\Sigma^d(p)\frac{1}{m-\not p} \\
-&\; \frac{1}{m-\not p}\Sigma^d(p)\frac{1}{m-\not p}\delta m\frac{1}{m-\not p} + \frac{1}{m-\not p}\Sigma^d(p)\frac{1}{m-\not p}\Sigma^d(p)\frac{1}{m-\not p} + \frac{1}{m-\not p}\delta m\frac{1}{m-\not p}\delta m\frac{1}{m-\not p} \\
=&\; \frac{1}{m-\not p + \Sigma^d(p) - \delta m}
\end{aligned} \tag{22.22}$$

$\Rightarrow$

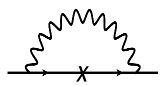
$$\mathcal{G}^{\rm reg}(p) \;=\; \frac{1}{m-\not p + \Sigma^{\rm reg}(p) - \delta m} \tag{22.23}$$

Now we must find δm from the condition that $\mathcal{G}^{\rm reg}(p)$ has a pole at $p^2 = m^2$ (at the physical mass m). Near $p^2 \simeq m^2$ The exact propagator $\mathcal{G}^{\rm reg}(p)$ (22.23) can be rewritten as

$$\begin{aligned}
\mathcal{G}^{\rm reg}(p) \;&=\; \frac{1}{m-\not p + \Sigma^{\rm reg}(p) - \delta m} \;=\; \frac{1}{m[1+\Sigma^{\rm reg}_1(p^2)] - \not p[1+\Sigma^{\rm reg}_2(p^2)] - \delta m} \\
&=\; \frac{m[1+\Sigma^{\rm reg}_1(p^2)] - \delta m + \not p[1+\Sigma^{\rm reg}_2(p^2)]}{\big(m[1+\Sigma^{\rm reg}_1(p^2)] - \delta m\big)^2 - p^2[1+\Sigma^{\rm reg}_2(p^2)]^2}
\end{aligned} \tag{22.24}$$

and it has a pole at $p^2 = m^2$ only if the denominator vanishes at $p^2 = m^2$:

$$\begin{aligned}
&\big(m[1+\Sigma^{\rm reg}_1(m^2)] - \delta m\big)^2 \;=\; m^2[1+\Sigma^{\rm reg}_2(m^2)]^2 \;\Rightarrow\; m[1+\Sigma^{\rm reg}_1(m^2)] - \delta m \;=\; m[1+\Sigma^{\rm reg}_2(m^2)] \\
&\Rightarrow\; \delta m \;=\; m\big[\Sigma^{\rm reg}_1(m^2) - \Sigma^{\rm reg}_2(m^2)\big]
\end{aligned} \tag{22.25}$$

Note that since $\Sigma_1(p^2)$ and $\Sigma_2(p^2)$ depend on δm^2 due to diagrams like  , the equation (22.25) should be solved anew in each order in perturbation theory.

In the lowest order we get

$$\begin{aligned}
\Sigma^{\rm reg}_1(p^2) \;&=\; \frac{e_0^2}{4\pi^2}\int_0^1 d\alpha\;\ln\frac{\mu^2}{m^2\alpha - p^2\bar\alpha\alpha - i\epsilon} - \frac{e_0^2}{8\pi^2} \\
\Sigma^{\rm reg}_2(p^2) \;&=\; \frac{e_0^2}{8\pi^2}\int_0^1 d\alpha\;\bar\alpha\ln\frac{\mu^2}{m^2\alpha - p^2\bar\alpha\alpha - i\epsilon} - \frac{e_0^2}{16\pi^2}
\end{aligned} \tag{22.26}$$

so

$$\delta m \;=\; m\big[\Sigma^{\rm reg}_1(m^2) - \Sigma^{\rm reg}_2(m^2)\big] \;=\; m\frac{e_0^2}{8\pi^2}\Big[\int_0^1 d\alpha\;(1+\alpha)\ln\frac{\mu^2}{m^2\alpha^2} - \frac{1}{2}\Big] \;=\; m\frac{e_0^2}{8\pi^2}\Big[\frac{3}{2}\ln\frac{\mu^2}{m^2} + 2\Big] \tag{22.27}$$

Thus, we have arranged that the exact propagator has a pole at $p^2 = m^2$. As we know from the KG theory, the residue at this pole is so-called Z-factor which enters the LSZ theorem (see the discussion in Sect 10.3). Let us find this factor (called Z_2 for historical reasons) in the leading order in perturbation theory. Near $p^2 = m^2$

$$\Sigma_i^{\rm reg}(p^2) \;=\; \Sigma_1^{\rm reg}(m^2) + (p^2-m^2)\frac{d}{dp^2}\Sigma_i^{\rm reg}(p^2)\Big|_{p^2=m^2} \;+\; O(p^2-m^2)^2 \tag{22.28}$$

so the numerator in the Eq. (22.24) behaves as

$$m[1+\Sigma_1^{\rm reg}(p^2)] - \delta m + \not{p}[1+\Sigma_2^{\rm reg}(p^2)] \;=\; (m+\not{p})\big[1+\Sigma_2^{\rm reg}(m^2)\big] \;+\; O(p^2-m^2) \tag{22.29}$$

and the denominator

$$\begin{aligned}\big(m[1+\Sigma_1^{\rm reg}(p^2)] - \delta m\big)^2 - p^2[1+\Sigma_2^{\rm reg}(p^2)]^2 \;=&\; (p^2-m^2)\Big\{2m\big(m[1+\Sigma_1^{\rm reg}(m^2)]-\delta m\big)\frac{\partial\Sigma_1^{\rm reg}}{\partial p^2}\Big|_{p^2=m^2}\\ &- [1+\Sigma_2^{\rm reg}(m^2)]^2 - 2m^2[1+\Sigma_2^{\rm reg}(m^2)]\frac{\partial\Sigma_2^{\rm reg}}{\partial p^2}\Big|_{p^2=m^2}\Big\} \;+\; O(p^2-m^2)^2\\ =&\; (m^2-p^2)[1+\Sigma_2^{\rm reg}(m^2)]\Big\{1+\Sigma_2^{\rm reg}(m^2) - 2m^2\big(\frac{\partial\Sigma_1^{\rm reg}}{\partial p^2} - \frac{\partial\Sigma_2^{\rm reg}}{\partial p^2}\big)\Big|_{p^2=m^2}\Big\}\end{aligned} \tag{22.30}$$

Thus, the exact propagator near the pole behaves as

$$\mathcal{G}(p) \;\stackrel{p^2\to m^2}{=}\; Z_2\frac{m+\not{p}}{m^2-p^2-i\epsilon} \;+\; \text{const} \tag{22.31}$$

with the residue determined by Eq. (22.30)

$$Z_2^{-1} \;=\; 1+\Sigma_2^{\rm reg}(m^2) - 2m^2\big(\frac{\partial\Sigma_1^{\rm reg}}{\partial p^2} - \frac{\partial\Sigma_2^{\rm reg}}{\partial p^2}\big)\Big|_{p^2=m^2} \tag{22.32}$$

In the leading order in perturbation theory we get from Eq. (22.26)

$$\begin{aligned}Z_2 \;=&\; 1 - \frac{e_0^2}{8\pi^2}\Big[\int_0^1 d\alpha\;\bar\alpha\ln\frac{\mu^2}{m^2\alpha^2} - \frac12 - 2\int_0^1 d\alpha\;\big(\frac{1}{\alpha}-\alpha\big)\Big] \;+\; O(e_0^4)\\ =&\; 1 - \frac{e_0^2}{16\pi^2}\ln\frac{\mu^2}{m^2} - \frac{e_0^2}{8\pi^2} + \frac{e_0^2}{4\pi^2}\int_0^1 d\alpha\;\big(\frac{1}{\alpha}-\alpha\big) \;+\; O(e_0^4)\end{aligned} \tag{22.33}$$

Note that the integral in the r.h.s. of this equation is "infrared divergent" as $\alpha \to 0$. To calculate it one needs to introduce a small photon mass λ^2 and then one gets $\ln\frac{m^2}{\lambda^2}$ instead of the IR divergence. This is a typical situation in theories with massless particles: the emission of a massless particle near the the mass shell is "infrared divergent" at small momenta so to regulate it one needs an additional "IR cutoff" like a small photon mass. Fortunately, in our case this IR divergence cancels with the corresponding term in Z_1 due to Ward identity so the physical electric charge is "infrared safe".

22.3 LSZ for electron scattering

Similarly to the scalar case, Z_2 is the coefficient of proportionality between $\hat\psi$ and $\hat\psi_{\rm in,out}$:

$$\hat\psi(x) \stackrel{t\to-\infty}{\to} Z_2^{\frac12}\hat\psi_{\rm in}(x), \qquad \hat\psi(x) \stackrel{t\to\infty}{\to} Z_2^{\frac12}\hat\psi_{\rm out}(x) \tag{22.34}$$

(the proof just repeats the derivation of Eq. (10.31)). The LSZ theorem (20.53) takes the form

$$\begin{aligned}
{}_{\rm out}\langle p_2,s_2;p_2',s_2'|p_1,s_1;p_1',s_1'\rangle_{\rm in} &= \lim_{p_i^2,\to m^2}\int d^4x_1d^4x_1'd^4x_2d^4x_2'\ e^{-ip_1x_1-ip_1'x_1'+ip_2x_2+ip_2'x_2'}\\
\times\ \frac{1}{(\sqrt{Z_2})^4}\langle\Omega|\mathrm{T}\{\bar u_\xi(p_2,s_2)(m-\not p_2)_{\xi\eta}\hat\psi_\eta(x_2)\bar u_\zeta(p_2',s_2')(m-\not p_2')_{\zeta\sigma}\hat\psi_\sigma(x_2')&\\
\times\ \hat{\bar\psi}_\lambda(x_1)(m-\not p_1)_{\lambda\rho}u_\rho(p_1,s_1)\hat{\bar\psi}_\omega(x_1')(m-\not p_1')_{\omega\chi}u_\chi(p_1',s_1')\}|\Omega\rangle&\\
= \frac{1}{(\sqrt{Z_2})^4}\lim_{p_i^2\to m^2}\bar u_\xi(p_2,s_2)(m-\not p_2)_{\xi\eta}\bar u_\zeta(p_2',s_2')(m-\not p_2')_{\zeta\sigma}&\\
\times\ (m-\not p_1)_{\lambda\rho}u_\rho(p_1,r_1)(m-\not p_1')_{\omega\chi}u_\chi(p_1',s_1')G_{\eta\sigma\lambda\omega}(p_1,p_1'\to p_2,p_2')&
\end{aligned}\tag{22.35}$$

where

$$\begin{aligned}
&G_{\eta\sigma\lambda\omega}(p_1,p_1'\to p_2,p_2') &(22.36)\\
&= \int d^4x_1d^4x_1'd^4x_2d^4x_2'\ e^{-ip_1x_1-ip_1'x_1'+ip_2x_2+ip_2'x_2'}\frac{\langle 0|\mathrm{T}\{\hat\psi_\eta(x_2)\hat\psi_\sigma(x_2')\hat{\bar\psi}_\lambda(x_1)\hat{\bar\psi}_\omega(x_1')e^{i\int dt\ \hat L_{\rm int}(t)}\}|0\rangle}{\langle 0|\mathrm{T}\{e^{i\int dt\ \hat L_{\rm int}(t)}\}|0\rangle}\\
&= \text{ sum of all Feynman diagrams with 4 electron tails} &(22.37)
\end{aligned}$$

Similarly to the scalar case (see Eq. (10.32) we can represent the sum of these diagrams as shown in Fig. 27
where $G^{\rm amp}$ is a sum of one-particle irreducible diagrams with four electron tails and $\mathcal G(p)$ is an exact Dirac propagator (22.4). We get

$$G_{\eta\sigma\lambda\omega}(p_1,p_1'\to p_2,p_2') = \mathcal G_{\eta\eta'}(p_2)\mathcal G_{\sigma\sigma'}(p_2')G^{\rm amp}_{\eta'\sigma'\lambda'\omega'}(p_1,p_1'\to p_2,p_2')\mathcal G_{\lambda'\lambda}(p_1)\mathcal G_{\omega'\omega}(p_1')$$

and therefore

$$\begin{aligned}
&\lim_{p_i^2\to m^2}(m-\not p_2)_{\xi\eta}(m-\not p_2')_{\zeta\sigma}G_{\eta\sigma\lambda\omega}(p_1,p_1'\to p_2,p_2')(m-\not p_1)_{\lambda\rho}(m-\not p_1')_{\omega\chi}\\
&= \lim_{p_i^2\to m^2}\big[(m-\not p_2)\mathcal G(p_2)\big]_{\xi\eta'}\big[(m-\not p_2')\mathcal G(p_2')\big]_{\zeta\sigma'}\big[\mathcal G(p_1)(m-\not p_1)\big]_{\lambda'\rho}\big[\mathcal G(p_1')(m-\not p_1')\big]_{\omega'\chi}G^{\rm amp}_{\eta'\sigma'\lambda'\omega'}(p_1,p_1'\to p_2,p_2')\\
&= Z_2^4G^{\rm amp}_{\xi\zeta\rho\chi}(p_1,p_1'\to p_2,p_2')\Big|_{p_i^2=m^2}
\end{aligned}\tag{22.38}$$

where we used Eq. (22.31). The matrix element of S-matrix takes the form

$$S(p_1,s_1;p_1',s_1'\to p_2,s_2;p_2',s_2') = (\sqrt{Z_2})^4\bar u(p_2,s_2)_\xi\bar u(p_2',s_2')_{\xi'}G^{\rm amp}_{\xi\xi'\eta\eta'}(p_1,p_1'\to p_2,p_2')\Big|_{p_i^2=m^2}u_\eta(p_1,s_1)u_{\eta'}(p_1',s_1') \tag{22.39}$$

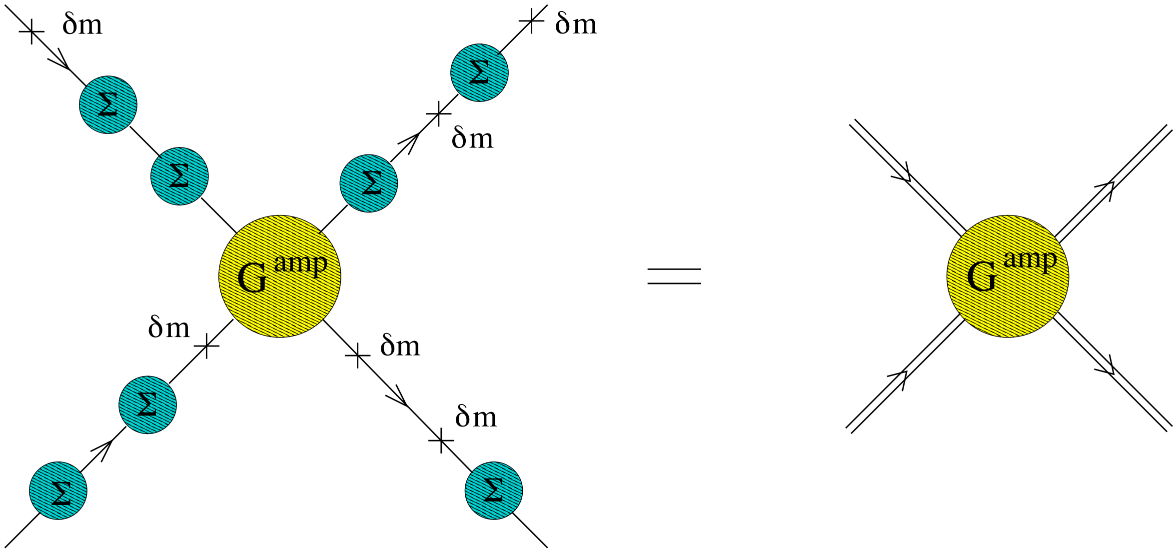


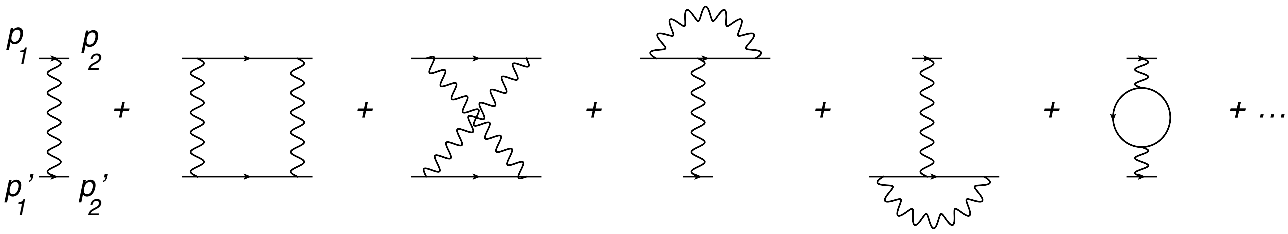
Figure 27. Electron-electron scattering. Double line denotes exact propagator $\mathcal{G}(p)$

or, in terms of $\mathcal{M}$-matrix

$$\mathcal{M}(p_1,s_1;p_1',s_1'\to p_2,s_2;p_2',s_2') \;=\; Z_2^2\bar{u}(p_2,s_2)_\xi\bar{u}(p_2',s_2')_{\xi'}\mathcal{G}^{\rm amp}_{\xi\xi'\eta\eta'}(p_1,p_1'\to p_2,p_2')\Big|_{p_i^2=m^2}u_\eta(p_1,s_1)u_{\eta'}(p_1',s_1') \tag{22.40}$$

where

$\mathcal{G}^{\rm amp}$ =



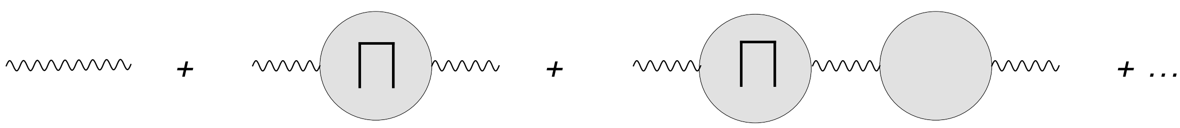
- diagrams with exchange $p_2 \leftrightarrow p_2'$ in the final state.

The last three diagrams on this figure are still divergent so we need to take care of these UV divergencies. Let us start with the last diagram displaying the e_0^2 correction to the photon propagator.

22.4 Photon propagator and Z_3

The diagrams for photon propagator are shown in Fig. 28.
Similarly to the Dirac propagator, let us group the diagrams in 1PI blocks as shown below:

$\mathcal{D}_{\mu\nu}(q)$ =



where

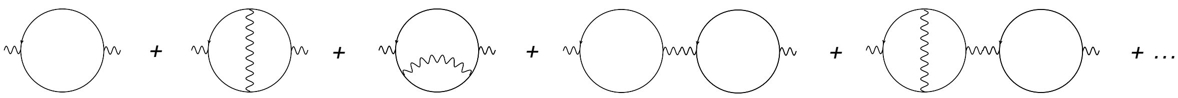
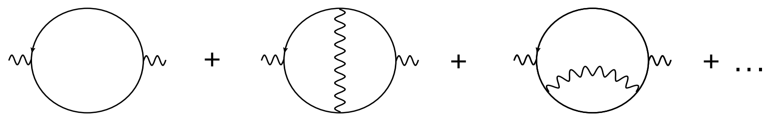


Figure 28. Feynman diagrams for exact photon propagator $\mathcal{D}_{\mu\nu}(p)$

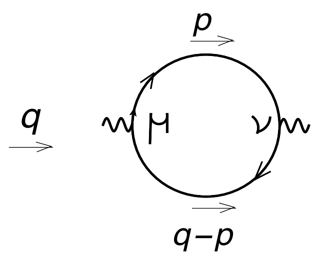
$$\Pi_{\mu\nu}(q) \;=$$



is the photon self-energy called "polarization operator" (for a reason discussed in the next lecture).

In the leading order in e_0^2 the polarization operator is given by the diagram

$$\Pi^{\text{l.o.}}_{\mu\nu}(q) \;=$$



$$= \; - \, e_0^2 \int \frac{d\!\!\bar{}^4 p}{i} \frac{\text{tr}\{\gamma^\mu (m + \not p)\gamma_\nu (m + \not p - \not q)\}}{(m^2 - p^2 - i\epsilon)[m^2 - (p-q)^2 - i\epsilon]}$$

In dimensional regularization [28]

$$\begin{aligned}
\Pi^{\text{l.o.}}_{\mu\nu}(q) \;&=\; - \, e_0^2 \tilde{\mu}^{4-d} \int \frac{d\!\!\bar{}^d p}{i} \frac{\text{tr}\{\gamma^\mu (m + \not p)\gamma_\nu (m + \not p - \not q)\}}{(m^2 - p^2 - i\epsilon)[m^2 - (p-q)^2 - i\epsilon]} \\
&= \; - \, 4 e_0^2 \tilde{\mu}^{4-d} \int \frac{d\!\!\bar{}^d p}{i} \frac{m^2 g_{\mu\nu} - p_\mu (q-p)_\nu - q_\nu (q-p)_\mu + g_{\mu\nu} p\cdot(q-p)}{(m^2 - p^2 - i\epsilon)[m^2 - (p-q)^2 - i\epsilon]} \\
&= \; - \, 4 e_0^2 \tilde{\mu}^{4-d} \int \frac{d\!\!\bar{}^d p}{i} \big[m^2 g_{\mu\nu} - p_\mu (q-p)_\nu - p_\nu (q-p)_\mu + g_{\mu\nu} p\cdot(q-p)\big] \int_0^1 d\alpha \frac{1}{[m^2 - (p - q\alpha)^2 - q^2 \bar{\alpha}\alpha - i\epsilon]^2} \\
&\overset{\text{shift } p \to p + q\alpha}{=} \; - \, 4 e_0^2 \tilde{\mu}^{4-d} \int \frac{d\!\!\bar{}^d p}{i} \int_0^1 d\alpha \frac{m^2 g_{\mu\nu} - (p + q\alpha)_\mu (q\bar{\alpha} - p)_\nu - (p + q\alpha)_\nu (q\bar{\alpha} - p)_\mu + g_{\mu\nu} (p + q\alpha)\cdot(q\bar{\alpha} - p)}{[m^2 - p^2 - q^2 \bar{\alpha}\alpha - i\epsilon]^2} \\
&\overset{\text{linear terms drop}}{=} \; - \, 4 e_0^2 \tilde{\mu}^{4-d} \int \frac{d\!\!\bar{}^d p}{i} \int_0^1 d\alpha \frac{(m^2 + q^2 \bar{\alpha}\alpha - p^2) g_{\mu\nu} + 2 p_\mu p_\nu - 2\bar{\alpha}\alpha q_\mu q_\nu}{[m^2 - p^2 - q^2 \bar{\alpha}\alpha - i\epsilon]^2}
\end{aligned} \tag{22.41}$$

Using formulas

$$\int \frac{d\!\!\bar{}^d p}{i} \frac{\Gamma(a)}{(M^2 - p^2 - i\epsilon)^a} \;=\; \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma\big(a - \frac{d}{2}\big)}{(M^2 - i\epsilon)^{a - \frac{d}{2}}}, \tag{22.42}$$

$$\int \frac{d\!\!\bar{}^d p}{i} \, p_\mu p_\nu \frac{\Gamma(a)}{(M^2 - p^2 - i\epsilon)^a} \;=\; \frac{g_{\mu\nu}}{d} \int \frac{d\!\!\bar{}^d p}{i} \, \frac{p^2 \Gamma(a)}{(M^2 - p^2 - i\epsilon)^a} \;=\; - \, \frac{g_{\mu\nu}}{d} \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma\big(a - \frac{d}{2} - 1\big)}{(M^2 - i\epsilon)^{a - \frac{d}{2} - 1}}$$

[28] In principle, the dimension of Dirac γ-matrix is $2^{\frac{d}{2}}$ but this is an overall factor which we can keep equal to 4 in practical calculations.

we obtain

$$
\begin{aligned}
&\Pi^{\text{l.o.}}_{\mu\nu}(q)\\
=&\; -4e_0^2\frac{\tilde{\mu}^{4-d}}{(4\pi)^{\frac{d}{2}}}\int_0^1 d\alpha\,\Big\{\frac{\Gamma\big(2-\frac{d}{2}\big)}{(m^2-q^2\bar{\alpha}\alpha-i\epsilon)^{2-\frac{d}{2}}}[(m^2+q^2\bar{\alpha}\alpha)g_{\mu\nu}-2\bar{\alpha}\alpha q_\mu q_\nu]+\frac{\Gamma\big(1-\frac{d}{2}\big)g_{\mu\nu}}{(m^2-q^2\bar{\alpha}\alpha-i\epsilon)^{1-\frac{d}{2}}}\big(\frac{d}{2}-1\big)\Big\}\\
=&\; -4e_0^2\frac{\tilde{\mu}^{4-d}}{(4\pi)^{\frac{d}{2}}}\int_0^1 d\alpha\,\Big\{\frac{\Gamma\big(2-\frac{d}{2}\big)}{(m^2-q^2\bar{\alpha}\alpha-i\epsilon)^{2-\frac{d}{2}}}[(m^2+q^2\bar{\alpha}\alpha)g_{\mu\nu}-2\bar{\alpha}\alpha q_\mu q_\nu]-\;g_{\mu\nu}\frac{(m^2-q^2\bar{\alpha}\alpha)\Gamma\big(2-\frac{d}{2}\big)}{(m^2-q^2\bar{\alpha}\alpha-i\epsilon)^{2-\frac{d}{2}}}\\
=&\; 8(q_\mu q_\nu-q^2g_{\mu\nu})e_0^2\frac{\tilde{\mu}^{4-d}}{(4\pi)^{\frac{d}{2}}}\int_0^1 d\alpha\;\frac{\bar{\alpha}\alpha\Gamma\big(2-\frac{d}{2}\big)}{(m^2-q^2\bar{\alpha}\alpha-i\epsilon)^{2-\frac{d}{2}}} \qquad (22.43)
\end{aligned}
$$

The regularized polarization operator (in the $\overline{\text{MS}}$ scheme) is obtained by subtraction of the pole at $d=4$

$$
\begin{aligned}
\Pi^{\text{reg}}_{\mu\nu}(q) \;=&\; 8(q_\mu q_\nu-q^2g_{\mu\nu})e_0^2\frac{\tilde{\mu}^{4-d}}{(4\pi)^{\frac{d}{2}}}\int_0^1 d\alpha\;\frac{\bar{\alpha}\alpha\Gamma\big(2-\frac{d}{2}\big)}{(m^2-q^2\bar{\alpha}\alpha-i\epsilon)^{2-\frac{d}{2}}}\;-\;\frac{e_0^2}{12\pi^2}\frac{q_\mu q_\nu-q^2g_{\mu\nu}}{2-\frac{d}{2}}\\
=&\;(q_\mu q_\nu-q^2g_{\mu\nu})\frac{e_0^2}{2\pi^2}\int_0^1 d\alpha\;\bar{\alpha}\alpha\ln\frac{\mu^2}{m^2-q^2\bar{\alpha}\alpha-i\epsilon} \qquad (22.44)
\end{aligned}
$$

so

$$
\begin{aligned}
\Pi^{\text{reg}}_{\mu\nu}(q) \;&=\; (q_\mu q_\nu-q^2g_{\mu\nu})\Pi^{\text{reg}}(q^2)\\
\Pi^{\text{reg}}(q^2) \;&=\; \frac{e_0^2}{2\pi^2}\int_0^1 d\alpha\;\bar{\alpha}\alpha\ln\frac{\mu^2}{m^2-q^2\bar{\alpha}\alpha-i\epsilon}\;+\;O(e_0^4) \qquad (22.45)
\end{aligned}
$$

The “transverse” structure of polarization operator

$$
\Pi_{\mu\nu} \;=\; (q_\mu q_\nu-g_{\mu\nu}q^2)\Pi(q^2) \qquad (22.46)
$$

is actually due to the gauge invariance of QED.

