Lorentz transformations

Reminder from AQM:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} \qquad \Lambda^{\mu}{}_{\nu} \equiv 4\times 4 \text{ matrix} \tag{351}$$

$$x'^2 = x^2 \Rightarrow \Lambda^{\mu}{}_{\alpha} \Lambda_{\mu}{}^{\beta} = \delta^{\beta}_{\alpha} \Leftrightarrow \Lambda^T \Lambda = 1 \tag{352}$$

$\Lambda^T \Lambda = 1$ means that $\det \Lambda = \overset{\pm}{\vee} 1$.

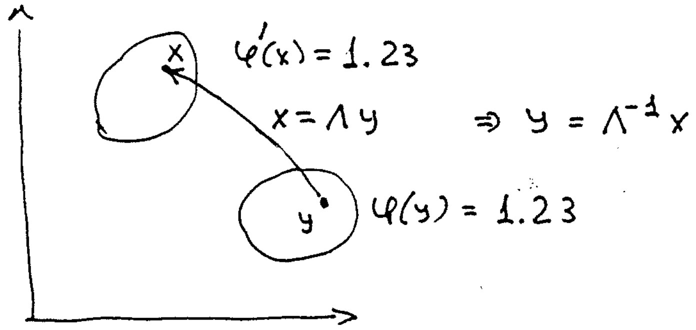
Lorentz transformations: rotations, boosts + + 3 discrete Lorentz transformations (P, T, and PT)

Lorentz invariance in a classical field theory

1. Scalar field

$$\varphi'(x) = \varphi(\Lambda^{-1} x) \qquad \text{„active rotation"} \tag{353}$$

(φ' is <u>not</u> a derivative, just the notation like p')



Action must be relativistic invariant (least action principle) $\Rightarrow \int d^4x\, \mathcal{L}(x) = \int d^4x'\, \mathcal{L}'(x')$ (354)

But

$$\int d^4x' = \int d^4(\Lambda x) = (\det \Lambda) \int d^4x = \int d^4x \tag{355}$$

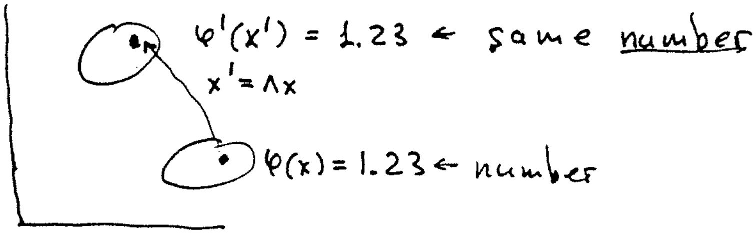
$\Rightarrow \mathcal{L}'(x') = \mathcal{L}(x)$ Lagrangian density must be relativistic invariant (356)

Let us check rel. invariance for the Klein-Gordon Lagrangian (36)

For the mass term rel. inv. is trivial

$$\varphi'(x') = \varphi(x) \Rightarrow$$

$$\Rightarrow \frac{m}{2}(\varphi'(x'))^2 = \frac{m}{2}\varphi^2$$



For the kinetic term

$$\frac{\partial}{\partial x'_\mu} = \Lambda^\mu{}_\nu \frac{\partial}{\partial x_\nu} \Rightarrow \qquad (357)$$

(Check: $\frac{\partial}{\partial x'_\mu} x'_\nu = \Lambda^\mu{}_\alpha \frac{\partial}{\partial x_\alpha}(\Lambda_\nu{}^\beta x_\beta) = \Lambda^\mu{}_\alpha \Lambda_\nu{}^\alpha = \delta^\mu_\nu$)

$$\Rightarrow \frac{\partial}{\partial x'_\mu}\varphi'(x')\frac{\partial}{\partial x'^\mu}\varphi'(x') = \left(\Lambda^\mu{}_\alpha \frac{\partial}{\partial x_\alpha}\right)\underbrace{\varphi'(x')}_{\overset{\parallel}{\varphi(x)}}\left(\Lambda_\mu{}^\beta \frac{\partial}{\partial x_\beta}\right)\underbrace{\varphi'(x')}_{\overset{\parallel}{\varphi(x)}} =$$

$$= \Lambda^\mu{}_\alpha \partial^\alpha \varphi(x)\Lambda_\mu{}^\beta \partial_\beta \varphi(x) = \partial^\mu \varphi(x)\partial_\mu \varphi(x) \Rightarrow \qquad (358)$$

$$\Rightarrow \mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2$$

is rel.-inv. (Same will be true of course for the Lagrangian for the self-interacting field $\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4$)

Equation of motion (Klein-Gordon equ.) is also rel.-inv.:

$$\frac{\partial^2}{\partial x'^\mu \partial x'_\mu}\varphi'(x') = \Lambda_\mu{}^\alpha \frac{\partial}{\partial x_\alpha}\Lambda^\mu{}_\beta \frac{\partial}{\partial x_\beta}\underbrace{\varphi'(x')}_{\overset{\parallel}{\varphi(x)}} = \Lambda^\mu{}_\beta \Lambda_\mu{}^\alpha \frac{\partial^2}{\partial x_\alpha \partial x_\beta}\varphi(x) =$$

$$= \frac{\partial^2}{\partial x_\alpha \partial x^\alpha}\varphi(x) \Rightarrow \qquad (359)$$

$$\frac{\partial}{\partial x'_\mu}\frac{\partial}{\partial x'^\mu}\varphi'(x') = \frac{\partial^2}{\partial x^\mu \partial x_\mu}\varphi(x) = -m^2\varphi(x) = -m^2\varphi'(x') \Rightarrow (\partial'^2 + \underset{\uparrow}{m^2})\varphi'(x') = 0$$

Form of the equation does not depend on a frame

Generators of Lorentz transformations

We will prove that

$$\Phi'(x) = \Phi(\Lambda^{-1}x) = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}\Phi(x)$$

We will demonstrate that the differential operator

$$e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}} \qquad (360)$$

where

$$J_{\mu\nu} = i(x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu}) \qquad (361)$$

generates Lorentz transformations

$$e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}} \varphi(x) = \varphi'(x) = \varphi(\Lambda^{-1}x) \qquad (362)$$

where the matrix of Lorentz transformation Λ is completely determined by 6 parameters $\omega_{\mu\nu}$

For boosts: take boost along x direction as an example

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & \theta & \\ -\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} \omega_{01} = \theta \\ \omega_{10} = -\theta \\ \text{other} = 0 \end{matrix} \quad \leftarrow \text{our guess} \qquad (363)$$

We must prove that

$$e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}} \phi(x^0,x^1) = e^{\omega_{01}J^{01}} \phi(x^0,x^1) = \phi(x^0 ch\theta - x^1 sh\theta, -x^0 sh\theta + x^1 ch\theta) \qquad (364)$$

where $\phi(x_0,x_1) = \varphi(x_0,x_1,x_2,x_3)$ at a certain x_2, x_3

Proof: let us write down the eqn. (364)

$$e^{\theta(x^0\frac{\partial}{\partial x_1} - x^1\frac{\partial}{\partial x_0})} \phi(x^0,x^1) = \phi(x^0 ch\theta - x^1 sh\theta, -x^0 sh\theta + x^1 ch\theta) \qquad (365)$$

differentiate l.h.s. and r.h.s. of eq. (365) with respect to θ and compare the differential equations

$$\frac{d}{d\theta}(\text{l.h.s.}) = (x^0\frac{\partial}{\partial x_1} - x^1\frac{\partial}{\partial x_0})(\text{l.h.s}) \qquad (366)$$

$$\frac{d}{d\theta}(\text{r.h.s.}) = \frac{\partial}{\partial\theta}\phi(x^0 ch\theta - x^1 sh\theta, -x^0 sh\theta + x^1 ch\theta) =$$

$$= (x^0 sh\theta - x^1 ch\theta)\left.\frac{\partial\phi(z^0,z^1)}{\partial z^0}\right|_{\substack{z_0 = x^0 ch\theta - x^1 sh\theta \\ z^1 = -x^0 sh\theta + x^1 ch\theta}} + (-x^0 ch\theta + x^1 sh\theta)\left.\frac{\partial\phi(z_0,z^1)}{\partial z^1}\right|_{\substack{z_0 = \cdots \\ z_1 = \cdots}} \qquad (367)$$

Now,

$$\frac{\partial}{\partial x^0}(\text{r.h.s}) = \frac{\partial}{\partial x^0}\phi(x^0\text{ch}\theta - x^1\text{sh}\theta, -x^0\text{sh}\theta + x^1\text{ch}\theta) =$$

$$= \text{ch}\theta\,\frac{\partial\phi(z^0,z^1)}{\partial z^0}\Bigg|_{\substack{z_0 = x^0\text{ch}\theta - x^1\text{sh}\theta \\ z_1 = -x^0\text{sh}\theta + x^1\text{ch}\theta}} - \text{sh}\theta\,\frac{\partial\phi(z^0,z^1)}{\partial z^1}\Bigg|_{\substack{z_0 = \dots \\ z_1 = \dots}} \qquad (368)$$

$$\frac{\partial}{\partial x^1}(\text{r.h.s.}) = \frac{\partial}{\partial x^1}\phi(x^0\text{ch}\theta - x^1\text{sh}\theta, -x^0\text{sh}\theta + x^1\text{ch}\theta) =$$

$$= -\text{sh}\theta\,\frac{\partial\phi(z^0,z^1)}{\partial z^0}\Bigg|_{\substack{z_0 = \dots \\ z_1 = \dots}} + \text{ch}\theta\,\frac{\partial\phi(z^0,z^1)}{\partial z^1}\Bigg|_{\substack{z_0 = \dots \\ z_1 = \dots}} \qquad (369)$$

From (368) and (369) we see that

$$x^0\frac{\partial}{\partial x_1}(\text{r.h.s}) - x^1\frac{\partial}{\partial x^0}(\text{r.h.s.}) = -x_0\frac{\partial}{\partial x^1}(\text{r.h.s.}) + x_1\frac{\partial}{\partial x^0}(\text{r.h.s.}) =$$

$$= x_0\,\text{sh}\theta\,\frac{\partial\phi(z^0,z^1)}{\partial z^0}\Bigg|_{\substack{z_0 = \dots \\ z_1 = \dots}} - x_0\,\text{ch}\theta\,\frac{\partial\phi(z^0,z^1)}{\partial z^1}\Bigg|_{\substack{z_0 = \dots \\ z_1 = \dots}} + x_1\,\text{ch}\theta\,\frac{\partial\phi(z^0,z^1)}{\partial z^0}\Bigg|_{\substack{z_0 = \dots \\ z_1 = \dots}}$$

$$x_1\,\text{sh}\theta\,\frac{\partial\phi(z^0,z^1)}{\partial z^1}\Bigg|_{\substack{z_0 = \dots \\ z_1 = \dots}} = (x^0\text{sh}\theta - x^1\text{ch}\theta)\,\frac{\partial\phi(z_0,z^1)}{\partial z_0}\Bigg|_{\substack{z_0 = \dots \\ z_1 = \dots}} +$$

$$+ (-x^0\text{ch}\theta + x^1\text{sh}\theta)\,\frac{\partial\phi(z^0,z^1)}{\partial z^1}\Bigg|_{\substack{z_0 = \dots \\ z_1 = \dots}} = (367) \Rightarrow$$

$$\Rightarrow \frac{d}{d\theta}(\text{r.h.s.}) = \left(x^0\frac{\partial}{\partial x_1} - x^1\frac{\partial}{\partial x_0}\right)(\text{r.h.s.}) \qquad (370)$$

Differential eqs. (366) and (370) coincide. Also, the value of (l.h.s.) and (r.h.s.) at $\theta = 0$ coincide (and $= \phi(x^0, x^1)$) $\Rightarrow$ l.h.s of eq (365) = r.h.s. of eq 365 ◢

Similarly, one can prove that for the rotation on φ around z axis ($\omega_{12} = -\omega_{21} = \varphi$, other ω's $= 0$)

$$e^{\omega_{12}J^{12}}\phi(x^1,x^2) = e^{\varphi\left(x^1\frac{\partial}{\partial x_2} - x^2\frac{\partial}{\partial x_1}\right)}\phi(x^1,x^2) =$$

$$= \phi(x^1\cos\varphi + x^2\sin\varphi, -x^1\sin\varphi + x^2\cos\varphi) \qquad (371)$$

For arbitrary combination of rotations and boosts the relation between Λ and ω is more complicated (see eq. (362))

Commutation relations between $J_{\mu\nu}$ operators:

$$[\frac{\partial}{\partial x_\mu}, x_\nu] = g_{\mu\nu} \Rightarrow$$

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}) \qquad (372)$$

Proof:

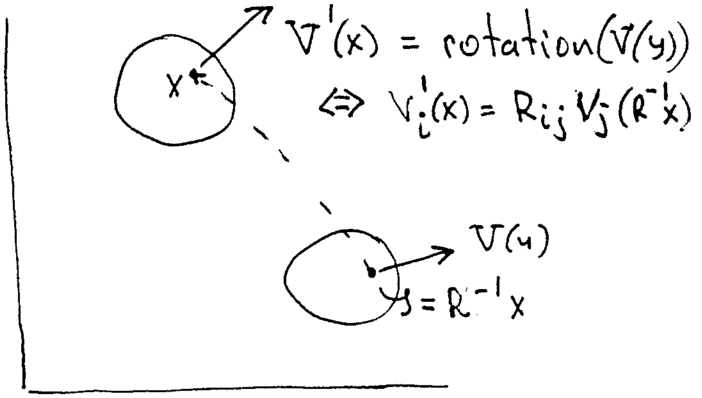
$$[J^{\mu\nu}, J^{\rho\sigma}] = \{i^2(x^\mu \frac{\overrightarrow{\partial}}{\partial x_\nu})(x^\rho \frac{\overrightarrow{\partial}}{\partial x_\sigma}) - i^2(x^\rho \frac{\overrightarrow{\partial}}{\partial x_\sigma}) x^\mu \frac{\overrightarrow{\partial}}{\partial x_\nu} - \mu \leftrightarrow \nu\} - (\rho \leftrightarrow \sigma)$$

$$= (- x^\mu g^{\nu\rho} \frac{\partial}{\partial x_\sigma} + x^\rho g^{\mu\sigma} \frac{\partial}{\partial x_\nu} - \mu \leftrightarrow \nu) - (\rho \leftrightarrow \sigma) = \text{r.h.s. of } (372)$$

II. Vector field $V^\mu(x)$

$$V^{\mu'}(x) = \Lambda^\mu{}_\nu V^\nu(\Lambda^{-1} x) \qquad (373)$$

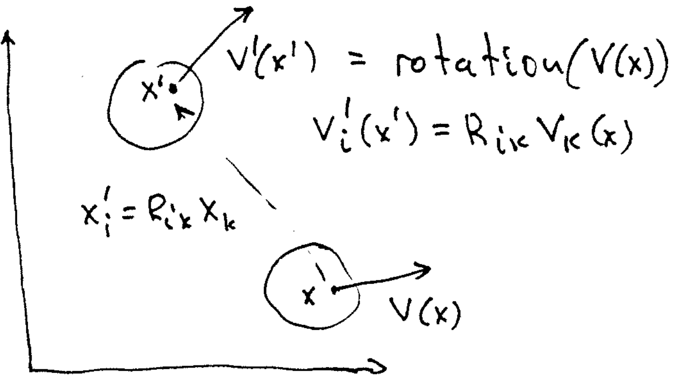
Illustration for the rotation:



Equivalent representation

$$V^{\mu'}(x') = \Lambda^\mu{}_\nu V^\nu(x) \qquad (374)$$

Illustation for the rotation:



Example: electromagnetic field $A^\mu(x)$

Let us prove that Maxwell's eqs are rel.-inv.

$$\partial_\mu F^{\mu\nu} = 0 \quad \text{- first pair of Maxwell's eqns} \qquad (375)$$

$$\Updownarrow$$

$$\partial^2 A^\nu - \partial^\nu \partial_\mu A^\mu = 0 \qquad (F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu) \qquad (376)$$

We must demonstrate that the transformed field $A_\mu'(x')$ satisfies the same eq. (376)

$$\left(\frac{\partial}{\partial x'}\right)^2 A'^\nu(x') - \frac{\partial}{\partial x'_\nu} \frac{\partial}{\partial x'_\mu} A^{\mu'}(x') = 0$$

Indeed,

$$\left(\frac{\partial}{\partial x'}\right)^2 A^{\nu'}(x') - \frac{\partial}{\partial x'_\nu}\frac{\partial}{\partial x'^\mu} A^{\mu'}(x') = \left(\frac{\partial}{\partial x}\right)^2 \Lambda^\nu{}_\sigma A^\sigma(x) - \left(\Lambda^\nu{}_\alpha \frac{\partial}{\partial x_\alpha}\right)\left(\Lambda_\mu{}^\beta \frac{\partial}{\partial x_\beta}\right)\cdot$$

$\shortparallel$ $\left(\frac{\partial}{\partial x}\right)^2$, see eq. (359)

$$\cdot \Lambda^\mu{}_\sigma A^\sigma(x) =$$

$$= \Lambda^\nu{}_\sigma \partial^2 A^\sigma(x) - \Lambda^\nu{}_\alpha \frac{\partial}{\partial x_\alpha} \underbrace{\Lambda_\mu{}^\beta \Lambda^\mu{}_\sigma}_{\delta^\beta_\sigma} \frac{\partial}{\partial x_\beta} A^\sigma(x) = \Lambda^\nu{}_\alpha\left(\partial^2 A\ (x) - \partial^\alpha \partial_\beta A^\beta(x)\right) = 0$$

⇒ the form of the Maxwell's eqn. (375) is the same in all frames.

