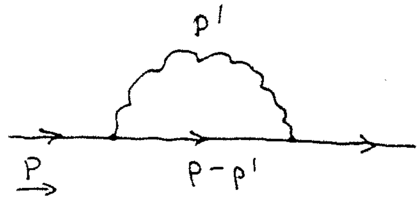
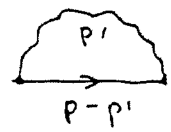
Renormalization (Peskin ch. 7, $e_{\text{Peskin}} = -e_{\text{here}}$)

A problem: some Feynman diagrams in QED are divergen

Example



$$= \frac{m+\not{p}}{m^2-p^2-i\epsilon}\underbrace{\left\{e^2\int\frac{\bar{d}^4p'}{i}\,\frac{\gamma_\mu(m+\not{p}-\not{p}')\gamma_\nu}{m^2-(p-p')^2-i\epsilon}\,\frac{g^{\mu\nu}}{p'^2+i\epsilon}\right\}}\frac{m+\not{p}}{m^2-p^2-i\epsilon}$$



$$(698)$$

$$\Sigma(p) = -$$

$$= -e^2\int\frac{\bar{d}^4p'}{i}\,\frac{4m+2(\not{p}'-p)}{m^2-(p'-p)^2-i\epsilon}\,\frac{1}{p'^2+i\epsilon} = \ ? \qquad (699)$$

Feynman formula

$$\frac{1}{AB} = \int_0^1 d\alpha\,\frac{1}{(A\bar{\alpha}+B\alpha)^2} \qquad (700)$$

$\bar{\alpha} \equiv 1-\alpha$ convenient notation

In general,

$$\frac{\Gamma(a)}{A^a}\,\frac{\Gamma(b)}{B^b} = \int_0^1 d\alpha\,\bar{\alpha}^{a-1}\alpha^{b-1}\,\frac{\Gamma(a+b)}{(A\bar{\alpha}+B\alpha)^{a+b}} \qquad (701)$$

Using Feynman formula (700), we get

$$\Sigma(p) = e^2\int_0^1 d\alpha\int\frac{\bar{d}^4p'}{i}\,\frac{4m+2(\not{p}'-\not{p})}{[m^2\alpha-(p-p')^2\alpha-p'^2\bar{\alpha}-i\epsilon]^2} =$$

$$= e^2\int_0^1 d\alpha\int\frac{\bar{d}^4p'}{i}\,\frac{4m+2(\not{p}'-\not{p})}{[m^2\alpha-(p'-p\alpha)^2-p^2\bar{\alpha}\alpha-i\epsilon]^2} = \text{shift } p'\to p'+p\alpha$$

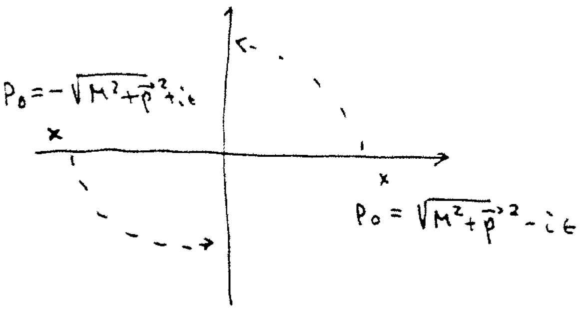
$$= e^2\int_0^1 d\alpha\int\frac{\bar{d}^4p'}{i}\,\frac{4m+2\not{p}'-2\not{p}\bar{\alpha}}{[m^2\alpha-p^2\bar{\alpha}\alpha-p'^2-i\epsilon]^2} =$$

$\int \bar{d}^4p\; p_\mu\, f(p^2) = 0$
linear term drops

$$= e^2\int_0^1 d\alpha\,(4m-2\not{p}\bar{\alpha})\int\frac{\bar{d}^4p'}{i}\,\frac{1}{[m^2\alpha-p^2\bar{\alpha}\alpha-p'^2-i\epsilon]^2} \qquad (702)$$

Suppose $p^2<0$, then $(M^2 = m^2\alpha + |p^2|\bar\alpha\alpha > 0)$

$$\int \frac{d^4p}{i} \frac{1}{(M^2-p'^2-i\epsilon)^2} = \int d^3p\, \frac{dp_0}{i} \frac{1}{(M^2+\vec p^{\,2}-p_0^2-i\epsilon)^2} =$$



= shift the contour of integration over p_0 to imaginary axis $p_0 = ip_4$

$$= \int d^3p\, dp_4 \frac{1}{(M^2+\vec p^{\,2}+p_4^2)^2} = \int d^4p \frac{1}{(M^2+p^2)^2}$$

the last integral is Euclidean, so (703)

$$\int d^4p = \frac{\pi^2}{16\pi^4}\int p^2 dp^2 \Rightarrow$$

$$\Rightarrow \int \frac{d^4p}{i}\frac{1}{(M^2-p^2-i\epsilon)^2} = \frac{1}{16\pi^2}\int p^2dp^2 \frac{1}{(M^2+p^2)^2} \sim \frac{1}{16\pi^2}\int_{M^2}^{\infty}\frac{dp^2}{p^2} \sim \frac{1}{16\pi^2}\ln\infty \quad (704)$$

diverges at the upper limit (UV divergence

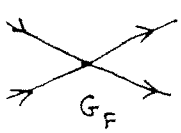
If we cut the integration at some large μ_{UV} („ultraviolet cutoff") we get

$$\Sigma(p) \underset{p^2<0}{=} e^2\int_0^1 d\alpha\,(4m-2p\bar\alpha)\int^{\mu_{UV}} d^4p' \frac{1}{(\alpha m^2 - p^2\bar\alpha\alpha + p'^2)^2} \simeq$$

$$\simeq e^2\int_0^1 d\alpha\,(4m-2p\bar\alpha)\frac{1}{16\pi^2}\ln\frac{\mu_{UV}^2}{m^2\alpha - p^2\bar\alpha\alpha} \quad (705)$$

The nessesity of a cutoff may indicate the existence of some new physics at large momenta.