Proof

1. Proof #1: from Ward identity $q^\mu\Pi_{\mu\nu}(q)\;=\;0\Rightarrow\Pi_{\mu\nu}(q)\;\sim\;(q_\mu q_\nu-g_{\mu\nu}q^2)$.
2. Proof # 2: from conservation of current $\partial^\mu\hat{j}_\mu(x)\;=\;0$:

Define

$$
\tilde{\Pi}_{\mu\nu}(q) \;=\; i\int d^4x\;e^{iqx}\langle\Omega|\mathrm{T}\{\hat{j}_\mu(x)\hat{j}_\nu(0)\}|\Omega\rangle \qquad (22.47)
$$

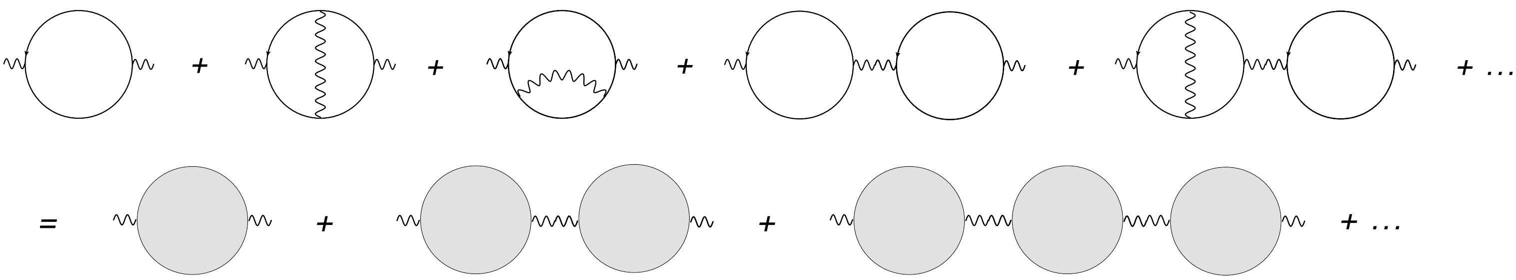
$$
\begin{aligned}
q^\mu \tilde{\Pi}_{\mu\nu} \;&=\; \int d^4x\ \big(\frac{\partial}{\partial x_\mu} e^{iqx}\big)\langle\Omega|\mathrm{T}\{\hat{j}_\mu(x)\hat{j}_\nu(0)\}|\Omega\rangle \;=\; -\int d^4x\ e^{iqx}\frac{\partial}{\partial x_\mu}\langle\Omega|\mathrm{T}\{\hat{j}_\mu(x)\hat{j}_\nu(0)\}|\Omega\rangle \\
&-\int d^4x\ e^{iqx}\frac{\partial}{\partial x_\mu}\Big[\theta(x_0)\langle\Omega|\hat{j}_\mu(x)\hat{j}_\nu(0)|\Omega\rangle + \theta(-x_0)\langle\Omega|\hat{j}_\nu(0)\hat{j}_\mu(x)|\Omega\rangle\Big] \\
&=\; -\int d^4x\ e^{iqx}\frac{\partial}{\partial x_0}\Big[\theta(x_0)\langle\Omega|\hat{j}_0(x)\hat{j}_\nu(0)|\Omega\rangle + \theta(-x_0)\langle\Omega|\hat{j}_\nu(0)\hat{j}_0(x)|\Omega\rangle\Big] \\
&-\int d^4x\ e^{iqx}\frac{\partial}{\partial x_i}\Big[\theta(x_0)\langle\Omega|\hat{j}_i(x)\hat{j}_\nu(0)|\Omega\rangle + \theta(-x_0)\langle\Omega|\hat{j}_\nu(0)\hat{j}_i(x)|\Omega\rangle\Big] \\
&=\; -\int d^3x\langle\Omega|[\hat{\bar{\psi}}\gamma_0\hat{\psi}(\vec{x}),\hat{\bar{\psi}}\gamma_\nu\hat{\psi}(\vec{0})]|\Omega\rangle - \int d^4x\ e^{iqx}\Big[\theta(x_0)\langle\Omega|\partial^\mu\hat{j}_\mu(x)\hat{j}_\nu(0)|\Omega\rangle + \theta(-x_0)\langle\Omega|\hat{j}_\nu(0)\partial^\mu\hat{j}_\mu(x)|\Omega\rangle\Big] \\
&=\; -\int d^3x\ e^{-i\vec{q}\cdot\vec{x}}\langle\Omega|\hat{\bar{\psi}}(\vec{x})\gamma_0\{\hat{\psi}(\vec{x}),\hat{\bar{\psi}}(\vec{0})\}\gamma_\nu\hat{\psi}(\vec{0}) - \hat{\bar{\psi}}(\vec{0})\gamma_\nu\{\hat{\psi}(\vec{0}),\hat{\bar{\psi}}(\vec{x})\}\gamma_0\hat{\psi}(\vec{x}) \\
&=\; -\langle\Omega|\hat{\bar{\psi}}(\vec{0})\gamma_\nu\hat{\psi}(\vec{0}) - \hat{\bar{\psi}}(\vec{0})\gamma_\nu\hat{\psi}(\vec{0})|\Omega\rangle \;=\; 0
\end{aligned}
\tag{22.48}
$$

and therefore

$$
\tilde{\Pi}_{\mu\nu}(q) \;=\; (q_\mu q_\nu - q^2 g_{\mu\nu})\tilde{\Pi}(q^2) \tag{22.49}
$$

It is easy to see that

$\tilde{\Pi}_{\mu\nu}(q) \;=$



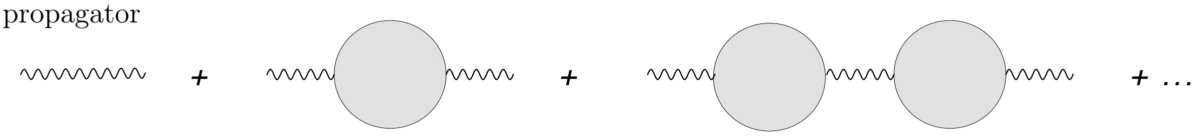
$$
\Rightarrow\ \tilde{\Pi}(q^2) \;=\; \frac{\Pi(q^2)}{1+\Pi(q^2)}
$$

where we used the formula

$$
(q_\mu q_\xi - q^2 g_{\mu\xi})(q^\xi q_\nu - q^2\delta^\xi_\nu) \;=\; -\,q^2(q_\mu q_\nu - q^2 g_{\mu\nu}) \tag{22.50}
$$

Photon propagator

$\mathcal{D}_{\mu\nu}(q) \quad =$



$$
\begin{aligned}
&=\; \frac{g_{\mu\nu}}{q^2} \;+\; \frac{g_{\mu\alpha}}{q^2}(q^\alpha q^\beta - q^2 g^{\alpha\beta})\tilde{\Pi}(q^2)\frac{g_{\beta\nu}}{q^2} \;=\; \frac{g_{\mu\nu}}{q^2} \;+\; (q^\mu q^\nu - q^2 g^{\mu\nu})\tilde{\Pi}(q^2) \\
&=\; \frac{g_{\mu\nu}}{q^2} \;+\; (q^\mu q^\nu - q^2 g^{\mu\nu})\frac{\Pi(q^2)}{1+\Pi(q^2)} \;=\; \frac{g_{\mu\nu}}{q^2[1+\Pi(q^2)]} + \frac{q_\mu q_\nu}{q^4}\frac{\Pi(q^2)}{1+\Pi(q^2)}
\end{aligned}
\tag{22.51}
$$

Due to Ward identity, the longitudinal part (second term in the r.h.s.) does not contribute to any of the S-matrix elements

$\Rightarrow$

we can use the exact photon propagator in the form

$$\mathcal{D}_{\mu\nu}(q) = \frac{g_{\mu\nu}}{q^2[1+\Pi(q^2)]} \tag{22.52}$$

As $q^2 \to 0$

$$\mathcal{D}_{\mu\nu} \stackrel{q^2\to 0}{\to} \frac{g_{\mu\nu}}{q^2}\frac{1}{1+\Pi(0)} \tag{22.53}$$

NB: $D_{\mu\nu}(q)$ has a pole at $q^2 = 0 \Rightarrow$ photon remains massless in all orders in perturbation theory. It is a consequence of gauge invariance.

After regularization

$$\begin{aligned}
\mathcal{D}^{\mathrm{reg}} &\stackrel{q^2\to 0}{\to} g_{\mu\nu}\frac{Z_3}{q^2} \\
Z_3 &= \frac{1}{1+\Pi^{\mathrm{reg}}(0)} = 1-\Pi^{\mathrm{reg}}(0) = 1-\frac{e_0^2}{12\pi^2}\ln\frac{\mu^2}{m^2} \\
\Rightarrow\ Z_3 &= 1-\frac{e_0^2}{12\pi^2}\ln\frac{\mu^2}{m^2} + O(e_0^4)
\end{aligned} \tag{22.54}$$

22.5 LSZ for Compton scattering

Similarly to the case of KG and Dirac fields (see Eqs. (10.31) and (22.34)) , $\sqrt{Z_3}$ is the coefficient of proportionality between Heisenberg operators and in-and out- operators

$$\hat{A}^i(x) \stackrel{t\to-\infty}{\to} Z_3^{\frac{1}{2}}\hat{A}^i_{\mathrm{in}}(x), \qquad \hat{A}^i(x) \stackrel{t\to\infty}{\to} Z_3^{\frac{1}{2}}\hat{A}^i_{\mathrm{out}}(x) \tag{22.55}$$

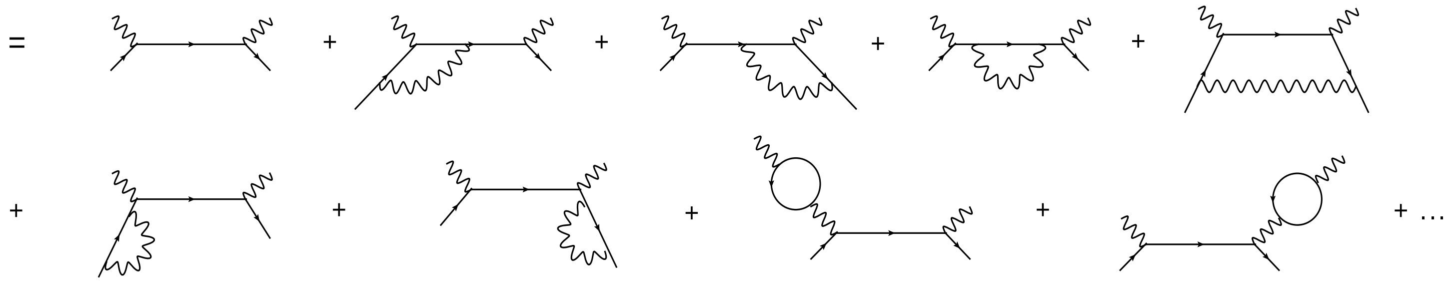
The LSZ formula (20.73) for Compton scattering (with Z-factors taken into account) has the form

$$\begin{aligned}
&{}_{\mathrm{out}}\langle p_2,s_2;k_2,\lambda_2|p_1,s_1;k_1,\lambda_1\rangle_{\mathrm{in}} \\
&= \lim_{p_i^2\to m^2,k_i^2\to 0} k_1^2k_2^2 e_\mu^{\lambda_2}(\vec{k}_2)e_\nu^{\lambda_1}(\vec{k}_1)\int d^4x_1d^4y_1d^4x_2d^4y_2\ e^{-ip_1x_1-ik_1z_1+ip_2x_2+ik_2z_2} \\
&\times \frac{1}{Z_2Z_3}\langle\Omega|\mathrm{T}\{\bar{u}_\xi(p_2,s_2)(m-\not{p}_2)_{\xi\eta}\hat{\psi}_\eta(x_2)\hat{A}^\nu(z_2)\hat{\bar{\psi}}_\lambda(x_1)(m-\not{p}_1)_{\lambda\rho}u_\rho(p_1,s_1)\hat{A}^\mu(z_1)\}|\Omega\rangle \\
&= \frac{1}{Z_2Z_3}\lim_{p_i^2\to m^2,k_i^2\to 0} k_1^2k_2^2 e_\mu^{\lambda_2}(\vec{k}_2)e_\nu^{\lambda_1}(\vec{k}_1)\bar{u}_\xi(p_2,s_2)(m-\not{p}_2)_{\xi\eta}(m-\not{p}_1)_{\lambda\rho}u_\rho(p_1,r_1)G^{\mu\nu}_{\eta\lambda}(p_1,k_1\to p_2,k_2)
\end{aligned} \tag{22.56}$$

where

$$\begin{aligned}
&G^{\mu\nu}_{\eta\lambda}(p_1,k_1\to p_2,k_2) \\
&= \int d^4x_1d^4z_1d^4x_2d^4z_2\ e^{-ip_1x_1-ik_1z_1+ip_2x_2+ik_2z_2}\frac{\langle 0|\mathrm{T}\{\hat{\psi}_\eta(x_2)\hat{\bar{\psi}}_\lambda(x_1)\hat{A}^\nu(z_2)\hat{A}^\mu(z_1)e^{i\int dt\ \hat{L}_{\mathrm{int}}(t)}\}|0\rangle}{\langle 0|\mathrm{T}\{e^{i\int dt\ \hat{L}_{\mathrm{int}}(t)}\}|0\rangle}
\end{aligned} \tag{22.57}$$

$$= \text{ sum of all Feynman diagrams with two electron and two photon tails} \tag{22.58}$$



Similarly to the case of electron scattering (see Fig. 27), one can sum up self-energy insertions and write the Compton amplitude as a product of four exact propagators and the four-particle 1PI Green function:

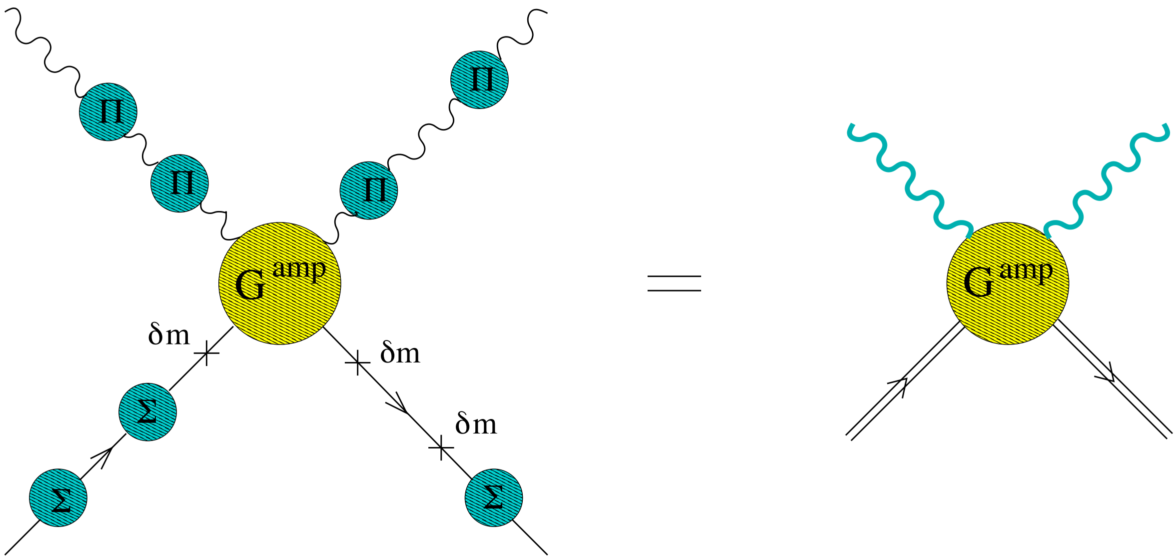


Figure 29. Compton scattering. Thick wavy line denotes the exact photon propagator $\mathcal{D}_{\mu\nu}(p)$ and double line the exact Dirac propagator $\mathcal{G}(p)$

Here G^{amp} is a sum of one-particle irreducible diagrams with two electron and two photon tails and $\mathcal{D}(p)$ is an exact photon propagator (22.4). We get

$$G^{\mu\nu}_{\eta\lambda}(p_1, k_1 \to p_2, k_2) \;=\; \mathcal{D}^{\nu\nu'}(k_2)\mathcal{D}^{\mu\mu'}(k_1)\mathcal{G}_{\eta\eta'}(p_2)(\mathcal{G}^{\text{amp}})^{\mu'\nu'}_{\eta'\lambda'}(p_1, p_1' \to p_2, p_2')\mathcal{G}_{\lambda'\lambda}(p_1)$$

and therefore

$$\begin{aligned}
&\lim_{p_i^2\to m^2, k^2\to 0} k_1^2 k_2^2 (m - \not{p}_2)_{\xi\eta} G^{\mu\nu}_{\eta\lambda}(p_1, k_1 \to p_2, k_2)(m - \not{p}_1)_{\lambda\rho} \\
&= \lim_{p_i^2\to m^2, k_i^2\to 0} k_1^2 \mathcal{D}_{\mu\mu'}(k_1) k_2^2 \mathcal{D}_{\nu\nu'}(k_2)\left[(m - \not{p}_2)\mathcal{G}(p_2)\right]_{\xi\eta'}\left[\mathcal{G}(p_1)(m - \not{p}_1)\right]_{\lambda'\rho}(G^{\text{amp}})^{\mu'\nu'}_{\eta'\lambda'}(p_1, p_1' \to p_2, p_2') \\
&\quad = Z_2^2 Z_3^2 (G^{\text{amp}})^{\mu\nu}_{\xi\rho}(p_1, k_1 \to p_2, k_2)\Big|_{p_i^2 = m^2, k_i^2 = 0} \qquad (22.59)
\end{aligned}$$

where we used Eqs. (22.31) and (23.27). The matrix element of S-matrix (22.56) takes the form

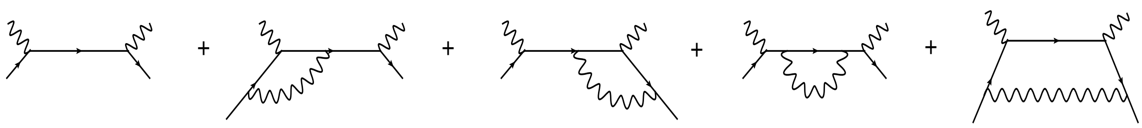
$$S(p_1,s_1;k_1,\lambda_1\to p_2,s_2;k_2,\lambda_2) \;=\; Z_2Z_3 e^{\lambda_1}_\mu(k_1)e^{\lambda_2}_\nu(k_2)\bar u(p_2,s_2)_\xi (G^{\rm amp})^{\mu\nu}_{\xi\eta}(p_1,k_1\to p_2,k_2)\Big|_{p_i^2=m^2,k_i^2=0}u_\eta(p_1,s_1) \tag{22.60}$$

or equivalently (cf. Eq. (22.64))

$$\mathcal{M}(p_1,s_1;p_1',s_1'\to p_2,s_2;p_2',s_2') \;=\; Z_2Z_3\bar u(p_2,s_2)_\xi \bar u(p_2',s_2')_{\xi'}\mathcal{G}^{\rm amp}_{\xi\xi'\eta\eta'}(p_1,p_1'\to p_2,p_2')\Big|_{p_i^2=m^2}u_\eta(p_1,s_1)u_{\eta'}(p_1',s_1') \tag{22.61}$$

where

$\mathcal{G}^{\rm amp}$ =



$+$ diagrams with exchange of incoming and outgoing photons.

22.6 Physical charge

It is natural to define physical charge of the electron as the coefficient in in Coulomb potential

$$V(r) \;=\; \frac{e^2}{4\pi r} \tag{22.62}$$

at large distances $r\to\infty$ (which corresponds to the limit $\vec q\to 0$ in the Fourier transform of Coulomb potential $V(\vec q) \;=\; \frac{e^2}{\vec q^2}$.)

The set of non-relativistic diagrams for electron-electron scattering is given in Fig. 30

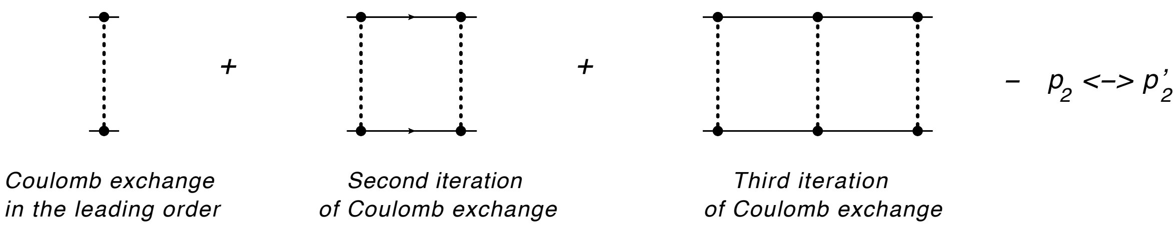
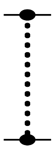


Figure 30. Non-relativistic diagrams for *ee* scattering. Dashed line denotes Coulomb potential (22.62)

Here 

$$= \; -\frac{\mathbf{e^2}}{\mathbf{\vec q^2}}\mathbf{4m^2}\delta_{\mathbf{s_1s_2}}\delta_{\mathbf{s_1's_2'}} \;=\; \text{Fourier transform of the Coulomb potential (22.62)} \tag{22.63}$$

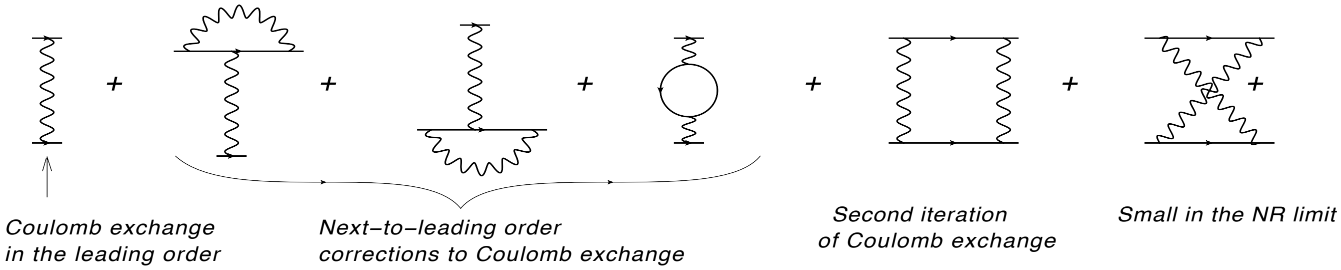
To express this physical charge in terms of the parameters in the Lagrangian we should consider the electron-electron scattering in QED, take the non-relativistic limit $\vec p_i\ll m$ and the limit $\vec{q} \to 0$ to get the large-distance behavior of Coulomb potential (in the c.m. frame $q = (0, \vec{q})$).

The matrix element of transition matrix (22.64) has the form

$$\mathcal{M}(p_1, s_1; p_1', s_1' \to p_2, s_2; p_2', s_2') = Z_2^2 \bar{u}(p_2, s_2)_\xi \bar{u}(p_2', s_2')_{\xi'} \mathcal{G}^{\text{amp}}_{\xi\xi'\eta\eta'}(p_1, p_1' \to p_2, p_2')\Big|_{p_i^2 = m^2} u_\eta(p_1, s_1) u_{\eta'}(p_1', s_1') \tag{22.64}$$

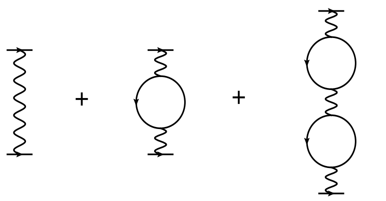
where in the non-relativistic limit

$\mathcal{G}^{\text{amp}} =$



- diagrams with exchange $p_2 \leftrightarrow p_2'$ in the final state.