Generators of Lorentz transformations for vector field

$$V^{\mu'}(x') = \left(e^{-\frac{i}{2}\omega_{\alpha\beta}J^{\alpha\beta}}\right)^\mu{}_\nu V^\nu(x) = \Lambda^\mu{}_\nu V^\nu(x) \qquad (377)$$

where

$$(J^{\alpha\beta})_{\mu\nu} = i\left(\delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\mu\right) \qquad (378)$$

Actually, the eq. (377) is the relation between the matrix $\Lambda^\mu{}_\nu$ and the parameters $\omega_{\alpha\beta}$

Illustration for the boost in x direction ($\omega_{01} = \omega_{10} = -\theta$, all other ω's $= 0$)

$$\left(e^{-\frac{i}{2}(\omega_{01}J^{01} + \omega_{10}J^{10})}\right)^\mu{}_\nu = \left(e^{-i\theta J^{01}}\right)^\mu{}_\nu = \left(1 - i\theta J^{01} - \frac{\theta^2}{2} J^{01}J^{01} + \dots\right)^\mu{}_\nu$$

$$(J^{01}J^{01})^\mu{}_\nu = (J^{01})^\mu{}_\alpha (J^{01})^\alpha{}_\nu = (\delta^0_\alpha g^{1\mu} - \delta^1_\alpha g^{0\mu})(\delta^0_\nu g^{1\alpha} - \delta^1_\nu g^{0\alpha})\, i^2$$

$$= -g^{0\mu}\delta^0_\nu + g^{1\mu}\delta^1_\nu = -\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}^\mu{}_\nu$$

In the subspace (x_0, x_1) $\quad J^{01} = i\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and $(J^{01})^2 = -\begin{pmatrix}1&0\\0&1\end{pmatrix}$

$$\Rightarrow \underline{1} - i\theta J^{01} - \frac{\theta^2}{2}J^2_{01} + \dots = \{1\} + \begin{pmatrix}0&\theta&\\ \theta&0&0\\ &0&0\;\,0\end{pmatrix} + \begin{pmatrix}\theta^2/2&0&\\ 0&\theta^2/2&0\\ &0&0\;\,0\end{pmatrix} +$$

$$+ \begin{pmatrix}0&\theta^3/3!&\\ \theta^3/3!&0&0\\ &0&0\;\,0\end{pmatrix} + \dots = \begin{pmatrix}\mathrm{ch}\,\theta & \mathrm{sh}\,\theta & 0 & 0\\ \mathrm{sh}\,\theta & \mathrm{ch}\,\theta & 0 & 0\\ 0&0&1&0\\ 0&0&0&1\end{pmatrix} = \Lambda^\mu{}_\nu \text{ (boost)} \qquad (379)$$

Combining eqs. (377) and (362) we get

$$V^{\mu'}(x) = \left(e^{-\frac{i}{2}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta}}\right)^{\mu}{}_{\nu} \; e^{-\frac{i}{2}\omega_{\lambda\rho}J^{\lambda\rho}} \; V^{\nu}(x) \qquad (380)$$

the first (differential) operator ↑ shifts the argument of V^ν to $x' = \Lambda^{-1}x$ and second (matrix) operator makes the rotation (or boost) of the field.

For the general case

$$\Phi'_a(x') = M_{ab}(\Lambda)\,\Phi_b(x) \iff \Phi'_a(x) = M_{ab}(\Lambda)\,\Phi_b(\Lambda^{-1}x) \qquad (381)$$

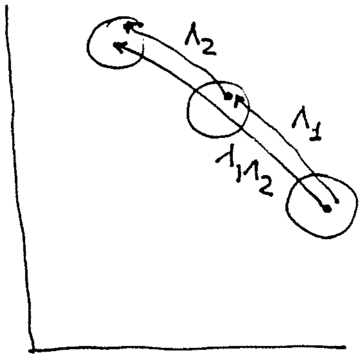
where $\Phi_a(x)$ is a certain field with n indices and M_{ab} is $n \times n$ matrix depending on Λ

Consistency:

$$x' = \Lambda_1 x \Rightarrow \Phi'_b(x') = M_{bc}(\Lambda_1)\,\Phi_c(x)$$

$$x'' = \Lambda_2 x' \Rightarrow \Phi''_a(x'') = M_{ab}(\Lambda_2)\,\Phi'_b(x')$$

$$x'' = (\Lambda_2\Lambda_1)x' \Rightarrow \Phi''_a(x'') = M_{ab}(\Lambda_2\Lambda_1)\,\Phi_b(x)$$



$$\Rightarrow M_{ab}(\Lambda_2\Lambda_1)\,\Phi_b(x) = M_{ab}(\Lambda_2)\,M_{bc}(\Lambda_1)\,\Phi_c(x) \Rightarrow$$

$$\Rightarrow M_{ab}(\Lambda_2\Lambda_1) = M_{ad}(\Lambda_2)\,M_{db}(\Lambda_1) \qquad (382)$$

$M(\Lambda_2\Lambda_1) = M(\Lambda_2)M(\Lambda_1) \Rightarrow M(\Lambda)$ - representation of Lorentz group

In terms of generators

$$M(\Lambda) = e^{-\frac{i}{2}\omega_{\alpha\beta}M^{\alpha\beta}} \qquad (383)$$

where $M^{\alpha\beta}$ are the generators in this representation (Mathematically, $M_{\alpha\beta}$ are called the elements of Lie algebra).

Commutation relations between the generators are the same for all representations

$$[M^{\mu\nu}, M^{\lambda\rho}] = i(g^{\nu\lambda}M^{\mu\rho} - g^{\mu\lambda}M^{\nu\rho} - g^{\nu\rho}M^{\mu\lambda} + g^{\mu\rho}M^{\nu\lambda}) \qquad (384)$$

For example, it is easy to check that

$$[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\lambda\rho}]^{\alpha}{}_{\beta} = (\mathcal{J}^{\mu\nu})^{\alpha}{}_{\gamma}(\mathcal{J}^{\lambda\rho})^{\gamma}{}_{\beta} - (\mathcal{J}^{\lambda\rho})^{\alpha}{}_{\gamma}(\mathcal{J}^{\mu\nu})^{\gamma}{}_{\beta} = -(\delta^{\mu}_{\gamma}g^{\nu\alpha} - \mu\leftrightarrow\nu)\cdot$$

$$\cdot(\delta^{\lambda}_{\beta}g^{\rho\gamma} - \lambda\leftrightarrow\rho) + (\delta^{\lambda}_{\gamma}g^{\rho\alpha} - \lambda\leftrightarrow\rho)(\delta^{\mu}_{\beta}g^{\nu\gamma} - \mu\leftrightarrow\nu) = \{(-g^{\mu\rho}g^{\nu\alpha}\delta^{\lambda}_{\beta} +$$

$$+ g^{\nu\lambda}g^{\rho\alpha}\delta^{\mu}_{\beta}) - (\mu\leftrightarrow\nu)\} - (\lambda\leftrightarrow\rho) = i(g^{\nu\lambda}\mathcal{J}^{\mu\rho} - g^{\mu\lambda}\mathcal{J}^{\nu\rho} - \lambda\leftrightarrow\rho) \qquad (385)$$

Proof of eq.(384):

Consider two infinitesimal Lorentz transformations with matrices Λ_1 and Λ_2.

For small Lorentz transformations we see from eq.(377) that

$$\Lambda^{\mu}{}_{\nu} = (e^{-\frac{i}{2}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta}})^{\mu}{}_{\nu} \simeq \delta^{\mu}_{\nu} - \frac{i}{2}\omega_{\alpha\beta}(\mathcal{J}^{\alpha\beta})^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}{}_{\nu} \qquad (386)$$

so

$$\Lambda_1 \simeq 1 + \omega_1 \quad , \quad \Lambda_2 = 1 + \omega_2 \quad \Rightarrow \Lambda_1\Lambda_2 - \Lambda_2\Lambda_1 \simeq \omega_1\omega_2 - \omega_2\omega_1$$

$$\Rightarrow M(\Lambda_1)M(\Lambda_2) - M(\Lambda_2)M(\Lambda_1) = M(\Lambda_1\Lambda_2) - M(\Lambda_2\Lambda_1) =$$

$$= M(1 + \omega_1 + \omega_2 + \omega_1\omega_2) - M(1 + \omega_1 + \omega_2 + \omega_2\omega_1) \qquad (387)$$

Compare l.h.s. and r.h.s. of eq.

$$M(\Lambda_1) = \exp -\frac{i}{2}\omega_{1\alpha\beta}M^{\alpha\beta} \simeq 1 - \frac{i}{2}\omega_{1\alpha\beta}M^{\alpha\beta} \qquad (388)$$

$$M(\Lambda_2) = \exp -\frac{i}{2}\omega_{2\alpha\beta}M^{\alpha\beta} \simeq 1 - \frac{i}{2}\omega_{2\alpha\beta}M^{\alpha\beta}$$

$$\Rightarrow M(\Lambda_1)M(\Lambda_2) - M(\Lambda_2)M(\Lambda_1) \simeq -\frac{1}{4}\omega_{1\alpha\beta}\omega_{2\lambda\rho}(M^{\alpha\beta}M^{\lambda\rho} - M^{\lambda\rho}M^{\alpha\beta})$$

$$= -\frac{1}{4}[M^{\alpha\beta}, M^{\lambda\rho}]\omega_{1\alpha\beta}\omega_{2\lambda\rho} \qquad (389)$$

On the other hand

$$M(1+\underbrace{\omega_1+\omega_2+\omega_1\omega_2}_{\text{small } \tilde{\omega}}) \simeq 1-\frac{i}{2}\tilde{\omega}_{\alpha\beta}M^{\alpha\beta} =$$

$$= 1-\frac{i}{2}(\omega_1+\omega_2+\omega_1\omega_2)_{\alpha\beta}M^{\alpha\beta}$$

Similarly (390)

$$M(1+\omega_1+\omega_2+\omega_2\omega_1) = 1-\frac{i}{2}(\omega_1+\omega_2+\omega_2\omega_1)_{\alpha\beta}M^{\alpha\beta}$$

$$\Rightarrow (\text{r.h.s.}) = -\frac{i}{2}(\omega_1\omega_2-\omega_2\omega_1)_{\alpha\beta}M^{\alpha\beta} = -\frac{i}{2}(\omega_1{}^{\alpha}{}_{\zeta}\,\omega_2{}^{\zeta\beta}-\omega_2{}^{\alpha}{}_{\zeta}\,\omega_1{}^{\zeta\beta})M_{\alpha\beta}$$

(l.h.s.)=(r.h.s) $\Rightarrow$ (391)

$$\Rightarrow \omega_{1\mu\nu}\omega_{2\lambda\rho}[M^{\mu\nu},M^{\lambda\rho}] = -\frac{i}{2}\omega_{1\mu\nu}\omega_{2\lambda\rho}(g^{\nu\lambda}M^{\mu\rho}-g^{\nu\rho}M^{\mu\lambda})$$

which is equivalent to comm. relation (384) ◄

Dirac eqn.

We must find the representation of Lorentz group corresponding to particles with spin ½

$$\psi'_\xi(x') = \underbrace{\left(e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}\right)_{\xi\eta}}_{n\times n \text{ matrix.}}\psi_\eta(x) \qquad [S^{\mu\nu}, S^{\alpha\beta}] = i((g^{\nu\alpha}S^{\mu\beta} - \alpha\leftrightarrow\beta) - \mu\leftrightarrow\nu) \tag{392}$$

There are (2×2) matrices

$$\Sigma^{\mu\nu} = \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu - \mu\leftrightarrow\nu)$$
$$\bar{\Sigma}^{\mu\nu} = \frac{i}{2}(\bar{\sigma}^\mu\sigma^\nu - \mu\leftrightarrow\nu) \tag{393}$$

where $\sigma^\mu = (1, \vec{\sigma})$ $\bar{\sigma}_\mu = (1, -\vec{\sigma})$ $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ – Pauli matrices

$$\sigma^1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad \sigma^2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \quad \sigma^3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

Corresponding representations of Lorentz group describe neutrino and antineutrino fields

$$\nu'_\xi(x') = \left(e^{-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}}\right)_{\xi\eta}\nu(x) \tag{394}$$

$$\bar{\nu}'(x') = \left(e^{-\frac{i}{2}\omega_{\mu\nu}\bar{\Sigma}^{\mu\nu}}\right)_{\xi\eta}\bar{\nu}(x) \tag{395}$$

As we know from AQM, such description is not invariant under parity transformation and therefore is not applicable to electrons.

Historically, Dirac have found the parity-even representation of Lorentz group which can be realized only 4×4 matrices

Idea: suppose we have 4x4 matrices γ_μ such that

$$\{\gamma_\mu, \gamma_\nu\} \equiv \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu} \tag{396}$$

Then define

$$S_{\mu\nu} \equiv \frac{i}{4}[\gamma_\mu, \gamma_\nu] \tag{397}$$

It is easy to check that

$$[S_{\mu\nu}, S_{\lambda\rho}] = i((g_{\nu\lambda}S_{\mu\rho} - \mu\to\nu) - \lambda\to\rho)), \tag{398}$$

so

$$(*) \quad \psi'_\xi(x') = \underset{\underset{4\times 4 \text{ matrix}}{\uparrow}}{\left(e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}\right)_{\xi\eta}} \underset{\underset{\text{„bispinor"}}{\uparrow}}{\psi_\eta}$$

is a representation of Lorentz group.

We will show that it corresponds to particles with spin ½ later.

Spinor representation of γ matrices

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad \Rightarrow \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \qquad (399)$$

↑ unit 2×2 matrix

Warning: in the AQM course $\gamma^\mu = \begin{pmatrix} 0 & \bar{\sigma}_\mu \\ \sigma_\mu & 0 \end{pmatrix}$. This is due to $(\psi) = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}\Big|_{\text{here}}$ and $(\psi) = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}\Big|_{\text{AQM}}$. Therefore, in order to compare with AQM course, all the 4×4 matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ should be replaced by $\begin{pmatrix} D & C \\ B & A \end{pmatrix}$.

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} D & C \\ B & A \end{pmatrix}\begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$$

Now, suppose $\psi(x)$ is a (bi)spinor field which transforms according to (*). What can be the form of a classical equation of motion ~~for~~ such field?

$(\partial^2 + m^2)\psi(x) = 0$ is possible

Because $[\partial_\mu, S_{\alpha\beta}] = 0$ (they act in different spaces) we would be able to prove that $\left(\frac{\partial}{\partial x'_\mu}\frac{\partial}{\partial x'^\mu} + m^2\right)\psi'(x') = \underset{= 0}{(\partial^2 + m^2)\psi(x)}$ just as in the scalar case.

But Dirac wanted a first-order differential equation (He thought that similarly to the Schrödinger eqn., a first-order diff. eqn for $\psi(x)$ would allow the probabilistic interpretation on one-particle level, which later proved to be wrong - we need both electrons and positrons for the rel.-inv. description, see AQM course)

This first-order equation is easy to guess:

$$i\gamma^\mu \frac{\partial}{\partial x^\mu}\psi(x) = m\psi(x) \qquad (400)$$

This is the famous Dirac equation. Let us prove that it is relativistic invariant ⟺ if $i\gamma_\mu \frac{\partial}{\partial x_\mu}\psi(x) = m\psi(x)$ then $i\gamma_\mu \frac{\partial}{\partial x'_\mu}\psi'(x') = m\psi'(x')$

Definition:

$$(\Lambda_{1/2})_{\xi\eta} \equiv \left(e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}\right)_{\xi\eta} \qquad (401)$$

Property:

$$(\Lambda_{1/2}^{-1})_{\xi\eta}(\gamma^\mu)_{\eta\rho}(\Lambda_{1/2})_{\rho\sigma} = \Lambda^\mu{}_\nu(\gamma^\nu)_{\xi\sigma} \qquad (402)$$

In explicit form this reads

$$\left(e^{\frac{i}{2}\omega_{\alpha\beta}S^{\alpha\beta}}\right)_{\xi\eta}(\gamma^\mu)_{\eta\rho}\left(e^{-\frac{i}{2}\omega_{\lambda\xi}S^{\lambda\xi}}\right)_{\rho\sigma} = \left(e^{-\frac{i}{2}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta}}\right)^\mu{}_\nu(\gamma^\nu)_{\xi\sigma} \qquad (403)$$

(recall that $\Lambda^\mu{}_\nu \equiv (\exp -\frac{i}{2}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta})^\mu{}_\nu$, see eq. (377))

Proof:

Expand l.h.s. and r.h.s. of eq. (403) in powers of ω

$$e^A B e^{-A} = \sum_{n=0}^\infty \frac{1}{n!}[\underbrace{A, .. [A}_{n},B]]] \Rightarrow \qquad (404)$$

$$\Rightarrow (\text{l.h.s.}) = e^{\frac{i}{2}(\omega S)}\gamma^\mu e^{-\frac{i}{2}(\omega S)} = \sum_{n=0}^\infty \frac{(i/2)^n}{n!}[\underbrace{(\omega S)[(\omega S), \ldots [(\omega S)}_{n}, \gamma^\mu]]] \qquad (405)$$

$[(\omega S), \gamma_\mu] = ?$

$$[\gamma^\mu, S^{\xi\eta}] = (\mathcal{J}^{\xi\eta})^\mu{}_\nu\gamma^\nu \Rightarrow$$

$$[(\omega S), \gamma^\mu] = -(\omega\mathcal{J})^\mu{}_\nu\gamma^\nu \qquad (406)$$

$$[(\omega S)[(\omega S), \gamma^\mu]] = -(\omega\mathcal{J})^\mu{}_\nu[(\omega S), \gamma^\nu] = -(\omega\mathcal{J})^\mu{}_\nu(-(\omega\mathcal{J})^\nu{}_\lambda\gamma^\lambda) = (\omega\mathcal{J})^\mu{}_\nu(\omega\mathcal{J})^\nu{}_\lambda\gamma^\lambda =$$
$$= ((\omega\mathcal{J})^2)^\mu{}_\lambda\gamma^\lambda$$

$$[(\omega S)[(\omega S)[(\omega S), \gamma^\mu]]] = ((\omega\mathcal{J})^2)^\mu{}_\lambda[(\omega S), \gamma_\lambda] = -((\omega\mathcal{J})^2)^\mu{}_\lambda(\omega\mathcal{J})^\lambda{}_\rho\gamma^\rho = -((\omega\mathcal{J})^3)^\mu{}_\nu\gamma^\nu$$

$$[\underbrace{(\omega S)[(\omega S) \ldots [(\omega S)}_{n}, \gamma^\mu]]] = (-1)^n((\omega\mathcal{J})^n)^\mu{}_\nu\gamma^\nu \Rightarrow \qquad (407)$$

$$\Rightarrow \sum_{n=0}^\infty \frac{(+i/2)^n}{n!}[\underbrace{(\omega S)[(\omega S) .. [(\omega S)}_{n}, \gamma^\mu]]] = \sum_{n=0}^\infty \frac{(-i/2)^n}{n!}((\omega\mathcal{J})^n)^\mu{}_\nu\gamma^\nu \equiv \underset{\text{r.h.s.}}{\left(e^{-\frac{i}{2}(\omega\mathcal{J})}\right)^\mu{}_\nu\gamma^\nu}$$

◢

Now let us prove that Dirac eqn is rel.-inv.

$$\left(i\gamma^\mu\frac{\partial}{\partial x'^\mu} - m\right)_{\xi\eta}\psi'_\eta(x') = \left(i\gamma^\mu\Lambda_\mu{}^\alpha\frac{\partial}{\partial x^\alpha} - m\right)_{\xi\rho}(\Lambda_{1/2})_{\rho\eta}\psi_\eta(x) = i\gamma^\mu_{\xi\rho}\Lambda_\mu{}^\alpha(\Lambda_{1/2})_{\rho\eta}\frac{\partial}{\partial x^\alpha}\psi(x) -$$
$$- m(\Lambda_{1/2})_{\xi\eta}\psi_\eta(x) = (\Lambda_{1/2})_{\xi\rho}\Big(i\underbrace{(\Lambda_{1/2}^{-1})_{\rho\sigma}(\gamma^\mu)_{\sigma\varphi}(\Lambda_{1/2})_{\varphi\eta}}_{\Lambda^\mu{}_\nu(\gamma^\nu)_{\rho\eta}}\Lambda_\mu{}^\alpha\frac{\partial}{\partial x^\alpha} - m\delta_{\rho\eta}\Big)\psi(x) = (\Lambda_{1/2})_{\xi\rho}\cdot \qquad (408)$$
$$\cdot\left(i\Lambda^\mu{}_\nu\Lambda_\mu{}^\alpha(\gamma^\nu)_{\rho\eta}\frac{\partial}{\partial x^\alpha} - m\delta_{\rho\eta}\right)\psi(x) = (\Lambda_{1/2})_{\xi\rho}\left(i(\gamma^\alpha)_{\rho\eta}\frac{\partial}{\partial x_\alpha} - m\delta_{\rho\eta}\right)\psi(x) = 0$$

◢

$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \Rightarrow (\partial^2 + m^2)\psi(x) = 0$

Proof : $(-i\gamma^\mu\partial_\mu - m)(i\gamma^\nu\partial_\nu - m)\psi(x) = 0 = (\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2)\psi(x) =$

$= (g^{\mu\nu}\partial_\mu\partial_\nu + m^2)\psi(x) = (\partial^2 + m^2)\psi(x).$ (409)

Dirac eqn must follow from some Lagrangian. Lagrangian must be a scalar. How to make a scalar from two spinors?