Example: four-fermion weak interactions of 50's



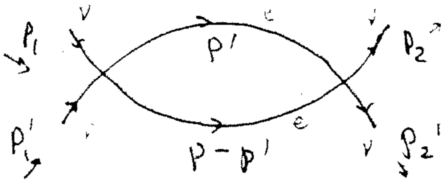
Fermi constant

$$\mathcal{L}_{int} = G_F(\bar\psi\gamma_\mu(1-\gamma_5)\nu)(\bar\nu\gamma_\mu(1-\gamma_5)\psi) \quad (706)$$

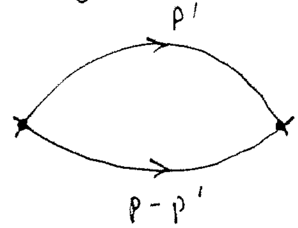
electron-positron field

neutrino field

Let us compute second-order scattering diagram in this theory



$p \equiv p_1 + p_1' = p_2 + p_2'$

amputated = 

$= F(p)$

"fish diagram"

$$F(p) = \int \frac{d^4p'}{i} \frac{(m+\not p')}{m^2 - p'^2 - i\epsilon} \otimes \frac{(m + \not p - \not p')}{m^2 - (p-p')^2}$$ use Feynman formula (700) =

$$= \int \frac{d^4p'}{i} \int_0^1 d\alpha \frac{(m + \not p' + \not p \alpha)}{(m^2 - p^2 \bar\alpha \alpha - p'^2)^2} \otimes (m + \not p \bar\alpha - \not p') = \text{take } p' \to \infty$$

$$\sim \int_0^1 d\alpha \int \frac{d^4p'}{i} \frac{\not p' \otimes \not p'}{(m^2 - p^2 \bar\alpha \alpha - p'^2 - i\epsilon)^2} \sim \int_0^1 d\alpha \int d^4p' \frac{\not p' \otimes \not p'}{(m^2 - p^2 \bar\alpha \alpha + p'^2)^2} =$$

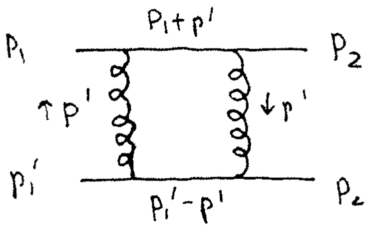
$$= \frac{1}{4} \gamma_\rho \otimes \gamma_\rho \int_0^1 d\alpha \int d^4p' \frac{p'^2}{(m^2 - p^2 \bar\alpha \alpha + p'^2)^2} \sim \gamma_\rho \otimes \gamma_\rho \frac{1}{16\pi^2} \int_{m^2}^{\mu^2} dp'^2 \sim$$

$$\sim \gamma_\rho \otimes \gamma_\rho \frac{1}{16\pi^2} \mu_{UV}^2 \qquad (707)$$

"quadratic divergence"

Now we know that the weak interactions are mediated by W (and Z) bosons with $m_W \simeq 80$ GeV so instead of the fish diagram we have

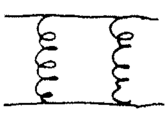
"weak coupling"



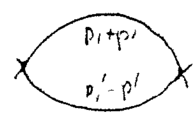
$$\sim \int \frac{d^4p'}{i} \frac{(m + p_1 + p') \otimes (m + p_1' - p')}{(m^2 - (p_1+p')^2)(m^2 - (p_1' - p')^2)} \frac{g_w^4}{(m_W^2 + p'^2)^2}$$

If we take the region of $\sim m^2 \ll m_W^2$, then W propagators

contract:



$$\to \frac{g_w^4}{m_W^4} \int d^4p' \frac{(m + p_1 + p') \otimes (m + p_1' - p')}{(m^2 - (p_1 + p'))^2 (m^2 - (p_1' - p')^2)} =$$



$= F(p)$

So, m_W serves as the ultraviolet cutoff for the four-fermion interaction (and $G_F = \frac{g_w^2}{2\sqrt{2}\, m_W^2}$)

The four-fermion model with Lagrangian (706) is an example of so-called „non-renormalizable" theory which is incomplete at large momenta. The indication for this incompleteness is the explicit dependence of physical cross sections on the UV cutoff μ (like in the above example).

The situation in QED (and other so-called „renormalizable" theories) is more subtle: one still needs the UV cutoff μ_{UV} for calculation of individual Feynman diagrams but the cross sections do not depend on μ.

How can it be?

$$\mathcal{L}_{QED} = \bar{\Psi}(i\partial\!\!\!/ - \underbrace{m_0}_{\text{„bare mass"}} + \overbrace{e_0}^{\text{„bare charge"}} A\!\!\!/)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{708}$$

Physical mass $m \equiv$ mass of the physical particles which propagate large distances and show up in our detectors.

Physical charge $e \equiv$ coefficient in front of Coulomb potential between two physical particles

$V(r) = \frac{e^2}{4\pi r}$ (from the experiment $\alpha \equiv \frac{e^2}{4\pi} \simeq \frac{1}{137}$)

A priori, we do not know the relation between m_0, e_0 and m, e. We have demonstrated that in the leading order in perturbation theory $m_0 = m$ and $e_0 = e$. It turns out, however, that already in the next-to-leading order this is no longer true. In general, both m and e can be expressed as an infinite series in coupling constant ($\equiv$ charge) e_0

$$m = m_0(1 + a_1 e_0^2 + a_2 e_0^4 + \dots) \tag{709}$$

$$e = e_0(1 + b_1 e_0^2 + b_2 e_0^4 + \dots) \tag{710}$$

(It is easy to see that the parameter of pert. expansion is e^2 rather than e.
As we shall see below, the coefficients a_i, b_i in these expansions logarithmically depend on the cutoff μ_{UV} (in general $a_n(b_n) = \sum_{k=0}^{n} c_k \ln^k \frac{\mu^2}{m_0^2}$ where c_k are numbers)

Solving the eqs. (709) and (710) one can re-express bare mass m_0 and bare charge e_0 in terms of physical e and m (with the coefficients being again logarithms of the UV cutoff)

$$m_0 = m(1 + \tilde{a}_1 e^2 + \tilde{a}_2 e^4 + \dots) \qquad (711)$$

$$e_0 = e(1 + \tilde{b}_1 e^2 + \tilde{b}_2 e^4 + \dots) \qquad (712)$$

Now, suppose we calculate a certain cross section using the set of Feynman rules following from the Lagrangian (708)

$$\sigma = e_0^4(\sigma_0^{(0)} + \sigma_1^{(0)} e_0^2 + \sigma_2^{(0)} e_0^4 + \dots) \qquad (713)$$

(Typically, a cross section starts from the term $\sim e_0^4$)

The $\sigma_k^{(0)}$'s in r.h.s. of eq. (713) are the functions of the momenta of scattered particles, m_0, and the UV cutoff μ. (It may be proved, that all the dependence on μ is logarithmical).