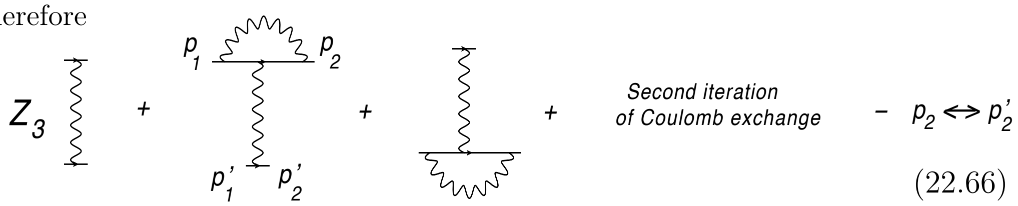
In $\vec{q} \to 0$ limit the sum of the corrections to Coulomb exchange gives free propagator multiplied by Z_3:



$$+ \ldots = \gamma^\mu \frac{\mathcal{D}_{\mu\nu}(q)}{q^2} \gamma_\nu \quad \overset{q^2 \to 0}{\to} \quad \gamma_\mu \frac{g_{\mu\nu} Z_3}{q^2} \gamma_\nu \tag{22.65}$$

and therefore

$\mathcal{G}^{\text{amp}} =$

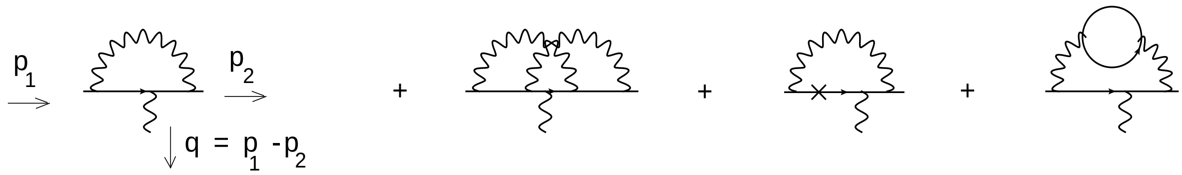


(22.66)

22.6.1 The exact QED vertex and Z_1

Definition:

$$\Lambda^\mu(p_1, p_2) = \tag{22.67}$$



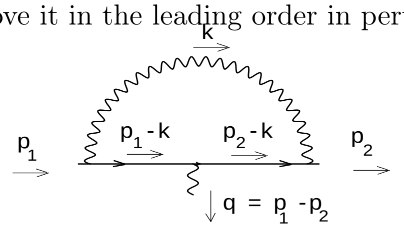
$= $ sum of all 3-point 1PI diagrams without γ^μ.

Property (another Ward identity)

$$\Lambda_\mu(p,p) \;=\; -\frac{\partial}{\partial p^\mu}\Sigma(p) \tag{22.68}$$

Let us prove it in the leading order in perturbation theory.

$\Lambda^\mu(p_1,p_2) \;=$



$$= e_0^2\int\frac{\text{đ}k}{i}\gamma_\alpha\frac{m+\slashed{p}_1-\slashed{k}}{m^2-(p_1-k)^2-i\epsilon}\gamma_\mu\frac{m+\slashed{p}_2-\slashed{k}}{m^2-(p_2-k)^2-i\epsilon}\gamma_\beta\frac{g^{\alpha\beta}}{k^2} \tag{22.69}$$

$$\Rightarrow \Lambda^\mu(p,p) \;=\; e_0^2\int\frac{\text{đ}k}{i}\gamma_\alpha\frac{m+\slashed{p}-\slashed{k}}{m^2-(p-k)^2-i\epsilon}\gamma_\mu\frac{m+\slashed{p}-\slashed{k}}{m^2-(p-k)^2-i\epsilon}\gamma_\beta\frac{g^{\alpha\beta}}{k^2+i\epsilon} \tag{22.70}$$

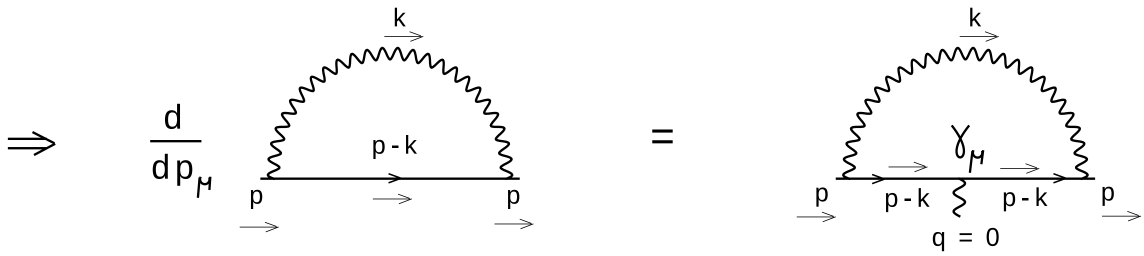
On the other hand

$$-\frac{\partial}{\partial p_\mu}\Sigma(p) \;=\; \frac{\partial}{\partial p_\mu}e_0^2\int\frac{\text{đ}k}{i}\gamma_\alpha\frac{m+\slashed{p}-\slashed{k}}{m^2-(p-k)^2-i\epsilon}\gamma_\beta\frac{g^{\alpha\beta}}{k^2+i\epsilon} \tag{22.71}$$

$$= e_0^2\int\frac{\text{đ}k}{i}\gamma_\alpha\Big[\frac{\gamma_\mu}{m^2-(p-k)^2-i\epsilon}+2(p-k)_\mu\frac{m+\slashed{p}-\slashed{k}}{[m^2-(p-k)^2-i\epsilon]^2}\Big]\gamma_\beta\frac{g^{\alpha\beta}}{k^2+i\epsilon}$$

$$= e_0^2\int\frac{\text{đ}k}{i}\gamma_\alpha\frac{\gamma_\mu(m^2-(p-k)^2)+2(p-k)_\mu(m+\slashed{p}-\slashed{k})}{[m^2-(p-k)^2-i\epsilon]^2}\gamma_\beta\frac{g^{\alpha\beta}}{k^2+i\epsilon} \;=\; \text{r.h.s. of Eq. (22.70)}$$

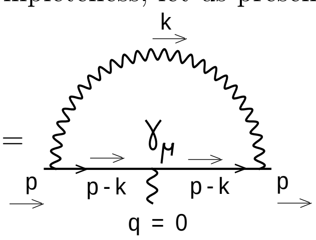
Graphically



Using this property, the Ward identity (22.68) can be easily proved in an arbitrary order in perturbation theory.

For completeness, let us present the explicit form of $\Lambda(p,p)$ in the leading order

$\Lambda_\mu^{\text{reg}}(p,p) \;=$

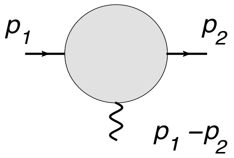


$= \qquad$ (22.72)

$$= \frac{e_0^2}{8\pi^2}\gamma_\mu\Big[\int_0^1 d\alpha\;\bar{\alpha}\ln\frac{\mu^2}{m^2\alpha-p^2\bar{\alpha}\alpha-i\epsilon}\Big] - \frac{e_0^2}{8\pi^2}p_\mu\int_0^1 d\alpha\frac{\bar{\alpha}\alpha(4m-2\slashed{p}\bar{\alpha})}{m^2\alpha-p^2\alpha\bar{\alpha}-i\epsilon} \;+\; O(2-\frac{d}{2})$$

22.6.2 Physical electric charge

From Eq. (22.79) we see that in the limit $q \to 0$ we need the contribution of the QED vertex $\Gamma_\mu(p_1, p_1)$. The QED vertex function is defined as follows:

$\Gamma_\mu(p_1, p_2) \ =$ 

$= \text{sum of all } 3-\text{point 1PI diagrams} \ = \ \gamma_\mu + \Lambda_\mu(p_1, p_2)$ (22.73)

The corresponding term in the $\mathcal{M}$-matrix (22.64) at $q \to 0$ is proportional to

$$\frac{1}{q^2}\bar{u}(p_1, s_2)\Gamma_\mu(p_1, p_1)u(p_1, s_1)\bar{u}(p'_1, s'_2)\gamma^\mu u(p_1, s_1) + \frac{1}{q^2}\bar{u}(p_1, s_2)\gamma_\mu(p_1, p_1)u(p_1, s_1)\bar{u}(p'_1, s'_2)\Gamma^\mu u(p_1, s_1) \tag{22.74}$$

From "Ward identity #2" (22.68) we get

$$\Gamma^{\text{reg}}_\mu(p, p) \ = \ \gamma_\mu + \Lambda^{\text{reg}}_\mu(p, p) \ = \ \gamma_\mu \ - \ \frac{\partial}{\partial p^\mu}\Sigma^{\text{reg}}(p) \tag{22.75}$$

$$= \ \gamma_\mu \ - \ \frac{\partial}{\partial p^\mu}\big[m\Sigma^{\text{reg}}_1(p^2) - \not{p}\Sigma^{\text{reg}}_2(p^2)\big] \ = \ \gamma_\mu\big[1 + \Sigma^{\text{reg}}_2(p^2)\big] - 2p_\mu\big[m\frac{\partial}{\partial p^2}\Sigma^{\text{reg}}_1(p^2) - \not{p}\frac{\partial}{\partial p^2}\Sigma^{\text{reg}}_2(p^2)\big]$$

and therefore

$$\bar{u}(p, r)\Gamma^{\text{reg}}_\mu(p, p)u(p, s) \ = \ \big[1 + \Sigma^{\text{reg}}_2(p^2)\big]\bar{u}(p, r)\gamma_\mu u(p, s) - 2p_\mu\bar{u}(p, r)\big[m\frac{\partial}{\partial p^2}\Sigma^{\text{reg}}_1(p^2) - \not{p}\frac{\partial}{\partial p^2}\Sigma^{\text{reg}}_2(p^2)\big]u(p, s)$$
$$= \ \big[1 + \Sigma^{\text{reg}}_2(p^2)\big]\bar{u}(p, r)\gamma_\mu u(p, s) - 2mp_\mu\big[\frac{\partial}{\partial p^2}\Sigma^{\text{reg}}_1(p^2) - \frac{\partial}{\partial p^2}\Sigma^{\text{reg}}_2(p^2)\big]\bar{u}(p, r)u(p, s)$$
$$= \ 2p_\mu\big[1 + \Sigma^{\text{reg}}_2(p^2)\big]\delta_{rs} - 4m^2p_\mu\big[\frac{\partial}{\partial p^2}\Sigma^{\text{reg}}_1(p^2) - \frac{\partial}{\partial p^2}\Sigma^{\text{reg}}_2(p^2)\big]\delta_{rs}$$
$$= \ 2p_\mu\delta_{ss'}\Big(\big[1 + \Sigma^{\text{reg}}_2(p^2)\big] - 4m^2\big[\frac{\partial}{\partial p^2}\Sigma^{\text{reg}}_1(p^2) - \frac{\partial}{\partial p^2}\Sigma^{\text{reg}}_2(p^2)\big]\Big) \ = \ 2p_\mu\delta_{rs}Z_2^{-1} \tag{22.76}$$

where we used Eq. (22.31) and normalization of spinors (25.22).

Historically, the coefficient of proportionality between the exact vertex Γ_μ on the mass shell and bare vertex γ_μ was called Z_1^{-1} so we have proved that

$$u(p, s')\Gamma^{\text{reg}}_\mu(p, p)u(p, s) \ \stackrel{p^2=m^2}{=} \ Z_1^{-1}u(p, s')\gamma_\mu u(p, s) \tag{22.77}$$

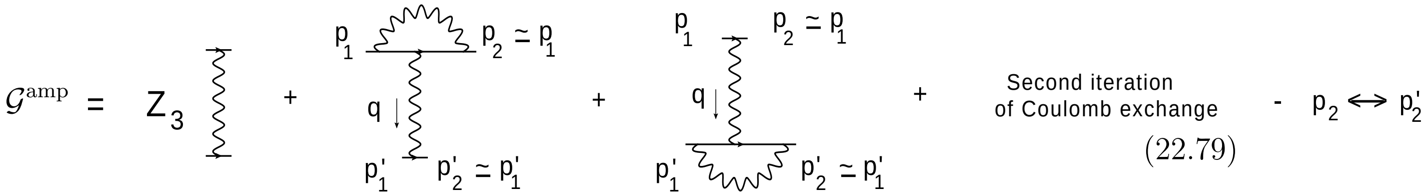
and therefore we've got the result

$$Z_1 \ = \ Z_2 \tag{22.78}$$

which is also called "Ward identity" [29].

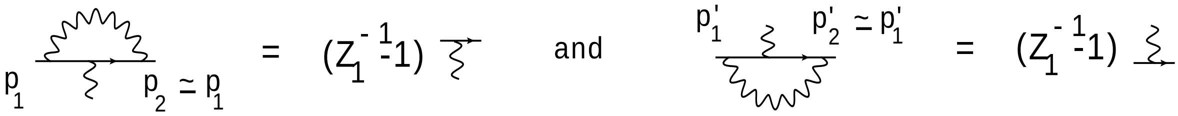
Now we are in a position to assemble the final one-loop result for the Coulomb potential. The matrix element of $\mathcal{M}$-matrix in the non-relativistic case is given by Eq. (22.74). In the limit $q = p_1 - p_2 \to 0$

[29] We have obtained this result keeping in mind the leading-order diagrams in Eq. (22.79) but the derivation itself is valid in all orders in perturbation theory.

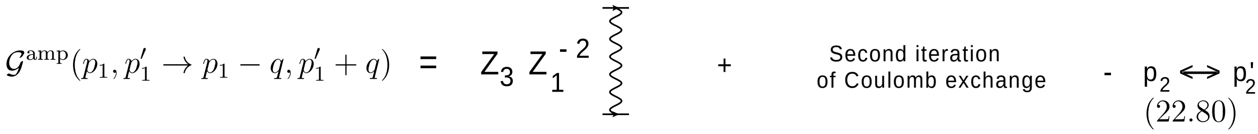


$$(22.79)$$

Since at $q^2 \to 0$

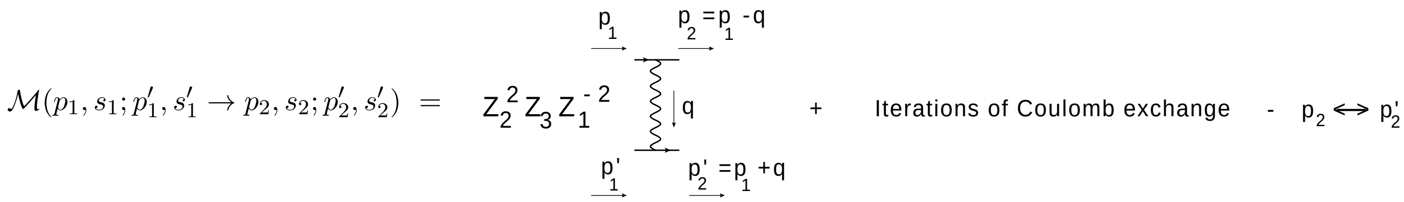


the equation (22.79) takes the form



$$(22.80)$$

Thus, the matrix element of $\mathcal{M}$-matrix of electron-electron scattering in the nonrelativistic limit and at $q \to 0$ takes the form



$$\begin{aligned}
&\overset{q\to 0}{=} \quad Z_2^2 Z_3 Z_1^{-2}\bar{u}(p_2, s_2)\gamma_\mu u(p_1, s_1)\bar{u}(p'_2, s'_2)\gamma^\mu u(p'_1, s'_1)\frac{e_0^2}{q^2} \;-\; (p_2 \leftrightarrow p'_2) \;+\; \text{iterations} \\
&= \; -4m^2\frac{e_0^2}{\vec{q}^2}Z_2^2 Z_3 Z_1^{-2} \;-\; (\vec{q} \leftrightarrow \vec{p}_1 - \vec{p}'_2) \;+\; \text{iterations} \qquad (22.81)
\end{aligned}$$

Comparing this to the non-relativistic expression for Coulomb potential (22.63) we see that the physical charge of the electron is expressed in terms of the parameters of the Lagrangian as

$$e^2 \;=\; Z_2^2 Z_3 Z_1^{-2} e_0^2 \;=\; Z_3 e_0^2 \qquad (22.82)$$

where we used "Ward identity # 3" $Z_1 = Z_2$, see Eq. (22.78).

In the leading order in perturbation theory Z_3 is given by Eq. (23.27) so

$$e^2 \;=\; Z_3 e_0^2 \;=\; e_0^2\big(1 - \frac{e_0^2}{12\pi^2}\ln\frac{\mu^2}{m^2}\big) + O(e_0^4) \qquad (22.83)$$

If we now express e_0^2 in terms of e^2

$$e_0^2 \;=\; e^2\big(1 + \frac{e^2}{12\pi^2}\ln\frac{\mu^2}{m^2}\big) + O(e^4) \qquad (22.84)$$

we will discover that the dependence on the UV cutoff μ disappears from physical cross sections. Let us illustrate that on the example of Compton scattering.

22.6.3 Compton scattering revisited

We rewrite the $\mathcal{M}$-matrix for Compton scattering

$$\mathcal{M}(p_1,s_1;p_1',s_1'\to p_2,s_2;p_2',s_2') \;=\; Z_2Z_3\bar{u}(p_2,s_2)_\xi\bar{u}(p_2',s_2')_{\xi'}\mathcal{G}^{\rm amp}_{\xi\xi'\eta\eta'}(p_1,p_1'\to p_2,p_2')\Big|_{p_i^2=m^2}u_\eta(p_1,s_1)u_{\eta'}(p_1',s_1') \tag{22.85}$$

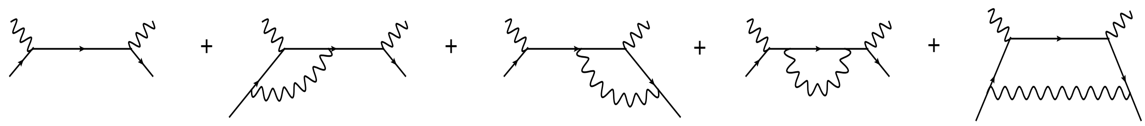
in terms of physical charge e

$$e_0^2 \;=\; Z_1^2Z_2^{-2}Z_3^{-1}e^2 \tag{22.86}$$

and get

$$\mathcal{M}(p_1,s_1;p_1',s_1'\to p_2,s_2;p_2',s_2') \;=\; Z_1^2Z_2^{-1}\bar{u}(p_2,s_2)_\xi\bar{u}(p_2',s_2')_{\xi'}\mathcal{G}^{\rm amp}_{\xi\xi'\eta\eta'}(p_1,p_1'\to p_2,p_2')\Big|_{p_i^2=m^2}u_\eta(p_1,s_1)u_{\eta'}(p_1',s_1') \tag{22.87}$$

where G^{amp} is given by the same set of diagrams but with physical charge e in each vertex.

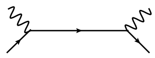
$\mathcal{G}^{\rm amp} \;=$ 

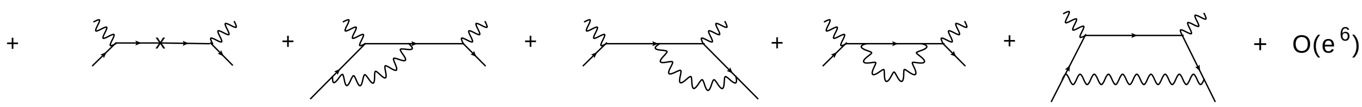
$+$ diagrams with exchange of incoming and outgoing photons.

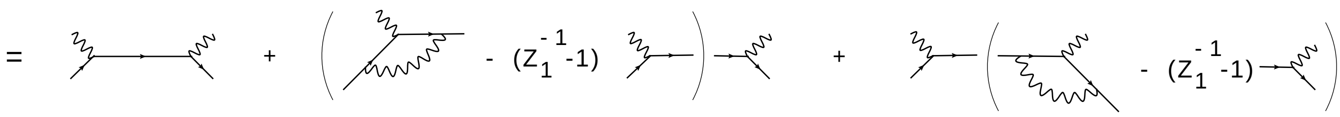
Now, rewriting $Z_1^2Z_2^{-1}$ as

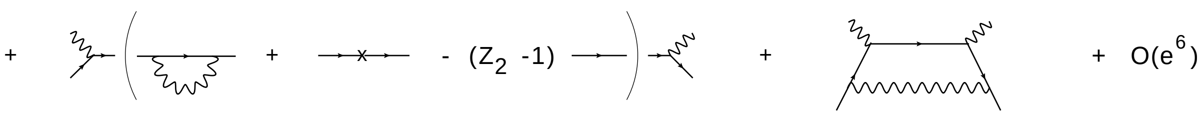
$$Z_1^2Z_2^{-1} \;=\; \frac{1}{[1+(Z_1^{-1}-1)]^2}\frac{1}{1+(Z_2-1)} \;\simeq\; 1+2(Z_1^{-1}-1)-(Z_2-1)\;+\;O(e^4) \tag{22.88}$$

we can redraw the diagrams for $\mathcal{M}$-matrix as follows (for simplicity, we do not display diagrams with exchange of incoming and outgoing photons)

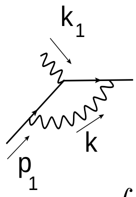
$\mathcal{M}(p_1,s_1;p_1',s_1'\to p_2,s_2;p_2',s_2') \;=\;$ $Z_1^2Z_2^{-1}$  (22.89)

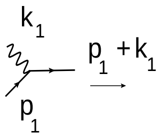






It is easy to see that the expression in each parentheses is UV finite (and the last diagram is finite by itself). Indeed, consider for example



$-\ (Z_1^{-1}-1)$


$$= e^3 \int \frac{d\!\!{}^-{}^4 k}{k^2+i\epsilon}\, \gamma_\alpha \frac{m+\not p_1-\not k}{m^2-(p_1-k)^2-i\epsilon}\gamma_\mu \frac{m+\not p_1+\not k_1-\not k}{m^2-(p_1+k_1-k)^2-i\epsilon}\gamma^\alpha \;-\; (Z_1^{-1}-1)e\gamma_\mu \tag{22.90}$$

$$= e^3 \int \frac{d\!\!{}^-{}^4 k}{k^2+i\epsilon}\, \gamma_\alpha \frac{m+\not p_1-\not k}{m^2-(p_1-k)^2-i\epsilon}\gamma_\mu \frac{m+\not p_1+\not k_1-\not k}{m^2-(p_1+k_1-k)^2-i\epsilon}\gamma^\alpha$$

$$- e^3 \int \frac{d\!\!{}^-{}^4 k}{k^2+i\epsilon}\, \gamma_\alpha \frac{m+\not p_1-\not k}{m^2-(p_1-k)^2-i\epsilon}\gamma_\mu \frac{m+\not p_1+\not k_1-\not k}{m^2-(p_1+k_1-k)^2-i\epsilon}\gamma^\alpha \Big|_{p_1^2=m^2,k_1^2\to 0}$$

where we have used the definition of Z_1 factor as a vertex on the mass shell and at zero momentum transfer (22.77). It is clear now that the UV divergence at $k\to\infty$, present in each of the integrals in the r.h.s. of this equation, cancels in their sum. Similarly one can demonstrate that the UV divergence in the self-energy diagram in the last line in r.h.s. of Eq. (22.89) is canceled by (Z_2-1) subtraction:

$$\begin{aligned}
\frac{1}{m-\not p}\big[\delta m - \Sigma^{\rm reg}(p)\big] - (Z_2-1) \;&=\; \frac{e_0^2}{8\pi^2}\frac{m+\not p}{m^2-p^2}\Big[\frac{3}{2}m\ln\frac{\mu^2}{m^2} + 2m + m - \frac{\not p}{2} \\
&- \int_0^1 d\alpha(2m-\not p\bar\alpha)\ln\frac{\mu^2}{m^2\alpha-p^2\alpha\bar\alpha-i\epsilon}\Big] + \frac{e_0^2}{8\pi^2}\Big[\int_0^1 d\alpha\ \bar\alpha\ln\frac{\mu^2}{m^2\alpha^2} - \frac{1}{2} - 2\int_0^1 d\alpha\ \big(\frac{1}{\alpha}-\alpha\big)\Big] \\
&= \;-\frac{e_0^2}{4\pi^2}\int_0^1 d\alpha\ \big(\frac{1}{\alpha}-\alpha\big) + \frac{e_0^2}{8\pi^2}\frac{m+\not p}{m^2-p^2}\int_0^1 d\alpha(2m-\not p\bar\alpha)\ln\big(1+\frac{\bar\alpha(m^2-p^2)}{\alpha m^2}\big)\Big]
\end{aligned} \tag{22.91}$$

where we used Eqs. (22.25), (22.33) and (22.26). We see now that the r.h.s. of the above equation does not depend on the UV cutoff μ (and does not have a pole as $p^2\to m^2$ which means that our δm is correct at this order of perturbation theory).

The general statement is that in each order in perturbation theory the matrix elements of $\mathcal{M}$-matrix become UV-finite after re-expressing e_0 in terms of physical charge according to Eq. (22.86). This property is called *renormalizability* of the theory. Most of the quantum field theories describing Nature are renormalizable (e.g. all ingredients of the Standard model have this property). A well-known exception is a theory of gravity which, being quantized in a usual way, leads to the non-renormalizable theory.