We know the answer from AQM: $\bar\psi\psi$ is a scalar (where $\bar\psi \overset{\text{def}}{\equiv} \psi^+\gamma_0$). Let us check this

$\psi'(x') = e^{-\frac{i}{2}(\omega S)}\psi(x) \Rightarrow \psi^{+\prime}(x') = \psi^+(x)\, e^{\frac{i}{2}(\omega S^+)} \Rightarrow \bar\psi'(x') = \bar\psi\gamma_0 e^{\frac{i}{2}(\omega S^+)}\gamma_0$

Property : $\gamma_0\gamma_\mu^+\gamma_0 = \gamma_\mu \Rightarrow \gamma_0(\omega S)^+\gamma_0 = \gamma_0\,\omega_{\mu\nu}\left(\frac{-i}{4}\right)(\gamma^{\nu+}\gamma^{\mu+} - \gamma^{\mu+}\gamma^{\nu+})\gamma_0 \cdot$

$= \omega_{\mu\nu}\frac{i}{4}(\gamma_0\gamma^{\mu+}\gamma_0\gamma_0\gamma^{\nu+}\gamma_0 - \mu\leftrightarrow\nu) = \frac{i}{4}\omega_{\mu\nu}[\gamma^\mu,\gamma^\nu] = (\omega S)$ (410)

$\Rightarrow \bar\psi'(x') = \bar\psi(x)\gamma_0 e^{i/2(\omega S^+)}\gamma_0 = \bar\psi(x)e^{+i/2(\omega S)}$ (411)

Thus,

$\psi'_\xi(x') = (\Lambda_{1/2})_{\xi\eta}\psi_\eta(x) \Rightarrow \bar\psi'_\xi(x') = \bar\psi_\eta(x)(\Lambda^{-1}_{1/2})_{\eta\xi}$ (412)

$\Rightarrow \bar\psi'_\xi(x')\psi'_\xi(x') = \bar\psi_\eta(x)(\Lambda^{-1}_{1/2})_{\eta\xi}(\Lambda_{1/2})_{\xi\delta}\psi_\delta(x) = \bar\psi_\eta(x)\psi_\eta(x) \Rightarrow$ (413)

$\Rightarrow \bar\psi\psi$ is a scalar

Similarly, one can prove that $\bar\psi\gamma_\mu\psi$ is a vector ! $\bar\psi'(x')\gamma^\mu\psi'(x') = \bar\psi(x)\underbrace{\Lambda^{-1}_{1/2}\gamma^\mu\Lambda_{1/2}}_{\Lambda^\mu{}_\nu\gamma^\nu}\psi(x) = \Lambda^\mu{}_\nu\,\bar\psi(x)\gamma^\nu\psi(x)$ (414)

$\bar\psi\psi$ - scalar

$\bar\psi\gamma^\mu\psi$ - vector $\Rightarrow \bar\psi(x)\gamma^\mu\frac{\partial}{\partial x^\mu}\psi(x)$ is a scalar. (Proof is similar:

$\bar\psi'(x')\gamma^\mu\frac{\partial}{\partial x'^\mu}\psi'(x') = \bar\psi(x)\Lambda^{-1}_{1/2}\gamma^\mu\Lambda_\mu{}^\nu\frac{\partial}{\partial x^\nu}\Lambda_{1/2}\psi(x) = \bar\psi(x)(\Lambda^{-1}_{1/2}\gamma^\mu\Lambda_{1/2})\Lambda_\mu{}^\nu\frac{\partial}{\partial x_\nu}\psi(x) =$ (415)

$= \bar\psi(x)\gamma^\alpha\Lambda^\mu{}_\alpha\Lambda_\mu{}^\nu\frac{\partial}{\partial x_\nu}\psi(x) = \bar\psi(x)\gamma^\nu\frac{\partial}{\partial x_\nu}\psi(x)$

$\gamma_\mu a^\mu \equiv \not a$

$\gamma_\mu a^\mu \equiv \not a$ - common notation (416)

Lagrangian for the Dirac field

$\mathcal{L} = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi$ $\qquad \frac{\partial L}{\partial\partial_\mu\psi} = i\bar\psi\gamma^\mu \quad \frac{\partial L}{\partial\psi} = m\bar\psi \quad \frac{\partial L}{\partial\bar\psi} = i\not\partial\psi - m$ (417)

$\frac{\partial L}{\partial\partial_\mu\bar\psi} = 0$

Euler - Lagrange eqs:

$\frac{\partial L}{\partial\bar\psi} = \partial_\mu\frac{\partial L}{\partial\partial_\mu\bar\psi} \Rightarrow i\not\partial\psi - m\psi = 0$ Dirac eqn (418)

$\frac{\partial L}{\partial\psi} = \partial_\mu\frac{\partial L}{\partial\partial_\mu\psi} \Rightarrow -m\bar\psi = i\bar\psi\gamma_i^\mu\overleftarrow{\partial}_\mu \Rightarrow \underset{\uparrow}{-i\bar\psi\overleftarrow{\not\partial} - m\bar\psi = 0}$ $\qquad \bar\psi\overleftarrow{\partial}_\mu \equiv \frac{\partial}{\partial x_\mu}\bar\psi(x)$ (419)

c.c. of Dirac eqn.

Weyl spinors

$$\psi = \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} \rightleftarrows \text{Weyl spinors}$$

$\uparrow$
Dirac bispinor

Dirac eqn: $i\gamma^\mu\partial_\mu\psi = m\psi \Leftrightarrow i\begin{pmatrix}0 & \sigma^\mu\\ \bar\sigma^\mu & 0\end{pmatrix}\partial_\mu\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} = m\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} \Rightarrow$

$$\Rightarrow \begin{array}{l} i\sigma^\mu\partial_\mu\psi_R = \psi_L m\\ i\bar\sigma^\mu\partial_\mu\psi_L = m\psi_R \end{array} \qquad (421)$$

For massless case $\quad i\sigma_\mu\partial^\mu\psi_R = 0 \quad \psi_R \equiv \bar\nu$ antineutrino

$\quad i\bar\sigma_\mu\partial^\mu\psi_L = 0 \quad \psi_L \equiv \nu$ neutrino (422

Reminder:

Plane waves. $\quad \psi(x) = u(p)e^{-ipx} \Rightarrow (\not p - m)u(p) = 0 \qquad (423)$

1. Particle at rest $\quad p_0 = (m,0,0,0)$

$$(m\gamma_0 - m)u(p_0) = 0 \Rightarrow m\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}u(p_0) = 0 \Rightarrow u(p_0) = \underset{\uparrow}{\sqrt{m}}\begin{pmatrix}\xi\\ \xi\end{pmatrix}$$

for convenient normalization $\xi^\dagger\xi = 1$

2. Boost in z direction $\quad \omega_{03} = \theta \quad \omega_{30} = -\theta$

$\Lambda_{1/2} = ?$

$$\Lambda = \begin{pmatrix}ch\theta & 0 & 0 & sh\theta\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ sh\theta & 0 & 0 & ch\theta\end{pmatrix}$$

$$S^{03} = \frac{i}{4}[\gamma^0,\gamma^3] = -\frac{i}{2}\begin{pmatrix}\sigma_z & 0\\ 0 & -\sigma_z\end{pmatrix}$$

$$e^{-\frac{i}{2}\omega_{\mu\nu}S_{\mu\nu}} = e^{-i\omega_{03}S^{03}} = e^{-\frac{1}{2}\theta\begin{pmatrix}\sigma_z & 0\\ 0 & -\sigma_z\end{pmatrix}} = \begin{pmatrix}e^{-\frac{\theta}{2}\sigma_z} & 0\\ 0 & e^{\frac{\theta}{2}\sigma_z}\end{pmatrix} = \Lambda_{1/2} \qquad (424)$$

$$\Rightarrow u(p) = \begin{pmatrix}ch\theta/2 - \sigma_z sh\theta/2 & 0\\ 0 & ch\theta/2 + \sigma_z sh\theta/2\end{pmatrix}\begin{pmatrix}\xi\\ \xi\end{pmatrix}\sqrt{m} =$$

$$\left.\begin{array}{ll} ch\theta = \frac{p_0}{m} & ch\frac{\theta}{2} = \sqrt{\frac{p_0+m}{2m}}\\ sh\theta = \frac{|p|}{m} & sh\frac{\theta}{2} = \sqrt{\frac{p_0-m}{2m}} \end{array}\right]$$

$$= \frac{1}{\sqrt{2(p_0+m)}}\begin{pmatrix}(p_0 + m - \sigma_z p_3)\,\xi\\ (p_0 + m + \sigma_z p_3)\,\xi\end{pmatrix} = \begin{pmatrix}\sqrt{p\sigma}(\xi)\\ \sqrt{p\bar\sigma}(\xi)\end{pmatrix} \qquad (425)$$

Formally $\quad \frac{1}{\sqrt{2(p_0+m)}}(p_0 + m - \vec\sigma\vec p) = \sqrt{p\cdot\sigma} \qquad (426)$

$\quad \frac{1}{\sqrt{2(p_0+m)}}(p_0 + m + \vec\sigma\vec p) = \sqrt{p\cdot\bar\sigma} \qquad (427)$

For arbitrary boost

$$u(p) = \begin{pmatrix}\sqrt{p\sigma}(\xi)\\ \sqrt{p\bar\sigma}(\xi)\end{pmatrix} = \frac{1}{\sqrt{2(p_0+m)}}\begin{pmatrix}(p_0 + m - \vec\sigma\vec p)\xi\\ (p_0 + m + \vec\sigma\vec p)\xi\end{pmatrix}$$

, see AQM (428).

Plane waves with definite helicity

$$u^h = \frac{1}{\sqrt{2(p_0+m)}}\begin{pmatrix}(p_0+m-|\vec{p}|)\,\omega^h \\ (p_0+m+|\vec{p}|)\,\omega^h\end{pmatrix} \qquad \omega^{(1)} = \begin{pmatrix} e^{-i\varphi}\cos\theta/2 \\ \sin\theta/2\end{pmatrix} \qquad \omega^{(2)} = \begin{pmatrix} -e^{-i\varphi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}\end{pmatrix}$$

$$\hat{h}\,u^h(p) = h\,u^h(p) \qquad \hat{h} \equiv \frac{1}{2}p_i\begin{pmatrix}\sigma_i & 0 \\ 0 & \sigma_i\end{pmatrix}$$ – helicity operator.

Negative - frequency plane waves

$$\psi(x) = v(p)\,e^{ipx} \tag{429}$$

Dirac eqn : $(\not{p}+m)\,v(p) = 0.$ (430)

Solutions:

$$v(p) = \frac{1}{\sqrt{2(p_0+m)}}\begin{pmatrix}(m_0+p_0-\vec{p}\vec{\sigma})\,\eta \\ (-m-p_0-\vec{p}\vec{\sigma})\,\eta\end{pmatrix} = \begin{pmatrix}\sqrt{p\sigma}\,\eta \\ -\sqrt{p\bar{\sigma}}\,\eta\end{pmatrix} \tag{431}$$

Explicit form:

1. For definite z - component of spin

$$v^{1/2}(p) = \begin{pmatrix}-\sqrt{p\sigma}\begin{pmatrix}0\\1\end{pmatrix} \\ \sqrt{p\bar{\sigma}}\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} \qquad v^{-1/2}(p) = \begin{pmatrix}\sqrt{p\sigma}\begin{pmatrix}1\\0\end{pmatrix} \\ -\sqrt{p\bar{\sigma}}\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} \tag{432}$$

2. For definite helicity

$$v^{[1/2]}(p) = \frac{1}{\sqrt{2(p_0+m)}}\begin{pmatrix}(-m-p_0-|\vec{p}|)\,\omega^{(2)} \\ (m+p_0-|\vec{p}|)\,\omega^{(2)}\end{pmatrix} \qquad v^{[-\frac{1}{2}]}(p) = \frac{1}{\sqrt{2(p_0+m)}}\begin{pmatrix}(m+p_0-|\vec{p}|)\,\omega^{(1)} \\ (-m-p_0-|\vec{p}|)\,\omega^{(1)}\end{pmatrix} \tag{433}$$

Useful properties:

Orthogonality:

$$\bar{u}^\lambda(p)\,u^{\lambda'}(p) = -\bar{v}^\lambda(p)\,v^{\lambda'}(p) = 2m\,\delta_{\lambda\lambda'} \tag{434}$$

$$\bar{u}^\lambda(p)\,\gamma_\mu\,u^{\lambda'}(p) = \bar{v}^\lambda(p)\,\gamma_\mu\,v^{\lambda'}(p) = 2p_\mu\,\delta_{\lambda\lambda'} \tag{435}$$

$$\bar{u}^\lambda(p)\,v^{\lambda'}(p) = \bar{v}^\lambda(p)\,u^{\lambda'}(p) = 0 \tag{436}$$

Completeness:

$$\sum_{\lambda=1,2} u^\lambda_\alpha(p)\,\bar{u}^\lambda_\beta(p) = (m+\not{p})_{\alpha\beta} \tag{437}$$

$$\sum_{\lambda=1,2} v^\lambda_\alpha(p)\,\bar{v}^\lambda_\beta(p) = (\not{p}-m)_{\alpha\beta} \tag{438}$$

4- vector of spin

$$\bar{u}(p,s)\,\gamma^\mu\gamma_5\,u(p,s) = -\bar{v}(p,s)\,\gamma^\mu\gamma_5\,v(p,s) = 2m\,s_\mu \qquad s^2 = -1,\ (s\cdot p) = 0 \tag{439}$$

$$u_\alpha(p,s)\,\bar{u}_\beta(p,s) = \left(\frac{1+\gamma_5\not{s}}{2}(\not{p}+m)\right)_{\alpha\beta} \qquad v_\alpha(p,s)\,\bar{v}_\beta(p,s) = \left(\frac{1+\gamma_5\not{s}}{2}(\not{p}-m)\right)_{\alpha\beta} \tag{440}$$

Dirac field bilinears

$\bar\psi\psi$ · scalar $\bar\psi\gamma_\mu\psi$ vector $\bar\psi\gamma_{\mu_1}\dots\gamma_{\mu_n}\psi$: who?