<u>Renormalizability of QED</u>: if one expresses the cross sections in terms of e and m rather than e_0 and m_0 the dependence on μ_{UV} drops.

In other words, in one plugs in the series (711) and (712) in the expansion of the cross section (713) one obtains the series

$$\sigma = e^4(\sigma_0 + \sigma_1 e^2 + \sigma_2 e^4 + \dots) \qquad (714)$$

where the coefficients σ_k are the finite functions of the momenta of the scattered particles and physical mass m.

Summary: renormalization program in QED

1. $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\partial\!\!\!/ + e_0 A\!\!\!/ - m_0)\Psi$

bare charge and bare mass

2. Calculate Feynman diagrams imposing an ultraviolet cutoff μ

3. Calculate m in terms of m_0 and e_0: $m = m_0(1 + a_1 e_0^2 + a_2 e_0^4 + \ldots)$

4. To relate e_0 to the physical charge e, calculate the diagrams for the non-relativistic Coulomb exchange. the result for the Coulomb potential will be

$$V(r) = \frac{1}{4\pi r}\underbrace{e_0^2(1 + b_1 e_0^2 + b_2 e_0^4 + \ldots)}$$

by definition, this is a (square of the) physical charge of the electron e^2 $\left(\frac{e^2}{4\pi} = \frac{1}{137}\right)$

4. Express m_0 and e_0 as a series in physical charge e (see eqs. (711), (712))

5. Write down Feynman diagrams as a series in physical charge e (and get rid of m_0 in favor of m).

The resulting expressions for the cross sections in terms of e and m will not depend on $\mu \Rightarrow$ will be finite.

So, the nessesity of the cutoff may indicate our incomplete understanding of the theory at large momenta, but the information about this new physics at large momenta is screened by the property of renormalizability (unlike the non-renormalizable theories where it was explicit like $G_F \simeq \frac{g_W^2}{M_W^2}$).

Renormalization program in QED at the one-loop level

1. Physical mass and LSZ theorem.

Technically, it is more convenient to calculate Feynman diagrams in terms of e_0 and physical mass m (rather than in terms of e_0 and m_0)

Similarly to the scalar theory (see eqs. (309-327)), we get

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\psi(i\partial\!\!\!/ - m_0 + e_0 A\!\!\!/)\psi =$$

$$= \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\psi(i\partial\!\!\!/ - m)\psi}_{\text{new } \mathcal{L}_0} + \underbrace{\delta m\,\bar\psi\psi + e_0\bar\psi A\psi}_{\text{new } \mathcal{L}_{int}} \qquad (715)$$

 $= \dfrac{1}{m - p\!\!\!/ - i\epsilon}$ ← physical mass

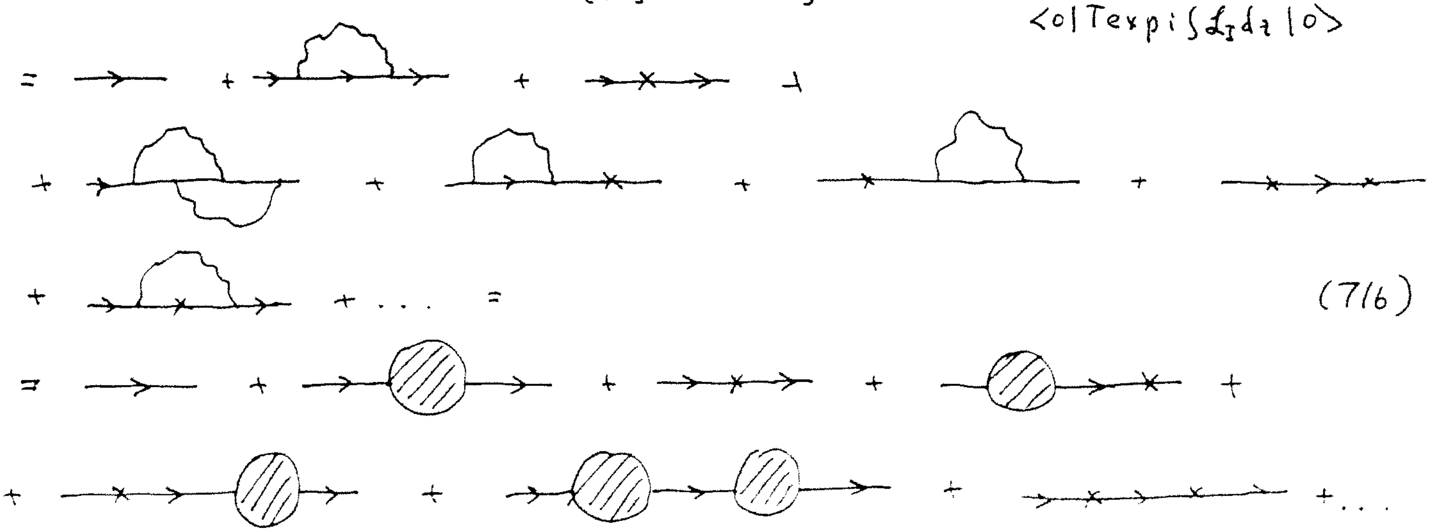
 - vertices (δm, $e_0\gamma^\mu$)

$\delta m = m(c_1 e_0^2 + c_2 e_0^4 + \ldots)$ - "mass counterterm"

 $= \dfrac{g_{\mu\nu}}{p^2 + i\epsilon}$ as usual

The coefficients $c_1, c_2, \ldots$ in the perturbative expansion of δm are fixed by the condition that the pole of the exact electron propagator $G(p)$ remains at $p^2 = m^2$.

$$G(p) = i\int dx\, e^{ipx}\langle\Omega|T\{\psi(x)\bar\psi(y)\}|\Omega\rangle = i\int dx\, e^{ipx}\frac{\langle 0|T\{\hat\psi_I(x)\hat{\bar\psi}_I(y)\, e^{i\int \hat{\mathcal{L}}_I dz}\}|0\rangle}{\langle 0|T\exp i\int \mathcal{L}_I dz|0\rangle}$$



(716)

where

= sum of the 1PI diagrams
"one-particle irreducible"

Notation

$\Sigma(p) = -$  $= -$  $+ O(e_0^4)$ – "self-energy" (717)

$\Sigma(p)$ is a 4×4 matrix depending on p.