22.7 Effective coupling constant

Let us calculate physical electric charge for a certain heavy fermion like muon. If the mass of this fermion is M we get from eq. (22.83)

$$e_M^2 \;=\; e_0^2\big(1-\frac{e_0^2}{12\pi^2}\ln\frac{\mu^2}{M^2}\big) + O(e_0^4) \tag{22.92}$$

It is more revealing to express the charge of heavy fermion in terms of physical charge of the electron rather than e_0. We get from Eqs. (22.83) and (22.93)

$$e_M^2 \;=\; e^2\big(1+\frac{e^2}{12\pi^2}\ln\frac{\mu^2}{m^2}\big)\big(1-\frac{e^2}{12\pi^2}\ln\frac{\mu^2}{M^2}\big) \;=\; e^2\big(1+\frac{e^2}{12\pi^2}\ln\frac{M^2}{m^2}\big) + O(e^4) \tag{22.93}$$

It can be demonstrated that a more accurate version of this equation looks like ($e(m) \equiv e$ - charge of the electron)

$$e^2(M) \;=\; \frac{e^2(m)}{1 - \frac{e^2(m)}{12\pi^2}\ln\frac{M^2}{m^2}} \;+\; O(e^4)(m)) \tag{22.94}$$

We see that the strength of the electromagnetic interaction increases with the mass of the interacting particles. This property is characteristic of all quantum field theories except non-Abelian gauge theories like QCD or a theory of weak interactions. In such theories there is an asymptitic freedom - the strength of the interaction decreases with the mass of the particle. In QCD, for example,

$$g^2(M) \;=\; \frac{g^2(m)}{1 + b\frac{g^2(m)}{16\pi^2}\ln\frac{M^2}{m^2} + O(g^4)} \tag{22.95}$$

where $b = 11 - \frac{2}{3}n_f$ ($n_f \equiv$ "number of active quarks" which is 3 for JLab energies). Unfortunately, before discussing QCD and asymptotic freedom in detail we need to elaborate on the renormalization program.

23 Renormalization in QED in terms of "renormalized fields"

There is technically more convenient renormalization program in QFT formulated in terms of so-called "renormalized fields" whose propagators have poles at physical masses with unit residues. (The relevant discussion can be found in Chapter 10 of *Peskin & Schroeder*).

As a starting point, we rewrite the QED Lagrangian changing the notations $\hat{\psi}$ to $\hat{\psi}_{(0)}$ and $\hat{A}_\mu$ to $\hat{A}^{(0)}_\mu$

$$\hat{\mathcal{L}} \;=\; -\frac{1}{4}\hat{F}^{(0)}_{\mu\nu}\hat{F}^{(0)\mu\nu} + \hat{\bar{\psi}}_{(0)}(i\partial\!\!\!/ - m_0 + e_0\hat{A\!\!\!/}^{(0)})\hat{\psi}_{(0)} \tag{23.1}$$

(from now on the fields in the Lagrangian (23.1) $\hat{\psi}_{(0)}$ and $\hat{A}_\mu$ to $\hat{A}^{(0)}_\mu$ will be called "unrenormalized fields"). Next, we rewrite this Lagrangian in terms of "renormalized fields" $\hat{\psi}$ and $\hat{A}_\mu$ defined as

$$\hat{\psi}(x) \;\stackrel{\rm def}{\equiv}\; Z_2^{-\frac{1}{2}}\hat{\psi}_{(0)}(x), \qquad \hat{A}^\mu(x) \;\stackrel{\rm def}{\equiv}\; Z_3^{-\frac{1}{2}}\hat{A}^\mu_{(0)}(x) \tag{23.2}$$

(the constants Z_2 and Z_3 will be specified below)

$$\hat{\mathcal{L}} \;=\; -\frac{1}{4}Z_3\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} + Z_2\hat{\bar{\psi}}(i\partial\!\!\!/ - m_0 + e_0 Z_3^{\frac{1}{2}}\hat{A\!\!\!/})\hat{\psi} \tag{23.3}$$

At the next step we rewrite this Lagrangian in terms of physical mass of the electron m and physical charge e

$$\hat{\mathcal{L}} \;=\; -\frac{1}{4}Z_3\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} + Z_2\hat{\bar{\psi}}(i\partial\!\!\!/ - m)\hat{\psi} + Z_1 e\hat{\bar{\psi}}\hat{A\!\!\!/}\hat{\psi} - \delta m\hat{\bar{\psi}}\hat{\psi} \tag{23.4}$$

where $\delta m \equiv m - m_0$ and

$$Z_1 \equiv \frac{e_0}{e}Z_2 Z_3^{\frac{1}{2}} \tag{23.5}$$

Finally, we split the Lagrangian (23.4) into three parts: free Lagrangian (with physical mass), interaction Lagrangian (with physical charge) and counterterm Lagrangian

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_{\rm int} + \mathcal{L}_{\rm ct} \\
\mathcal{L}_0 &= -\frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} + \hat{\bar{\psi}}(i\partial\!\!\!/ - m)\hat{\psi}, \qquad \mathcal{L}_{\rm int} = e\hat{\bar{\psi}}\hat{A\!\!\!/}\hat{\psi}, \\
\mathcal{L}_{\rm ct} &= \frac{1}{4}(Z_3-1)\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} + (Z_2-1)\hat{\bar{\psi}}(i\partial\!\!\!/ - m)\hat{\psi} + (Z_1-1)e\hat{\bar{\psi}}\hat{A\!\!\!/}\hat{\psi},
\end{aligned} \tag{23.6}
$$

Now we calculate Feynman diagrams with this Lagrangian using some suitable cutoff for UV divergencies (usually dimensional regularization) and adjust parameters Z_1, Z_2, and Z_3 in any order of perturbation theory in such a way that the terms coming from the counterterm Lagrangian cancel the divergencies of Feynman diagrams obtained with usual Lagrangian $\hat{\mathcal{L}}_0 + \hat{\mathcal{L}}_{\rm int}$. [30]

Note that we build the interaction representation using $\mathcal{L}_0$ so for the purpose of obtaining the set of Feynman diagrams both $\hat{\mathcal{L}}_{\rm int}$ and $\hat{\mathcal{L}}_{\rm ct}$ constitute an interaction terms so, for example, for Compton scattering amplitude one obtains

$$
\langle\Omega|\mathrm{T}\{\hat{\psi}_\eta(x_2)\hat{\bar{\psi}}_\lambda(x_1)\hat{A}^\nu(z_2)\hat{A}^\mu(z_1)\}|\Omega\rangle = \lim_{d\to 4}\frac{\langle 0|\mathrm{T}\{\hat{\psi}_\eta(x_2)\hat{\bar{\psi}}_\lambda(x_1)\hat{A}^\nu(z_2)\hat{A}^\mu(z_1)e^{i\int d^dz[\hat{\mathcal{L}}_{\rm int}(x)+\hat{\mathcal{L}}_{\rm ct}(x)]}\}|0\rangle}{\langle 0|\mathrm{T}\{e^{i\int d^dz[\hat{\mathcal{L}}_{\rm int}(x)+\hat{\mathcal{L}}_{\rm ct}(x)]}\}|0\rangle} \tag{23.7}
$$

where all the fields in the r.h.s. are in the interaction representation. The ladder operators are defined in a usual way as coefficients of the expansion of fields in the interaction representation in plane waves so the commutation relations between ladder operators will be the same. Thus, the Feynman rules for QED are as given on p. 145 (with physical electron mass m and physical charge e) plus new interaction vertices coming from the counterterm Lagrangian

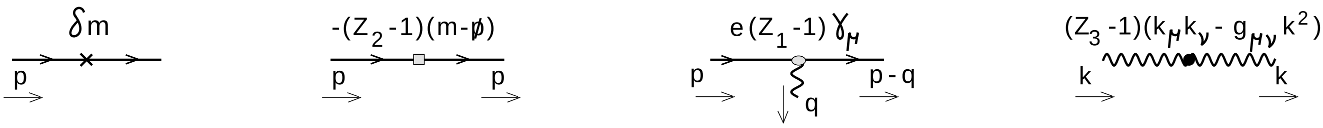


Figure 31. Vertices coming from the counterterm Lagrangian.

Now we shall specify the equations for counterterm coefficients Z_1, Z_2 and Z_3. As we mentioned above, Z_2 and Z_3 are to be determined from the conditions that the residues at the poles of exact Dirac and photon propagators are equal to 1 and δm should be obtained from requirement that the pole of Dirac propagator remains at $p^2 = m^2$, same as before

[30] In a general QFT with divergent diagrams one can always add counterterms to the Lagrangian in such a way that they cancel the UV divergencies appearing in Feynman diagrams with original Lagrangian. However, for renormalizable theory the number of this counterterms is finite (three for QED, as seen from Eq. (23.41) whereas for a non-renormalizable theory the number of these counterterms is increasing with the order of perturbation theory making this theory an unusable theory with infinitely many coupling constants.

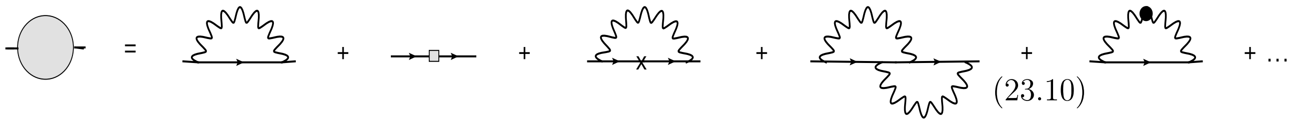
(see Eq. (22.27):

$$\mathcal{G}(p) \overset{p^2\to m^2}{\rightarrow} \frac{m+\not{p}}{m^2-p^2-i\epsilon}, \qquad \mathcal{D}(k) \overset{k^2\to 0}{\rightarrow} \frac{1}{k^2+i\epsilon} \tag{23.8}$$

Let us find δm, Z_2 and Z_3. First, to get δm we express the exact Dirac propagator as in Eq. (22.4)

$$\begin{aligned} \mathcal{G}(p) &= \frac{1}{m-\not{p}+\Sigma(p)-\delta m} = \frac{1}{m[1+\Sigma_1(p^2)]-\not{p}[1+\Sigma_2(p^2)]-\delta m} \\ &= \frac{m[1+\Sigma_1(p^2)]-\delta m+\not{p}[1+\Sigma_2(p^2)]}{\big(m[1+\Sigma_1(p^2)]-\delta m\big)^2-p^2[1+\Sigma_2(p^2)]^2} \end{aligned} \tag{23.9}$$

but with self-energy $\Sigma(p) = m\Sigma_1(p^2)-\not{p}\Sigma_2(p^2)$ including "new" vertices coming from counterterms in Fig. 31 proportional to (Z_1-1), (Z_2-1), and (Z_3-1)

$\Sigma(p) =$  (23.10)

(δm counterterm is treated separately, as before, see Eq. (22.4)). From the condition that the denominator in Eq. (23.9) vanishes at $p^2=m^2$ we have the same equation

$$\delta m = m\big[\Sigma_1(m^2)-\Sigma_2(m^2)\big] \tag{23.11}$$

but the first-order expressions for $\Sigma_1(m^2)$ and $\Sigma_2(m^2)$ should now include the contribution of (Z_2-1) counterterm. Fortunately, this counterterm is proportional to $m-\not{p}$ so it brings equal contributions to Σ_1 and Σ_2 which cancel in their difference in the r.h.s. of Eq. (23.11) and therefore we can recycle the old result (23.21) with physical charge e in place of e_0 (see also Eq. (22.26))

$$\begin{aligned} \Sigma_1^{\rm old}(p^2) &= \frac{e^2}{2\pi^2(4-d)} + \frac{e^2}{4\pi^2}\int_0^1 d\alpha\ \ln\frac{\mu^2}{m^2\alpha-p^2\bar{\alpha}\alpha-i\epsilon} - \frac{e^2}{8\pi^2} \\ \Sigma_2^{\rm old}(p^2) &= \frac{e^2}{8\pi^2(4-d)} + \frac{e^2}{8\pi^2}\int_0^1 d\alpha\ \bar{\alpha}\ln\frac{\mu^2}{m^2\alpha-p^2\bar{\alpha}\alpha-i\epsilon} - \frac{e^2}{16\pi^2} \end{aligned} \tag{23.12}$$

and get

$$\delta m = m\big[\Sigma_1(m^2)-\Sigma_2(m^2)\big] = \frac{3me^2}{8\pi^2(4-d)} + m\frac{e^2}{8\pi^2}\Big[\frac{3}{2}\ln\frac{\mu^2}{m^2}+2\Big] \tag{23.13}$$

Next, let us find Z_2 from the first requirement in Eq. (23.8). From Eqs. (22.31) and (22.32) we know that

$$\mathcal{G}(p) \overset{p^2\to m^2}{=} \Big[1+\Sigma_2(m^2)-2m^2\Big(\frac{\partial\Sigma_1}{\partial p^2}-\frac{\partial\Sigma_2}{\partial p^2}\Big)\Big|_{p^2=m^2}\Big]^{-1}\frac{m+\not{p}}{m^2-p^2-i\epsilon} + \text{const} \tag{23.14}$$

so the first condition Eq. (23.8) yields

$$\Sigma_2(m^2) = 2m^2\Big(\frac{\partial\Sigma_1}{\partial p^2} - \frac{\partial\Sigma_2}{\partial p^2}\Big)\Big|_{p^2=m^2} \tag{23.15}$$

The first-order self-energy is given by two first diagrams in the r.h.s. of Eq. (23.10). The first diagram was calculated in Sect. 22.2.3 (see Eq. (23.12)) while the second is $-(Z_2-1)(m-\not p)$ so we get

$$\begin{aligned}
\Sigma_1(p^2) &= \frac{e^2}{2\pi^2(4-d)} + \frac{e^2}{4\pi^2}\int_0^1 d\alpha\ \ln\frac{\mu^2}{m^2\alpha - p^2\bar\alpha\alpha - i\epsilon} - \frac{e^2}{8\pi^2} + (Z_2-1)\\
\Sigma_2(p^2) &= \frac{e^2}{8\pi^2(4-d)} + \frac{e^2}{8\pi^2}\int_0^1 d\alpha\ \bar\alpha\ln\frac{\mu^2}{m^2\alpha - p^2\bar\alpha\alpha - i\epsilon} - \frac{e^2}{16\pi^2} + (Z_2-1)
\end{aligned} \tag{23.16}$$

which leads to [31]

$$Z_2 - 1 = \frac{e^2}{8\pi^2(d-4)} - \frac{e^2}{16\pi^2}\Big(\ln\frac{\mu^2}{m^2}+2\Big) + \frac{e^2}{4\pi^2}\int_0^1 d\alpha\ \Big(\frac{1}{\alpha}-\alpha\Big) \tag{23.17}$$

(cf. Eq. (22.33)). Let us now find Z_1 from the requirement that the exact electron-elector-photon vertex gives exactly $e\gamma_\mu$ at the mass shell and zero momentum transfer:

$$\bar u(p,s')\Gamma_\mu(p,p)u(p,s') = e\bar u(p,s')\gamma_\mu u(p,s') = 2p_\mu\delta_{ss'}e \tag{23.18}$$

In the first order in e^2 the exact vertex is given by the diagrams in Fig. 32. If we keep

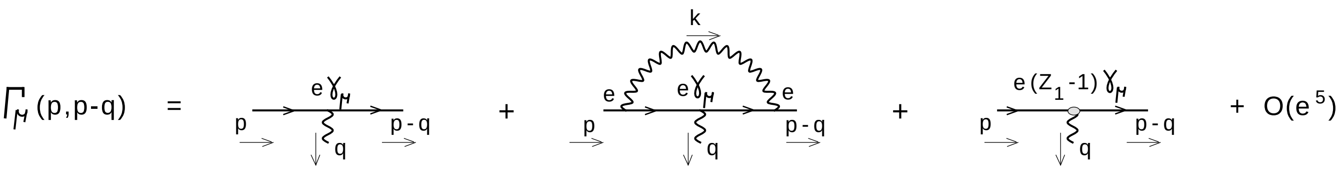


Figure 32. Exact vertex function in the first two orders

the notation $\Lambda(p,p-q)$ for the set of diagrams in Eq. (22.67) (Λ_μ = sum of all 3-point 1PI diagrams without γ^μ and without $(Z_1-1)\gamma_\mu$) in the leading order we obtain from Eq. (23.18)

$$\bar u(p,s')\Lambda_\mu(p,p)u(p,s) \stackrel{p^2=m^2}{=} -(Z_1-1)\bar u(p,s')\gamma_\mu u(p,s) = -2p_\mu\delta_{ss'}(Z_1-1) \tag{23.19}$$

Due to Ward identity (22.68) [32]

$$\Lambda_\mu(p,p) = -\frac{\partial}{\partial p^\mu}\big[\Sigma(p)-(Z_2-1)(m-\not p)\big] = \gamma_\mu(1-Z_2)+\gamma_\mu\Sigma_2(p^2)-2p_\mu\big[m\Sigma'_1(p^2)-\not p\Sigma'_2(p^2)\big] \tag{23.20}$$

[31] As we mentioned after Eq. (22.33), the integral over α in the r.h.s. of this equation is “infrared divergent” so to calculate it one needs to introduce a small photon mass λ^2 and then one gets const $= \ln\frac{m^2}{\lambda^2}+\frac{9}{4}$. Fortunately, due to Ward identity this IR divergence cancels with the corresponding term in Z_1 so the physical electric charge is “infrared safe”.

[32] we have replaced $\Sigma(p)$ by $\Sigma(p)+(Z_2-1)(m-\not p)$ in Ward identity since we redefined Σ to include the (Z_2-1) counterterm so $\Sigma^{\rm reg}_{\rm old}$ which enters the Ward identity (22.68) is $\Sigma^{\rm reg}_{\rm old} = \Sigma_{\rm new}-(Z_2-1)(m-\not p)$

(see Eq. (22.75)) so

$$\bar u(p,r)\Lambda_\mu(p,p)u(p,s) \;=\; [1-Z_2+\Sigma_2(p^2)]\bar u(p,r)\gamma_\mu u(p,s) - 2p_\mu \bar u(p,r)\big[m\Sigma'_1(p^2) - \not p\Sigma'_2(p^2)\big]u(p,s)$$
$$= \; 2p_\mu\delta_{ss'}\Big(1-Z_2+\Sigma_2(p^2) - 4m^2\big[\Sigma'_1(p^2)-\Sigma'_2(p^2)\big]\Big) \qquad (23.21)$$

(see Eq. (22.76). At $p^2=m^2$ the two last terms in the r.h.s. cancel due to Eq. (23.15) and we obtain

$$\bar u(p,r)\Lambda_\mu(p,p)u(p,s) \;\overset{p^2=m^2}{=}\; -2p_\mu\delta_{ss'}(Z_2-1) \qquad (23.22)$$

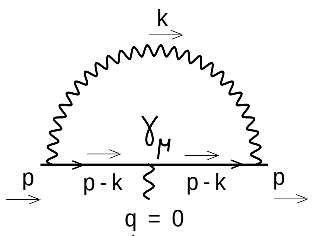
which gives due to Eq. (23.19)

$$Z_1-1 \;=\; Z_2-1 \quad\Rightarrow\quad Z_1 \;=\; Z_2 \qquad (23.23)$$

- same result as before. Thus, we do not need a new parameter for the vertex counterterm: the Ward identity ensures that the vertex counterterm is $e(Z_2-1)\gamma_\mu$. The condition $\Gamma_\mu(m,m) \;=\; \gamma_\mu$ preserves the property that the physical charge is equal to e in all orders in perturbation theory. In practical calculation our condition $\Gamma_\mu(m,m) \;=\; \gamma_\mu$ means that $\Gamma_\mu(p,q)=\Gamma_\mu(p,q)-\Gamma_\mu(m,m)+\gamma_\mu$ and therefore instead of the (UV divergent) diagrams for the vertex $\Gamma(p,q)$ we can calculate the difference of the diagrams for $\Gamma(p,q)$ and $\Gamma(m,m)$ which is UV safe so the dependence of μ disappears.

For future use, let us present the explicit form if $\Lambda_\mu(p,p)$ in the leading order (cf. Eq. (22.72)

$\Lambda_\mu(p,p) \;=$



$= \qquad (23.24)$

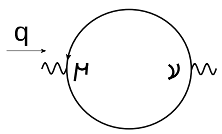
$$= \frac{e^2}{8\pi^2}\gamma_\mu\Big[\frac{1}{4-d}-\frac12+\int_0^1 d\alpha\;\bar\alpha\ln\frac{\mu^2}{m^2\alpha-p^2\bar\alpha\alpha-i\epsilon}\Big] - \frac{e^2}{8\pi^2}p_\mu\int_0^1 d\alpha\frac{\bar\alpha\alpha(4m-2\not p\bar\alpha)}{m^2\alpha-p^2\alpha\bar\alpha-i\epsilon} \;+\; O(2-\frac{d}{2})$$

Finally, let us discuss Z_3 and the effective charge. From conservation of current $\partial^\mu j_\mu(x) \;=\; 0$ we obtained the result (22.52)

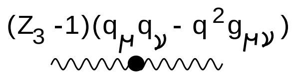
$$\mathcal{D}_{\mu\nu}(q) \;=\; \frac{g_{\mu\nu}}{q^2[1+\Pi(q^2)]} \qquad (23.25)$$

which ensures that the pole stays at $q^2=0$ in any order of perturbation theory keeping the photon massless. To get the expression for Z_3 in e^2 order we consider two first diagrams for the polarization operator

$\Pi_{\mu\nu}(q) \;=$



$+$



$+ \; O(e^4) \qquad (23.26)$

and use the second requirement (23.8) that the residue in the pole should be kept equal to 1 which means that $\Pi(q^2)\big|_{q^2=0} = 0$. We obtain

$$\Pi(q^2)\big|_{q^2=0} \;=\; \Pi^{\rm old}(q^2)\big|_{q^2=0} + (Z_3-1) \;=\; 0 \quad\Rightarrow\quad (Z_3-1) \;=\; -\,\Pi^{\rm old}(0) \tag{23.27}$$

and recycling the old result (22.43) we get

$$Z_3 - 1 \;=\; \frac{e^2}{12\pi^2}\Big[\frac{2}{d-4} - \ln\frac{\mu^2}{m^2}\Big] \tag{23.28}$$

Finally, to get the relation between physical charge and e_0 in the Lagrangian we recall the definition (23.5) and "Ward identity #3" $Z_1 = Z_2$ and obtain

$$e_0^2 \;=\; Z_3^{-1}e^2 \;\simeq\; e^2\Big(1 - \frac{e^2}{12\pi^2}\Big[\frac{2}{d-4} - \ln\frac{\mu^2}{m^2}\Big]\Big) \;+\; O(e^4) \tag{23.29}$$

To compare physical charge of a heavy fermion with mass M to the charge of the electron e one should repeat our whole renormalization program for this heavy fermion. The result will be the Eq. (23.29) with the different physical mass M:

$$e_0^2 \;\simeq\; e^2(M)\Big(1 - \frac{e^2(M)}{12\pi^2}\Big[\frac{2}{d-4} - \ln\frac{\mu^2}{M^2}\Big]\Big) \;+\; O(e^4(M)) \tag{23.30}$$

Since e_0 in the Lagrangian is the same we get

$$\frac{e^2}{e^2(M)} \;=\; \frac{1 - \frac{e^2(M)}{12\pi^2}\Big[\frac{2}{d-4} - \ln\frac{\mu^2}{M^2}\Big]}{1 - \frac{e^2}{12\pi^2}\Big[\frac{2}{d-4} - \ln\frac{\mu^2}{m^2}\Big]} \;+\; O(e^4) \;=\; 1 - \frac{e^2}{12\pi^2}\ln\frac{M^2}{m^2} \;+\; O(e^4) \tag{23.31}$$

which gives the same rule for running coupling constant

$$e^2(M) \;=\; \frac{e^2(m)}{1 - \frac{e^2(m)}{12\pi^2}\ln\frac{M^2}{m^2}} \;+\; O(e^4) \tag{23.32}$$

as we obtained in the previous Section, see Eq. (22.94)

23.0.1 Final set of Feynman rules for reduced Green functions in QED in terms of renormalized fields

In each order in perturbation theory, the counterterms should be obtained from the conditions that

- the pole of exact Dirac propagator stays at $p^2 = m^2$ (defines δm) with the residue 1 (defines $Z_2 - 1$)
- the pole of exact photon propagator at $q^2 = 0$ has residue 1 (defines $Z_3 - 1$)
- the electron-electron vertex in the non-relativistic limit at momentum transfer $q \to 0$ is equal to physical charge e (defines $Z_1 - 1$)

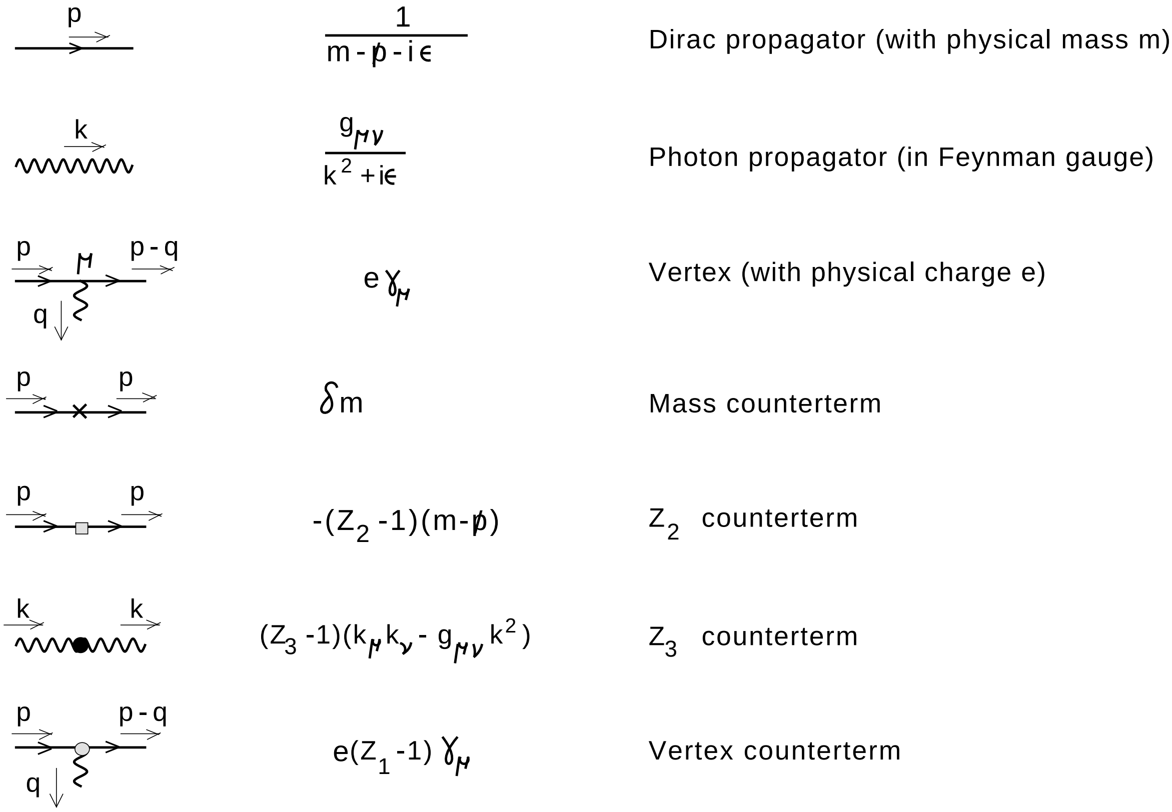


Figure 33. Feynman rules for diagrams with renormalized fields

It can be proved that the Feynman diagrams obtained from the rules in Fig. 33 are UV finite - the UV divergencies are canceled by counterterms order by order in perturbation theory.