Complete set of 4×4 matrices:

1	1
γ^μ	4
$\gamma^{[\mu}\gamma^{\nu]} \equiv -i\sigma^{\mu\nu}$	6
$\gamma^{[\mu}\gamma^\nu\gamma^{\lambda]}$	4
$\gamma^{[\mu}\gamma^\nu\gamma^\lambda\gamma^{\rho]}$	1
	16

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \Rightarrow \gamma^{[\mu}\gamma^\nu\gamma^\lambda\gamma^{\rho]} = -i\varepsilon^{\mu\nu\lambda\rho}\gamma^5 \qquad \gamma_5 \equiv \gamma^5 \tag{441}$$

Useful formula

$$\gamma^\mu\gamma^\nu\gamma^\lambda = g^{\mu\nu}\gamma^\lambda + g^{\nu\lambda}\gamma^\mu - g^{\mu\lambda}\gamma^\nu + i\varepsilon^{\mu\nu\lambda\rho}\gamma_\rho\gamma_5 \tag{442}$$

$$\Rightarrow \gamma^{[\mu}\gamma^\nu\gamma^{\lambda]} = -i\varepsilon^{\mu\nu\lambda\rho}\gamma_\rho\gamma_5 \tag{443}$$

Bilinears

$\bar\psi\psi$	scalar
$\bar\psi\gamma_\mu\psi$	vector
$\bar\psi\sigma^{\mu\nu}\psi$	tensor
$\bar\psi\gamma^\mu\gamma^5\psi$	pseudo-vector
$\bar\psi\gamma^5\psi$	pseudo-scalar

For example

$$\bar\psi'(x')\sigma^{\mu\nu}\psi'(x') = \frac{i}{2}\bar\psi'(x')\gamma^\mu\gamma^\nu\psi'(x') - \mu\leftrightarrow\nu = \frac{i}{2}\bar\psi(x)\underbrace{\Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2}}_{\Lambda^\mu{}_\alpha\gamma^\alpha}\underbrace{\Lambda_{1/2}^{-1}\gamma^\nu\Lambda_{1/2}}_{\Lambda^\nu{}_\beta\gamma^\beta}\psi(x) - \mu\leftrightarrow$$

$$= \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\frac{i}{2}\bar\psi(x)(\gamma^\alpha\gamma^\beta - \alpha\leftrightarrow\beta)\psi(x) = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\bar\psi(x)\sigma^{\alpha\beta}\psi(x) \tag{444}$$

Fierz identities

$$\Gamma_{\alpha\beta} \underset{\uparrow}{=} \frac{1}{4}(\gamma^\mu)_{\alpha\beta}\mathrm{Tr}\{\Gamma\gamma_\mu\} + \frac{1}{4}\delta_{\alpha\beta}\mathrm{Tr}\{\Gamma\} - \frac{1}{4}(\gamma^\mu\gamma^5)_{\alpha\beta}\mathrm{Tr}\{\Gamma\gamma_\mu\gamma_5\} + \frac{1}{8}(\sigma^{\mu\nu})_{\alpha\beta}\mathrm{Tr}\{\Gamma\sigma_{\mu\nu}\} + \frac{1}{4}(\gamma^5)_{\alpha\beta}\mathrm{Tr}\{\Gamma\gamma_5\} \tag{445}$$

completeness of 16 matrices (*)

Example

$$\bar\psi\gamma^\mu\underbrace{\psi\bar\chi}_{\Gamma}\gamma_\mu\chi = \bar\psi\gamma_\mu\Gamma\gamma^\mu\chi = \bar\psi\chi\,\mathrm{Tr}\Gamma + \frac{1}{4}\bar\psi\overbrace{\gamma^\mu\gamma^\alpha\gamma_\mu}^{-2\gamma^\alpha}\chi\,\mathrm{Tr}(\Gamma\gamma_\alpha) - \frac{1}{4}\bar\psi\overbrace{\gamma^\mu\gamma^\alpha\gamma^5\gamma_\mu}^{2\gamma^\alpha\gamma^5}\psi\,\cdot$$

$$\cdot\,\mathrm{Tr}(\Gamma\gamma_\alpha\gamma_5) + \frac{1}{8}\bar\psi\underbrace{\gamma^\mu\sigma_{\lambda\rho}\gamma_\mu}_{=0}\chi\,\mathrm{Tr}(\Gamma\sigma^{\lambda\rho}) + \frac{1}{4}\bar\psi\gamma^5\chi\,\mathrm{Tr}(\Gamma\gamma_5) = \bar\psi\chi\,\mathrm{Tr}\Gamma - \frac{1}{2}\bar\psi\gamma^\alpha\chi\,\mathrm{Tr}(\Gamma\gamma_\alpha) -$$

$$-\frac{1}{2}\bar\psi\gamma^\alpha\gamma^5\chi\,\mathrm{Tr}(\Gamma\gamma_\alpha\gamma_5) - \psi\gamma^5\chi\,\mathrm{Tr}(\Gamma\gamma_5) = -(\bar\psi\chi)(\bar\chi\psi) + \frac{1}{2}(\bar\psi\gamma^\alpha\chi)(\bar\chi\gamma_\alpha\psi) + \frac{1}{2}(\bar\psi\gamma^\alpha\gamma^5\chi)(\bar\chi\gamma_\alpha\gamma_5\psi) +$$

$$+(\bar\psi\gamma^5\chi)(\bar\chi\gamma_5\psi) \tag{446}$$

Quantization of the Dirac field

Dirac eqn.:

$(i\partial\!\!\!/ - m)\psi(x) = 0$

Solution in terms of plane waves (see AQM, f-la (6.5.29

$$\psi_\xi(x) = \sum_{\lambda=1,2}\int\frac{d^3p}{\sqrt{2E_p}}\Big(\underbrace{u^\lambda_\xi(p)e^{-ipx}}_{\text{plane waves}}a(p,\lambda) + \underbrace{v^\lambda_\xi(p)e^{ipx}}\,b^*(p,\lambda)\Big)\Big|_{p_0=E_p} \qquad (447)$$

coefficient functions

where the spinors $u(p)$ and $v(p)$ satisfy the equations

$$(p\!\!\!/ - m)u(p) = 0 \qquad (448)$$

$$(p\!\!\!/ + m)v(p) = 0 \qquad (449)$$

(for any λ). Dirac conjugation of eq. (447) reads

$$\bar\psi(x) = \sum_{\lambda=1,2}\int\frac{d^3p}{\sqrt{2E_p}}\left(\bar v^\lambda(p)e^{-ipx}b(p,\lambda) + \bar u^\lambda(p)e^{ipx}a^*(p,\lambda)\right)\Big|_{p_0=E_p} \qquad (450)$$

Lagrangian is $\mathcal{L} = \bar\psi(i\partial\!\!\!/ - m)\psi$ (see eq. (417)).

General formula $H = \sum p\dot q - L$

Canonical coordinate is $\psi(x) \Rightarrow$ canonical momentum corresponding to this coordinate is

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\partial_0\psi(x)} = i\psi^+(x) \qquad (451)$$

$$\Rightarrow\ H = \int d^3x\ i\psi^+(x)\partial_0\psi(x) - \int d^3x\ i\,\psi^+(x)\gamma_0(\gamma_0\partial^0 + \vec\gamma\cdot\vec\nabla - im)\psi(x)$$

$$= \int d^3x\ \psi^+(x)(-i\gamma_0\,\vec\gamma\cdot\vec\nabla + m\gamma_0)\psi(x) \qquad (452)$$

(Sign: $\vec\gamma = (\gamma^1,\gamma^2,\gamma^3)$, $\vec\nabla = \left(\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^2},\frac{\partial}{\partial x^3}\right) \Rightarrow \gamma^k\partial_k = \vec\gamma\cdot\vec\nabla$)

To quantize the Dirac field, we must promote $\psi(x)$ and $\pi(x) = i\psi^+(x)$ to operators, satisfying the canonical commutational relations. Let us try (we'll have a trouble very soon)

Let us try

Schrodinger picture

$$\hat\psi(\vec x) = \int\frac{\text{đ}^3p}{\sqrt{2E_p}}\sum_s\left(\hat a_p^s\, u(\vec p,s)\, e^{+i\vec p\vec x} + \hat b_p^{s+}\, v(\vec p,s)\, e^{-i\vec p\vec x}\right) \qquad (453)$$

$$\hat{\bar\psi}(\vec x) = \int\frac{\text{đ}^3p}{\sqrt{2E_p}}\sum_s\left(\hat b_p^s\, \bar v(\vec p,s)\, e^{+i\vec p\vec x} + \hat a_p^{s+}\, \bar u(\vec p,s)\, e^{-i\vec p\vec x}\right). \qquad (454)$$

- If we impose commutation relations

$$[\hat a_p^s, \hat a_{p'}^{s'+}] = (2\pi)^3\delta(\vec p-\vec p')\delta_{ss'}\ , \quad [\hat b_p^s, \hat b_p^{s'+}] = -(2\pi)^3\delta(\vec p-\vec p')\delta_{ss'} \qquad (455)$$

we would get

$$[\hat\psi(\vec x), \hat\psi^+(\vec y)] = \int\frac{\text{đ}^3p}{\sqrt{2E_p}}\int\frac{\text{đ}^3p'}{\sqrt{2E_{p'}}}\sum_{s,s'}\Big([\hat a_p^s,\hat a_{p'}^{s'+}]\, u(p,s)\,\bar u(p',s')\, e^{i\vec p\vec x - i\vec p'\vec y} -$$

$$- [\hat b_{p'}^{s'},\hat b_p^{+s}]\, v(p,s)\,\bar v(p',s')\, e^{-i\vec p\vec x + i\vec p'\vec y}\Big) = \int\frac{\text{đ}^3p}{2E_p}\Big(\sum_s u(p,s)\bar u(p,s)\, e^{i\vec p(\vec x-\vec y)} +$$

$$+ \sum_s v(p,s)\bar v(p,s)\, e^{-i\vec p(\vec x-\vec y)}\Big) = \int\frac{\text{đ}^3p}{2E_p}\left((\not p+m)e^{i\vec p(\vec x-\vec y)} + (\not p-m)e^{-i\vec p(\vec x-\vec y)}\right) =$$

$$= \int\frac{\text{đ}^3p}{2E_p}\left[(E_p\gamma^0-\vec p\vec\gamma+m)\, e^{i\vec p(\vec x-\vec y)} + (E_p\gamma^0-\vec p\vec\gamma-m)\, e^{-i\vec p(\vec x-\vec y)}\right] = \int\frac{\text{đ}^3p}{2E_p}2E_p\, e^{i\vec p(\vec x-\vec y)} = \delta^3(\vec x-\vec y) \qquad (456)$$

Hamiltonian:

$$H = \int d^3x\ \bar\psi(-i\vec\gamma\cdot\vec\nabla+m)\psi = \int d^3x\int\frac{\text{đ}^3p}{\sqrt{2E_p}}\sum_s\left(b_p^s\,\bar u(p,s)\, e^{i\vec p\vec x} + a_p^{s+}\,\bar u(p,s)\, e^{-i\vec p\vec x}\right)\cdot$$

$$\int\frac{d^3p'}{\sqrt{2E_{p'}}}\sum_{s'}\Big(\underset{E_p\gamma_0}{\underset{\shortparallel}{(\vec\gamma\vec p'+m)}}\, u(p',s')\, a_{p'}^{s'}\, e^{i\vec p'\vec x} + \underset{-E_{p'}\gamma_0}{\underset{\shortparallel}{(-\vec\gamma\vec p'+m)}}\, v(p',s')\, b_{p'}^{s'+}\, e^{-i\vec p'\vec x}\Big) =$$

$$= \int\frac{\text{đ}^3p}{2}\sum_{s,s'}\Big[\overset{= v^+(\vec p,s)u(-\vec p,s')=0}{\cancel{b_p^s a_p^{s'}\,\bar u(\vec p,s)\gamma_0 u(-\vec p,s')}} + a_p^{s+}a_p^{s'}\underset{2E_p\delta_{ss'}}{\underset{\shortparallel}{\bar u(p,s)\gamma_0 u(p,s')}} - b_p^s b_{p'}^{s'+}\overbrace{\bar v(p,s)\gamma_0 v(p,s')}^{2E_p\delta_{ss'}}$$

$$-\ \underset{u^+(\vec p,s)v(-\vec p,s')=0}{\underset{\shortparallel}{\cancel{a_p^{s+}b_p^{+s'}\,\bar u(p,s)\gamma_0 v(-p,s')}}}\Big] \qquad (457)$$

$$= \int\text{đ}^3p\ E_p\left(a_p^{s+}a_p^s - b_p^s b_p^{s+}\right) \rightarrow \int\text{đ}^3p\ E_p\left(a_p^{s+}a_p^s \overset{!}{-} b_p^{+s}b_p^s\right) = \underline{\text{bad}}$$

Correct quantization of Dirac field:

$$\{\psi(\vec{x}), \psi^\dagger(\vec{y})\} = \delta^3(\vec{x}-\vec{y}) \qquad \{A,B\} \equiv AB + BA. \qquad (458)$$

In terms of creation and annihilation operators:

$$\{a_p^s, a_{p'}^{s'}\} = (2\pi)^3\delta(\vec{p}-\vec{p}\,')\,\delta_{ss'} \qquad \{b_p^s, b_{p'}^{s'}\} = (2\pi)^3\delta(\vec{p}-\vec{p}\,')\,\delta_{ss'} \qquad \begin{matrix}\text{other}\\ \{,\}\text{s}\\ \text{are zero}\end{matrix} \qquad (459)$$

Everything is the same, but

$$H = \sum_s \int d^3p\, E_p\,(a_p^{s+}a_p^s - b_p^s b_p^{+s}) = \sum_s \int d^3p\, E_p\,(a_p^{s+}a_p^s \underset{\underset{\text{OK.}}{\uparrow}}{\overset{!}{+}} b_p^{s+} b_p^s) + \begin{matrix}\text{infinite}\\ \text{constant}\end{matrix} \qquad (460)$$

From this point – as in the case of (complex) scalar field

$|0\rangle \qquad a_p^s|0\rangle = 0 \qquad b_p^s|0\rangle = 0 \qquad \sqrt{2E_p}\,a_p^{+s}|0\rangle = |p,s\rangle$ – one-electron state

$\sqrt{2E_p}\,b_p^{+s}|0\rangle = |p,s)$ – one-positron state

$(a_p^{+s})^2|0\rangle = \frac{1}{2}\{a_p^{+s}, \underset{\leftarrow 0}{a_p^{s+}}\}|0\rangle = 0 \Rightarrow$ Fermi-Dirac statistics

(spin integer – bosons
half-integer – fermions)
„Pauli spin-statistics theorem"

Quantization (the right one)

Schrödinger picture: $\psi(x) \to \hat{\psi}(\vec{x})$, $\pi(x) = i\psi^\dagger(x) \to i\hat{\psi}^\dagger(\vec{x})$

$$\hat{\psi}(\vec{x}) = \int \frac{d^3p}{\sqrt{2E_p}} \sum_s \left(\hat{a}_p^s\, u(\vec{p},s)\, e^{i\vec{p}\vec{x}} + \hat{b}_p^{s+}\, v(\vec{p},s)\, e^{-i\vec{p}\vec{x}}\right) \qquad (461)$$

$$\hat{\bar{\psi}}(\vec{x}) = \int \frac{d^3p}{\sqrt{2E_p}} \sum_s \left(\hat{b}_p^s\, \bar{v}(\vec{p},s)\, e^{i\vec{p}\vec{x}} + \hat{a}_p^{s+}\, \bar{u}(\vec{p},s)\, e^{-i\vec{p}\vec{x}}\right) \qquad (462)$$

Canonical <u>anti</u>commutation relations

$$\left.\begin{aligned} \{\hat{\psi}(\vec{x}), \hat{\psi}^+(\vec{y})\} &= \delta(\vec{x}-\vec{y}) \\ \{\hat{\psi}(\vec{x}), \hat{\psi}(\vec{y})\} &= 0 \\ \{\hat{\psi}^+(\vec{x}), \hat{\psi}^+(\vec{y})\} &= 0 \end{aligned}\right\} \Longleftrightarrow \left\{\begin{aligned} &\{\hat{a}_p^s, \hat{a}_{p'}^{+s'}\} = (2\pi)^3\delta(\vec{p}-\vec{p}\,')\,\delta_{ss'} \\ &\{\hat{b}_p^s, \hat{b}_{p'}^{+s'}\} = (2\pi)^3\delta(\vec{p}-\vec{p}\,')\,\delta_{ss'} \\ &\{a,b\} = \{a,a\} = \{a^+,a^+\} = \{b^+,b^+\} = \{b^+,a^+\} = \end{aligned}\right. \qquad (463$$

Hamiltonian is

$$\hat{H} = \int d^3x\, \hat{\bar{\psi}}(x)(-i\vec{\gamma}\cdot\vec{\nabla} + m)\hat{\psi}(x) = \int d^3p \sum_s E_p\,(\hat{a}_p^{s+}\hat{a}_p^s + \hat{b}_p^{s+} b_p^s) \qquad (464$$

Using the relations (463) it is easy to prove that

$$\begin{aligned} &[\hat{H}, \hat{a}_p^s] = -E_p\,\hat{a}_p^s \qquad &&[\hat{H}, b_p^s] = -E_p\,\hat{b}_p^s \\ &[\hat{H}, \hat{a}_p^{s+}] = E_p\,\hat{a}_p^{s+} \qquad &&[\hat{H}, b_p^{s+}] = E_p\,\hat{b}_p^{s+} \end{aligned} \qquad (465)$$

Heisenberg picture :

$$\hat\psi(\vec x,t) \equiv e^{i\hat H t}\psi(\vec x)e^{-i\hat H t}$$
$$\hat\psi^+(\vec x,t) \equiv e^{i\hat H t}\psi^+(\vec x)e^{-i\hat H t} \qquad (466)$$

(of course, $\hat H^+ = \hat H$). From eqs (465) we see that

$$e^{i\hat H t}\hat a_p^s e^{-i\hat H t} = e^{-iE_p t}\hat a_p^s$$
$$e^{i\hat H t}\hat a_p^{s+} e^{-i\hat H t} = e^{iE_p t}\hat a_p^{s+} \qquad (467)$$

(proof repeats the derivation of eqs. (155) and (156)). Similarly,

$$e^{i\hat H t}\hat b_p^s e^{-i\hat H t} = e^{-iE_p t}\hat b_p^s$$
$$e^{i\hat H t}\hat b_p^{s+} e^{-i\hat H t} = e^{iE_p t}\hat b_p^{s+} \qquad (468)$$

Therefore ($x \equiv (\vec x,t)$ as always)

$$\hat\psi(\vec x,t) = \int\frac{d^3p}{\sqrt{2E_p}}\sum_s\left(\hat a_p^s u(p,s)e^{-ipx} + \hat b_p^{s+}v(p,s)e^{ipx}\right)\Big|_{p_0=E_p} \qquad (469)$$

and

$$\hat{\bar\psi}(\vec x,t) = \int\frac{d^3p}{\sqrt{2E_p}}\sum_s\left(\hat b_p^s \bar v(p,s)e^{-ipx} + \hat a_p^{+s}\bar u(p,s)e^{ipx}\right)\Big|_{p_0=E_p} \qquad (470)$$

Equal-time (anti)commutators

$$\{\hat\psi(\vec x,t),\hat\psi^+(\vec x',t)\} = \delta(\vec x-\vec x')$$
$$\{\hat\psi(\vec x,t),\hat\psi(\vec x',t)\} = \{\hat\psi^+(\vec x,t),\hat\psi^+(\vec x',t)\} = 0 \qquad (471)$$

Proof

$$\{\hat\psi_\xi(\vec x,t),\hat\psi^+_\eta(\vec x',t)\} = \int\frac{d^3p}{\sqrt{2E_p}}\frac{d^3p'}{\sqrt{2E'_p}}\sum_s\sum_{s'}\Big(\{\hat a_p^s,\hat a^{+s'}_{p'}\}u_\xi(p,s)\bar u_\sigma(p',s')\gamma_{0_{\sigma\eta}}e^{-i(E_p-E'_p)t+i\vec p\vec x-}$$
$$+\{\hat b_p^{s+},\hat b_{p'}^{s'}\}v_\xi(p,s)\bar v_{\sigma}(p',s')\gamma_{0_\eta}\exp i(E_p-E_p')t-i\vec p\vec x+i\vec p'\vec x'\Big\} = \int\frac{d^3p}{2E_p}\sum_s\Big\{u_\xi(p,s)\bar u_\sigma(p,s)\gamma_{0_{\sigma\eta}}\cdot$$
$$\cdot e^{i\vec p(\vec x-\vec x')} + v_\xi(p,s)\bar v_\sigma(p,s)\gamma_{0_{\sigma\eta}}e^{-i\vec p(\vec x-\vec x')}\Big] = \int\frac{d^3p}{2E_p}\Big[(\not p+m)\gamma_0 e^{i\vec p(\vec x-\vec x')} + (\not p-m)\gamma_0 e^{-i\vec p(\vec x-\vec x')}\Big]_{p_0=E_p}$$
$$= \int\frac{d^3p}{2E_p}\Big[(p_0\gamma_0-\cancel{\vec p\cdot\vec\gamma+m})\gamma_0 e^{i\vec p(\vec x-\vec x')} + (p_0\gamma_0+\cancel{\vec p\cdot\vec\gamma-m})\gamma_0 e^{i\vec p(\vec x-\vec x')}\Big\}_{p_0=E_p} = \delta(\vec x-\vec x')$$

◢

States:

$|0\rangle$ - vacuum (472)

Vacuum is a state annihilated by both a_p and b_p

$$a_p|0\rangle = b_p|0\rangle = 0. \qquad (473)$$

One-particle states

$a_p^{+s}|0\rangle$ - one-electron state
$b_p^{+s}|0\rangle$ - one-positron state } will see below

For now: energy of both states is $E_p = \sqrt{m^2+\vec{p}^{\,2}}$. Indeed

$$\hat{H}\, a_p^{+s}|0\rangle = [\hat{H}, a_p^{+s}]|0\rangle = E_p\, a_p^{+s}|0\rangle \qquad (474)$$

$$\hat{H}\, b_p^{+s}|0\rangle = [H, b_p^{+s}]|0\rangle = E_p\, b_p^{+s}|0\rangle \qquad (475$$

(cf. eq. (135)). Also, the momentum of these states is $\vec{p}$.