Let us calculate it

$$\Sigma(p) = -e_0^2 \int \frac{d^4p'}{i} \frac{4m + 2(\not p' - \not p)}{(m^2 - (p'-p)^2 - i\epsilon)(p'^2 + i\epsilon)} + O(e_0^4) \qquad \text{(see eq. (699))}$$

From previous lecture we know that this integral diverges (logarithmically) at large p so we need some UV cutoff.

Rigorous definition of regularized Feynman diagrams: dimensional regularization and MS scheme
"minimal subtraction"

Step 1: calculate $\Sigma(p)$ at arbitrary (integer) dimension of the space-time d. Define

$$\tilde{\Sigma}^d(p) \equiv -e_0^2 \int \frac{d^dp'}{i} \frac{4m + 2(\not p' - \not p)}{(m^2 - (p-p')^2 - i\epsilon)(p'^2 + i\epsilon)} \qquad \int d^dp \equiv \int dp_0 \int d^{d-1}p \qquad (718)$$

= use Feynman f-la (700) =

$$e_0^2 \int_0^1 d\beta \int \frac{d^dp'}{i} \frac{4m + 2\not p' - 2\not p\bar\beta)}{[m^2\beta - p^2\bar\beta\beta - p'^2 - i\epsilon]^2} =$$

linear term in the numerator drops due to $\int d^dp \; p_\mu f(p^2) = 0$

$$= e_0^2 \int_0^1 d\beta \,(4m - 2\not p\bar\beta) \int \frac{d^dp'}{i} \frac{1}{[m^2\beta - p^2\bar\beta\beta - i\epsilon - p'^2]^2} \qquad \bar\beta \equiv 1-\beta$$

= rotation of the contour of integration over p_0', see previous lecture: $p_0' \to ip_0'$

$$= e_0^2 \int_0^1 d\beta\,(4m - 2\not p\bar\beta) \int \frac{\cancel{i}dp_0' d^{d-1}p'}{\cancel{i}} \frac{1}{[m^2\beta - p^2\bar\beta\beta - i\epsilon + p'^2]} =$$

$$= e_0^2 \int_0^1 d\beta\,(4m - 2\not p\bar\beta) \int d^dp' \frac{1}{[m^2\beta - p^2\bar\beta\beta - i\epsilon + p'^2]^2} \qquad (719)$$

There is a formula

$$\int d^d p \, f(p^2) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int dp \, p^{d-1} f(p^2) \tag{720}$$

so

$$\int \frac{d^d p}{} \frac{\Gamma(a)}{(M^2+p^2)^a} = \frac{\pi^{d/2}}{\Gamma(d/2)} \int \frac{dp^2 (p^2)^{\frac{d}{2}-1}}{(2\pi)^d} \frac{\Gamma(a)}{(M^2+p^2)^a} = \frac{\pi^{d/2}}{(2\pi)^d} \frac{\Gamma(a-d/2)}{(M^2)^{a-d/2}} \tag{721}$$

and therefore

$$\tilde{\Sigma}^d(p) = e_0^2 \frac{\pi^{d/2}}{(2\pi)^d} \int_0^1 d\beta \, (4m - 2\bar{\beta}\not{p}) \frac{\Gamma(2-d/2)}{(m^2\beta - p^2\bar{\beta}\beta - i\epsilon)^{2-d/2}} \tag{722}$$

Formally, the integral (718) is defined only for $d=1,2,$ and 3 but the r.h.s. of eq. (722) gives us an opportunity to define $\tilde{\Sigma}^d(p)$ at any (real or complex) d by analytical continuation. At the point $d=4$ Γ-function has a pole reflecting the fact that the original integral (699) is divergent.

Step 2.

Define

$$\Sigma^d(p) = \tilde{\mu}^{4-d} \tilde{\Sigma}^d(p) \tag{723}$$

$\tilde{\mu}$ — UV cutoff

and expand $\Sigma^d(p)$ as a function of d around the pole at $d=4$

$$\Gamma(2-d/2) = \frac{\Gamma(3-d/2)}{2-d/2} = \frac{1}{2-d/2}\left(1 + (2-d/2)\psi(1) + O(2-d/2)^2\right) \tag{724}$$

$\psi(1) = -c$

$c = 0.577\ldots$ Euler constant

$$\frac{\tilde{\mu}^{4-d}(4\pi)^{2-d/2}}{(m^2\beta - p^2\bar{\beta}\beta - i\epsilon)^{2-d/2}} = 1 + (2-d/2)\ln\frac{4\pi\tilde{\mu}^2}{m^2\beta - p^2\bar{\beta}\beta - i\epsilon} + O(2-d/2)^2 \tag{725}$$

$$\Rightarrow \Sigma^d(p) = \frac{e_0^2}{16\pi^2}\int_0^1 d\beta\,(4m - 2\not{p}\bar{\beta})\frac{(4\pi\tilde{\mu}^2)^{2-d/2}\,\Gamma(2-d/2)}{(m^2\beta - p^2\bar{\beta}\beta - i\epsilon)^{2-d/2}} =$$

$$= \frac{e_0^2}{16\pi^2}\int_0^1 d\beta\,(4m - 2\not{p}\bar{\beta})\left(\frac{1}{2-d/2} + \ln\frac{\tilde{\mu}^2 4\pi e^{-c}}{m^2\beta - p^2\bar{\beta}\beta - i\epsilon} + O(2-d/2)\right) =$$

$$= \frac{e_0^2}{16\pi^2}\left\{\frac{4m - \not{p}}{2-d/2} + \int_0^1 d\beta\,(4m - 2\not{p}\bar{\beta})\ln\frac{\mu^2}{m^2\beta - p^2\bar{\beta}\beta - i\epsilon}\right\} \tag{726}$$