Let us summarize the leading-order counterterms

$$\begin{aligned}
\frac{\delta m}{m} &= \;=\; \frac{3e^2}{8\pi^2(4-d)} + \frac{e^2}{8\pi^2}\Big[\frac{3}{2}\ln\frac{\mu^2}{m^2} + 2\Big] \;+\; O(e^4) \\
Z_1 - 1 &= Z_2 - 1 \;=\; \frac{e^2}{8\pi^2(d-4)} - \frac{e^2}{16\pi^2}\Big(\ln\frac{\mu^2}{m^2} + \frac{9}{4} + \ln\frac{m^2}{\lambda^2}\Big) \\
Z_3 - 1 &= \frac{e^2}{12\pi^2}\Big[\frac{2}{d-4} - \ln\frac{\mu^2}{m^2}\Big]
\end{aligned} \tag{23.33}$$

where λ is an IR cutoff ("photon mass"), see the footnote on p. 176. With these countertems all one-loop Feynman diagrams are finite. For example, it is easy to see that the one-loop diagrams for Compton scattering (22.89) are finite since the subtractions in parentheses, which were coming before from the expansion of $Z_1 Z_2^{-2} Z_3$ factor in Eq. 22.89, will be provided now by the counterterms in Fig. 33.

Note also that LSZ theorem is Eq. (20.73) with renormalized fields $\hat{\psi}$ and $\hat{A}$ (and without any Z-factors) so the Feynman rules for reduced Green functions in Fig. 33 are supplied by usual rules for matrix elements of $\mathcal{M}$-matrix:

Matrix element of $\mathcal{M}$-matrix is a reduced amputated Green function on a mass shell multiplied by: $\bar{u}(p,s)$ for each outgoing electron, $u(p,s)$ for each incoming electron, $v(p,s)$ for each outgoing positron, $\bar{v}(p,s)$ for each incoming positron, and $e^{\lambda}_{\mu}(k)$ for each incoming or outgoing photon.

23.0.2 Renormalization at a Euclidean point

The renormalization scheme which we worked out in previous Sections works well in many theories, but unfortunately not in QCD. The problem is that the fields in QQCD Lagrangian are quarks and gluons but there are no "physical" quarks and gluons - the physical states are hadrons (bound states of quarks and gluons). There is, however, a modification of the renormalization procedure that works for QCD and in preparation for QCD discussion in next Section we will develop it here using the familiar example of QED. (A detailed analysis can be found in *Peskin* Ch. 12.2).

The basic idea that instead of defining coupling constant(s) and Z-factors at the physical mass $p^2 = m^2$ we perform the renormalization at a certain Euclidean point $p^2 = -M^2$ and repeat the program of previous Section: rewrite this Lagrangian in terms of "renormalized fields" $\hat{\psi}$ and $\hat{A}_\mu$ defined as

$$\hat{\psi}(x) \ \stackrel{\text{def}}{\equiv}\ Z_2^{-\frac{1}{2}}\hat{\psi}_{(0)}(x), \qquad \hat{A}^\mu(x) \ \stackrel{\text{def}}{\equiv}\ Z_3^{-\frac{1}{2}}\hat{A}^\mu_{(0)}(x) \tag{23.34}$$

so that

$$\hat{\mathcal{L}} \ = \ -\frac{1}{4}Z_3\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} + Z_2\hat{\bar{\psi}}(i\not{\partial} - m_0 + e_0 Z_3^{\frac{1}{2}}\hat{A})\hat{\psi} \tag{23.35}$$

and the counterterms are obtained from the requirement that

$$\mathcal{G}(p) \ \stackrel{p^2\to -M^2}{\to}\ \frac{m_M + \not{p}}{m_M^2 - p^2}, \qquad \mathcal{D}(k) \ \stackrel{k^2\to -M^2}{\to}\ \frac{1}{k^2}, \tag{23.36}$$

and from the condition that the electron-electron-photon vertex [33]

$$\Gamma_\mu(p,p) \ \stackrel{p^2=-M^2}{=}\ e_M\gamma_\mu \ + \ \text{other structures} \tag{23.37}$$

where $m_M \equiv m(M)$ and $e_M \equiv e(M)$ are some parameters (depending on M) related to physical charge and physical mass by formulas

$$m_M \ = \ m\big(1 + a_1 e^4 + a_2 e^6 + \ldots\big), \qquad e_M^2 \ = \ e^2 + b_1 e^4 + b_2 e^6 + \ldots \tag{23.38}$$

(The coefficients a_n and b_n will turn out to be UV finite, typically $\big(\ln\frac{M^2}{m^2}\big)^n$).

Next, as before, we rewrite bare Lagrangian (23.35) in terms of m_M and e_M as follows

$$\hat{\mathcal{L}} \ = \ -\frac{1}{4}Z_3\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} + Z_2\hat{\bar{\psi}}(i\not{\partial} - m_M)\hat{\psi} + Z_1 e_M\hat{\bar{\psi}}\hat{A}\hat{\psi} - \delta m\hat{\bar{\psi}}\hat{\psi} \tag{23.39}$$

[33] As we saw from Ward identity (22.68), the exact vertex $\Gamma_\mu(p,p)$ has the general matrix structure $a\gamma_\mu + bp_\mu + cp_\mu\not{p}$, see Eq. (22.75). In previous Section, when we performed subtractions on the mass shell , we used the condition $\bar{u}(p,s)\Gamma_\mu(p,p)u(p,s') \ = \ e_{\text{phys}}\bar{u}(p,s)\gamma_\mu(p,p)u(p,s')$ to fix the coefficient in front of the proper structure. Now at $p^2 = -M^2$ we do not have spinors $u(p,s)$ (they are defined as solutions of Dirac equation for $p^2 = m^2_{\text{phys}}$) so we will require that the coefficient in front of the γ_μ structure is equal to e_M, in other words $\Gamma_\mu(p,p) = e_M\gamma_\mu + bp_\mu + cp_\mu p$ (the structure γ_μ can be singled out for example by taking $\text{tr}\{\Gamma_\mu \ \not{\eta}\}$ where the vector $\eta = (1,0,0,0)$ is orthogonal tp p).

where $\delta m \equiv m_M - m_0$ and

$$Z_1 \equiv \frac{e_0}{e_M} Z_2 Z_3^{\frac{1}{2}}, \tag{23.40}$$

and split the Lagrangian (23.39) into three parts: free Lagrangian (with mass m_M), interaction Lagrangian (with charge e_M) and counterterm Lagrangian

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_{\rm int} + \mathcal{L}_{\rm ct} \\
\mathcal{L}_0 &= -\frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} + \hat{\bar{\psi}}(i\partial\!\!\!/ - m_M)\hat{\psi}, \qquad \mathcal{L}_{\rm int} = e_M \hat{\bar{\psi}}\hat{A\!\!\!/}\hat{\psi}, \\
\mathcal{L}_{\rm ct} &= \frac{1}{4}(Z_3 - 1)\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} + (Z_2 - 1)\hat{\bar{\psi}}(i\partial\!\!\!/ - m_M)\hat{\psi} + (Z_1 - 1)e_M\hat{\bar{\psi}}\hat{A\!\!\!/}\hat{\psi},
\end{aligned} \tag{23.41}$$

Feynman rules for this Lagrangian (for reduced Green functions) are presented in Fig. (34)

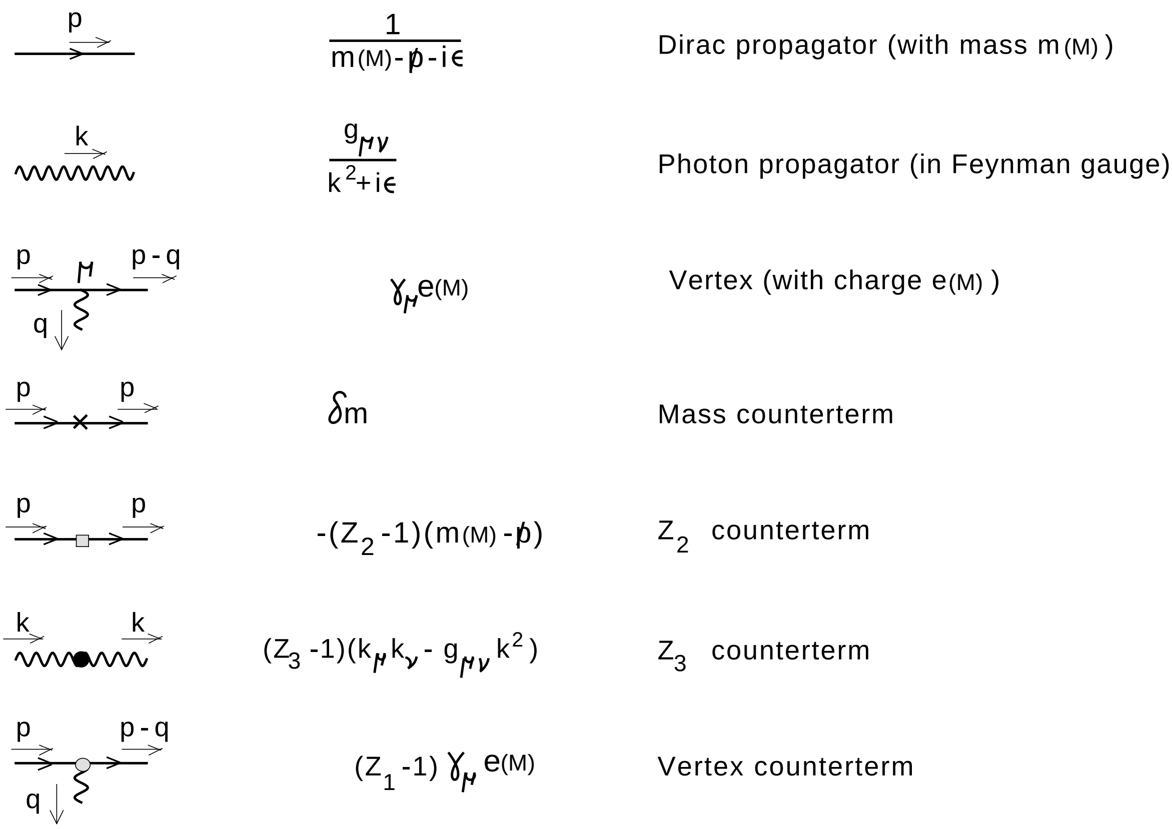


Figure 34. Feynman rules for diagrams with renormalized fields

Now we calculate Feynman diagrams with this Lagrangian using some suitable cutoff for UV divergencies (usually dimensional regularization) and adjust parameters Z_1, Z_2, and Z_3 in any order of perturbation theory in such a way that the conditions (23.39) and (23.40) are satisfied. Similarly to the previous case (renormalization on physical mass and charge) the terms coming from the counterterm Lagrangian cancel the divergencies of Feynman diagrams obtained with usual Lagrangian $\hat{\mathcal{L}}_0 + \hat{\mathcal{L}}_{\rm int}$.

Let us demonstrate how this scheme works at the one-loop level.

First, we need to calculate counterterms. We start with Z_2. The exact Dirac propagator is given by Eq. (23.9)

$$\mathcal{G}(p) \;=\; \frac{1}{m_M - \not{p} + \Sigma_1(p) - \delta m} \;=\; \frac{m_M[1+\Sigma_1(p^2)] - \delta m + \not{p}[1+\Sigma_2(p^2)]}{\big(m_M[1+\Sigma_1(p^2)] - \delta m\big)^2 - p^2[1+\Sigma_2(p^2)]^2} \tag{23.42}$$

where $\Sigma(p)$ is given by the same diagrams as in Eq. (23.10) (with m_M and e_M in place of m and e) so we can recycle Eq. (23.16):

$$\begin{aligned}
\Sigma_1(p^2) \;&=\; \frac{e_M^2}{2\pi^2(4-d)} + \frac{e_M^2}{4\pi^2}\int_0^1 d\alpha\;\ln\frac{\mu^2}{m_M^2\alpha - p^2\bar{\alpha}\alpha - i\epsilon} - \frac{e_M^2}{8\pi^2} \;+\; (Z_2-1)\\
\Sigma_2(p^2) \;&=\; \frac{e_M^2}{8\pi^2(4-d)} + \frac{e_M^2}{8\pi^2}\int_0^1 d\alpha\;\bar{\alpha}\ln\frac{\mu^2}{m_M^2\alpha - p^2\bar{\alpha}\alpha - i\epsilon} - \frac{e_M^2}{16\pi^2} \;+\; (Z_2-1)
\end{aligned} \tag{23.43}$$

Now we will find δm and Z_2 from the condition $\mathcal{G}(p) \overset{p^2\to -M^2}{\to} \frac{m_M+\not{p}}{m_M^2+M^2}$, see Eq. (23.36). The expansion of the numerator of Eq. (23.42) near $p^2=-M^2$ yields

$$\begin{aligned}
&m_M[1+\Sigma_1(p^2)] - \delta m + \not{p}[1+\Sigma_2(p^2)] \overset{p^2\to -M^2}{=} m_M[1+\Sigma_1(-M^2)] - \delta m + \not{p}[1+\Sigma_2(-M^2)] \;+\; O(p^2+M^2)\\
&=\; (m_M+\not{p})[1+\Sigma_2(-M^2)] + m_M[\Sigma_1(-M^2) - \Sigma_2(-M^2)] - \delta m \;+\; O(p^2+M^2)
\end{aligned} \tag{23.44}$$

so from the requirement that the numerator should be proportional to $m_M+\not{p}$ at $p^2=-M^2$ we see that

$$\frac{\delta m}{m_M} \;=\; \Sigma_1(-M^2) - \Sigma_2(-M^2) \;=\; \frac{e_M^2}{8\pi^2}\Big[\frac{3}{4-d} - \frac{1}{2} + \int_0^1 d\alpha\,(1+\alpha)\ln\frac{\mu^2}{m^2\alpha + M^2\bar{\alpha}\alpha}\Big] + O(e_M^4) \tag{23.45}$$

Next, the denominator in Eq. (23.42) behaves as

$$\big(m_M[1+\Sigma_1(p^2)] - \delta m\big)^2 - p^2[1+\Sigma_2(p^2)]^2 \;=\; (m_M^2+M^2)[1+\Sigma_2(-M^2)]^2 \;+\; O(p^2+M^2) \tag{23.46}$$

so

$$\mathcal{G}(p) \overset{p^2\to -M^2}{\to} \frac{m_M+\not{p}}{m_M^2+M^2}\,\frac{1}{1+\Sigma_2(-M^2)} \;+\; O(p^2+M^2) \tag{23.47}$$

and therefore to get the condition (23.36) we need $\Sigma_2(-M^2) \;=\; 0$ which gives

$$Z_2 - 1 \;=\; -\frac{e_M^2}{8\pi^2(4-d)} - \frac{e_M^2}{8\pi^2}\int_0^1 d\alpha\;\bar{\alpha}\ln\frac{\mu^2}{m_M^2\alpha + M^2\bar{\alpha}\alpha} + \frac{e_M^2}{16\pi^2} \tag{23.48}$$

Let us now find the vertex counterterm. The exact vertex up to the first order in e_M^2 is given by diagrams in Fig. 32. As in previous Section, we denote the sum of all three-point 1PI diagrams without $e_M\gamma_\mu$ and without $(Z_1-1)e_M\gamma_\mu$ by $\Lambda(p,p-q)$ and get from Ward identity (22.68)

$$\Lambda_\mu(p,p) \;=\; -\frac{\partial}{\partial p^\mu}\big[\Sigma(p) - (Z_2-1)(m_M - \not{p})\big] \;=\; \gamma_\mu(1-Z_2) + \gamma_\mu\Sigma_2(p^2) - 2p_\mu\big[m_M\Sigma'_1(p^2) - \not{p}\Sigma'_2(p^2)\big] \tag{23.49}$$

see Eq. (23.20) and the footnote # 32 on page 176. In the explicit form (see Eq. (23.24))

$$\Lambda_\mu(p,p) \tag{23.50}$$

$$= \frac{e_M^2}{8\pi^2}\gamma_\mu\Big[\frac{1}{4-d} - \frac{1}{2} + \int_0^1 d\alpha\;\bar\alpha \ln\frac{\mu^2}{m_M^2\alpha - p^2\bar\alpha\alpha - i\epsilon}\Big] - \frac{e_M^2}{8\pi^2}p_\mu\int_0^1 d\alpha\frac{\bar\alpha\alpha(4m_M - 2\not{p}\bar\alpha)}{m_M^2\alpha - p^2\alpha\bar\alpha - i\epsilon} \;+\; O(2-\frac{d}{2})$$

Since at $p^2 = -M^2$ from Eq. (23.47) we have $\Sigma_2(-M^2) = 0$, the equation (23.49) reduces to

$$\Lambda_\mu(p,p) \;\stackrel{p^2\to -M^2}{\to}\; = \gamma_\mu(1-Z_2) - 2p_\mu\big[m_M{\Sigma'}_1(-M^2) - \not{p}{\Sigma'}_2(-M^2)\big] \tag{23.51}$$

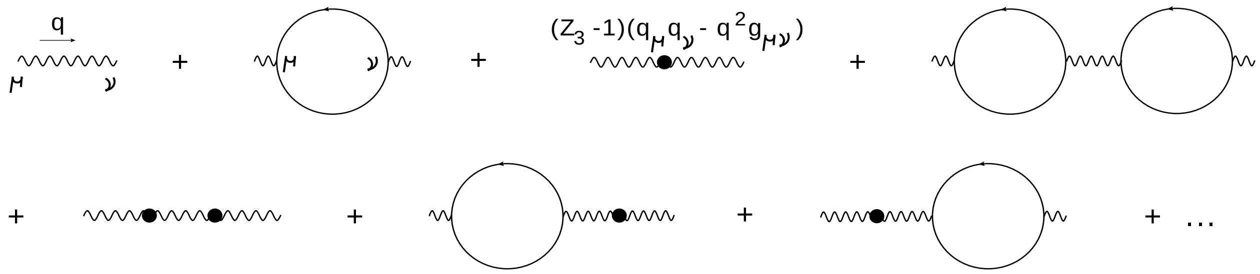
This equation agrees with Eqs. (23.50) and (23.48), (23.43) in the first order in perturbation theory. Now, from the condition

$$e_M\gamma_\mu + e_M(Z_1-1)\gamma_\mu + \Lambda(p,p) \;\equiv\; \Gamma_\mu(p,p) \;\stackrel{p^2=-M^2}{=}\; e_M\gamma_\mu \;+\; \text{other structures} \tag{23.52}$$

we get $Z_1 = Z_2$ as before.

Finally, to get the $Z_3 - 1$ counterterm we should use the second requirement (23.36). The exact photon propagator is

$\mathcal{D}_{\mu\nu}(q) \;=$



$$= \frac{g_{\mu\nu}}{q^2} \;+\; \frac{g_{\mu\alpha}}{q^2}(q^\alpha q^\beta - q^2 g^{\alpha\beta})\big[\Pi^{\rm old}(q^2) + (Z_3-1)\big]\frac{g_{\beta\nu}}{q^2} \tag{23.53}$$

$$+\; \frac{g_{\mu\alpha}}{q^2}(q^\alpha q^\beta - q^2 g^{\alpha\beta})\big[\Pi^{\rm old}(q^2) + (Z_3-1)\big]\frac{g_{\beta\gamma}}{q^2}(q^\gamma q^\delta - q^2 g^{\gamma\delta})\big[\Pi^{\rm old}(q^2) + (Z_3-1)\big]\frac{g_{\delta\nu}}{q^2} \;+\; ...$$

$$= \frac{g_{\mu\nu}}{q^2} \;+\; (q^\mu q^\nu - q^2 g^{\mu\nu})\frac{\Pi^{\rm old}(q^2) + (Z_3-1)}{1 + \Pi^{\rm old}(q^2) + (Z_3-1)} \;=\; \frac{g_{\mu\nu}}{q^2[1 + \Pi^{\rm old}(q^2) + (Z_3-1)]} \;+\; q_\mu q_\nu \times \text{smth}$$

where $\Pi^{\rm old}(q^2)$ is given by Eq. (22.43) (with "new" e_M and m_M)

$$\Pi^{\rm old}_{\mu\nu}(q) \;=\; 8e_M^2\frac{\tilde\mu^{4-d}}{(4\pi)^{\frac{d}{2}}}\int_0^1 d\alpha\;\frac{\bar\alpha\alpha\Gamma\big(2-\frac{d}{2}\big)}{(m_M^2 - q^2\bar\alpha\alpha - i\epsilon)^{2-\frac{d}{2}}} \;+\; O(e_M^4)$$

$$= \frac{e_M^2}{6\pi^2(4-d)} + \frac{e_M^2}{2\pi^2}\int_0^1 d\alpha\;\bar\alpha\alpha\ln\frac{\mu^2}{(m^2 - q^2\bar\alpha\alpha - i\epsilon)} \;+\; O(e_M^4) \tag{23.54}$$

We see that in order so to satisfy second of the conditions (23.36) we need

$$Z_3 - 1 \;=\; -\,\Pi^{\rm old}(q^2)\big|_{q^2=-M^2} \;=\; \frac{e_M^2}{6\pi^2(d-4)} - \frac{e_M^2}{2\pi^2}\int_0^1 d\alpha\;\bar\alpha\alpha\ln\frac{\mu^2}{m^2 + M^2\bar\alpha\alpha} \;+\; O(e_M^4) \tag{23.55}$$

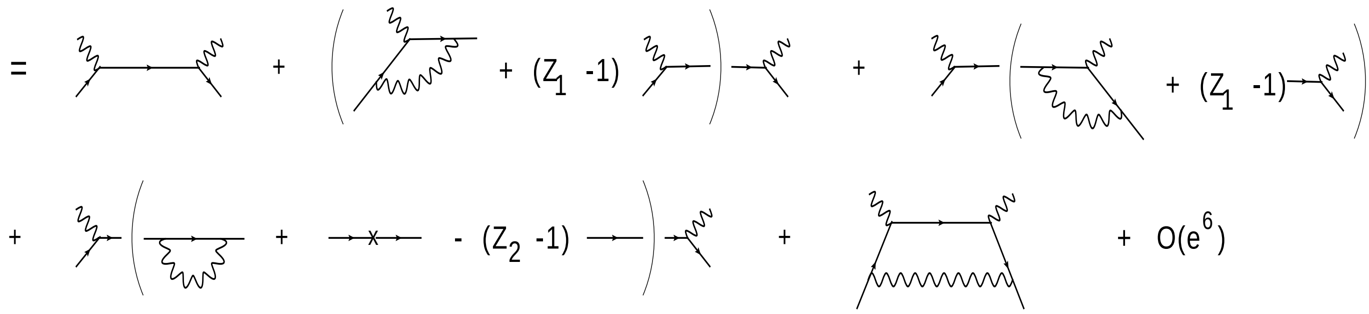
Let us present all counterterms in this scheme

$$
\begin{aligned}
Z_1 - 1 \;=\; Z_2 - 1 \;&=\; \frac{e_M^2}{8\pi^2(d-4)} - \frac{e_M^2}{8\pi^2}\int_0^1 d\alpha\;\bar{\alpha}\ln\frac{\mu^2}{m_M^2\alpha + M^2\bar{\alpha}\alpha} + \frac{e_M^2}{16\pi^2} \;+\; O(e_M^4) \\
Z_3 - 1 \;&=\; \frac{e_M^2}{6\pi^2(d-4)} - \frac{e_M^2}{2\pi^2}\int_0^1 d\alpha\;\bar{\alpha}\alpha\ln\frac{\mu^2}{m^2 + M^2\bar{\alpha}\alpha} \;+\; O(e_M^4)
\end{aligned}
\tag{23.56}
$$

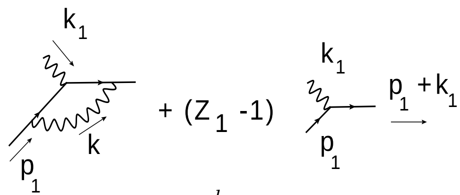
If one now takes $M^2 = 0$ in the above equations one obtains the counterterms for regularization on the mass shell (23.17) and (23.27). Note that here counterterm Z_2 is IR - finite, quite unlike the renormalization on the mass shell where Z_2 given by Eq. (23.33) needs IR cutoff. This is the reason why the renormalization with subtractions at the Euclidean point is very convenient for dealing with theories with massless particles (like QCD) since in such scheme all counterterms are IR-finite so the UV and IR divergencies are disentangled.