Momentum operator in Dirac theory

Tensor of energy-momentum: $T_{\mu\nu} = \partial_\nu\phi \frac{\partial \mathcal{L}}{\partial\partial_\mu\phi} - g_{\mu\nu}\mathcal{L}$ - general formula

For Dirac field $\frac{\partial\mathcal{L}}{\partial\partial_\mu\psi} = i\bar\psi\gamma_\mu$, $\frac{\partial\mathcal{L}}{\partial\partial_\mu\bar\psi} = 0 \Rightarrow$

$$\Rightarrow T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\partial_\nu\psi + \underset{=0}{\cancel{\frac{\partial\mathcal{L}}{\partial\partial_\mu\bar\psi}\partial_\nu\bar\psi}} - g_{\mu\nu}\mathcal{L} = i\bar\psi\gamma_\mu\partial_\nu\psi - g_{\mu\nu}\,(i\not\partial - m)\psi = i\bar\psi\gamma_\mu\partial_\nu\psi \qquad (476)$$

Symmetrical form

$$T_{\mu\nu} = \frac{i}{4}\bar\psi(\gamma_\mu\overleftrightarrow{\partial}_\nu + \gamma_\nu\overleftrightarrow{\partial}_\mu)\psi \qquad (477)$$

Similarly to the scalar case (see eq. (137))

$$\partial_\mu T^{\mu\nu} = \frac{i}{4}\bar\psi(\underset{-im}{\underset{\shortparallel}{\not\partial}}\overleftrightarrow{\partial}_\nu + \underset{im}{\underset{\shortparallel}{\overleftarrow{\not\partial}}}\overleftrightarrow{\partial}_\nu - \gamma_\nu(\underset{m^2}{\underset{\shortparallel}{\partial^2}} - \underset{m^2}{\underset{\shortparallel}{\overleftarrow{\partial}^2}}))\psi = 0 \qquad (472)$$

Momentum conservation:

$$\partial_\nu T^{\nu i} = 0 \Rightarrow \int d^3x\,(\partial_0 T^{0i} + \underset{\uparrow \text{ by parts}}{\cancel{\partial_k T^{ki}}}) = 0 \Rightarrow \frac{d}{dt}\underbrace{\int d^3x\, T^{0i}}_{P_i \text{ - momentum}} = 0 \qquad (479)$$

(cf. eq. (141))

$$\Rightarrow P^i = \int d^3x\, T^{0i} = \frac{i}{4}\int d^3x\; \bar\psi(t,\vec{x})(\gamma_0\overleftrightarrow{\partial}^i + \gamma^i\overleftrightarrow{\partial}_0)\psi(t,\vec{x}) \; - \qquad (480)$$

momentum of classical Dirac field

The corresponding operator is

$$\hat{P}^i = \int d^3x \, \frac{i}{4} \hat{\bar{\psi}}(\vec{x},t)(\gamma_0 \overleftrightarrow{\partial}^i + \gamma^i \overleftrightarrow{\partial}^0)\psi(\vec{x},t) =$$

$$= \frac{i}{4}\int d^3x \int \frac{\bar{d}^3p}{\sqrt{2E_p}} \frac{\bar{d}^3p'}{\sqrt{2E_p'}} \sum_s \sum_{s'} (\hat{a}_p^{s+}\bar{u}(\vec{p},s) e^{+iE_pt} + \hat{b}_{-p}^s \bar{v}(-\vec{p},s) e^{-iE_pt}) e^{-i\vec{p}\vec{x}} (\gamma_0 \overleftrightarrow{\partial}^i + \gamma^i \overleftrightarrow{\partial}^0)(\hat{a}_{p'}^{s'} u(\vec{p}',s')\cdot$$

$$\cdot e^{-iE_p't} + \hat{b}_{-p'}^{+s'} v(-\vec{p}',s) e^{iE_p't}) e^{+i\vec{p}'x} =$$

$$= \int \frac{\bar{d}^3p}{2E_p} \sum_{s,s'} \Big[\frac{p_i}{2} (\hat{a}_p^{s+}\bar{u}(p,s) e^{iE_pt} + \hat{b}_{-p}^s \bar{v}(-p,s) e^{-iE_pt})\gamma_0 (a_p^{s'} u(\vec{p},s') e^{-iE_pt} + \hat{b}_{-p}^{+s'} v(-p,s) e^{iE_pt}$$

$$+ \frac{E_p}{4}(\hat{a}_p^{s+}\bar{u}(\vec{p},s) e^{iE_pt} + \hat{b}_{-p}^s \bar{v}(-\vec{p},s) e^{-iE_pt})\gamma^i (\hat{a}_p^{s'} u(\vec{p},s') e^{-iE_pt} - \hat{b}_{-p}^{+s'} v(-\vec{p},s') e^{iE_pt}) +$$

$$+ \frac{E_p}{4}(+\hat{a}_p^{s+}\bar{u}(\vec{p},s) e^{iE_pt} - \hat{b}_{-p}^s \bar{v}(-\vec{p},s) e^{-iE_pt})\gamma^i (\hat{a}_p^{s'} u(\vec{p},s') e^{-iE_pt} + \hat{b}_{-p}^{+s'} v(-\vec{p},s') e^{iE_pt} \Big] \tag{481}$$

Using the formulas

$$\bar{v}(\vec{p},s)\gamma_0 u(-\vec{p},s') = v^+(\vec{p},s) u(-\vec{p},s') = 0$$
$$\bar{u}(\vec{p},s)\gamma_0 v(-\vec{p},s') = u^+(p,s) v(-\vec{p},s') = 0 \tag{482}$$

we get

$$\hat{p}^i = \int \frac{\bar{d}^3p}{2E_p} \sum_{s,s'} \Big[\frac{p_i}{2} \overbrace{\bar{u}(\vec{p},s)\gamma_0 u(\vec{p},s')}^{2E_p\delta_{ss'}} \hat{a}_p^{+s} a_p^{s'} + \frac{p_i}{2} \overbrace{\bar{v}(-\vec{p},s)\gamma_0 v(-\vec{p},s)}^{2E_p\delta_{ss'}} b_{-p}^s b_{-p}^{+s'} +$$

$$+ \frac{E_p}{2} \underbrace{\bar{u}(p,s)\gamma^i u(p,s')}_{2p^i\delta_{ss'}} \hat{a}_p^{+s}\hat{a}_p^{s'} - \frac{E_p}{2} \bar{v}\underbrace{(-\vec{p},s)\gamma^i v(-\vec{p},s')}_{-2p^i\delta_{ss'}} \hat{b}_{-\vec{p}}^s \hat{b}_{-\vec{p}}^{+s'} \Big] =$$

$$= \int \frac{\bar{d}^3p}{2E_p} \sum_s \Big[2E_p \frac{p_i}{2} (\hat{a}_p^{+s} a_p^s - b_{-p}^s b_{-p}^{+s}) + 2p_i \frac{E_p}{2} (\hat{a}_p^{+s}\hat{a}_p^s - \hat{b}_{-\vec{p}}^s \hat{b}_{-p}^{+s}) \Big] = \tag{483}$$

$$= \int \bar{d}^3p \sum_s (\hat{a}_p^{+s} a_p^s - \hat{b}_p^s \hat{b}_p^{+s}) p_i = \int \bar{d}^3p \sum_s (a_p^{+s} a_p^s + b_p^{+s} b_p^s) p_i \cancel{+ \infty\ \text{const}}$$

Thus,

$$\hat{P}^i = \int \bar{d}^3p \, p_i \sum_s (\hat{a}_p^{+s} a_p^s + \hat{b}_p^{+s} b_p^s) \qquad \text{(does not depend on t)} \tag{484}$$

(cf. eq. (145)). Now it is easy to see that the momentum of the states (474) and (475) is $\vec{p}$:

$$[P^i, a_p^{+s}] = p^i \hat{a}_p^{+s} \tag{485}$$

$$\hat{P}^i a_p^{+s}|0\rangle = [\hat{P}^i, \hat{a}_p^{+s}]|0\rangle = p^i \hat{a}_p^{+s}|0\rangle$$

Similarly

$$[P^i, b_p^{+s}] = p^i \hat{b}_p^{+s} \Rightarrow \hat{P}^i b_p^{+s}|0\rangle = p^i b_p^{+s}|0\rangle \tag{486}$$

$\Rightarrow$ Both states $a_p^{+s}|0\rangle$ and $b_p^{+s}|0\rangle$ are the eigenstates of the momentum operator with eigenvalue p_i.

Combining eqs. (465) and (485) we get

$$[P^\mu, a_p^{s+}] = p^\mu a_p^{s+} \qquad [P^\mu, a_p^s] = -p^\mu a_p^s$$

$$[P^\mu, b_p^{s+}] = p^\mu b_p^{s+} \qquad [P^\mu, b_p^s] = -p^\mu b_p^s \qquad (487$$

(where $p_0 = E_p$). Similarly to eq. (166)

$$(487) \Rightarrow \quad \hat\psi(x) = e^{iPx}\,\hat\psi(0)\,e^{-i\hat P x}$$

$$\hat{\bar\psi}(x) = e^{i\hat P x}\,\hat{\bar\psi}(0)\,e^{-i\hat P x} \qquad (488$$

Thus, the momentum operator $\hat P^\mu$ generates the shift of the field:

$$e^{i\hat P a}\,\hat\psi(x)\,e^{-i\hat P a} = \hat\psi(x+a) \qquad (489)$$

It is instructive to find an operator which generates the Lorentz transformations (rotations and boosts). The classical Dirac field transforms according to (392)

$$\psi'(x') = \Lambda_{1/2}\,\psi(x) \qquad (490)$$

where $x'_\mu = \Lambda_\mu{}^\nu x_\nu$. The QM definition of a classical field is

$$\langle\Phi|\,\hat\psi(x)\,|\Phi\rangle = \psi(x) \qquad (491)$$

where $|\Phi\rangle$ is the state where we measure the value of the Dirac field. Under Lorentz transformation, the state transforms as follows

$$|\Phi\rangle \to |\Phi'\rangle = U|\Phi\rangle \qquad (492)$$

where $U = U(\Lambda)$ is a certain unitary operator (unitary because $\langle\Phi'|\Phi'\rangle = \langle\Phi|U^+U|\Phi\rangle = \langle\Phi|\Phi\rangle = 1$ so $U^+U = 1$)

therefore, from (489)-(491) we get

$\langle\Phi'|\,\hat\psi(x')\,|\Phi'\rangle$ = classical field after Lorentz transformatio

= $\Lambda_{1/2}$ (classical field before Lorentz transformation) =

$$= \Lambda_{1/2}\langle\Phi|\hat\psi(x)|\Phi\rangle \Rightarrow \langle\Phi|U^+\hat\psi(x')U|\Phi\rangle = \Lambda_{1/2}\langle\Phi|\hat\psi(x)|\Phi\rangle \Rightarrow$$

$$\Rightarrow U^+\hat\psi(x')U = \Lambda_{1/2}\hat\psi(x) \Rightarrow U\hat\psi(x)U^+ = \Lambda_{1/2}^{-1}\hat\psi(x') \qquad (493)$$

In explicit form

$$U\hat{\psi}(x)U^{+} = \Lambda_{1/2}^{-1}\,\psi(\Lambda x) \qquad (494$$

so the operator $U(\Lambda)$ generates the Lorentz rotation of the field in the same way as $e^{i\hat{P}a}$ generates translations (see eq. (488)). Similarly, we can try an ansatz

$$U(\Lambda) = e^{-\frac{i}{2}\omega_{\alpha\beta}\hat{M}^{\alpha\beta}} \qquad (495)$$

We obtain

$$e^{-\frac{i}{2}\omega_{\alpha\beta}M^{\alpha\beta}}\psi(x)\,e^{\frac{i}{2}\omega_{\alpha\beta}M^{\alpha\beta}} = e^{\frac{i}{2}\omega_{\alpha\beta}S^{\alpha\beta}}\psi(x') = \\ = e^{\frac{i}{2}\omega_{\alpha\beta}S^{\alpha\beta}}\,e^{\frac{i}{2}\omega_{\alpha\beta}J^{\alpha\beta}}\psi(x) \qquad (496)$$

where we have used the definition of the generators of Lorentz transformati (362):

$$\varphi(\Lambda^{-1}x) = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}\varphi(x) \Rightarrow \varphi(\Lambda x) = e^{\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}\varphi(x) \qquad (497$$

Let us expand the formula (496)

$$e^{-\frac{i}{2}\omega_{\alpha\beta}\hat{M}^{\alpha\beta}}\hat{\psi}(x)\,e^{\frac{i}{2}\omega_{\alpha\beta}M^{\alpha\beta}} = e^{\frac{i}{2}\omega_{\alpha\beta}(S^{\alpha\beta}+J^{\alpha\beta})}\hat{\psi}(x) \qquad (498$$

at small ω. We get:

$$\left.\begin{array}{l} (\text{l.h.s.}) = \psi(x) - \frac{i}{2}\omega_{\alpha\beta}[\hat{M}^{\alpha\beta},\psi(x)] \\ (\text{r.h.s}) = \psi(x) + \frac{i}{2}\omega_{\alpha\beta}(S^{\alpha\beta}+J^{\alpha\beta})\psi(x) \end{array}\right\} \Rightarrow [\hat{M}^{\alpha\beta},\hat{\psi}] = -(S^{\alpha\beta}+J^{\alpha\beta})\hat{\psi} \qquad (499$$

Similarly to the operator of momentum (480) we may try to find $\hat{M}_{\alpha\beta}$ in the form

$$\hat{M}^{\alpha\beta} = \int d^3x\;\psi^{+}(\vec{x})\,\Gamma^{\alpha\beta}\,\psi(\vec{x}) \qquad (500)$$

where $\Gamma^{\alpha\beta}$ is a certain matrix and/or differential operator. Let us consider commutation relation (499) at $t=0$.