$\mu^2 = \tilde{\mu}^2 4\pi e^{-c}$ — UV cutoff in the „$\overline{MS}$ scheme"

Step 3: „minimal subtraction" (MS) prescription

Define $\Sigma^{reg}(p)$ as $\Sigma^{d}(p)$ - (pole at $d=4$)

$$\Sigma^{reg}(p) = \frac{e_0^2}{16\pi^2}\int_0^1 d\beta\,(4m - 2\not p\bar\beta)\ \ln\frac{\mu^2}{m^2\beta - p^2\bar\beta\beta - i\epsilon} + O(e_0^4) \tag{727}$$

In general

$$\Sigma^{reg}(p) = m\,\Sigma_1(p^2) - \not p\,\Sigma_2(p^2) \tag{728}$$

where

$$\Sigma_{1,2}(p^2) = \sum_{n=2}^{\infty} e_0^{2n} f_n\left(\frac{\mu^2}{p^2}, \frac{p^2}{m^2}\right)$$

and f_n's are scalar logarithmical functions.

Let us return now to the calculation of exact propagator $G(p)$ in (716)

$$G(p) = \frac{1}{m-\not p} - \frac{1}{m-\not p}\Sigma(p)\frac{1}{m-\not p} + \frac{1}{m-\not p}\delta m\frac{1}{m-\not p} - \frac{1}{m-\not p}\Sigma(p)\frac{1}{m-\not p}\delta m\frac{1}{m-\not p} -$$

$$- \frac{1}{m-\not p}\delta m\frac{1}{m-\not p}\Sigma(p)\frac{1}{m-\not p} + \frac{1}{m-\not p}\Sigma(p)\frac{1}{m-\not p}\Sigma(p)\frac{1}{m-\not p} + \frac{1}{m-\not p}\delta m\frac{1}{m-\not p}\delta m\frac{1}{m-\not p} + \ldots =$$

$$= \frac{1}{m - \not p + \Sigma(p) - \delta m} \tag{729}$$

(cf. eq. (311) for scalar theory)

$$\Rightarrow\ G^{reg}(p) = \frac{1}{m - \not p + \Sigma^{reg}(p) - \delta m - i\epsilon} \tag{730}$$

Now we must find δm from the condition that $G^{reg}(p)$ has a pole at $p^2 = m^2$

$$G^{reg}(p) = \frac{1}{m - \delta m + m\Sigma_1 - \not p(1+\Sigma_2) - i\epsilon} = \frac{m - \delta m + m\Sigma_1 + \not p(1+\Sigma_2)}{(m-\delta m + m\Sigma_1)^2 - p^2(1+\Sigma_2)^2 - i\epsilon} =$$

$$= \frac{1}{1+\Sigma_2}\left(\not p + \frac{m-\delta m + m\Sigma_1}{1+\Sigma_2}\right)\frac{1}{\left(\frac{m-\delta m+m\Sigma_1}{1+\Sigma_2}\right)^2 - p^2 - i\epsilon} \tag{731}$$

so $G^{reg}(p)$ has a pole at $p^2 = m^2$ if

$$\frac{m - \delta m + m\Sigma_1(m^2)}{1+\Sigma_2(m^2)} = m \tag{732}$$

which gives

$$\delta m = m(\Sigma_1(m^2) - \Sigma_2(m^2)) \tag{733}$$

Because the expressions for $\Sigma_1(m^2)$ and $\Sigma_2(m^2)$ depend on δm^2 due to the diagrams of the  type, the eq. (733) must be solved anew in each order in pert. theory.

In the lowest order we get

$$\Sigma_1(p^2) = \frac{e_0^2}{16\pi^2}\, 4 \int_0^1 d\beta \, \ln \frac{\mu^2}{m^2\beta - p^2\bar{\beta}\beta - i\epsilon} \qquad (734)$$

$$\Sigma_2(p^2) = \frac{e_0^2}{16\pi^2}\, 2 \int_0^1 d\beta\, \bar{\beta} \ln \frac{\mu^2}{m^2\beta - p^2\bar{\beta}\beta - i\epsilon} \qquad (735)$$

so

$$\delta m = m\left(\Sigma_1(m^2) - \Sigma_2(m^2)\right) = \frac{e_0^2}{8\pi^2} m \int_0^1 d\beta\, (2-\bar{\beta}) \ln \frac{\mu^2}{m^2\beta^2} = m\frac{e_0^2}{8\pi^2}\left(\frac{3}{2}\ln\frac{\mu^2}{m^2} + \frac{5}{2}\right) \qquad (736)$$

the residue of $G(p)$ at $p^2 = m^2$ is the so-called Z-factor (cf. eqs. (320)-(327) for scalar theory)

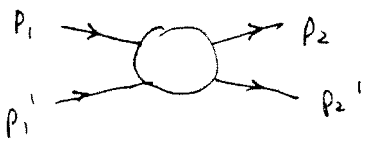
$$Z_2 \simeq \frac{1}{1 + \Sigma_2(m^2)} \simeq 1 - \frac{e_0^2}{8\pi^2}\int_0^1 d\beta\, \bar{\beta} \ln \frac{\mu^2}{m^2\beta^2} = 1 - \frac{e_0^2}{16\pi^2}\left(\ln\frac{\mu^2}{m^2} - 3\right) + O(e_0^4) \qquad (737)$$

Thus,

$$G(p) \xrightarrow[p^2 \to m^2]{} \frac{Z_2 (m + \not{p})}{m^2 - p^2 - i\epsilon} \qquad (738)$$

Similarly to the scalar case, the constant Z_2 is the coefficient of proportionality between $\psi(x)$ and $\psi_{in}(x)$ (or $\psi_{out}(x)$) which enters the LSZ theorem.