It is easy to see that the counterterms (23.56) subtract all UV divergencies in one-loop diagrams obtained with Fig. 34 Feynman rules. For example, in the case of Compton scattering (22.89) we get

$$
\mathcal{M}(p_1, s_1; p_1', s_1' \to p_2, s_2; p_2', s_2') \;=\; \tag{23.57}
$$



It is easy to see that the expression in each parentheses is UV finite (and the last diagram is finite by itself). Indeed, consider for example



$$
\begin{aligned}
&= e_M^3\int\frac{d\!\!\bar{}^d k}{i(k^2+i\epsilon)}\;\gamma_\alpha\frac{m_M + \not{p}_1 - \not{k}}{m_M^2 - (p_1-k)^2 - i\epsilon}\gamma_\mu\frac{m_M + \not{p}_1 + \not{k}_1 - \not{k}}{m_M^2 - (p_1+k_1-k)^2 - i\epsilon}\gamma^\alpha \;+\; (Z_1 - 1)e\gamma_\mu \\
&= e_M^3\int\frac{d\!\!\bar{}^d k}{i(k^2+i\epsilon)}\;\gamma_\alpha\frac{m + \not{p}_1 - \not{k}}{m^2 - (p_1-k)^2 - i\epsilon}\gamma_\mu\frac{m_M + \not{p}_1 + \not{k}_1 - \not{k}}{m_M^2 - (p_1+k_1-k)^2 - i\epsilon}\gamma^\alpha \\
&\qquad\qquad - e_M^3\gamma_\mu\Big[\frac{1}{8\pi^2(d-4)} - \frac{1}{8\pi^2}\int_0^1 d\alpha\;\bar{\alpha}\ln\frac{\mu^2}{m_M^2\alpha + M^2\bar{\alpha}\alpha} + \frac{1}{16\pi^2}\Big]
\end{aligned}
$$

$$
\begin{aligned}
= & \; e_M^3 \int \frac{d\!\!{}^-{}^d k}{i(k^2+i\epsilon)} \Big[\gamma_\alpha \frac{m_M+\not p_1-\not k}{m_M^2-(p_1-k)^2-i\epsilon}\gamma_\mu \frac{m_M+\not p_1+\not k_1-\not k}{m_M^2-(p_1+k_1-k)^2-i\epsilon}\gamma^\alpha \\
& \qquad - \gamma_\alpha \frac{m_M+\not p_1-\not k}{m_M^2-(p_1-k)^2-i\epsilon}\gamma_\mu \frac{m_M+\not p_1+\not k_1-\not k}{m_M^2-(p_1+k_1-k)^2-i\epsilon}\gamma^\alpha \Big|_{p_1^2=M^2,k_1^2\to 0}\Big] \\
& + e_M\Lambda(p_1,p_1)\Big|_{p_1^2=M^2} - e_M^3\gamma_\mu\Big[\frac{1}{8\pi^2(d-4)} - \frac{1}{8\pi^2}\int_0^1 d\alpha\;\bar\alpha \ln\frac{\mu^2}{m_M^2\alpha+M^2\bar\alpha\alpha} + \frac{1}{16\pi^2}\Big]
\end{aligned}
$$

where we have added and subtracted

$$
\Lambda_\mu(p,p) \;=\; e_M^2 \int \frac{d\!\!{}^-{}^d k}{i(k^2+i\epsilon)}\; \gamma_\alpha \frac{m_M+\not p_1-\not k}{m_M^2-(p_1-k)^2-i\epsilon}\gamma_\mu \frac{m_M+\not p_1-\not k}{m_M^2-(p_1-k)^2-i\epsilon}\gamma^\alpha
$$

at $p^2 = M^2$. It is clear now that the integral over k in the r.h.s. id UV divergent at $d \to 4$ since the leading term at $k \to \infty$ cancels in the difference of the integrands. As to the last term in the r.h.s. of this equation, from the explicit form of $\Lambda_\mu(p,p)$ (23.50) it is equal to

$$
-\frac{e_M^2}{8\pi^2}p_\mu \int_0^1 d\alpha \frac{\bar\alpha\alpha(4m_M - 2\not p\bar\alpha)}{m_M^2\alpha + M^2\alpha\bar\alpha - i\epsilon}
$$

which is finite.

Srimilarly, repeating the calculation in Eq. (22.91) one can demonstrate that the UV divergence in the self-energy diagram in the last line in r.h.s. of Eq. (23.57) is canceled by (Z_2-1) subtraction so the amplitude of Compton scattering (23.57) is finite.

Finally, let us discuss relation between $e(M)$ and physical charge e. From Eq. (23.40) (and $Z_1 = Z_2$) we see that

$$
e_M^2 \;=\; e_0^2 Z_3 \;=\; e_0^2\Big[1 + \frac{e_0^2}{6\pi^2(d-4)} - \frac{e_0^2}{2\pi^2}\int_0^1 d\alpha\;\bar\alpha\alpha \ln\frac{\mu^2}{(m^2+M^2\bar\alpha\alpha)} \;+\; O(e_0^4)\Big] \tag{23.58}
$$

On the other hand, in the previous Section we've got the relation between e_0 and physical charge e

$$
e_0^2 \;=\; e^2\Big(1 - \frac{e^2}{12\pi^2}\Big[\frac{2}{d-4} - \ln\frac{\mu^2}{m^2}\Big]\Big) \;+\; O(e^4) \tag{23.59}
$$

(see Eq. (23.29). Comparing these two equations we obtain

$$
e_M^2 \;=\; e^2\Big[1 + \frac{e^2}{2\pi^2}\int_0^1 d\alpha\;\bar\alpha\alpha \ln\frac{m^2+M^2\bar\alpha\alpha}{m^2} \;+\; O(e^4)\Big] \tag{23.60}
$$

Again, we see that at large M the "effective charge" $e(M)$ increases.

Part XXII

24 QCD

QCD is a theory of interacting quarks and gluons. The eight gluons are described by A_μ^a - 8 real massless vector fields (like 8 different photons). It will be convenient to use matrix notation: $A_\mu \equiv A^a_\mu t^a$ where t^a are 8 Gell-Mann matrices- Hermitian matrices with properties $\mathrm{Tr} t^a = 0$, $\mathrm{Tr} t^a t^b = \frac{1}{2}\delta^{ab}$ (for the explicit form, see any textbook).

The matrices t^a are the generators of SU(3)group - the group of unitary 3×3 matrices Ω with $\det\Omega = 1$. An arbitrary SU(3) matrix can be parametrized as $\exp(i\sum_1^8 \omega^a t^a)$ where ω^a are real numbers. The group SU_3 is non-Abelian since in general $\Omega_1\Omega_2 \neq \Omega_2\Omega_1$.

The three quarks are described by the three-component SU(3) spinor ψ^k_ξ (the quark of each color k has the additional Lorentz (bi)spinor index ξ similarly to the gluon which has color index a and vector index μ). Also, there are different quarks which can be described by an additional index called flavor. For now, six quarks are known: u (up), d)down), s(strange), c (charm), b (beauty) , and t (top).

The QCD is an example of so-called Yang-Mills theories which are generalizations of QED to the case of non-Abelian gauge group (another example is the Weinberg-Salam theory of weak interactions).

24.1 Lagrangian and non-Abelian gauge symmetry

Let us recall gauge invariance in QED. The Lagrangian (density) for QED is given by Eq. (20.1)

$$\mathcal{L}(x) \;=\; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\psi(i\not{D} - m)\psi, \qquad D_\mu\psi(x) \;\equiv\; \big(\partial_\mu - ieA_\mu(x)\big)\psi(x) \tag{24.1}$$

It is invariant under Abelian gauge transformations (20.5)

$$\begin{aligned} \psi(x) &\;\to\; e^{i\alpha(x)}\psi(x) \\ \bar\psi(x) &\;\to\; e^{-i\alpha(x)}\bar\psi(x) \\ A_\mu(x) &\;\to\; A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x) \end{aligned} \tag{24.2}$$

(Abelian because multiplication by $e^{i\alpha}$ forms and Abelian group U_1).

The QCD Lagrangian is similar to Eq. (24.1)

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\; G_{\mu\nu}G^{\mu\nu} + \sum_{\text{flavors}} \bar\psi_f(i\not{D} - m_q)\psi_f \tag{24.3}$$

where

$$\begin{aligned} &G_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu], \\ &D_\mu \equiv \partial_\mu - igA_\mu \end{aligned} \tag{24.4}$$

and g is called QCD coupling constant. The Gell-Mann matrices t^a satisfy the relation

$$[t^a, t^b] = if^{abc}t^c \tag{24.5}$$

where f^{abc} are SU(3) structure constants. They are totally antisymmetric in all indices and satisfy the Jacobi identity

$$f^{abm}f^{cdm} + f^{acm}f^{dbm} + f^{adm}f^{bcm} \;=\; 0 \tag{24.6}$$

Using Eq. (24.6) one can rewrite $G_{\mu\nu}$ as

$$G_{\mu\nu} \;=\; \sum_{a=1}^{8} t^a G^a_{\mu\nu}, \quad G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{24.7}$$

In what follows I will omit the summation index (as usual, the summation of the repeated indices will be implied).

Non-Abelian gauge invariance:

$$\begin{aligned}
\psi(x) \;&\to\; \Omega(x)\psi(x) \\
\bar\psi(x) \;&\to\; \bar\psi(x)\Omega^\dagger(x) \\
A_\mu(x) \;&\to\; \Omega(x)A_\mu(x)\Omega^\dagger(x) + \frac{i}{g}\Omega(x)\partial_\mu\Omega^\dagger(x)
\end{aligned} \tag{24.8}$$

Let us prove that the QCD Lagrangian (24.3) is invariant under the above transformations: $\mathcal{L}_{\rm QCD} \to \mathcal{L}_{\rm QCD}$.

First, we prove that $G_{\mu\nu}(x) \to \Omega(x)G_{\mu\nu}(x)\Omega^\dagger(x)$:

$$\begin{aligned}
G^{\mu\nu}(x) \;\to\;& \partial_\mu\big(\Omega A_\nu\Omega^\dagger + ig^{-1}\Omega\partial_\nu\Omega^\dagger\big) \\
&-ig\big(\Omega A_\mu\Omega^\dagger + ig^{-1}\Omega\partial_\mu\Omega^\dagger\big)\big(\Omega A_\nu\Omega^\dagger + ig^{-1}\Omega\partial_\nu\Omega^\dagger\big) - (\mu\leftrightarrow\nu) \\
=\;& \big(\partial_\mu\Omega\big)A_\nu\Omega^\dagger + \Omega\big(\partial_\mu A_\nu\big)\Omega^\dagger + \Omega A_\nu\partial_\mu\Omega^\dagger + ig^{-1}\big(\partial_\mu\Omega\big)\partial_\nu\Omega^\dagger \\
&+ ig^{-1}\Omega\partial_\mu\partial_\nu\Omega^\dagger - ig\Omega A_\mu A_\nu\Omega^\dagger + \Omega\big(\partial_\mu\Omega^\dagger\big)\Omega A_\nu\Omega^\dagger \\
&+ \Omega A_\mu\partial_\nu\Omega^\dagger + ig^{-1}\Omega\big(\partial_\mu\Omega^\dagger\big)\Omega\partial_\nu\Omega^\dagger - (\mu\leftrightarrow\nu) \\
=\;& \Omega\big(\partial_\mu A_\nu - igA_\mu A_\nu\big)\Omega^\dagger - (\mu\leftrightarrow\nu) = \Omega G_{\mu\nu}\Omega^\dagger
\end{aligned}$$

(we used the property $\Omega\partial_\mu\Omega^\dagger = -\big(\partial_\mu\Omega\big)\Omega^\dagger$)

Next, we prove that $D_\mu\psi(x) \to \Omega(x)D_\mu\psi(x)$:

$$\begin{aligned}
&D_\mu\psi(x) \to \big[\partial_\mu - ig\Omega A_\mu(x)\Omega^\dagger(x) + \Omega(x)\big(\partial_\mu\Omega^\dagger(x)\big)\big]\Omega(x)\psi(x) \\
&= \Omega(x)\partial_\mu\psi(x) - ig\Omega(x)A_\mu(x)\psi(x) = \Omega(x)D_\mu\psi(x)
\end{aligned}$$

Finally,

$$\begin{aligned}
&{\rm Tr}\; G^{\mu\nu}(x)G_{\mu\nu}(x) \;\to\; {\rm Tr}\;\Omega(x)G^{\mu\nu}(x)\Omega^\dagger(x)\Omega(x)G_{\mu\nu}(x)\Omega^\dagger(x) = {\rm Tr}\; G^{\mu\nu}(x)G_{\mu\nu}(x), \\
&\bar\psi(x)\not{D}\psi(x) \to \bar\psi(x)\Omega^\dagger(x)\gamma^\mu\Omega(x)D_\mu\psi(x) = \bar\psi(x)\not{D}\psi(x) \\
&m\bar\psi(x)\psi(x) \to m\bar\psi(x)\Omega(x)\Omega^\dagger(x)\psi(x) = m\bar\psi(x)\psi(x)
\end{aligned}$$

so the Lagrangian (24.3) is invariant under transformations (24.8).

Classical theory: non-linear equations

$$\begin{aligned}
&\big(D^\mu G_{\mu\nu}\big)^a = -g\sum_{\rm flavors}\bar\psi_q t^a\gamma_\nu\psi, \\
&(i\not{D} - m_q)\psi_q(x) = 0, \quad \bar\psi(i\overleftarrow{\not{D}} + m_q)\psi_q(x) = 0
\end{aligned} \tag{24.9}$$

(here $\big(D^\mu G_{\mu\nu}\big)^a \equiv \partial^\mu G^a_{\mu\nu} + gf^{abc}A^{b\mu}G^c_{\mu\nu}$).

To solve the non-linear equations (24.9) is very difficult. Up to now, only a few explicit solutions of $D^\mu G_{\mu\nu}=0$ (for "pure gluodynamics") are known, the most famous example is the so-called instanton

$$A^a_\mu \;=\; \frac{1}{g}\eta^a_{\mu\nu}\frac{(x-x_0)^\nu}{(x-x_0)^2+\rho^2}$$

with the finite action $8\pi^2$ (and "topological charge" $\frac{1}{8\pi^2}\epsilon^{\mu\nu\alpha\beta}\int d^4x\ G^a_{\mu\nu}(x)G^a_{\alpha\beta}(x)=8\pi^2$).

24.1.1 Energy-momentum tensor in QCD

From Noether theorem on gets (cf Eqs. (19.44) and (14.28))

$$\begin{aligned} T^{\mu\nu} &= \sum\frac{\partial\mathcal{L}}{\partial_\mu A^a_\alpha}\partial^\nu A^{a\alpha}+\frac{\partial\mathcal{L}}{\partial\partial_\nu\psi}\partial^\mu\psi+\partial^\mu\bar\psi\frac{\partial\mathcal{L}}{\partial\partial_\nu\bar\psi}-g^{\mu\nu}\mathcal{L} \qquad (24.10)\\ &= -G^{a\mu\alpha}\partial^\nu A^a_\alpha+i\bar\psi\gamma_\nu D_\mu\psi-g^{\mu\nu}\mathcal{L}\\ &= -G^{a\mu\alpha}G^{a\nu}{}_\alpha+\frac{g^{\mu\nu}}{4}G^{a\xi\eta}G^a_{\xi\eta}+\frac{i}{4}\bar\psi(\gamma_\mu\overset{\leftrightarrow}{D^\nu}+\gamma_\nu\overset{\leftrightarrow}{D^\mu})\psi\ +\ \text{total derivative}\end{aligned}$$

so the symmetric form of energy-momentum tensor is

$$T^{\mu\nu}\;=\;-2\mathrm{Tr}\ G^{a\mu\alpha}(x)G^{a\nu}{}_\alpha(x)+\frac{g^{\mu\nu}}{2}\mathrm{Tr}\ G^{a\xi\eta}(x)G^a_{\xi\eta}(x)+\frac{i}{4}\bar\psi(x)(\gamma_\mu\overset{\leftrightarrow}{D^\nu}+\gamma_\nu\overset{\leftrightarrow}{D^\mu})\psi(x) \qquad (24.11)$$

It is easy to see that energy-momentum tensor (24.11) is gauge invariant since it is made of fields $G^{\mu\nu}$, ψ and $D_\mu\psi$ which are only "rotating":

$$\begin{aligned} &G^{\mu\nu}(x)\ \to\ \Omega(x)G^{\mu\nu}(x)\Omega^\dagger(x), \qquad (24.12)\\ &\psi(x)\ \to\ \Omega(x)\psi(x),\quad D^\mu\psi(x)\ \to\ \Omega(x)D^\mu\psi(x),\quad \bar\psi(x)\ \to\ \bar\psi(x)\Omega(x),\quad \bar\psi(x)\overset{\leftarrow}{D}{}^\mu\ \to\ \bar\psi(x)\overset{\leftarrow}{D}{}^\mu\end{aligned}$$

24.2 QCD quantization

Announcement of the result:
Perturbation theory - like QED: free Lagrangian ≡ 8 issues of electrodynamics labeled by $a=1\div 8$ plus three free Dirac fields
⇒ Feynman rules are the same, except now we have the self-interaction of gluons.
This is almost true - Ward identity in QCD is different ⇒ ghosts.

QCD Lagrangian can be written as a sum of three terms

$$\mathcal{L}\;=\;\mathcal{L}_F+\mathcal{L}_D+\mathcal{L}_{\rm int} \qquad (24.13)$$

$$\mathcal{L}_F\;=\;-\frac{1}{4}\,F^a_{\mu\nu}F^{a\mu\nu},\qquad F_{\mu\nu}\equiv\partial_\mu A_\nu-\partial_\nu A_\mu \qquad (24.14)$$

$$\mathcal{L}_D\;=\;\sum_{\text{flavors}}\bar\psi_f(i\partial\!\!\!/-m_f)\psi_f \qquad (24.15)$$

$$\mathcal{L}_{\rm int}\;=\;2ig\,\mathrm{Tr}\left\{\partial^\mu A^\nu[A_\mu,A_\nu]\right\}+g\sum_{\text{flavors}}\bar\psi_q\,A\!\!\!/\psi_q-\frac{g^2}{2}\mathrm{Tr}\left\{[A_\mu,A_\nu][A^\mu,A^\nu]\right\} \qquad (24.16)$$

Or, another representaion of $\mathcal{L}_{\rm int}$:

$$\mathcal{L}_{\rm int}\;=\;g\sum_{\text{flavors}}\bar\psi_q\,A\!\!\!/\psi_q\;-\;gf^{abc}\,(\partial^\mu A^{a\nu})A^b_\mu A^c_\nu\;-\;\frac{g^2}{4}f^{abn}f^{cdn}A^a_\mu A^b_\nu A^{c\mu}A^{d\nu} \qquad (24.17)$$

(Recall $[t^a, t^b] = if^{abc}t^c$).

As in QED, we choose the Coulomb gauge: the canonical coordinates are the quark fields $\psi_\xi^{kf}(x)$ and potentials $A_i^a(x)$ satisfying the Coulomb gauge condition

$$\partial^i A_i^a i(x) \;=\; -\vec{\nabla}\cdot\vec{A}^a(x) \;=\; 0 \tag{24.18}$$

and the canonical momenta are (cf. Eq. (20.15))

$$\begin{aligned}
\pi_f^k(t,\vec{x}) \;&\equiv\; \frac{\partial\mathcal{L}}{\partial\dot{\psi}_f^k}(t,\vec{x}) \;=\; i\psi_f^{k\dagger}(t,\vec{x}) \qquad (24.19)\\
\pi_i^a(t,\vec{x}) \;&\equiv\; \frac{\partial\mathcal{L}}{\partial\dot{A}^{ai}}(t,\vec{x}) \;=\; -G_{0i}^a(t,\vec{x}) \;=\; -\dot{A}_i^a(t,\vec{x}) + \partial_i A_0^a(t,\vec{x}) - gf^{abc}A_0^bA_i^c(x) \;\equiv\; \mathcal{E}_i^a(t,\vec{x})
\end{aligned}$$

The gluon field strength $\mathcal{E}_a^i = -G_a^{0i}$ is sometimes called the "chromoelectric field" (and $\mathcal{B}_i \equiv \frac{1}{2}\epsilon_{0ijk}G^{jk}$ "chromomagnetic field")

$$\begin{aligned}
\hat{H}(t) \;&=\; \int d^3x\,\left[\pi_i^a(t,\vec{x})\dot{A}^{ai}(t,\vec{x}) + i\psi_f^{k\dagger}(t,\vec{x})\dot{\psi}_f^k(t,\vec{x}) - \mathcal{L}(t,\vec{x})\right] \qquad (24.20)\\
&=\; \int d^3x\Big[\frac{1}{2}[\hat{\vec{\mathcal{E}}}^a\cdot\hat{\vec{\mathcal{E}}}^a(t,\vec{x}) + \vec{\mathcal{B}}^a\cdot\vec{\mathcal{B}}^a(t,\vec{x})] + \;\psi_f^{k\dagger}[m\delta^{kl} - i\delta^{kl}\vec{\gamma}\cdot\vec{\nabla} - gt_{kl}^a\vec{\gamma}\cdot\vec{A}^a]\psi_f^l(t,\vec{x})\Big]
\end{aligned}$$

(cf. Eq. (20.17) for QED). As in QED, the scalar potential $A_0^a(x) \equiv \Phi^a(x)$ is not an independent variable but is expressed through equation of motion (Gauss law)

$$\begin{aligned}
D^iG_{i0} \;&=\; -g\bar{\psi}t^a\gamma_0\psi \qquad (24.21)\\
\Leftrightarrow\;[\delta^{ab}\nabla^2 - 2gf^{abc}(\vec{A}^c\cdot\vec{\nabla}) - g^2f^{acn}f^{bdn}\vec{A}^m\cdot\vec{A}^d]A_0^b \;&=\; g\bar{\psi}t^a\gamma_0\psi - gf^{abc}\vec{A}^b\cdot\dot{\vec{A}}^c
\end{aligned}$$

as

$$A_0^a(t,\vec{x}) \;=\; \Phi^a(t,\vec{x}) \;=\; -g\sum_{\text{flavors}}\int d^3x'\frac{\psi^{k\dagger}t_{kl}^a\psi^l(t,\vec{x}') - f^{abc}\vec{A}^b\cdot\dot{\vec{A}}^c(t,\vec{x}')}{4\pi|\vec{x}-\vec{x}'|} + O(g^2) \tag{24.22}$$

Classical QCD Hamiltonian in Coulomb gauge is a sum of 8 QED Hamiltonians (20.20) with the new interaction terms due to gluon self-interactions (24.16)

$$\begin{aligned}
\hat{H}(t) \;=\; &\int d^3x\Big[\frac{1}{2}[\hat{\vec{E}}_{\rm tr}^a\cdot\hat{\vec{E}}_{\rm tr}^a(t,\vec{x}) + \vec{B}^a\cdot\vec{B}^a(t,\vec{x})] + \frac{1}{2}\vec{\nabla}\Phi^a(t,\vec{x})\cdot\vec{\nabla}\Phi^a(t,\vec{x}) \qquad (24.23)\\
&+ \psi_f^{k\dagger}[m\delta^{kl} - i\delta^{kl}\vec{\gamma}\cdot\vec{\nabla} - gt_{kl}^a\vec{\gamma}\cdot\vec{A}^a]\psi_f^l(t,\vec{x}) + gf^{abc}\partial^\mu A^{a\nu}A_\mu^bA_\nu^c(t,\vec{x}) + \;g^2f^{abn}f^{cdn}A_\mu^aA_\nu^bA^{c\mu}A^{d\nu}(t,\vec{x})\Big]
\end{aligned}$$

where $a = 1\div 8$ for gluons and $k = 1,2,3$ for quarks (and flavor index $f = 1,2...$ for quarks)

24.3 Interaction picture

To quantize QCD in the interaction picture we repeat the procedure of Sect. 20.3 for QED. To this end we need to separate the Hamiltonian (24.23) in the free part and interaction part. As a first step, we separate field strength $G^a_{\mu\nu}$ (chromoelectric and chromomagnetic fields) in the Abelian part and non-Abelian parts:

$$\begin{aligned} G^a_{\mu\nu} &= F^a_{\mu\nu} + g f^{abc} A^b_\mu A^c_\nu, \qquad F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \mu \leftrightarrow \nu \\ \mathcal{E}^{ai} &= E^{ai} - g f^{abc} A^b_0 A^c_i, \qquad E^a_i \equiv -\dot{A}^i + \partial^i A_0, \qquad \mathcal{B}^{ai} = B^{ai} + g\epsilon_{0ijk} A^{bj} A^{ck}, \qquad B^{ai} \equiv \epsilon_{0ijk}\partial^j A^{ak} \end{aligned} \tag{24.24}$$

Next, as in QED, we separate the Hamiltonian (24.23) in four parts (cf. Eq. (20.36)):

$$\begin{aligned} H &= H_{\rm quark} + H_{\rm gluon} + H_{\rm int} + H_{\rm Coul}, \\ H_{\rm quark} &= \sum_{\rm flavors}\int d^3x\ \psi^{kf\dagger}(t,\vec{x})(m - i\vec{\gamma}\cdot\vec{\nabla})\psi^{kf}(t,\vec{x}) \\ H_{\rm gluon} &= \frac{1}{2}\int d^3x\ [\dot{\vec{A}}^a(t,\vec{x})\cdot\dot{\vec{A}}^a(t,\vec{x}) + \vec{B}^a(t,\vec{x})\cdot\vec{B}^a(t,\vec{x})] \\ H_{\rm int} &= \int d^3x\ \big[-g\hat{\bar{\psi}}^{kf}\vec{\gamma}\cdot\vec{A}^a t^a_k l\hat{\psi}^{lf}(t,\vec{x}) + g f^{abc}(\partial^i A^{aj})A^b_i A^c_j(t,\vec{x}) + \frac{g^2}{4} f^{abn} f^{cdn} A^a_i A^b_j A^{ci} A^{dj}(t,\vec{x})\big] \\ H_{\rm Coul} &= \frac{1}{2}\int d^3x\ (\vec{D}A^a_0(t,\vec{x}))^2 \;=\; -\frac{g}{2}\int d^3x\ A^a_0(\vec{x})[\psi^\dagger t^a\psi(t,\vec{x}) - f^{abc}\vec{A}^b\cdot\dot{\vec{A}}^c(t,\vec{x})] \\ &= \int d^3x d^3y\ [\psi^{k\dagger} t^a_{kl}\psi^l(t,\vec{x}) - f^{abc}\vec{A}^b\cdot\dot{\vec{A}}^c(t,\vec{x})]\frac{g^2}{8\pi|\vec{x}-\vec{y}|}[\psi^{k\dagger} t^a_{kl}\psi^l(t,\vec{y}) - f^{abc}\vec{A}^b\cdot\dot{\vec{A}}^c(t,\vec{y})] \;+\; O(g^3) \end{aligned} \tag{24.25}$$

where in the last line we used equation (24.27).