We get

$$[\hat{M}^{\alpha\beta}, \hat{\psi}_\varrho(\vec{x})] = \int d^3x' \, [\hat{\psi}^+_\xi(\vec{x}')(\Gamma^{\alpha\beta})_{\xi\eta} \hat{\psi}_\eta(\vec{x}'), \hat{\psi}_\varrho(\vec{x})] =$$

$$= \overset{?}{-} \int d^3x' \, \underset{= \delta^3(x-x')\delta_{\varrho\xi}}{\{\hat{\psi}_\varrho(\vec{x}), \hat{\psi}^+_\xi(\vec{x}')\}} (\Gamma^{\alpha\beta})_{\xi\eta} \hat{\psi}_\eta(\vec{x}') = \overset{?}{-} (\Gamma^{\alpha\beta})_{\varrho\eta} \hat{\psi}_\eta(\vec{x}')$$

So,

$$[\hat{M}^{\alpha\beta}, \psi(\vec{x})] = \overset{?}{-} \Gamma^{\alpha\beta} \psi(\vec{x}) \qquad (501)$$

Comparing this to eq. (499) we see that $\Gamma_{\alpha\beta} = J_{\alpha\beta} + S_{\alpha\beta}$

$$\Rightarrow \quad \hat{M}_{\alpha\beta} = \int d^3x \, \hat{\psi}^+(\vec{x})(J_{\alpha\beta} + S_{\alpha\beta})\hat{\psi}(\vec{x}) = \int d^3x \, \bar{\hat{\psi}} \gamma_0 (J_{\alpha\beta} + S_{\alpha\beta}) \hat{\psi}(\vec{x}) \qquad (502)$$

In explicit form

$$\hat{M}_{\alpha\beta} = \int d^3x \, \bar{\psi}(\vec{x}) \gamma_0 (i(x_\alpha \partial_\beta - x_\beta \partial_\alpha) + \tfrac{1}{2}\sigma_{\alpha\beta}) \psi(\vec{x}) \qquad (503)$$

Using Dirac equation, one can demonstrate that

$$\int d^3x \, \bar{\psi}(\vec{x},t) \gamma_0 (i(x_\alpha \partial_\beta - x_\beta \partial_\alpha) + \tfrac{1}{2}\sigma_{\alpha\beta}) \psi(\vec{x},t) \qquad (504)$$

does not actually depent on t.

$$\frac{\partial}{\partial t}(504) = \int d^3x \, \bar{\psi}(\vec{x},t) (\gamma_0 \frac{\partial}{\partial t} - \gamma_0 \overleftarrow{\frac{\partial}{\partial t}})(i(x_\alpha \partial_\beta - x_\beta \partial_\alpha) + \tfrac{1}{2}\sigma_{\alpha\beta}) \psi(\vec{x},t)$$

$$= \int d^3x \, \bar{\psi}(\vec{x},t)(im - \gamma_\kappa \overleftarrow{\partial}^\kappa + \gamma_0 \frac{\partial}{\partial t})(i(x_\alpha \partial_\beta - x_\beta \partial_\alpha) + \frac{\sigma_{\alpha\beta}}{2}) \psi(\vec{x},t) = \text{by parts} =$$

$$= \int d^3x \, \bar{\psi}(\vec{x},t)(im + \gamma^\mu \partial_\mu)(i(x_\alpha \partial_\beta - x_\beta \partial_\alpha) + \frac{\sigma_{\alpha\beta}}{2}) \psi(\vec{x},t) \doteq$$

$$= \int d^3x \, \bar{\psi}(\vec{x},t) \; i\gamma^\mu (g_{\mu\alpha}\partial_\beta - g_{\mu\beta}\partial_\alpha) + \tfrac{1}{2}[\gamma^\mu, \sigma_{\alpha\beta}]\partial_\mu + (i(x_\alpha \partial_\beta - x_\beta \partial_\alpha) + \frac{\sigma_{\alpha\beta}}{2}) \cdot$$

$$\cdot (im + \underset{\to 0}{\cancel{\gamma_\mu \partial^\mu}})\} \psi(\vec{x},t) = 0$$

where we used the identity $[\gamma^\mu, \sigma_{\alpha\beta}] = 2i(\delta^\mu_\alpha \gamma_\beta - \delta^\mu_\beta \gamma_\alpha)$

Thus,

$$\int d^3x \, \bar{\psi}(\vec{x},t) \gamma_0 (i(x_\alpha \partial_\beta - x_\beta \partial_\alpha) + \tfrac{1}{2}\sigma_{\alpha\beta}) \psi(\vec{x},t) = \hat{M}_{\alpha\beta} \qquad (504)$$

The operator $M_{\alpha\beta}$ generates Lorentz transformations

$$e^{-\frac{i}{2}\omega_{\alpha\beta}M^{\alpha\beta}} \psi(x) \, e^{\frac{i}{2}\omega_{\alpha\beta}M^{\alpha\beta}} = \Lambda^{-1}_{1/2} \psi(\Lambda x) \qquad (505)$$

(Actually, we have proved it only for small ω (see eq. (498)), but it can be generalized to arbitrary ω)

Spatial components of $M_{\alpha\beta}$ generate rotations ⇒ they should be related to operator of angular momentum

Define

$$\hat{J}_i \equiv \frac{1}{2}\epsilon_{ijk} M^{jk} = \epsilon_{ijk} \int d^3x\, \hat{\psi}^+(\vec{x}) \left(i x^j \frac{\partial}{\partial x_k} + \frac{1}{4}\sigma^{jk} \right) \hat{\psi}(\vec{x}) = \qquad (506)$$

Because

$$\epsilon_{ijk}\sigma^{jk} = \epsilon_{ijk}\frac{i}{2}[\gamma^j,\gamma^k] = \frac{i}{2}\epsilon_{ijk}\begin{pmatrix} -[\sigma_j,\sigma_k] & 0 \\ 0 & -[\sigma_j,\sigma_k] \end{pmatrix} = 2\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \qquad (507)$$

the operator (506) takes the form: $\left(\Sigma_i \equiv \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}\right)$, σ_i - Pauli matrix

$$\hat{J}_i = \int d^3x \left[\epsilon_{ijk}\, x^j\, \hat{\psi}^+(\vec{x})\, i\frac{\partial}{\partial x_k} \hat{\psi}(\vec{x}) + \hat{\psi}^+(\vec{x}) \frac{\Sigma_i}{2} \hat{\psi}(\vec{x}) \right] = \hat{L}_i + \hat{S}_i \qquad (508)$$

Let us demonstrate that the first and the second terms in (508) are the operators for the orbital momentum and spin, respectively.

For orbital momentum we will use the classical analog For the classical mechanics

$$\vec{L} = \sum \vec{r} \times \vec{p} \rightarrow \int d^3x\ \vec{x} \times \vec{p}(\vec{x},t) \qquad (509)$$

where $\vec{p}(\vec{x},t)$ is the density of the momentum. For the classical Dirac field

$$P^i = \int d^3x\ \bar{\psi}(\vec{x},t)\, i\gamma_0 \partial^i \psi(\vec{x},t) = \int d^3x\ \psi^+(\vec{x},t)\, i\partial^i \psi(\vec{x},t) \qquad (510)$$

(here we use the non-symmetric form of $T_{\mu\nu}$ (476)) and therefore the density of the momentum is

$\vec{p}(\vec{x},t) = \psi^+(\vec{x},t)\, i\vec{\nabla}\psi(\vec{x},t)$ so

$$\vec{L} = \int d^3x\ \vec{x} \times \psi^+(\vec{x},t)\, i\vec{\nabla}\psi(\vec{x},t) \qquad (511)$$

When we promote ψ and ψ^+ to operators $(\psi(\vec{x},t) \rightarrow \hat{\psi}(\vec{x})$, $\psi^+(\vec{x},t) \rightarrow \hat{\psi}^+(\vec{x}))$ we obtain exactly the first term in r.h.s. eq. (508)

To demonstrate that the second term in r.h.s. (508) is the spin operator is a bit more difficult since spin has no analogy in the classical mechanics

Consider Dirac particle at rest $|\vec{p}=0\rangle = \sqrt{2m}\, a^+_0 |0\rangle$
The orbital momentum of such particle is of course 0:

$$\begin{aligned}\hat{\vec{L}}|\vec{p}=0\rangle &= \sqrt{2m}\int d^3x\ \hat{\psi}^+(\vec{x})\,\vec{x}\times i\vec{\nabla}\,\psi(\vec{x})\ a^{+\lambda}_0|0\rangle \sim\\ &\sim \int d^3x\ \hat{\psi}^+(\vec{x})\,\vec{x}\times i\vec{\nabla}\int d^3p \sum_{\lambda'} a^{\lambda'}_p u^{\lambda'}(p)\, e^{-ipx}\, a^{+\lambda}_0|0\rangle = \\ &= \int d^3x\ \hat{\psi}^+(\vec{x})\ d^3p\ \vec{x}\times\vec{p}\sum_{\lambda'} u^{\lambda'}(p)\,\underbrace{[a^{\lambda'}_p, a^{+\lambda}_0]}_{\sim\,\delta^3(\vec{p})} = 0\end{aligned} \qquad (512)$$

Let us apply now the operator $\vec{S}$. For example,

$$\hat{S}_3 = \int d^3x\ \psi^+(\vec{x})\frac{\Sigma_3}{2}\psi(\vec{x}) = \int\frac{d^3p}{2E_p}\sum_{r,r'}\left(\hat{a}^{+r'}_p u^+(\vec{p}) + \hat{b}^{r'}_{-p} v^{+r'}(-\vec{p})\right)\frac{\Sigma_3}{2}\cdot$$
$$\cdot\left(a^r_p u^r(\vec{p}) + b^{+r}_{-p} v^r(-\vec{p})\right)$$

and therefore

$$\hat{S}_3\,\hat{a}^{+\lambda}_0|0\rangle = \int\frac{d^3p}{2E_p}\sum_{r,r'}\hat{a}^{+r'}_p\hat{a}^r_p\left(u^{+r'}(\vec{p})\frac{\Sigma_3}{2}u^r(\vec{p})\right)\hat{a}^{+\lambda}_0|0\rangle$$
$$\{\hat{a}^r_p, a^{+\lambda}_0\} = (2\pi)^3\delta(\vec{p})\,\delta_{\lambda r}$$
$$= \frac{1}{2m}\sum_{r'}\left(u^{+r'}(0)\frac{\Sigma_3}{2}u^\lambda(0)\right)a^{+r'}_0|0\rangle \qquad (513)$$

For the spinor particle at rest $u^\lambda(0) = \sqrt{m}\begin{pmatrix}\xi^\lambda\\ \xi^\lambda\end{pmatrix}$ (see eq. (425)) so

$$\hat{S}_3\, a^{+\lambda}_0|0\rangle = \sum_r\left(\xi^{r+}\frac{\sigma_3}{2}\xi^\lambda\right)a^{+\lambda}_0|0\rangle \qquad (514)$$

For $\xi^1 = \begin{pmatrix}1\\0\end{pmatrix}$

$$\hat{S}_3\,\hat{a}^{+1}_0|0\rangle = \frac{1}{2}\hat{a}^{+1}_0|0\rangle \quad \Rightarrow \text{spin } \tfrac{1}{2} \text{ up} \uparrow \qquad (515)$$

For $\xi^2 = \begin{pmatrix}0\\1\end{pmatrix}$

$$\hat{S}_3\,\hat{a}^{+2}_0|0\rangle = -\frac{1}{2}\hat{a}^{+2}_0|0\rangle \quad \Rightarrow \text{spin } \tfrac{1}{2} \text{ down} \downarrow \qquad (516)$$

In the case of antifermion

$$\hat{S}_3\, b^{+\lambda}_0|0\rangle = \int\frac{d^3p}{2E_p}\sum_{r,r'}\left(v^{+r'}(-\vec{p})\frac{\Sigma_3}{2}v^r(-\vec{p})\right)\overbrace{b^{r'}_{-p}\,b^{+r}_{-p}}^{-(2\pi)^3\delta(\vec{p})\,\delta_{\lambda r'}}\,b^{+\lambda}_0|0\rangle$$
$$= \frac{-1}{2m}\sum_r\left(v^{+r}(0)\frac{\Sigma_3}{2}v^\lambda(0)\right)b^{+r}_0|0\rangle = -\sum_r\left(\xi^{+r}\frac{\sigma_3}{2}\xi^\lambda\right)b^+|0\rangle \qquad (517)$$

$$\xi = \begin{pmatrix}1\\0\end{pmatrix}:\quad \hat{S}_3\, b^{+1}_0|0\rangle = -\frac{1}{2}b^{+1}_0|0\rangle \quad\Rightarrow \text{spin } 1/2 \text{ down} \downarrow \qquad (518)$$

$$\xi = \begin{pmatrix}0\\1\end{pmatrix}:\quad \hat{S}_3\, b^{+2}_0|0\rangle = \frac{1}{2}b^{+2}_0|0\rangle \quad\Rightarrow \text{spin } 1/2 \text{ up} \uparrow \qquad (519)$$

AQM - - -

Electric charge

$j_\mu = \bar{\psi}\gamma_\mu\psi$ is conserved $\Rightarrow \frac{dQ}{dt} = 0$ where $Q = \int d^3x\, \psi^+(x)\psi(x)$ (520

in terms of ladder operators

$$Q = \int d^3p \sum_\lambda (a_p^{+\lambda} a_p^\lambda + b_{-p}^\lambda b_{-p}^{\lambda +}) = \int d^3p \sum_s (a_p^{+\lambda} a_p^\lambda - b_p^{+\lambda} b_p^\lambda) + \underline{\text{inf. constant}} \quad (521)$$

$Q\, a_p^{+\lambda}|0\rangle = a_p^{+\lambda}|0\rangle$ charge (+1)

$Q\, b_p^{+\lambda}|0\rangle = - b_p^{+\lambda}|0\rangle$ charge (-1) (522)

Wavefunctions

$\langle 0|\psi(x)|p,\lambda\rangle = u^\lambda(p)\, e^{-ipx}$ – wavefunction of a fermion

$\langle 0|\bar{\psi}(x)|\bar{p},\lambda\rangle = \bar{v}^\lambda(p)\, e^{-ipx}$ –\\– antifermion (523)

Dirac propagator

$S_F(x-y) = \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle$

$$T\{\psi(x)\bar{\psi}(y)\} \overset{def}{\equiv} \theta(x_0-y_0)\psi(x)\bar{\psi}(y) - \theta(y_0-x_0)\bar{\psi}(y)\psi(x) \quad (524)$$

Explicit form:

$$S_F(x-y)_{\alpha\beta} = \langle 0|\sum_\lambda \int \frac{d^3p}{\sqrt{2E_p}} a_p^\lambda e^{-ipx} u_\alpha^\lambda(p) \int \frac{d^3p'}{\sqrt{2E_{p'}}} \sum_{\lambda'} a_{p'}^{+\lambda'} e^{ipy} \bar{u}_\beta^{\lambda'}(p')\, \theta(x_0-y_0) - \theta(y_0-x_0) \cdot$$

$$\cdot \sum_{\lambda'} \int \frac{d^3p'}{\sqrt{2E_{p'}}} b_{p'} e^{-ip'y} \bar{v}_\beta^\lambda(p') \sum_\lambda \int \frac{d^3p}{\sqrt{2E_p}} b_p^+ e^{ipx} v_\alpha^\lambda(p)|0\rangle = \theta(x_0-y_0)\int \frac{d^3p}{2E_p} e^{-ip(x-y)}$$

$$\sum_\lambda u_\alpha^\lambda(p)\bar{u}_\beta^\lambda(p) - \theta(y_0-x_0)\int \frac{d^3p}{2E_p} e^{ip(x-y)} \sum_\lambda v_\alpha^\lambda(p)\bar{v}_\beta^\lambda(p) \Big|_{p_0=E_p} =$$

$$= \theta(x_0-y_0)\int \frac{d^3p}{2E_p} e^{-ip_0(x_0-y_0)+i\vec{p}(\vec{x}-\vec{y})} (\not{p}+m)_{\alpha\beta} + \theta(y_0-x_0)\int \frac{d^3p}{2E_p} (-\not{p}+m)_{\alpha\beta} e^{ip(x-y)} / =$$

$$= \theta(x_0-y_0)\int \frac{d^3p}{2E_p} e^{-ip(x-y)} (\not{p}+m)\Big|_{p_0=E_p} + \theta(y_0-x_0)\int \frac{d^3p}{2E_p} (\not{p}+m) e^{-ip(x-y)}\Big|_{p_0=-E_p} \quad (525$$

$$= \int \frac{d^4p}{(2\pi)^4 i} \frac{m+\not{p}}{m^2-p^2-i\epsilon} e^{-ip(x-y)}$$

↑ Feynman propagator

Wick's theorem for fermions

$$T\{\psi(x)\bar\psi(y)\} \equiv \theta(x_0-y_0)\,\psi(x)\bar\psi(y) - \theta(y_0-x_0)\,\bar\psi(y)\psi(x)$$

$$\langle 0|T\{\psi(x)\bar\psi(y)\}|0\rangle = S_F(x-y) = \int \frac{d^4p}{i}\,\frac{e^{-ip(x-y)}}{m^2-p^2-i\epsilon}(m+\not{p}) \quad \text{- Feynman propagator}$$

Generalization of the definition of T-product: $\underbrace{\text{for bosons, but}}_{\text{same as}}$ (-1) for every fermion interchange

Example:

$$T\{\hat\psi(x)\hat{\bar\psi}(y)\hat\psi(z)\} = \hat\psi(x)\hat{\bar\psi}(y)\hat\psi(z)\,\theta(x_0>y_0>z_0) - \hat{\bar\psi}(y)\hat\psi(x)\hat\psi(z)\,\theta(y_0>x_0>z_0) + \\ + \hat{\bar\psi}(y)\hat\psi(z)\hat\psi(x)\,\theta(y_0>z_0>x_0) - \hat\psi(x)\hat\psi(z)\hat{\bar\psi}(y)\,\theta(x_0>z_0>y_0) + \hat\psi(z)\hat\psi(x)\hat{\bar\psi}(y)\,\theta(z_0>x_0>y_0) - \\ - \hat\psi(z)\hat{\bar\psi}(y)\hat\psi(x)\,\theta(z_0>y_0>x_0) \qquad (526)$$

Normal product for fermions: again (same as for bosons + (-1) for every fermion interchange)

Example:

$$N(a_pa_{p'}a_q^+) = (-1)^2a_q^+a_pa_{p'} = a_q^+a_pa_{p'}\,(= -\,a_q^+a_{p'}a_p)$$

$$N(a_pa_{p'}a_q^+a_{q'}^+) = \cancel{(-1)^2}N(a_q^+a_pa_{p'}a_{q'}^+) = N(a_q^+a_pa_{p'}a_{q'}^+) = \cancel{(-1)^2}N(a_q^+a_{q'}^+a_pa_{p'}) = \\ = a_q^+a_{q'}^+a_pa_{p'}\left(\begin{array}{l}\leqslant -a_{q'}^+a_qa_pa_{p'} \\ \leqslant a_{q'}^+a_q^+a_{p'}a_p\end{array}\right) \qquad (527)$$

$$N(a_pa_{p'}a_q^+a_{p''}a_{q'}^+) = \cancel{(-1)^2}N(a_q^+a_pa_{p'}a_{p''}a_{q'}^+) = (-1)^3N(a_q^+a_{q'}^+a_pa_{p'}a_{p''}) = \\ = -\,a_q^+a_{q'}^+a_pa_{p'}a_{p''}$$

and so on.

Contraction:
$$T\{\psi(x)\bar\psi(y)\} \equiv N(\psi(x)\bar\psi(y)) + \overbrace{\psi(x)\bar\psi(y)}$$
$$\overbrace{\psi(x)\bar\psi(y)} = -\,\overbrace{\bar\psi(y)\psi(x)} = S_F(x-y) \qquad (528)$$
$$\overbrace{\psi(x)\psi(y)} = \overbrace{\bar\psi(x)\bar\psi(y)} = 0 \qquad \left(\text{as usual, } \overbrace{\psi(x)\psi(y)} = T\{\psi(x)\psi(y)\} - \\ - N(\psi(x)\psi(y))\right)$$

For example,

$$N(\overbrace{\psi(x)\psi(y)\bar\psi(z)}\bar\psi(t)) \equiv -\,N(\overbrace{\psi(x)\bar\psi(z)}\psi(y)\bar\psi(t)) = -\,S_F(x-z)\,N(\psi(y)\bar\psi(t))$$

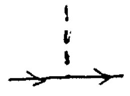
Wick's theorem:

$$T\{\psi(x_1)\psi(x_2)\bar\psi(x_3)\dots\psi(x_n)\} = N(\psi(x_1)\psi(x_2)\bar\psi(x_3)\dots\psi(x_n) + \text{all possible contractions}) \qquad (529)$$

Same as ~~for~~ bosons (but do not forget (-1) factors in the definitions of T and N!)