LSZ for the electron-electron scattering (see eq. (537)):

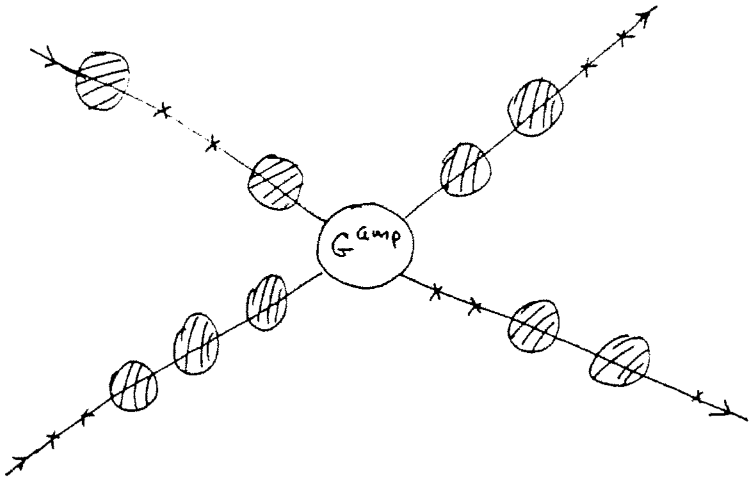


$${}_{out}\langle p_2, s_2; p_2', s_2' | p_1 s_1; p_1', s_1' \rangle_{in} = \frac{1}{(\sqrt{Z_2})^4} \lim_{p_i^2 \to m^2} \left(\bar{u}(p_2,s_2)(m - \not{p}_2)\right)_3 \left(\bar{u}(p_2',s_2')(m-\not{p}_2')\right)_{3'} \cdot$$

$$\cdot\, \underset{\uparrow \text{ spinor indices}}{G_{33'22'}}(p_2, p_1', p_2, p_2') \left((m - \not{p}_1) u(p_1, s_1)\right)_2 \left((m-\not{p}_1') u(p_1', s_1')\right)_{2'} \qquad (739)$$

$G(p_1,p_1',p_2,p_2')$ = sum of all Feynman d-ms with 4 electron tails =

=



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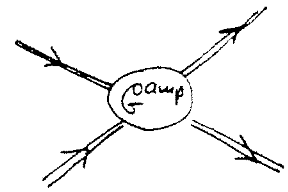



$= \mathcal{G}(p) -$

= exact electron propagator (716)

$$G(p_1,p_1',p_2,p_2') = \mathcal{G}(p_1)\,\mathcal{G}(p_1')\, G^{amp}(p_1,p_1',p_2,p_2')\,\mathcal{G}(p_2)\,\mathcal{G}(p_2') \Rightarrow$$

$\Rightarrow$

$$\lim_{p_i^2\to m^2} (\not{p}_2 - m)(\not{p}_2' - m)\, G(p_1,p_1',p_2,p_2')(\not{p}_1 - m)(\not{p}_1' - m) =$$

$$= \lim_{p_i^2\to m^2} (m-\not{p}_2)\mathcal{G}(p_2)(m-\not{p}_2')\,\mathcal{G}(p_2')\; G^{amp}(p_1,p_1',p_2,p_2')\;\mathcal{G}(p_1)(m-\not{p}_1)\;\mathcal{G}(p_1')(m-\not{p}_1') \quad (740)$$

From eq. (738) we get $(m-\not{p}_i)\,\mathcal{G}(p_i) \xrightarrow[p_i^2\to m^2]{} Z_2$, so

$$(740) = Z_2^4\, G^{amp}(p_1,p_1',p_2,p_2')\Big|_{p_i^2=m^2} \Rightarrow$$

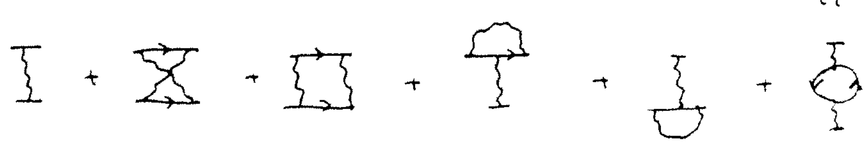
$$\Rightarrow S(p_1 s_1; p_1', s_1' \to p_2, s_2; p_2', s_2') = (\sqrt{Z_2})^4\, \bar u(p_2,s_2)_3\, \bar u(p_2' s_2')_{3'}\, G^{amp}_{33'\eta\eta'}(p_1 p_1' p_2 p_2')\, u_\eta(p_1 s_1)\, u_{\eta'}(p_1' s_1')$$

spinor indices (741)

or, in terms of M-matrix

$$M(p_1,s_1; p_1', s_1' \to p_2, s_2; p_2', s_2') = (\sqrt{Z_2})^4\, \bar u(p_2,s_2)_3\, \bar u(p_2',s_2')_{3'}\, G^{amp}_{33'\eta\eta'}(p_1,p_1',p_2,p_2')\, u_\eta(p_1,s_1)\, u_{\eta'}(p_1',s_1')$$

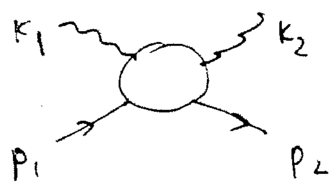
where

G^{amp} =  $-$

$-$ (diagrams with the exchange $p_2 \rightleftarrows p_2'$ in the final state) $+ O(e_0^6)$

Photon propagator and Z_2.

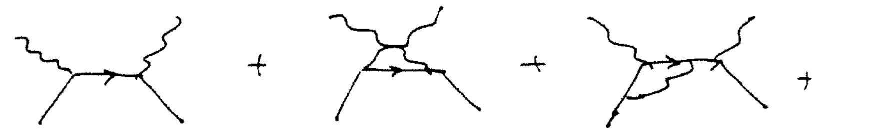
Example: LSZ for Compton scattering

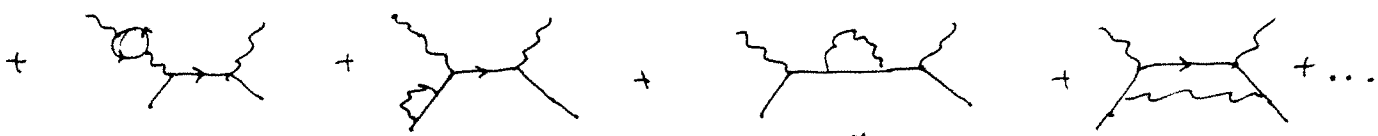


$$S(p_1,k_1 \to p_2,k_2) = -\left(\frac{1}{\sqrt{Z_3}}\right)^2 \left(\frac{1}{\sqrt{Z_2}}\right)^2 \lim_{\substack{p_i^2\to m^2 \\ k_i^2 \to 0}} \bar{u}_\zeta(p_2,s_2)\, e^{\lambda_2}_\mu(k_2)\, (m-\not{p}_2)_{\zeta\delta}\, k_2^2$$
$$G^{\mu\nu}_{\delta\varrho}(k_1, p_1 \to k_2,p_2)\, (m-\not{p}_1)_{\varrho\eta}\, k_1^2\, u_\eta(p_1,s_1)\, e^{\lambda_1}_\nu(k_1) \qquad (742)$$