To quantize the theory with the Hamiltonian (24.25) we, as usually, promote canonical coordinates at $t=0$ to operators $\psi^{kf}(0,\vec{x}) \to \hat{\psi}^{kf}(\vec{x})$ and $A^i(t,\vec{x}) \to \hat{A}^a_i(\vec{x})$ satisfying canonical commutation relations (20.22)

$$\begin{aligned} \{\hat{\psi}^{kf}_\xi(\vec{x}), \hat{\psi}^{lh}_\eta\dagger(\vec{y})\} &= \delta_{\xi\eta}(\vec{x}-\vec{y})\delta^{kl}\delta^{fh}, \qquad \{\hat{\psi}^{kf}_\xi(\vec{x}), \hat{\psi}^{lg}_\eta(\vec{y})\} = \{\hat{\psi}^{kf\dagger}_\xi(\vec{x}), \hat{\psi}^{lg\dagger}_\eta(\vec{y})\} = 0 \\ [\hat{A}^{ai}(\vec{x}), (\hat{E}_{\rm tr})^{bj}(\vec{y})] &= \delta^{\rm tr}_{ij}\delta^{ab}\delta(\vec{x}-\vec{y}), \qquad [\hat{A}^{ai}(\vec{x}), \hat{A}^{bj}(\vec{y})] = [\hat{E}^{ai}_{\rm tr}(\vec{x}), \hat{E}^{bj}_{\rm tr}(\vec{y})] = 0, \\ [\hat{A}^{ai}(\vec{x}), \hat{\psi}^{kf}_\xi(\vec{y})] &= [\hat{A}^{ai}(\vec{x}), \hat{\psi}^\dagger_{kf}\xi(\vec{y})] = [(\hat{E}^{\rm tr})^a_i(\vec{x}), \hat{\psi}^{kf}(\vec{x}')] = [(\hat{E}^{\rm tr})^a_i(\vec{x}), \hat{\psi}^{kf}(\vec{x}')] = 0 \end{aligned} \tag{24.26}$$

Note that we imposed CCR between canonical coordinates and transverse Abelian parts of canonical momenta $(\hat{E}^{\rm tr})^a_i(\vec{x}) = -\dot{\hat{A}}^a_i$. As in QED, the scalar potential $\hat{A}^a_0$ is not and independent dynamical variable since it is determined by quantum version of Eq. (24.27).

$$\hat{A}^a_0(\vec{x}) = \hat{\Phi}^a(\vec{x}) = -g\sum_{\rm flavors}\int d^3x'\,\frac{\hat{\psi}^{k\dagger}(\vec{x}')t^a_{kl}\hat{\psi}^l(\vec{x}') + f^{abc}\vec{A}^b(\vec{x}')\cdot\vec{E}^{\rm tr}(\vec{x}')}{4\pi|\vec{x}-\vec{x}'|} + O(g^2) \tag{24.27}$$

The expansion in ladder operators is constructed similarly to Eq. (20.39):

$$\begin{aligned} \hat{\psi}^{kf}_\xi(x) &= \sum_s\int\frac{d\!\!{}^-{}^3p}{\sqrt{2E_p}}\big[u_\xi(\vec{p},s)e^{i\vec{p}\cdot\vec{x}}\hat{a}^{skf}_{\vec{p}} + v_\xi(\vec{p},s)e^{-i\vec{p}\cdot\vec{x}}\hat{b}^{skf\dagger}_{\vec{p}}\big]\Big|_{E_p=\sqrt{m_f^2+\vec{p}^2}}, \\ \hat{\bar{\psi}}^{kf}_{I\xi}(\vec{x}) &= \sum_s\int\frac{d\!\!{}^-{}^3p}{\sqrt{2E_p}}\big[\bar{v}_\xi(\vec{p},s)e^{i\vec{p}\cdot\vec{x}}\hat{b}^{skf}_{\vec{p}} + \bar{u}_\xi(\vec{p},s)e^{-i\vec{p}\cdot\vec{x}}\hat{a}^{skf\dagger}_{\vec{p}}\big]\Big|_{E_p=\sqrt{m_f^2+\vec{p}^2}}, \\ (\hat{A}_I)^a_i(x) &= e^{it\hat{H}_{\rm gluon}}\hat{A}^a_i(\vec{x})e^{-it\hat{H}_{\rm gluon}} = \sum_{\lambda=1,2}\int\frac{d\!\!{}^-{}^3k}{\sqrt{2\omega_k}}\ e^\lambda_i(\vec{k})\big(\hat{a}^{a\lambda}_{\vec{k}}e^{i\vec{k}\cdot\vec{x}} + \hat{a}^{a\lambda\dagger}_{\vec{k}}e^{-i\vec{k}\cdot\vec{x}}\big)\Big|_{\omega_k=|\vec{k}|} \end{aligned} \tag{24.28}$$

Similarly to QED, the commutation relations between ladder operators

$$[\hat a^{a\lambda}_{\vec k},\hat a^{b\lambda'\dagger}_{\vec k'}] ~=~ (2\pi)^3\delta(\vec k-\vec k')\delta^{ab}\delta^{\lambda\lambda'} \tag{24.29}$$
$$\{\hat a^{kfs}_{\vec p},\hat a^{lf's'\dagger}_{\vec p'}\} ~=~ \{\hat b^{kfs}_{\vec p},\hat b^{lf's'\dagger}_{\vec p'}\} ~=~ (2\pi)^3\delta^{kl}\delta^{ff'}\delta_{ss'}\delta(\vec p-\vec p')$$

(with all other (anti)commutators being 0) lead to CCRs (24.26)

The quantum version of Coulomb-gauge QCD Hamiltonian (24.25) reads

$$\begin{aligned}
\hat H ~&=~ \hat H_{\text{quark}}+\hat H_{\text{gluon}}+\hat H_{\text{int}}+\hat H_{\text{Coul}}, \qquad (24.30)\\
\hat H_{\text{quark}} ~&=~ \int d^3x~\hat\psi^{k\dagger}(\vec x)(m-i\vec\gamma\cdot\vec\nabla)\hat\psi^k(\vec x)\\
\hat H_{\text{gluon}} ~&=~ \frac12\int d^3x~[\dot{\hat{\vec A}}{}^a(\vec x)\cdot\dot{\hat{\vec A}}{}^a(\vec x)+\vec B^a(\vec x)\cdot\hat{\vec B}{}^a(\vec x)]\\
\hat H_{\text{int}} ~&=~ \int d^3x~\big[-g\bar{\hat\psi}\vec\gamma\cdot\hat{\vec A}\hat\psi(\vec x)+gf^{abc}(\partial^i\hat A^{aj})A^b_i\hat A^c_j(\vec x)+\frac{g^2}{4}f^{abn}f^{cdn}\hat A^a_i\hat A^b_j\hat A^{ci}\hat A^{dj}(\vec x)\big]\\
\hat H_{\text{Coul}} ~&=~ \frac12\int d^3x~(\vec D\hat A^a_0(\vec x))^2 ~=~ -\frac g2\int d^3x~\hat A^a_0(\vec x)\big[\hat\psi^\dagger t^a\hat\psi(\vec x)-f^{abc}\hat{\vec A}{}^b\cdot\dot{\hat{\vec A}}{}^c(\vec x)\big]\\
=~ &\int d^3xd^3y~[\hat\psi^{k\dagger}t^a_{kl}\hat\psi^l(\vec x)-f^{abc}\hat{\vec A}{}^b\cdot\dot{\hat{\vec A}}{}^c(\vec x)]\frac{g^2}{8\pi|\vec x-\vec y|}[\hat\psi^{k\dagger}t^a_{kl}\hat\psi^l(\vec y)-f^{abc}\hat{\vec A}{}^b\cdot\dot{\hat{\vec A}}{}^c(\vec y)] ~+~ O(g^3)
\end{aligned}$$

The Heisenberg operators are defined by usual formulas

$$\begin{aligned}
&\hat A^a_\mu(t,\vec x) ~\equiv~ e^{i\hat Ht}\hat A^a_\mu(\vec x)e^{-i\hat Ht}, \qquad \hat\pi^a_i(t,\vec x) ~=~ e^{i\hat Ht}\hat\pi^a_i(\vec x)e^{-i\hat Ht} ~=~ e^{i\hat Ht}\hat E^a_i(\vec x)e^{-i\hat Ht} ~\equiv~ \hat E^a_i(t,\vec x)\\
&\hat\psi^k(t,\vec x) ~\equiv~ e^{i\hat Ht}\hat\psi^k(\vec x)e^{-i\hat Ht}, \qquad \bar{\hat\psi}{}^k(t,\vec x) ~\equiv~ e^{i\hat Ht}\bar{\hat\psi}{}^k(\vec x)e^{-i\hat Ht} \qquad (24.31)
\end{aligned}$$

It is easy to see that the Heisenberg operators satisfy the equal-time commutation relations (24.26) (cf. Eq. (7.14)), and, repeating the argument of Eq. (7.15) we see that $\hat H(t) ~\equiv~ \int d^3x\hat H(\psi(t,\vec x),A(t,\vec x))$ does not depend on time. In addition, after some algebra it can be demonstrated that Heisenberg operators satisfy the same equations of motion (24.9) as their classical counterparts (cf. Eq. (20.34)):

$$\big(D^\mu\hat G_{\mu\nu}\big)^a = -g\sum_f\bar{\hat\psi}{}^{kf}(t^a)_{kl}\gamma_\nu\hat\psi^{lf},$$
$$(i\hat{\not{\mathcal D}}-m_f)\hat\psi^{kf}(x)=0, \qquad \bar{\hat\psi}(i\overleftarrow{\hat{\not{\mathcal D}}}+m_f)\psi^{kf}(x)=0 \tag{24.32}$$

where $\hat{\mathcal D}_\mu\hat\psi^k_f(x)\equiv[\partial_\mu\delta^{kl}-igt^a\hat A^a(x)]\hat\psi^l_f(x)$. (As we noted above, the flavor index f is just a label so it can be put up or down as seems convenient).

To build the interaction representation, we define "perturbative Hamiltonian" as

$$\hat H_0 ~\stackrel{\text{def}}{=}~ \hat H_{\text{quark}}+\hat H_{\text{gluon}} \tag{24.33}$$

then $\hat H ~=~ \hat H_0+\hat H_{\text{int}}+\hat H_{\text{Coul}}$.

Operators in the interaction representation are defined as usual (note that Eq. (24.26) $\Rightarrow~[\hat H_{\text{quark}},\hat H_{\text{gluon}}] ~=~ 0$)

$$\begin{aligned}
\hat A^{ai}_I(z) ~&=~ e^{iz_0\hat H_0}\hat A^{ai}(\vec z)e^{-iz_0\hat H_0} ~=~ e^{iz_0\hat H_{\text{gluon}}}\hat A^i(\vec z)e^{-iz_0\hat H_{\text{gluon}}}\\
\hat\psi_I(z) ~&=~ e^{i\hat H_0z_0}\hat\psi(\vec z)e^{-i\hat H_0z_0} ~=~ e^{iz_0\hat H_{\text{quark}}}\hat\psi(\vec z)e^{-iz_0\hat H_{\text{quark}}}\\
\bar{\hat\psi}{}^{kf}_I(z) ~&=~ e^{i\hat H_0z_0}\bar{\hat\psi}{}^{kf}(\vec z)e^{-i\hat H_0z_0} ~=~ e^{iz_0\hat H_{\text{quark}}}\bar{\hat\psi}{}^{kf}(\vec z)e^{-iz_0\hat H_{\text{quark}}} \qquad (24.34)
\end{aligned}$$

These operators can be expressed in terms of ladder ladder operators similarly to Eq. (20.39):

$$\begin{aligned}
\hat\psi^{kf}_{I\xi}(x) &= e^{it\hat H_{\rm quark}}\hat\psi^{kf}(\vec x)e^{-it\hat H_{\rm quark}} = \sum_s\int\frac{d\!\!{}^-{}^3p}{\sqrt{2E_p}}\big[u_\xi(\vec p,s)e^{-ipx}\hat a^{skf}_{\vec p}+v_\xi(\vec p,s)e^{ipx}\hat b^{skf\dagger}_{\vec p}\big]\Big|_{p_0=E_p},\\
\hat{\bar\psi}^{kf}_{I\xi}(\vec x) &= e^{it\hat H_{\rm quark}}\hat{\bar\psi}^{kf}(\vec x)e^{-it\hat H_{\rm quark}} = \sum_s\int\frac{d\!\!{}^-{}^3p}{\sqrt{2E_p}}\big[\bar v_\xi(\vec p,s)e^{-ipx}\hat b^{skf}_{\vec p}+\bar u_\xi(\vec p,s)e^{ipx}\hat a^{skf\dagger}_{\vec p}\big]\Big|_{p_0=E_p},\\
(\hat A_I)^a_i(x) &= e^{it\hat H_{\rm gluon}}\hat A^a_i(\vec x)e^{-it\hat H_{\rm gluon}} = \sum_{\lambda=1,2}\int\frac{d\!\!{}^-{}^3k}{\sqrt{2\omega_k}}\,e^\lambda_i(\vec k)\big(\hat a^{a\lambda}_{\vec k}e^{-ikx}+\hat a^{a\lambda\dagger}_{\vec k}e^{ikx}\big)\Big|_{k_0=\omega_k=|\vec k|}
\end{aligned}\tag{24.35}$$

Now we define $|\Omega\rangle$ as a true QCD vacuum (= lowest-energy eigenstate of Hamiltonian (24.30) and $|0\rangle$ as a "perturbative vacuum" which is lowest-energy eigenstate of H_0. Since H_0 can be represented as

$$\begin{aligned}
\hat H_0 &= \hat H_{\rm gluon}+\hat H_{\rm quark}\\
&= \int d\!\!{}^-{}^3k\;\omega_k\sum_{\lambda,a}\hat a^{a\lambda\dagger}_{\vec k}\hat a^{a\lambda}_{\vec k}\Big|_{\omega_k=|\vec k|}+\sum_{s,k,flavors}\int d\!\!{}^-{}^3p\;E^f_p\big(\hat a^{skf\dagger}_{\vec p}\hat a^{skf}_{\vec p}+\hat b^{skf\dagger}_{\vec p}\hat b^{skf}_{\vec p}\big)\Big|_{E^f_p=\sqrt{m_f^2+\vec p^2}}
\end{aligned}\tag{24.36}$$

(cf. Eqs (19.37) and (14.16)) we see that the perturbative vacuum is annihilated by $\hat a^{a\lambda}_{\vec k}$, $\hat a^{skf}_{\vec p}$, and $\hat b^{skf}_{\vec p}$:

$$\hat a^{a\lambda}_{\vec k}|0\rangle = \hat a^{skf}_{\vec p}|0\rangle = \hat b^{skf}_{\vec p}|0\rangle = 0\tag{24.37}$$

Similarly to QED, the states

$$|k,a\rangle = \sqrt{2\omega_k},\qquad |k,s,f\rangle = \sqrt{2E_f}\hat a^{a\lambda\dagger}_{\vec k}|0\rangle,\qquad |k,s,f\rangle = \sqrt{2E_f}\hat b^{skf}_{\vec p}|0\rangle\tag{24.38}$$

are one-gluon, one-quark and one-antiquark states. Similarly, one can define states of multiple free quarks and gluons as an eigenstates of $\hat H_0$. [34]

The Green functions in QCD are defined as matrix elements of T-product of operators switched between true-vacuum states

$$\begin{aligned}
&G\big(x,...x^{(m)},y,...y^{(n)},z,...z^{(l)}\big)\\
&= \langle\Omega|\mathrm{T}\{\prod_{m=1}^{m'}\hat\psi^{k_{m'}}_{f_{m'}}(x^{(m')})\prod_{n'=1}^{n}\hat{\bar\psi}^{k_{n'}}_{f_{n'}}(y^{(n')})\prod_{l'=1}^{l}\hat A^{a_{l'}}_{\mu_{l'}}(z^{(l')})\}|\Omega\rangle
\end{aligned}\tag{24.39}$$

To find the perturbative expansion of these Green functions we will use our old trick (9.29)

$$|\Omega\rangle = \lim_{\epsilon\to0}\lim_{\tau\to\infty}\frac{1}{e^{-iE_0T}\langle\Omega|0\rangle}\hat U(0,-T)|0\rangle\Big|_{T=\tau(1-i\epsilon)}\tag{24.40}$$

[34] There are no free quarks or gluons in Nature. The observed particles are hadrons which are (unseparable) bound states of quarks and gluons. Despite that, sometimes it is convenient to pretend that there is no confinement and calculate cross sections of production of quarks and gluons. For example, the amplitude of e^+e^- annihilation into hadrons can be calculated at high energies as as a square of $\mathcal{M}$-matrix of transitions between electron-positron and quark-antiquark (and gluon in higher orders in g^2) states. The usual lore is that quarks and gluons transfer to hadrons with probability 1 and the kinematical properties of a high-energy process are not affected by this "quarks+gluons → hadrons" transition

which is correct in any theory with non-degenerate vacuum. Repeating the usual steps, we get

$$\langle\Omega|\mathrm{T}\{\prod_{m=1}^{m'}\hat{\psi}^{k_{m'}}_{f_{m'}}(x^{(m')})\prod_{n'=1}^{n}\hat{\bar{\psi}}^{k_{n'}}_{f_{n'}}(y^{(n')})\prod_{l'=1}^{l}\hat{A}^{a_{l'}}_{\mu_{l'}}(z^{(l')})\}|\Omega\rangle \tag{24.41}$$
$$= \frac{\langle 0|\mathrm{T}\{\prod_{m=1}^{m'}\hat{\psi}^{k_{m'}}_{If_{m'}}(x^{(m')})\prod_{n'=1}^{n}\hat{\bar{\psi}}^{k_{n'}}_{If_{n'}}(y^{(n')})\prod_{l'=1}^{l}\hat{A}^{a_{l'}}_{I\mu_{l'}}(z^{(l')})e^{-i\int dt\ (\hat{H}_{\mathrm{I}}(t)+\hat{H}_{\mathrm{C}}(t))}\}|0\rangle}{\langle 0|\mathrm{T}\{e^{-i\int dt\ (\hat{H}_{\mathrm{I}}(t)+\hat{H}_{\mathrm{C}}(t))}\}|0\rangle}$$

where $\hat{H}_I$ and $\hat{H}_C$ are $\hat{H}_{\mathrm{int}}$ and $\hat{H}_{\mathrm{Coulomb}}$ made from interaction-representation operators (24.35).

Now we can expand the r.h.s. of Eq. (24.48) in powers of e ($\Leftrightarrow$ in powers of $\hat{H}_I$ and $\hat{H}_C$) and use Wick's theorem to get all possible Feynman diagrams. The contractions are:
1. Quark propagator

$$\widehat{\hat{\psi}^{kf}_{\xi}(x)\hat{\bar{\psi}}}{}^{lf'}_{\eta}(y) \;=\; \langle 0|\mathrm{T}\{\hat{\psi}^{kl}_{\xi}(x)\hat{\bar{\psi}}^{lf'}_{\eta}(y)\}|0\rangle \;=\; \delta^{kl}\delta^{ff'}\int\frac{đ^4p}{i}e^{-ip(x-y)}\frac{m_f+\not{p}}{m_f^2-p^2-i\epsilon} \tag{24.42}$$

and
2. Propagator of transverse gluon (cf Eq. (20.48))

$$\begin{aligned}
\widehat{\hat{A}^i_a(x)\hat{A}}{}^j_b(y) \;&=\; \langle 0|\mathrm{T}\{\hat{A}^i_a(x)\hat{A}^j_b(y)\}|0\rangle \;=\; \delta_{ab}\langle 0|\mathrm{T}\{\hat{\vec{A}}^a_i(x)\hat{\vec{A}}^b_j(y)\}|0\rangle \\
&=\; \delta_{ab}\int\frac{đ^3k}{2\omega_k}\sum_{\lambda=1,2}\vec{e}^{\,\lambda}_i(\vec{k})\vec{e}^{\,\lambda}_j(\vec{k})\big(\theta(x_0-y_0)e^{-ik(x-y)}+\theta(y_0-x_0)e^{ik(x-y)}\big) \\
&=\; \delta^{ab}\int\frac{đ^4k}{i}\frac{1}{k^2+i\epsilon}e^{-ik(x-y)}\Big(g^{ij}+\frac{k^ik^j}{\vec{k}^2}\Big) \;=\; \delta^{ab}D^{ij}_{\mathrm{tr}}(x-y)
\end{aligned} \tag{24.43}$$

(in what follows we omit index "I" of the interaction representation). This transverse propagator can be expressed as (20.50)

$$\begin{aligned}
D^{\mu\nu}_{\mathrm{tr}}(x-y) \;&=\; D^{\mu\nu}_F(x-y)+D^{\mu\nu}_{\mathrm{Ward}}(x-y)+D^{\mu\nu}_{\mathrm{inst}}(x-y) \\
D^{\mu\nu}_F(x-y) \;&=\; \int\frac{đ^4k}{i}\frac{g^{\mu\nu}}{k^2+i\epsilon}e^{-ik(x-y)} \\
D^{\mu\nu}_{\mathrm{W}}(x-y) \;&=\; \int\frac{đ^4k}{i}\frac{g^{\mu\nu}}{k^2+i\epsilon}e^{-ik(x-y)}\Big(\frac{k^\mu k^\nu}{\vec{k}^2}-\frac{k^0}{\vec{k}^2}(k^\mu\eta^\nu+k^\nu\eta^\mu)\Big) \\
D^{\mu\nu}_{\mathrm{inst}}(x-y) \;&=\; \int\frac{đ^4k}{i}\frac{\eta^\mu\eta^\nu}{\vec{k}^2}e^{-ik(x-y)} \;=\; -i\delta(x_0-y_0)\int đ^3k\frac{\eta^\mu\eta^\nu}{\vec{k}^2}e^{i\vec{k}(\vec{x}-\vec{y})} \;=\; -i\delta(x_0-y_0)\frac{\eta^\mu\eta^\nu}{4\pi|\vec{x}-\vec{y}|}
\end{aligned} \tag{24.44}$$

where $\eta=(1,0,0,0)$. In QED, the contribution of $D^{\mu\nu}_{\mathrm{inst}}$ canceled with the terms arising from the expansion of Coulomb term $\hat{H}_C$ in Eq. (24.48) and the contribution of $D^{\mu\nu}_{\mathrm{W}}$ vanished due to Ward identity (20.65). In QCD the situation is more subtle: the Ward identity is (slightly) different so the contribution of longitudinal unphysical gluons is not canceled in the gluon loops.

It was proved (using functional integrals) that a good way to memorize these need-to-be-subtracted longitudinal remnants in gluon loops is to introduce a new term in the

Lagrangian

$$\Delta\mathcal{L} \;=\; \bar{c}(-\partial_\mu D^\mu)^{mn}c^n \;=\; -\,\bar{c}^n(x)\partial^2 c^n(x) + ig\bar{c}^m(x)t^a_{mn}\partial^\mu\big(A^a_\mu(x)c^n(x)\big) \qquad (24.45)$$

with the condition that new "ghost" massless scalar particle c live only in loops (and there are eight of them, $a = 1 \div 8$). In addition, there is a factor (-1) for any ghost loop so one may consider the ghost particle as a scalar fermion (the spin-statistics theorem is not applicable to ghosts since they are not physical particles).