Yukawa theory

$$\mathcal{L} = \underbrace{\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{M^2}{2}\varphi^2}_{\text{Klein-Gordon}} + \underbrace{\bar\psi(i\not\partial - m)\psi}_{\text{Dirac}} \underbrace{- g\varphi\bar\psi\psi}_{\text{interaction}} \qquad \bar\psi\psi \equiv \bar\psi_\xi\psi_\xi \qquad (530)$$


g - coupling constant

Quantization: similar to self-interacting scalar theory

$$H = \underbrace{H_{KG} + H_D}_{H_0} + g\int d^3x\ \varphi(\vec x,t)\bar\psi(\vec x,t)\psi(x,t) \qquad (531)$$

↖ classical Hamiltonian

$$\hat H = \hat H_{KG} + \hat H_D + g\int d^3x\ \hat\varphi(\vec x)\hat{\bar\psi}(\vec x)\hat\psi(\vec x) = \hat H_0 + \hat H_{int} \qquad (531)$$

quantum ↗ Hamiltonian

Interaction representation

$$\hat\varphi_I(z) = e^{i\hat H_0 t}\hat\varphi(\vec z)e^{-i\hat H_0 t} = e^{i\hat H_{KG}t}\hat\varphi(\vec z)e^{-i\hat H_{KG}t} \qquad z \equiv \vec z, t$$

$$\hat\psi_I(z) = e^{i\hat H_0 t}\hat\psi(\vec z)e^{-i\hat H_0 t} = e^{i\hat H_D t}\hat\psi(\vec z)e^{-i\hat H_D t} \qquad (532)$$

$$\hat{\bar\psi}_I(z) = e^{i\hat H_0 t}\hat{\bar\psi}(\vec z)e^{-iH_0 t} = e^{i\hat H_D t}\hat{\bar\psi}(\vec z)e^{-iH_D t}$$

$\Rightarrow$

$$\hat\varphi_I(z) = \int\frac{d^3p}{\sqrt{2E_p}}\left(a_p e^{-ipz} + a_p^+ e^{ipz}\right)\Big|_{p_0 = E_p = \sqrt{M^2+\vec p^2}}$$

$$\hat\psi_I(z) = \int\frac{d^3p}{\sqrt{2E_p}}\sum_s\left(a_p^s u(p,s)e^{-ipz} + b_p^{+s}v(p,s)e^{ipz}\right)\Big|_{p_0 = E_p = \sqrt{m^2+\vec p^2}} \qquad (533)$$

$$\hat{\bar\psi}_I(z) = \int\frac{d^3p}{\sqrt{2E_p}}\sum_s\left(b_p^s\bar v(p,s)e^{-ipz} + a_p^+\bar u(p,s)e^{ipz}\right)\Big|_{p_0 = E_p = \sqrt{m^2+\vec p^2}}$$

(see eqs (153) and (461-462)).

Similarly to the eq.( )

$$\langle\Omega|T\{\psi(x_1)\ldots\psi(x_m)\bar\psi(x_{m+1})\ldots\bar\psi(x_n)\varphi(y_1)\varphi(y_2)\ldots\varphi(y_\ell)\}|\Omega\rangle = \qquad (534)$$

↑ Heisenberg operators

$|\Omega\rangle$ - „true vacuum" ≡ lowest-energy eigenstate of $\hat H$

$$\frac{\langle 0|T\left[\hat\psi_I(x_1)\ldots\hat\psi_I(x_m)\hat{\bar\psi}_I(x_{m+1})\ldots\hat\psi_I(x_n)\hat\varphi_I(y_1)\ldots\hat\varphi_I(y_\ell)\ \exp\left\{-i\int d^4z\ g\hat\varphi_I(z)\bar\psi_I(z)\psi_I(z)\right\}\right]|0\rangle}{\langle 0|T\{\exp - i\int dz\ g\ \hat\varphi_I(z)\hat{\bar\psi}_I(z)\psi_I(z)\}|0\rangle}$$

↗ operators in the interaction representation

$|0\rangle$ = „perturbative vacuum" ≡ lowest-energy eigenstate of $\hat H_0$

How to calculate the r.h.s. (of eq. (534)): expand in powers of g and go ahead using the (anti)commutation relations (129) and (463)

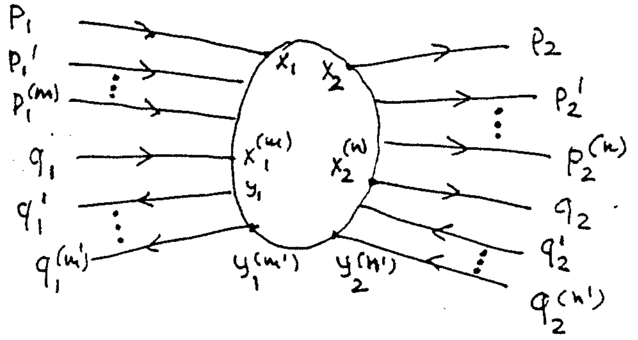
$$[a_{p'}, a^+_p] = (2\pi)^3 \delta(\vec{p}-\vec{p}\,')$$
$$\{a^s_p, a^{+\,s'}_{p'}\} = (2\pi)^3 \delta(\vec{p}-\vec{p}\,')\delta_{ss'} \; ; \; \{b^s_p, b^{s'+}_{p'}\} = (2\pi)^3 \delta(\vec{p}-\vec{p}\,')\delta_{ss'} \qquad (535)$$

and the fact that

$$a_p|0\rangle = a^s_p|0\rangle = b^s_p|0\rangle = 0 \quad \text{and} \quad \langle 0|a^+_p = \langle 0|a^{+s}_p = \langle 0|b^{+s}_p = 0. \qquad (536)$$

(As usual, the denominator will cancel the vacuum bubbles, cf. eq. (302)).

For calculation of cross sections we need the LSZ theorem



$$_{out}\langle p_2 s_2; p_2' s_2'; \dots p_2^{(n)}, s_2^{(n)}; q_2, r_2; q_2', r_2'; \dots q_2^{(n')}, r_2^{(n')} | p_1, s_1; p_1', s_1'; \dots p_1^{(m)}, s_1^{(m)}; q_1, r_1; \dots q_1^{(m')}, r_1^{(m')}\rangle_{in} =$$
$$= i^{m'+n'-m-n} \int \prod_1^m dx_1 \prod_1^{m'} dy_1 \prod_1^n dx_2 \prod_1^{n'} dy_2 \; \exp\Big\{-i\sum_1^m p_1^{(k)} x_1^{(k)} - i\sum_1^{m'} q_1^{(k)} y_1^{(k)} +$$
$$+ i\sum_1^n p_2^{(k)} x_2^{(k)} + i\sum_1^{n'} q_2^{(k)} y_2^{(k)}\Big\} \, .$$
$$\cdot \langle \Omega | T\Big\{ \prod_1^n \bar{u}(p_2^{(k)}, s_2^{(k)})(i\not{\partial} - m)\psi(x_2^{(k)}) \prod_1^{m'} \bar{v}(p_1^{(k)}, r_1^{(k)})(i\not{\partial} - m)\psi(y_1^{(k)}) \, \cdot$$
$$\cdot \prod_1^m \bar{\psi}(x_1^{(k)})(-i\overleftarrow{\not{\partial}} - m)u(p_1^{(k)}, s_1^{(k)}) \prod_1^{n'} \bar{\psi}(y_2^{(k)})(-i\overleftarrow{\not{\partial}} - m)v(p_2^{(k)}, r_2^{(k)})\Big\}|\Omega\rangle \qquad (537)$$

Proof: similar to eq. (221).

Meaning is as usual: S-matrix is the amputated Green function on the mass shell.

LSZ ⇔ Peskin's mnemonic rule (cf. eq. (328))

$$ {}_{out}\langle p_2,s_2; p_2',s_2'; \dots p_2^{(n)} s_2^{(n)}; q_2,r_2; \dots q_2^{(n')} r_2^{(n')} | p_1,s_1; \dots p_1^{(m)}, s_1^{(m)}; q_1,r_1; \dots q_1^{(m')} r_1^{(m')} \rangle_{in} = \tag{538}$$

$$= \langle p_2,s_2; \dots p_2^{(n)} s_2^{(n)}; q_2,r_2; \dots q_2^{(n')} r_2^{(n')} | T\{\exp - i\int dz\, g \varphi_I(z) \bar\psi_I(z) \psi_I(z)\} | p_1,s_1; \dots p_1^{(m)} s_1^{(m)}; q_1 r_1 \dots q_1^{(m')} r_1^{(m'}$$

where

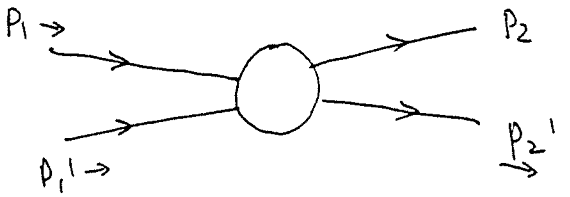
connected d-ms only

$$|p_1 s_1, \dots q_1^{(m')} r_1^{(m')}\rangle = \prod_1^m \sqrt{2E_1^{(k)}}\, a^{+\,s_1^{(k)}}_{p_1^{(k)}} \prod_1^{m'} \sqrt{2E_1^{(k)}}\, b^{r_1^{(k)}+}_{q_1^{(k)}} |0\rangle \tag{539}$$

$$\langle p_2,s_2, \dots q_2^{(n')} r_2^{(n')}| = \langle 0| \prod_1^n \sqrt{2E_2^{(k)}}\, a^{s_2^{(k)}}_{p_2^{(k)}} \prod_1^{n'} \sqrt{2E_2^{(k)}}\, b^{r_2^{(k)}}_{q_2^{(k)}}$$

-states in the interaction representation

To warm up, consider the elastic fermion-fermion scattering in the first nontrivial order of pert. theory



$$S(p_1,s_1; p_1',s_1' \rightarrow p_2,s_2; p_2',s_2') = {}_{out}\langle p_2,s_2; p_2',s_2' | p_1,s_1; p_1',s_1'\rangle_{in} =$$

$$= \langle p_2 s_2; p_2',s_2' | T\exp(-ig\int dz\; \varphi_I \bar\psi_I \psi_I(z)) | p_1 s_1; p_1' s_1'\rangle = \tag{540}$$

$$= \sqrt{2E_2}\sqrt{2E_2'}\sqrt{2E_1}\sqrt{2E_1'}\; \langle 0| \hat a^{s_2}_{p_2} \hat a^{s_2'}_{p_2'}\; T\{\exp(-ig\int dz\, \varphi_I(z)\bar\psi_I(z)\psi_I(z))\}\, a^{+\,s_1'}_{p_1} a^{+\,s_2}_{p_2} |0\rangle$$

Let us expand in powers of coupling constant g.

First term:

$$S^{(1)} = -ig\int dz\, \langle 0| \hat a^{s_2}_{p_2} \hat a^{s_2'}_{p_2'}\; T\{\varphi_I(z)\, \bar\psi_I(z)\, \psi_I(z)\}\; \hat a^{+s_1}_{p_1} \hat a^{+}{}_{p_1'}^{s_1'} |0\rangle \tag{541}$$

$$\| $$

$$\sim \int d^3_p\, (a_p e^{-ipz} + a^+_p e^{ipz})$$

$$= -ig\int dz\, \langle 0| \hat a^{s_2}_{p_2} \hat a^{s_2'}_{p_2'}\; N\{\varphi_I(z)\, \bar\psi_I(z)\, \psi_I(z) + \text{contractions}\}\, a^{+s_1'}_{p_1} a^{+s_2}_{p_2} |0\rangle = 0$$

since $a_p|0\rangle = 0$ or $\langle 0| a^+_p = 0$

Second term of the expansion (we omit the label „I")

$$S^{(2)} = \langle 0 | a^{s_2}_{p_2} a^{s_2'}_{p_2'} (-g^2/2) \int dz\, dz'\, T\{ \varphi(z) \bar\psi \psi(z)\, \varphi(z') \bar\psi \psi(z') \}\, a^{+s_1}_{p_1} a^{+s_1'}_{p_1'} |0\rangle \cdot \sqrt{2E_2 2E_2' 2E_1 2E_1'}$$

$$= \langle 0 | a^{s_2}_{p_2} a^{s_2'}_{p_2'} \left(-\frac{g^2}{2}\right) \int dz\, dz'\, N\big[\varphi \bar\psi \psi(z)\, \varphi \bar\psi \psi(z') + \bar\psi\psi(z) \overline{\varphi(z)\, \varphi(z')}\, \bar\psi \psi(z') +$$

$$+ \text{all other possible contractions}\big]\, a^{+s_1}_{p_1} a^{+s_1'}_{p_1'} |0\rangle \qquad (542)$$

$$= -\frac{g^2}{2} \int dz\, dz'\, D_F(z-z') \langle 0 | a^{s_2}_{p_2} a^{s_2'}_{p_2'} N(\bar\psi(z)\psi(z)\, \bar\psi(z')\psi(z'))\, a^{+s_1}_{p_1} a^{+s_1'}_{p_1'} |0\rangle$$

$$+ (\text{other terms} = 0) =$$

$$= -g^2 \int dz\, dz'\, D_F(z-z') \langle 0 | a^{s_2}_{p_2} a^{s_2'}_{p_2'} N(\bar\psi_\alpha(z)\psi_\alpha(z)\, \bar\psi_\beta(z')\psi_\beta(z'))\, a^{+s_1}_{p_1} a^{+s_1'}_{p_1'} |0\rangle \sqrt{16 E_2 E_2' E_1 E_1'}$$

where

$$\overline{\psi(z')\, a^{+s_1}_{p_1}} \overset{def}{\equiv} \{\psi(z'), a^{+s_1}_{p_1}\} = \int \frac{d^3p'}{\sqrt{2E_{p'}}} \sum_{s'} \Big(\{a^{s'}_{p'}, a^{+s_1}_{p_1}\} e^{-ip'z} u(p',s') + \underbrace{\cancel{\{b^{+s'}_{p'}, a^{+s_1}_{p_1}\}}}_{=0} e^{ip'z} v(p',s') \Big)$$

$$= \frac{1}{\sqrt{2E_{p_1}}} u(p_1,s_1) e^{-ip_1 z'} \Big|_{p_{10}=E_{p_1}} \qquad (543)$$

$$\overline{a^{s_2}_{p_2} \bar\psi}(z) \overset{def}{\equiv} \{a^{s_2}_{p_2}, \bar\psi(z)\} = \int \frac{d^3p'}{\sqrt{2E_{p'}}} \sum_{s'} \Big(\underbrace{\cancel{\{a^{s_2}_{p_2}, b^{s'}_{p'}\}}}_{=0} e^{-ip'z} \bar v(p',s') + \{a^{s_2}_{p_2}, a^{+s'}_{p'}\} e^{ip'z} \cdot$$

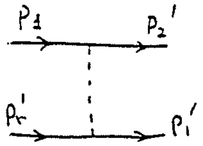
$$\cdot \bar u(p',s') \Big) = \frac{1}{\sqrt{2E_2}} \bar u(p_2,s_2) \exp(ip_2 z_2) \Big|_{p_{20}=E_2} \qquad (544)$$

Collecting ⊖ signs from the permutations, we get

$$S^{(2)} = -g^2 \int dz\, dz'\, e^{-i(p_1-p_2')z'} e^{i(p_2-p_1')z}\, \bar u_\alpha(p_2,s_2) u_\alpha(p_1',s_1')\, \bar u_\beta(p_2',s_2') u_\beta(p_1,s_1)\, D_F(z-z')$$

$$= -g^2 (2\pi)^4 \delta(p_1+p_1'-p_2-p_2') \Big(\underbrace{\int dz\, e^{-iqz} D_F(z)}_{=\frac{1}{i(M^2-u-i\epsilon)}} \Big) \bar u(p_2,s_2) u(p_1',s_1')\, \bar u(p_2',s_2') u(p_1,s_1) \qquad (545)$$

$$= (2\pi)^4 i\, \delta(p_1+p_1'-p_2-p_2') \frac{(\bar u^{s_2}_{p_2} u^{s_1'}_{p_1'})(\bar u^{s_2'}_{p_2'} u^{s_1}_{p_1})}{M^2-u-i\epsilon}$$



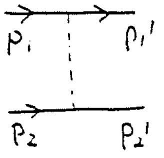
$$u \equiv (p_1-p_2')^2 = (p_1'-p_2)^2$$

Another term:

$$-g^2 \int dz\, dz'\, D_F(z-z') \langle 0 | a^{s_2}_{p_2} a^{s_2'}_{p_2'} N(\bar\psi_\alpha(z)\psi_\alpha(z)\, \bar\psi_\beta(z')\psi_\beta(z'))\, a^{+s_1}_{p_1} a^{+s_1'}_{p_1'} |0\rangle = \qquad (546)$$

$$= +g^2 \int dz\, dz'\, e^{-i(p_1-p_1')z'} e^{i(p_2-p_2')z}\, \bar u_\alpha(p_2,s_2) u_\alpha(p_1,s_1)\, \bar u_\beta(p_2',s_2') u_\beta(p_1',s_1')\, D_F(z-z')$$

$$= -(2\pi)^4 i\, \delta(p_1+p_1'-p_2-p_2') \frac{(\bar u^{s_2}_{p_2} u^{s_1}_{p_1})(\bar u^{s_2'}_{p_2'} u^{s_1'}_{p_1'})}{M^2-t-i\epsilon}$$

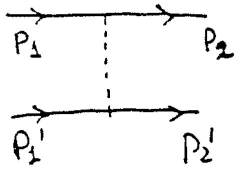


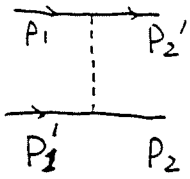
$$t \equiv (p_1-p_1')^2 = (p_2-p_2')^2$$

Thus,

$$S^{(2)}(p_2,s_2,p_2',s_2';p_1,s_1,p_1',s_1') = (2\pi)^4 i\delta(p_1+p_1'-p_2-p_2)\,M^{(2)}(p_2,s_2,p_2',s_2';p_1,s_1,p_1',s_1')$$

$$M^{(2)} = -\left[\frac{(\bar u(p_2,s_2)u(p_1,s_1))(\bar u(p_2',s_2')u(p_1',s_1')}{M^2-t-i\epsilon} - \frac{(\bar u(p_2,s_2)u(p_1',s_1'))(\bar u(p_2',s_2')u(p_1,s_1))}{M^2-u-i\epsilon}\right] \quad (547)$$

= 

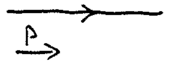
− 

note the relative ⊖ sign between the two diagrams !