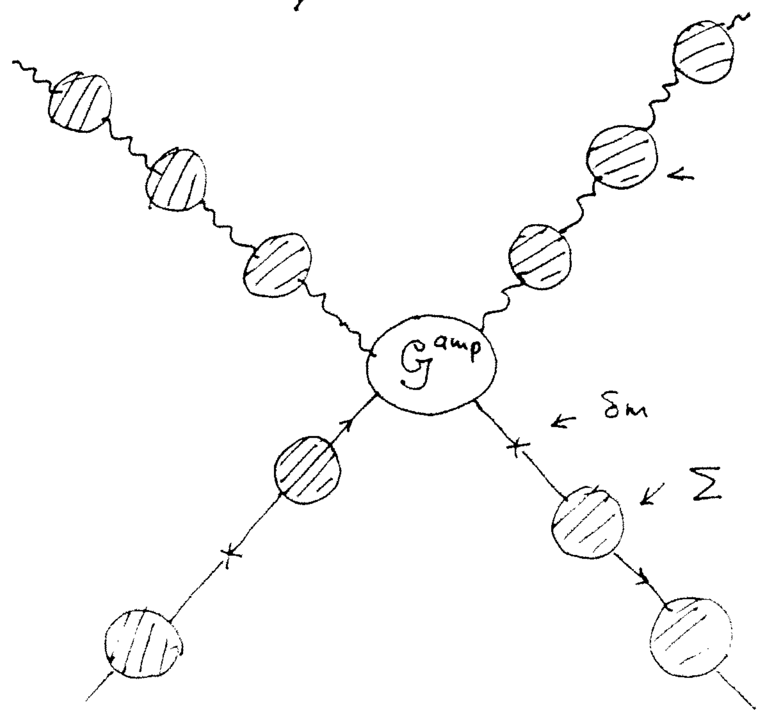
$$\Rightarrow M(p_1 s_1; k_1\lambda_1 \to p_2 s_2; k_2,\lambda_2) = \frac{1}{Z_2 Z_3} \lim_{\substack{p_i^2\to m^2 \\ k_i^2 \to 0}} e^{\lambda_2}_\mu(k_2)\, k_2^2\, \bar{u}_\zeta(p_2,s_2)(m\ \not{p}_2)_{\zeta\eta} \cdot$$
$$\cdot\, G^{\mu\nu}_{\delta\varrho}(k_1,p_1;k_2,p_2)\,(m\ \not{p}_1)_{\varrho\eta}\, u_\eta(p_1,s_1)\, k_1^2\, e^{\lambda_1}_\nu(k_1) \qquad (743)$$

$G^{\mu\nu}_{\delta\varrho}(k_1,p_1;k_2,p_2) =$





$=$



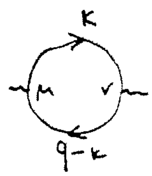
$\Pi_{\mu\nu}$ = sum of the 1PI diagrams:

Σ = sum of the 1PI diagrams

$\Pi_{\mu\nu}(q)$ = „polarization operator" =  +

\+  +  +  + ...

Let us calculate $\Pi_{\mu\nu}(p)$ in the leading order in pert. theory

$$\Pi_{\mu\nu}(q) = $$  $$\int \frac{d^4k}{i} \frac{1}{(m^2-k^2-i\epsilon)(m^2-(q-k)^2-i\epsilon)} \mathrm{Tr}\{\gamma_\mu(m+\not{k})\gamma_\nu(m+\not{k}-\not{q})\}$$

$$= -4e_0^2 \int \frac{d^4k}{i} \left(m^2 g_{\mu\nu} - k_\mu(q-k)_\nu - k_\nu(q-k)_\mu + (k,q-k)g_{\mu\nu}\right) \frac{1}{(m^2-k^2-i\epsilon)(m^2-(q-k)^2-i\epsilon)} \qquad (744)$$

In dimensional regularization

$$(k,q-k) \equiv k\cdot(q-k)$$

$$\Pi^d_{\mu\nu} = -4e_0^2 \int \frac{d^dk}{i} (m^2 g_{\mu\nu} - k_\mu(q-k)_\nu - k_\nu(q-k)_\mu + (k,q-k)g_{\mu\nu}) \frac{\mu^{4-d}}{(m^2-k^2-i\epsilon)(m^2-(q-k)^2-i\epsilon)}$$

$$= -4e_0^2 \int \frac{d^dk}{i} (m^2 g_{\mu\nu} - k_\mu(q-k)_\nu - k_\nu(q-k)_\mu + (k,q-k)g_{\mu\nu}) \int_0^1 d\beta \frac{\mu^{4-d}}{(m^2-(k-q\beta)^2-q^2\bar\beta\beta-i\epsilon)^2}$$

$$-4e_0^2 \int_0^1 d\beta \int \frac{d^dk}{i} \{(m^2+(k+q\beta, q\bar\beta-k))g_{\mu\nu} - (k+q\beta)_\mu(q\bar\beta-k)_\nu - (k+q\beta)_\nu(q\bar\beta-k)_\mu\} \frac{\mu^{4-d}}{(M_\beta^2-k^2-i\epsilon)^2}$$

$$M_\beta^2 \equiv m^2 - q^2\bar\beta\beta$$

= linear terms drop =

$$= -4e_0^2 \int \frac{d^dk}{i} \int_0^1 d\beta \frac{(m^2-k^2+q^2\bar\beta\beta)g_{\mu\nu} - 2\bar\beta\beta q_\mu q_\nu + 2k_\mu k_\nu}{(m^2-q^2\bar\beta\beta-k^2-i\epsilon)^2} \mu^{4-d} \qquad (745)$$