Summarizing, we can use the photon propagator $D^F_{\mu\nu}$ for gluons, take away H_C from the exponential in Eq. (24.48), and introduce the ghost particles (24.45) which can live only in loops. A good way to memorize this is to write down the "Lagrangian in the Feynman gauge"

24.3.1 QCD Lagrangian for practical calculations in Feynman gauge

The Lagrangian for QCD practitioners is $\mathcal{L}_{\rm QCD}$ + (gauge-fixing term $= -\frac{1}{2}\partial^\mu A^a_\mu\partial^\nu A^a_\nu$) + ghost term (24.45)

$$\mathcal{L}_F = -\frac{1}{2}{\rm Tr}\; G_{\mu\nu}G^{\mu\nu} + \sum_{\rm flavors}\bar{\psi}_f(i\not{D} - m_q)\psi_f - \frac{1}{2}\partial^\mu A^a_\mu\partial^\nu A^a_\nu + \bar{c}(-\partial_\mu D^\mu)^{mn}c^n \qquad (24.46)$$

It can be rewritten as a sum of free Lagrangian and interaction Lagrangian

$$\begin{aligned}
\mathcal{L}(x) \;&=\; \mathcal{L}_0(x) + \mathcal{L}_{\rm int}(x) \qquad (24.47)\\
\mathcal{L}_0(x) \;&=\; \frac{1}{2}A^a_\mu\partial^2 A^{a\mu}(x) + \sum_f \bar{\psi}^{kf}(i\not{\partial} - m_f)\psi^{kf} - \bar{c}^a\partial^2 c^a\\
\mathcal{L}_{\rm int}(x) \;&=\; g\sum_{\rm flavors}\bar{\psi}^k_f A^a t^a_{kl}\psi^l_f \;-\; gf^{abc}(\partial^\mu A^{a\nu})A^b_\mu A^c_\nu \;-\; \frac{g^2}{4}f^{abn}f^{cdn}A^a_\mu A^b_\nu A^{c\mu}A^{d\nu} + ig\bar{c}^m t^a_{mn}\partial^\mu\big(A^a_\mu c^n\big)
\end{aligned}$$

The interaction representation of Green functions is

$$\begin{aligned}
&\langle\Omega|{\rm T}\{\prod_{m=1}^{m'}\hat{\psi}^{k_{m'}}_{f_{m'}}(x^{(m')})\prod_{n'=1}^{n}\hat{\bar{\psi}}^{k_{n'}}_{f_{n'}}(y^{(n')})\prod_{l'=1}^{l}\hat{A}^{a_{l'}}_{\mu_{l'}}(z^{(l')})\}|\Omega\rangle \qquad (24.48)\\
&= \frac{\langle 0|{\rm T}\{\prod_{m=1}^{m'}\hat{\psi}^{k_{m'}}_{If_{m'}}(x^{(m')})\prod_{n'=1}^{n}\hat{\bar{\psi}}^{k_{n'}}_{If_{n'}}(y^{(n')})\prod_{l'=1}^{l}\hat{A}^{a_{l'}}_{I\mu_{l'}}(z^{(l')})e^{i\int d^4z\;(\hat{\mathcal{L}}_{\rm int}(z))}\}|0\rangle}{\langle 0|{\rm T}\{e^{i\int d^4z\;\hat{\mathcal{L}}_{\rm in}(z)}\}|0\rangle}
\end{aligned}$$

where all the operators in the r.h.s. are in the interaction representation, $\mathcal{L}_{\rm int}$ is given by Eq. (24.47) and the propagators

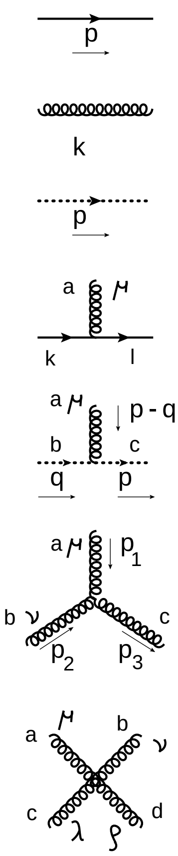
$$\begin{aligned}
\widehat{\hat{\psi}^{kf}_\xi(x)\hat{\bar{\psi}}^{lf'}_\eta}(y) \;&=\; \langle 0|{\rm T}\{\hat{\psi}^{kl}_\xi(x)\hat{\bar{\psi}}^{lf'}_\eta(y)\}|0\rangle \;=\; \delta^{kl}\delta^{ff'}\int\frac{{d\!\!\!{}^-}^4p}{i}e^{-ip(x-y)}\frac{m_f + \not{p}}{m^2 - p^2 - i\epsilon} \qquad (24.49)\\
\widehat{\hat{A}^\mu_a(x)\hat{A}^\nu_b}(y) \;&=\; \langle 0|{\rm T}\{\hat{A}^i_a(x)\hat{A}^j_b(y)\}|0\rangle \;=\; \delta^{ab}\int\frac{{d\!\!\!{}^-}^4k}{i}\frac{g^{\mu\nu}}{k^2 + i\epsilon}e^{-ik(x-y)} \;=\; \delta^{ab}D^{\mu\nu}_F(x-y)\\
\widehat{\hat{c}^k(x)\hat{\bar{c}}^l}(y) \;&=\; \delta^{ab}\int\frac{{d\!\!\!{}^-}^4k}{i}\frac{-1}{k^2 + i\epsilon}e^{-ik(x-y)} \qquad (24.50)
\end{aligned}$$

follow from the Feynman-gauge Lagrangian $\mathcal{L}_0$ in th Eq. (24.47). In addition, the operators $\hat{c}$ and $\hat{\bar{c}}$ are fermions, so they anticommute (and there is (-1) for any ghost loop as usual for fermions).

24.3.2 Feynman rules for QCD

Let us summarize Feynman rules for reduced Green functions in QCD. (As usual, the reduced Green function is defined as $G(p_1, ...p_n) = (-i)^{n-1}(2\pi)^4\delta(\sum p_i)\mathcal{G}(p_1, ..., p_n)$)

Feynman rules for QCD



$\dfrac{1}{m - \not{p} - i\epsilon}$	Quark propagator
$\dfrac{g^{\mu\nu}\delta^{ab}}{k^2 + i\epsilon}$	Gluon propagator in Feynman gauge
$\dfrac{\delta^{ab}}{-p^2 - i\epsilon}$	Ghost propagator
$g\gamma_\mu t^a_{kl}$	Quark-gluon vertex
$igf^{abc}p_\mu$	Ghost-gluon vertex
$-igf^{abc}[(p_1 - p_2)_\lambda g_{\mu\nu} + (p_2 - p_3)_\mu g_{\nu\lambda} + (p_3 - p_1)_\nu g_{\lambda\mu}]$	Triple gluon vertex
$-g^2[f^{abm}f^{cdm}(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}) + f^{acm}f^{bdm}(g_{\mu\nu}g_{\lambda\rho} - g_{\mu\rho}g_{\nu\lambda}) + f^{adm}f^{bcm}(g_{\mu\nu}g_{\lambda\rho} - g_{\mu\lambda}g_{\nu\rho})]$	Four-gluon vertex

Figure 35. Feynman rules for reduced Green functions in QCD

As usually, there is also $\int\frac{\not{d}k}{i}$ for each gluon loop and $-\int\frac{\not{d}p}{i}$ for each quark and ghost loop.

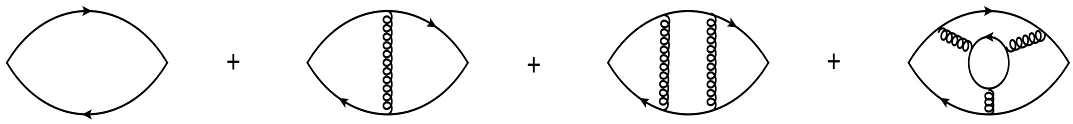
Thus, as in QED, we have perturbation theory for Green functions. However, unlike QED, we do not observe quarks and gluons (in the initial or final stages of a scattering process). Instead, we see only hadrons in our detectors. This property is called "color confinement".

Color confinement: only particles which are singlets with respect to color SU(3) group can be observed. Colored particles are confined within their interaction range ($\sim$ 1 fermi).

How to calculate the mass of, say, ρ-meson in QCD: take some current $j_\mu(x)$ with quantum numbers of ρ-meson (e.g. $\bar{u}\gamma_\mu u(x) - \bar{d}\gamma_\mu d(x)$ for ρ^0) and calculate

$$i\int d^4x \langle\Omega|\mathrm{T}\{j_\mu(x)j_\nu(0)\}|\Omega\rangle \;= \tag{24.51}$$

$$=$$



$$+ \ldots \;=\; (p_\mu p_\nu - g_{\mu\nu})\Pi(p^2)$$

The "polarization operator" $\Pi(p^2)$ must have a pole at $p^2 \to m_\rho^2$ and the residue in this pole is $\langle\Omega|\rho\rangle|^2$. If somebody some day will calculate $\Pi(p^2)$ exactly, he/she will find a pole

$$\Pi(p^2) \;\; \stackrel{p^2\to m_\rho^2}{\to} \;\; \frac{\langle\Omega|j_\mu(0)|\rho\rangle|^2}{m_\rho^2 - p^2 - i\epsilon} \tag{24.52}$$

and determine the mass of the ρ-meson. For now, no one has an idea tow to do that.

Instead: approximate calculation in the Euclidean space using lattice simulations.

$$\Pi(p^2 = -P^2) \;\simeq\; \frac{\text{const}}{m_\rho^2 + P^2} \qquad \Rightarrow \qquad \Pi(x) \;\simeq\; e^{-m_\rho|x|} \tag{24.53}$$

Thus, one calculates the correlation function (24.51) in the Euclidean theory using lattice simulations of the functional integral for this correlation function and matches the behavior of this function in the coordinate space to $e^{-m_\rho|x|}$.

To calculate scattering amplitudes of hadron-hadron scattering in QCD is much more complicated. A typical meson-baryon scattering in QCD looks like

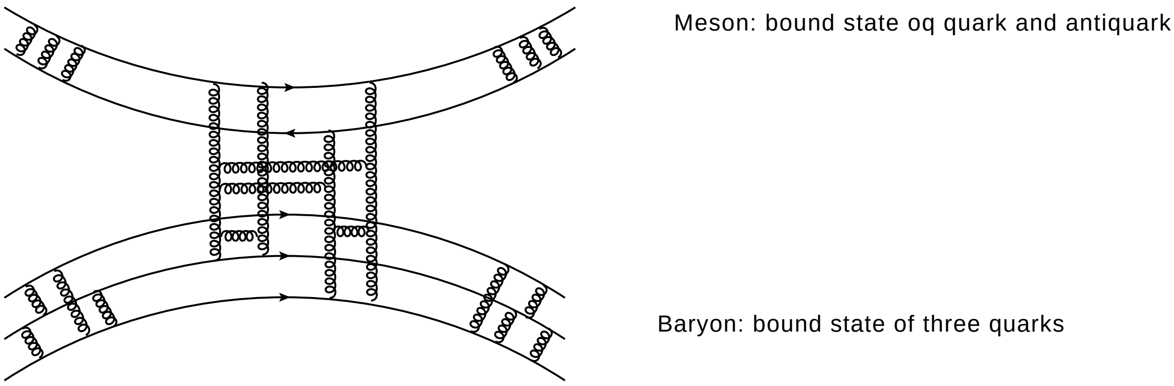


Figure 36. A cartoon of meson-baryon scattering in QCD

To get this amplitude we use factorization theorems to reduce it to simple matrix elements which can be calculated on the lattice.

24.4 Asymptotic freedom in QCD

25 Appendix

25.1 Dirac matrices and spinors in spinor representation

The set of Dirac matrices in the spinor representation is:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \tag{25.1}$$

Here $\sigma^\mu = (\sigma_0, \vec\sigma)$, $\bar\sigma^\mu = (\sigma_0, -\vec\sigma)$, where σ_0 is a unit matrix and σ_x, σ_y, and σ_z are Pauli matrices. In the explicit form:

$$\gamma^0 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}, \tag{25.2}$$

where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{25.3}$$

The γ_5 matrix has the form:

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{25.4}$$

and it anticommute with all matrices γ^μ:

$$\gamma^\mu\gamma_5 = -\gamma_5\gamma^\mu \tag{25.5}$$

Master property of γ-matrices:

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu} \tag{25.6}$$

Consequences:

$$\begin{aligned}
\not a\ \not a &= a_\mu a^\mu \equiv a^2 \qquad \text{for any 4 − vector a} \\
\not a\gamma_\mu\ \not a &= 2a_\mu\ \not a - \gamma_\mu a^2 \\
\gamma_\mu\gamma_\alpha\gamma^\mu &= -2\gamma_\alpha \\
\gamma_\mu\gamma_\alpha\gamma_\beta\gamma^\mu &= 4g_{\alpha\beta} \\
\gamma_\mu\gamma_\alpha\gamma_\beta\gamma_\xi\gamma^\mu &= -2\gamma_\xi\gamma_\beta\gamma_\alpha
\end{aligned} \tag{25.7}$$

Traces:

$$\begin{aligned}
\mathrm{Tr}\{\gamma_\mu\gamma_\nu\} &= 4g_{\mu\nu} \\
\mathrm{Tr}\{\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho\} &= 4(g_{\mu\nu}g_{\lambda\rho} + g_{\mu\rho}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\rho}) \\
\mathrm{Tr}\{\gamma_\mu\gamma_\nu\gamma_5\} &= 0 \\
\mathrm{Tr}\{\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho\gamma_5\} &= 4i\epsilon_{\mu\nu\lambda\rho} \\
\mathrm{Tr}\{\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho\gamma_\xi\gamma_\eta\} &= 4\Big(g_{\mu\nu}(g_{\lambda\rho}g_{\xi\eta} + g_{\rho\xi}g_{\lambda\eta} - g_{\lambda\xi}g_{\rho\eta}) - g_{\mu\lambda}(g_{\nu\rho}g_{\xi\eta} + g_{\rho\xi}g_{\nu\eta} - g_{\nu\xi}g_{\rho\eta}) \\
&+ g_{\mu\rho}(g_{\nu\lambda}g_{\xi\eta} - g_{\nu\xi}g_{\lambda\eta} + g_{\lambda\xi}g_{\nu\eta}) - g_{\mu\xi}(g_{\rho\eta}g_{\nu\lambda} + g_{\nu\eta}g_{\lambda\rho} - g_{\nu\rho}g_{\lambda\eta}) \\
&+ g_{\mu\eta}(g_{\lambda\rho}g_{\nu\xi} + g_{\rho\xi}g_{\nu\lambda} - g_{\lambda\xi}g_{\nu\rho})\Big)
\end{aligned} \tag{25.8}$$

where ϵ is totally antisymmetric symbol ($\epsilon_{0123} = 1 = -\epsilon^{0123}$). Trace of any odd number of γ-matrices is zero.

Useful formula:

$$\epsilon_{\mu\nu\alpha\beta}\epsilon^{\alpha\beta\lambda\rho} = -2\left(\delta_\mu^\lambda\delta_\nu^\rho - \delta_\nu^\lambda\delta_\mu^\rho\right) \tag{25.9}$$

Complex conjugation:

$$\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0, \qquad \gamma_5^\dagger = \gamma_5 \tag{25.10}$$

and therefore

$$\begin{aligned}\left(\bar{u}(p)\gamma_{\mu_1}...\gamma_{\mu_n}u(p')\right)^\dagger &= \bar{u}(p')\gamma_{\mu_n}...\gamma_{\mu_1}u(p)\\ \left(\bar{v}(p)\gamma_{\mu_1}...\gamma_{\mu_n}v(p')\right)^\dagger &= \bar{v}(p')\gamma_{\mu_n}...\gamma_{\mu_1}v(p)\end{aligned} \tag{25.11}$$

The explicit form of the spinors with definite z- component of the spin in the rest frame $\lambda = \pm\frac{1}{2}$ is:

$$\begin{aligned}u^{(\frac{1}{2})}(p) &= \begin{pmatrix}\sqrt{p\sigma}\binom{1}{0}\\ \sqrt{p\bar{\sigma}}\binom{1}{0}\end{pmatrix} = \frac{1}{\sqrt{2(p_0+m)}}\begin{pmatrix}(m+p_0-\vec{p}\cdot\vec{\sigma})\binom{1}{0}\\ (m+p_0+\vec{p}\cdot\vec{\sigma})\binom{1}{0}\end{pmatrix}\\ u^{(-\frac{1}{2})}(p) &= \begin{pmatrix}\sqrt{p\sigma}\binom{0}{1}\\ \sqrt{p\bar{\sigma}}\binom{0}{1}\end{pmatrix} = \frac{1}{\sqrt{2(p_0+m)}}\begin{pmatrix}(m+p_0-\vec{p}\cdot\vec{\sigma})\binom{0}{1}\\ (m+p_0+\vec{p}\cdot\vec{\sigma})\binom{0}{1}\end{pmatrix}\end{aligned} \tag{25.12}$$

$$\begin{aligned}v^{(\frac{1}{2})}(p) &= \begin{pmatrix}-\sqrt{p\sigma}\binom{0}{1}\\ \sqrt{p\bar{\sigma}}\binom{0}{1}\end{pmatrix} = \frac{1}{\sqrt{2(p_0+m)}}\begin{pmatrix}(-m-p_0+\vec{p}\cdot\vec{\sigma})\binom{0}{1}\\ (m+p_0+\vec{p}\cdot\vec{\sigma})\binom{0}{1}\end{pmatrix}\\ v^{(-\frac{1}{2})}(p) &= \begin{pmatrix}\sqrt{p\sigma}\binom{1}{0}\\ -\sqrt{p\bar{\sigma}}\binom{1}{0}\end{pmatrix} = \frac{1}{\sqrt{2(p_0+m)}}\begin{pmatrix}(m+p_0-\vec{p}\cdot\vec{\sigma})\binom{1}{0}\\ (-m-p_0-\vec{p}\cdot\vec{\sigma})\binom{1}{0}\end{pmatrix}\end{aligned} \tag{25.13}$$

and

$$\begin{aligned}\bar{u}^{(\frac{1}{2})}(p) &= \frac{1}{\sqrt{2(p_0+m)}}\left((1,0)(m+p_0+\vec{p}\cdot\vec{\sigma});(1,0)(m+p_0-\vec{p}\cdot\vec{\sigma})\right)\\ \bar{u}^{(-\frac{1}{2})}(p) &= \frac{1}{\sqrt{2(p_0+m)}}\left((0,1)(m+p_0+\vec{p}\cdot\vec{\sigma});(1,0)(m+p_0-\vec{p}\cdot\vec{\sigma})\right)\end{aligned} \tag{25.14}$$

$$\begin{aligned}\bar{v}^{(\frac{1}{2})}(p) &= \frac{1}{\sqrt{2(p_0+m)}}\left((0,1)(m+p_0+\vec{p}\cdot\vec{\sigma});(0,1)(-m-p_0+\vec{p}\cdot\vec{\sigma})\right)\\ \bar{v}^{(-\frac{1}{2})}(p) &= \frac{1}{\sqrt{2(p_0+m)}}\left((1,0)(-m-p_0-\vec{p}\cdot\vec{\sigma});(1,0)(m+p_0-\vec{p}\cdot\vec{\sigma})\right)\end{aligned} \tag{25.15}$$

Here $u^{(\lambda}(p)$ and $\bar{v}^\lambda(p)$ are the spinors corresponding to electron and positron (respectively) with spin λ (in the rest frame) and $\bar{u}^{(\lambda}(p)$,$v^\lambda(p)$ are the Dirac conjugate spinors.

The spinors for the states with definite helicity $h = \pm\frac{1}{2}$ are:

$$u^{[\frac{1}{2}]}(p) = \frac{1}{\sqrt{2(p_0+m)}}\begin{pmatrix}(m+p_0-|\vec{p}|)\omega^{(1)}\\ (m+p_0+|\vec{p}|)\omega^{(1)}\end{pmatrix}, \quad u^{[-\frac{1}{2}]}(p) = \frac{1}{\sqrt{2(p_0+m)}}\begin{pmatrix}(m+p_0+|\vec{p}|)\omega^{(2)}\\ (m+p_0-|\vec{p}|)\omega^{(2)}\end{pmatrix} \tag{25.16}$$

$$v^{[\frac{1}{2}]}(p) = \frac{1}{\sqrt{2(p_0+m)}} \begin{pmatrix} (-m-p_0-|\vec{p}|)\omega^{(2)} \\ (m+p_0-|\vec{p}|)\omega^{(2)} \end{pmatrix}, \quad v^{[-\frac{1}{2}]}(p) = \frac{1}{\sqrt{2(p_0+m)}} \begin{pmatrix} (m+p_0-|\vec{p}|)\omega^{(1)} \\ (-m-p_0-|\vec{p}|)\omega^{(1)} \end{pmatrix} \tag{25.17}$$

where two-component spinor ω has the form:

$$\omega^{(1)} = \begin{pmatrix} e^{-i\alpha}\cos\left(\frac{\theta}{2}\right) \\ e^{i(\phi-\alpha)}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}, \qquad \omega^{(2)} = \begin{pmatrix} -e^{-i\alpha}\sin\left(\frac{\theta}{2}\right) \\ e^{i(\phi-\alpha)}\cos\left(\frac{\theta}{2}\right) \end{pmatrix} \tag{25.18}$$

where θ and ϕ are the polar and asimuthal angle of the momentum $\vec{p}$ and the phase α is arbitrary (it is convenient to choose $\alpha = \phi$ as in Eq. (13.53)).

Let us present also the explicit form of the Dirac conjugate spinors with definite helicity:

$$\begin{aligned} \bar{u}^{[\frac{1}{2}]}(p) &= \frac{1}{\sqrt{2(p_0+m)}} \left(\omega^{(1)\dagger}(m+p_0+|\vec{p}|), \quad \omega^{(1)\dagger}(m+p_0-|\vec{p}|)|\right) \\ \bar{u}^{[-\frac{1}{2}]}(p) &= \frac{1}{\sqrt{2(p_0+m)}} \left(\omega^{(2)\dagger}(m+p_0-|\vec{p}|), \quad \omega^{(2)\dagger}(m+p_0+|\vec{p}|)|\right) \end{aligned} \tag{25.19}$$

and

$$\begin{aligned} \bar{v}^{[\frac{1}{2}]}(p) &= \frac{1}{\sqrt{2(p_0+m)}} \left(\omega^{(2)\dagger}(m+p_0-|\vec{p}|), \quad \omega^{(2)\dagger}(-m-p_0-|\vec{p}|)\right) \\ \bar{v}^{[-\frac{1}{2}]}(p) &= \frac{1}{\sqrt{2(p_0+m)}} \left(\omega^{(1)\dagger}(-m-p_0-|\vec{p}|), \quad \omega^{(1)\dagger}(m+p_0-|\vec{p}|)\right) \end{aligned} \tag{25.20}$$

where

$$\begin{aligned} \omega^{1\dagger} &= \left(e^{i\alpha}\cos\frac{\theta}{2}, \quad e^{i(\alpha-\phi)}\sin\frac{\theta}{2}\right) \\ \omega^{2\dagger} &= \left(-e^{i\alpha}\sin\frac{\theta}{2}, \quad e^{i(\alpha-\phi)}\cos\frac{\theta}{2}\right) \end{aligned} \tag{25.21}$$

Properties of spinors:
1. Orthogonality

$$\begin{gathered} \bar{u}^{\lambda}(p)u^{\lambda'}(p) = 2m\delta_{\lambda\lambda'} = -\bar{v}^{\lambda}(p)v^{\lambda'}(p) \\ \bar{u}^{\lambda}(p)\gamma^{\mu}u^{\lambda'}(p) = \bar{v}^{\lambda}(p)\gamma^{\mu}v^{\lambda'}(p) = 2p^{\mu}\delta_{\lambda\lambda'} \\ \bar{u}^{\lambda}(p)v^{\lambda'}(p) = 0 = \bar{v}^{\lambda}(p)u^{\lambda'}(p) \end{gathered} \tag{25.22}$$

2.Completeness

$$\begin{gathered} \sum\nolimits_{\lambda=1,2} \left(u^{\lambda}_{\alpha}(p)\bar{u}^{\lambda}_{\beta}(p) - v^{\lambda}_{\alpha}(p)\bar{v}^{\lambda}_{\beta}(p)\right) = \delta_{\alpha\beta} \\ \sum\nolimits_{\lambda=1,2} u^{\lambda}_{\alpha}(p)\bar{u}^{\lambda}_{\beta}(p) = (m+\not{p})_{\alpha\beta} \\ \sum\nolimits_{\lambda=1,2} v^{\lambda}_{\alpha}(p)\bar{v}^{\lambda}_{\beta}(p) = (\not{p}-m)_{\alpha\beta} \end{gathered} \tag{25.23}$$

If s_μ is a four-vector of spin of the particle, then

$$\bar{u}(p,s)\gamma^{\mu}\gamma_5 u(p,s) = -\bar{v}(p,s)\gamma^{\mu}\gamma_5 v(p,s) = 2ms_{\mu} \tag{25.24}$$

and also

$$u_\alpha(p,s)\bar{u}_\beta(p,s) = \left(\frac{1+\gamma_5\ \not{s}}{2}(\not{p}+m)\right)_{\alpha\beta}, \quad v_\alpha(p,s)\bar{v}_\beta(p,s) = \left(\frac{1+\gamma_5\ \not{s}}{2}(\not{p}-m)\right)_{\alpha\beta} \tag{25.25}$$

For the particle with helicity $\frac{1}{2}$ the 4-vector of spin is $s^\mu(p, h=\frac{1}{2}) = (\frac{|\vec{p}|}{m}, \frac{\vec{p}p_0}{|\vec{p}|m})$ and for the particle with helicity $-\frac{1}{2}$ it is $s^\mu(p, h=-\frac{1}{2}) = (-\frac{|\vec{p}|}{m}, -\frac{\vec{p}p_0}{|\vec{p}|m})$

Check of $\bar{u}^s(\vec{p})\gamma^0 v^{s'}(-\vec{p}) \ = \ \bar{v}^s(\vec{p})\gamma^0 u^{s'}(-\vec{p}) \ = \ 0$:

$$\bar{u}^{(\frac{1}{2})}(\vec{p})\gamma^0 v^{(-\frac{1}{2})}(-\vec{p}) \ = \ \frac{1}{2(p_0+m)}\big((1,0)(m+p_0+\vec{p}\cdot\vec{\sigma});(1,0)(m+p_0-\vec{p}\cdot\vec{\sigma})\big)\begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix}\begin{pmatrix} (m+p_0+\vec{p}\cdot\vec{\sigma})\binom{1}{0} \\ (-m-p_0+\vec{p}\cdot\vec{\sigma})\binom{1}{0} \end{pmatrix} \ = \ 0$$

$$\bar{v}^{(\frac{1}{2})}(\vec{p})\gamma^0 u^{(-\frac{1}{2})}(-\vec{p}) \ = \ \frac{1}{2(p_0+m)}\big((0,1)(m+p_0+\vec{p}\cdot\vec{\sigma});(0,1)(-m-p_0+\vec{p}\cdot\vec{\sigma})\big)\begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix}\begin{pmatrix} (m+p_0+\vec{p}\cdot\vec{\sigma})\binom{1}{0} \\ (m+p_0-\vec{p}\cdot\vec{\sigma})\binom{1}{0} \end{pmatrix} \ = \ 0 \tag{25.26}$$