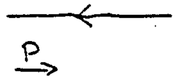
Set of Feynman rules ~~for~~ Yukawa theory
For usual Green functions $G(p_1,..p_1^{(m)};p_2,..p_2^{(n)})$
(in the momentum space)

Propagators 

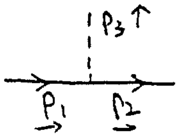
$\equiv \frac{1}{i(M^2-p^2-i\epsilon)}$



$\equiv \frac{m+\not p}{i(m^2-p^2-i\epsilon)}$ ⟹

⟹ 

$\equiv \frac{m-\not p}{i(m^2-p^2-i\epsilon)}$

Vertex 

$-ig\,(2\pi)^4\,\delta(p_1-p_2-p_3)$

External lines

$\overline{\phi\,|q}\rangle \equiv \sqrt{2E_q}\,[\phi,\ a_q^+]|0\rangle \to 1$ $\qquad \langle \overline{q|\phi} \to 1$

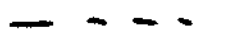
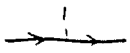
$\overline{\psi\,|p,s}\rangle \equiv \sqrt{2E_p}\,\{\psi, a_p^{+s}\}|0\rangle \to u(p,s)$
incoming electron

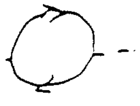
$\langle \overline{p,s|\bar\psi} \equiv \sqrt{2E_p}\,\{a_p^s,\bar\psi\}|0\rangle \to \bar u(p,s)$
outgoing electron

$\overline{\bar\psi\,|p,s}\rangle \equiv \sqrt{2E_p}\,\{\bar\psi, b_p^{+s}\}|0\rangle \to \bar v(p,s)$
incoming positron

$\langle \overline{p,s|\psi} \equiv \sqrt{2E_p}\,\{b_p^s,\psi\} \to v(p,s)$
outgoing positron

For modified Green functions $\mathcal{G}(p_1,..p_1^{(m)};p_2,..p_2^{(n)})$

 $= \frac{1}{M^2-p^2-i\epsilon}$  $= \frac{m+\not p}{m^2-p^2-i\epsilon}$  $= g$

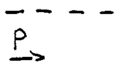
$\int \frac{d^4p}{(2\pi)^4 i}$ for each loop $\qquad$ (−1) for each fermion loop 

Feynman rules for Yukawa theory

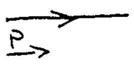
Ⓘ For usual Green functions

$$G(p_1,\dots p_1^{(m_1)}, k_1\dots k_1^{(n_1)}; p_2,\dots p_2^{(m_2)}, k_2,\dots k_2^{(n_2)}) = \Pi\int dx_1^{(i)} e^{ip_1^{(j)}x_1^{(j)}} \Pi\int dy_1^{(j)} e^{ik_1^{(j)}y_1^{(j)}}$$
$$\Pi\int dx_2^{(i)} e^{-ip_2^{(j)}x_2^{(j)}} \int \Pi dy_2^{(j)} e^{-ik_2^{(j)}y_2^{(j)}} \langle \Omega | T\{\psi(x_1)\bar\psi(x_1')\dots\psi(x_1^{(m_1)})\,\varphi(y_1)\dots\varphi(y_1^{(n_1)})$$
$$\bar\psi(x_2)\psi(x_2')\dots\psi(x_2^{(m_2)})\,\varphi(y_2)\dots\varphi(y_2^{(n_2)})\,|\Omega\rangle \qquad (548)$$

1) Propagators



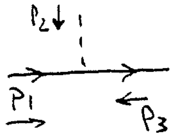
$$\frac{1}{i(M^2-p^2-i\epsilon)}$$



$$\frac{m+\not p}{i(m^2-p^2-i\epsilon)}$$

arrow in the direction of <u>negative</u> charge flow (different notation in comparison to AQM course)

2) Vertex



$$-ig(2\pi)^4\delta(p_1+p_2+p_3)$$

3) Integration over all momenta (except the external legs)

4) No symmetry coefficients ⑤ extra (-1) for fermion loop

Ⅱ For modified Green functions

some integer number

$$G(p_1,\dots k_1^{(n_1)}; p_2,\dots k_2^{(n_2)}) \equiv (2\pi)^4\, i^{\#}\, \delta(\Sigma p_1^{(j)}+\Sigma k_1^{(j)}-\Sigma p_2^{(j)}-\Sigma k_2^{(j)})\cdot \mathcal{G}(p_1,\dots k_1^{(n_1)}; p_2,\dots k_2^{(n_2)})$$

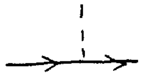
1) Propagators

$$\frac{1}{M^2-p^2-i\epsilon}$$

$$\frac{m+\not p}{m^2-p^2-i\epsilon}$$

p↑↑ flow of negative charge

2) Vertex



$$-g$$

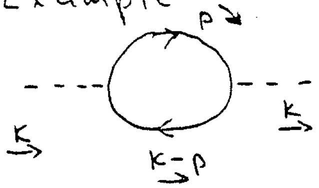
3) Conservation of momentum in each vertex $\oplus$
$\int\frac{d^4p}{(2\pi)^4 i}$ for each undetermined momentum

4) No symmetry coefficient

5) Extra (-1) for each fermiol loop

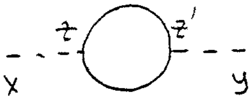
Why (-1) for each fermion loop?

Example



$$\mathcal{G}(k) = \frac{1}{(M^2-k^2-i\epsilon)^2}(-1?)\,g^2\int\frac{\not{d}p}{i}\,\frac{Tr\{(m+\not{p})(m-\not{k}+\not{p})\}}{(m^2-p^2-i\epsilon)(m^2-(k-p)^2-i\epsilon)} \qquad (549)$$

Let us start from the beginning



$$G(x,y) = \langle 0|\,T\{\varphi(x)\,\varphi(y)\,(\tfrac{-g^2}{2})\int dz\,\varphi(z)\bar\Psi\Psi(z)\int dz'\,\varphi\bar\Psi\Psi(z')\}|0\rangle$$

$$= \int dz\,dz'\,g^2\langle 0|\,\overbrace{\varphi(x)\,\varphi(z)}\;\overline{\bar\Psi(z)\underbrace{\Psi(z)\;\bar\Psi(z')}\Psi(z')}\;\overbrace{\varphi(z')\,\varphi(y)}\,|0\rangle =$$

$$-g^2\int dz\,dz'\;D_F(x-z)\,D_F(z'-y)\;(-1)\;\overbrace{\Psi_\alpha(z)\,\bar\Psi_\beta(z')}\;\overbrace{\Psi_\beta(z')\,\bar\Psi_\alpha(z)} = +g^2\int dz\,dz'\;D_F(x-z)\cdot$$

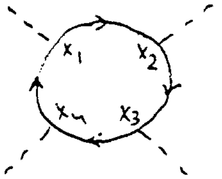
$$\cdot D_F(z'-y)\,S_{F_{\alpha\beta}}(z-z')\,S_{F_{\beta\alpha}}(z'-z) = +g^2\int dz\,dz'\int\frac{\not{d}}{i}\,\frac{e^{-ik_1(x-z)}}{M^2-k_1^2-i\epsilon}\int\frac{\not{d}k_2}{i}\,\frac{e^{-ik_2(z'-y)}}{M^2-k_2^2-i\epsilon}$$

$$\int\frac{\not{d}p_1}{i}\,e^{-ip_1(z-z')}\int\frac{\not{d}p_2}{i}\,e^{-ip_2(z'-z)}\;\frac{Tr\,(m+\not{p}_1)(m+\not{p}_2)}{(m^2-p_1^2-i\epsilon)(m^2-p_2^2-i\epsilon)} = \qquad \begin{matrix}p_1\to p\\ p_2 = p-k\end{matrix}$$

$$= -ig^2\int\not{d}k\,\frac{1}{(M^2-k^2-i\epsilon)^2}\,e^{-ik(x-y)}\int\frac{\not{d}p}{i}\,(-1)\,\frac{Tr\,(m+\not{p})(m+\not{p}-\not{k})}{(m^2-p^2-i\epsilon)(m^2-(p-k)^2-i\epsilon)}$$

$$\Rightarrow\; \mathcal{G}(k) = \frac{1}{(M^2-k^2-i\epsilon)^2}\,(-g^2)\int\frac{\not{d}p}{i}\,\frac{Tr\,(m+\not{p})(m+\not{p}-k)}{(m^2-p^2-i\epsilon)(m^2-(p-k)^2-i\epsilon)} \qquad (550)$$

In general,



$$\sim\; \underbrace{\bar\Psi\overbrace{\Psi(x_1)\;\bar\Psi}\overbrace{\Psi(x_2)\,\bar\Psi}\overbrace{\Psi(x_3)\;\bar\Psi}}\Psi(x_4) =$$

$$= (-1)\;\overbrace{\Psi(x_1)\bar\Psi(x_2)}\;\overbrace{\Psi(x_2)\bar\Psi(x_3)}\;\overbrace{\Psi(x_3)\bar\Psi(x_4)}\;\overbrace{\Psi(x_4)\bar\Psi(x_1)} =$$

$$= (-1)\;Tr\;S_F(x_1-x_2)\;S_F(x_2-x_3)\;S_F(x_3-x_4)\;S_F(x_4-x_1) \qquad (551)$$

Feynman rules for
the matrix elements of S-matrix (or M-matrix)

$S = 1 + (2\pi)^4 i\delta(\Sigma p_i)\,M$

Incoming and outcoming fermions:

$\overbrace{\Psi|p},s\rangle \to u(p,s)$ for incoming electron; $\uparrow$ electron state

$\langle p,\overbrace{s|\bar\Psi} \Rightarrow \bar u(p,s)$ for outgoing electron; $\uparrow$ electron state

$\overbrace{\Psi|\widetilde{p},}s\rangle \to \bar v(p,s)$ for incoming positron; $\uparrow$ positron state

$\langle\widetilde{p},s|\bar\Psi \to v(p,s)$ for outgoing positron; $\uparrow$ positron state

For scalar particles, $\overbrace{\varphi|p},s\rangle \to 1$ and $\langle p,\overbrace{s|\varphi} \to 1$

Final recipe for cross sections

$$d\sigma = \frac{1}{I} \prod \int \frac{d^3 p_2^{(i)}}{\sqrt{2E_2^{(i)}}} \prod \int \frac{d^3 k_2^{(j)}}{\sqrt{2E_2^{(j)}}} (2\pi)^4 \delta(p_1 + p_1' - \sum p_2^{(i)} - \sum k_2^{(j)}) \, |M(p_1, p_1'; p_2 .. k_2^{(n_2)})|^2 \quad (552)$$

↑ general formula for $2 \to n$ cross section

some int er number

$$M(p_1, p_1'; p_2 \dots k_2^{(n_2)}) = i^{\#} G^{amp}_{conn}(p_1, p_1'; p_2, .. k_2^{(n_2)}) \cdot \quad (553)$$

- $u(p,s)$ for incoming electron (fermion)
  $\bar{u}(p,s)$ for outgoing electon (fermion)
  $\bar{v}(p,s)$ for incoming positron (antifermion)
  $v(p,s)$ for outgoing positron (antifermion)

(Amputation means removing the legs like

from the diagram)

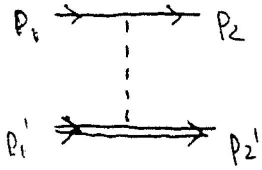
Simple application - Yukawa potential

Non-relativistic limit : $|\vec{p}| \ll m$

Distinguishable fermions $u(p,s)$ and $U(p,s)$

different particles (like p and n)

Let us compare first-order result for the trasition matrix (≡ cross section) with the Born approximation f-la from quantum mechanics.



Feynman diagram

μ - mass of the scalar (π) mesor

$$G(p_1, p_1'; p_2, p_2') = \frac{g^2}{\mu^2 - (p_1 - p_2)^2 - i\epsilon} \Rightarrow$$

$$\Rightarrow M(p_1, p_1'; p_2, p_2') = g^2 \frac{(\bar{u}(p_2, s_2) u(p_1, s_1))(\bar{U}(p_2', s_2') U(p_1', s_1'))}{\mu^2 - t - i\epsilon} \quad (554)$$

In the nonrel. limit $p_1 = (m, \vec{p}_1)$, $p_1' = (m, \vec{p}_2')$, $p_2 = (m, \vec{p}_2)$, $p_2' = (m, \vec{p}_2')$

mass of fermions

$t = (p_1 - p_2)^2 = -(\vec{p}_1 - \vec{p}_2)^2$

$$u(p,s) = \sqrt{m} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix} \Rightarrow \bar{u}(p_2, s_2) u(p_1, s_1) = 2m \, \xi^+_{s_2} \xi_{s_1} = 2m \, \delta_{s_2 s_1}$$

Similarly $\bar{U}(p_2', s_2') U(p_1', s_1') = 2m \delta_{s_2' s_1'}$ (555)

Thus,

$$M(p_2,p_2';p_1,p_1') = \frac{4m^2g^2}{\mu^2+(\vec{p}_2-\vec{p}_1)^2}\delta_{s_2s_1}\delta_{s_2's_1'} \quad \leftarrow \text{ spin independent} \qquad (556)$$

and the cross section is

$$\frac{d\sigma}{d\Omega} = \frac{|M|^2}{64\pi^2 s} = \frac{g^4m^2}{16\pi^2}\frac{1}{(\mu^2+4|\vec{p}_1|^2\sin^2\theta/_2)^2} \qquad (557)$$

Let us compare this answer with the non-relativistic result for the scattering from the Yukawa potential

$$V(\vec{r}) = V_0\frac{e^{-\alpha|\vec{r}|}}{|\vec{r}|}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{nrl} = \left(\frac{mV_0}{\alpha^2+4|\vec{p}_1|^2\sin^2\theta/_2}\right)^2 \qquad (558)$$

We see that

$$\alpha = \mu \qquad \mu (= m_\pi) \text{ is the range of Yukawa potential } (\simeq 1\text{fm})$$

$$V_0^2 = \frac{g^4}{16\pi^2} \Rightarrow V_0 = \underset{\uparrow}{-}\frac{g^2}{4\pi} \qquad (559)$$

corr. to attractive potential

In order to check the sign, we must return to the nrl formula

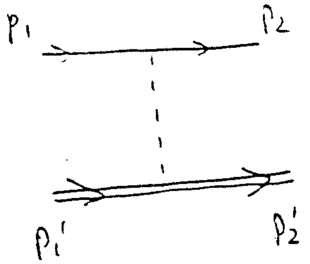
$$\langle\vec{p}_2|T|\vec{p}_1\rangle = -\bar{V}(\vec{q})\,2\pi\,\delta(E_2-E_1) \qquad \vec{q} = \vec{p}_2-\vec{p}_1 \qquad \Rightarrow$$

$$\Rightarrow \bar{V}(\vec{q}) = \frac{-g^2}{\vec{q}^2+\mu^2} \Rightarrow \bar{V}(\vec{r}) = -g^2\int d^3q\,\frac{e^{i\vec{q}\vec{r}}}{\mu^2+|\vec{q}|^2} = \qquad (560)$$

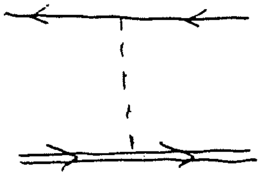
$$= -\frac{g^2}{4\pi^2}\int_0^\infty dq\, q^2\,\frac{e^{iq|\vec{r}|}-e^{-iq|\vec{r}|}}{iqr}\,\frac{1}{\mu^2+q^2} = -\frac{g^2}{4\pi^2 ir}\int_{-\infty}^{\infty}dq\,\frac{qe^{iqr}}{\mu^2+q^2} = -\frac{g^2}{4\pi r}e^{-\mu r}$$

Yukawa potential is attractive for the antiparticles also.

Let us compare $M(p_2,p_2';p_1,p_1')$ for particle-particle and particle-antiparticle scattering



$$M^{ff}(p_1,p_1';p_2 p_2') = g^2 \frac{\bar{u}(p_2 s_2)\, u(p_1 s_1)\ \bar{U}(p_2' s_2')\, U(p_1' s_1')}{M^2+\vec{q}^{\,2}} \qquad (561)$$



$$M^{f\bar{f}}(p_1,p_1';p_2,p_2') = -g^2 \frac{\bar{v}(p_1 s_1)\, v(p_2 s_2)\ \bar{U}(p_2' s_2')\, U(p_1' s_1')}{\mu^2+\vec{q}^{\,2}} \qquad (562$$

Different sign corresponds to

$$\langle 0|\, a_{p_2'} a_{p_2}\ \bar{\psi}\psi\ \bar{\psi}\psi\ a^+_{p_1} a^+_{p_1'}\, |0\rangle \quad \text{versus} \quad \langle 0|\, a_{p_2'} b_{p_2}\ \bar{\psi}\psi\ \bar{\psi}\psi\ b^+_{p_1} a^+_{p_1'}\, |0\rangle$$

But this sign is compensated by the (-) sign in the product of v-spinors

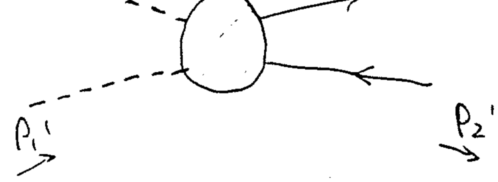
$$\bar{v}(p_1,s_1)\, v(p_2,s_2) \simeq m(\xi^{+s_1},-\xi^{+s_1})\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\xi^{s_2}\\-\xi^{s_2}\end{pmatrix} = -2m\,\delta_{s_1 s_2} \qquad (563)$$

So in the nonrelativistic limit

$$M^{f\bar{f}}(p_1,p_1';p_2,p_2') = M^{ff}(p_1,p_1',p_2,p_2') \qquad (564)$$

$\Rightarrow$ Yukawa potential is always attractive (both for particles and antiparticles)

HW 3: Find the differential cross section (in c.m. frame) for the process $\pi\pi \to e^+e^-$ in the lowest order in perturbation theory



$$\frac{d\sigma}{d\Omega} = ?$$