Using formulas

$$\int \frac{d^dp}{i} \frac{\Gamma(a)}{(M^2-p^2-i\epsilon)^a} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(a-d/2)}{(M^2-i\epsilon)^{a-d/2}} \qquad (746)$$

$$\int \frac{d^dp}{i} p_\mu p_\nu \frac{\Gamma(a)}{(M^2-p^2-i\epsilon)^a} = \frac{g_{\mu\nu}}{d} \int \frac{d^dp}{i} p^2 \frac{\Gamma(a)}{(M^2-p^2-i\epsilon)^a} = -\frac{g_{\mu\nu}}{2(4\pi)^{d/2}} \frac{\Gamma(a-1-\frac{d}{2})}{(M^2-i\epsilon)^{a-1-d/2}}$$

we get

$$\Pi^{d}_{\mu\nu}(q) = -4e_0 \frac{\mu^{4-d}}{(4\pi)^{d/2}} \int_0^1 d\beta \left\{ \frac{\Gamma(2-d/2)}{(M_\beta^2-i\epsilon)^{2-d/2}}\left[(m^2+q^2\bar\beta\beta)g_{\mu\nu} - 2\bar\beta\beta q_\mu q_\nu\right] + \frac{\Gamma(1-d/2)}{(M_\beta^2-i\epsilon)^{1-d/2}}\right.$$

$$\left.\cdot\left(\frac{d}{2}g_{\mu\nu} - g_{\mu\nu}\right)\right\} = -\frac{4e_0^2}{(4\pi)^{d/2}}\int_0^1 d\beta \frac{\Gamma(2-d/2)\mu^{4-d}}{(M_\beta^2-i\epsilon)^{2-d/2}}\left[(m^2+q^2\bar\beta\beta)g_{\mu\nu} - 2\bar\beta\beta q_\mu q_\nu - g_{\mu\nu}\underbrace{M_\beta^2}_{m^2-q^2\bar\beta\beta}\right] =$$

$$= \frac{4e_0^2\mu^{4-d}}{(4\pi)^{d/2}}(q_\mu q_\nu - q^2 g_{\mu\nu})\int_0^1 d\beta\, 2\bar\beta\beta \frac{\Gamma(2-d/2)}{(M_\beta^2-i\epsilon)^{2-d/2}} = \text{expand near } d=4$$

$$= (q_\mu q_\nu - q^2 g_{\mu\nu})\left\{\frac{e_0^2}{4\pi^2}\frac{2}{3(4-d)} + \frac{e_0^2}{2\pi^2}\int_0^1 d\beta\, \bar\beta\beta \ln\frac{\mu^2}{m^2-q^2\bar\beta\beta-i\epsilon}\right\} \qquad (747)$$

$$\Rightarrow \Pi^{reg}_{\mu\nu}(q) = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi^{reg}(q^2) \qquad \Pi^{reg}(q^2) = \frac{e_0^2}{12\pi^2}\int_0^1 d\beta\, \bar\beta\beta \ln\frac{\mu^2}{m^2-q^2\bar\beta\beta-i\epsilon} \qquad (748)$$

The structure

$$\Pi_{\mu\nu}(q) = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(q^2) \qquad (749)$$

is due to the gauge invariance.

Proof :

#1 : from Ward identity $\quad q_\mu \Pi_{\mu\nu}(q) = 0 \Rightarrow \Pi_{\mu\nu} \sim q_\mu q_\nu - q^2 g_{\mu\nu}$

#2 : from $\partial_\mu \hat{j}^\mu(x) = 0$ (conservation of current)

Define $\quad \tilde\Pi_{\mu\nu}(q) = i\int dx\, e^{iqx} \langle\Omega|T\{\hat{j}_\mu(x)\hat{j}_\nu(0)\}|\Omega\rangle \qquad (750)$

$$q^\mu\tilde\Pi_{\mu\nu}(q) = \int dx \left(\frac{\partial}{\partial x_\mu}e^{iqx}\right)\langle\Omega|T\{\hat{j}_\mu(x)\hat{j}_\nu(0)\}|\Omega\rangle = -\int d^4x\, e^{iqx}\frac{\partial}{\partial x_\mu}\langle\Omega|T\{\hat{j}_\mu(x)$$

$$\hat{j}_\nu(0)\}|\Omega\rangle = -\int dx\, e^{iqx}\left[\frac{\partial}{\partial x_0}\langle\Omega|\theta(x_0)\hat{j}_0(x)\hat{j}_\nu(0) + \theta(-x_0)\hat{j}_\nu(0)\hat{j}_0(x)|\Omega\rangle + \right.$$

$$\left.+\frac{\partial}{\partial x_i}\langle\Omega|T\{\hat{j}_i(x)\hat{j}_\nu(0)\}|\Omega\rangle\right] = -\int dx\, e^{iqx}\delta(x_0)\langle\Omega|[\hat{j}_0(\vec{x}),\hat{j}_0(0)]|\Omega\rangle$$

$$-\int dx\, e^{iqx}\langle\Omega|T\{\underbrace{\partial^\mu\hat{j}_\mu(x)}_{0},\hat{j}_\nu(0)\}|\Omega\rangle = \qquad \{\hat\psi(\vec{x}),\hat\psi^+(y)\} = \delta^3(\vec{x}-\vec{y})$$

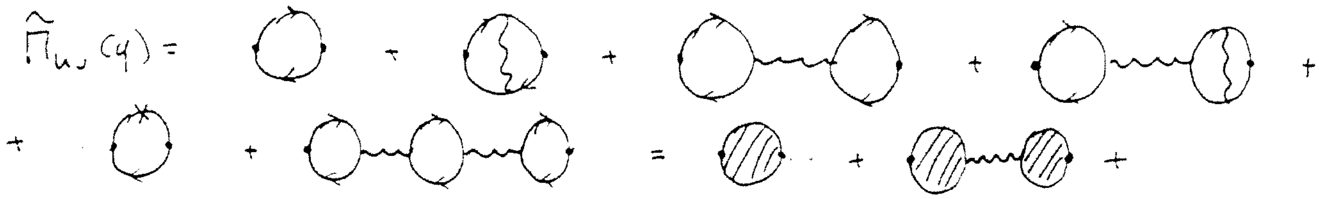
$$= -\int d^3x\, e^{-i\vec{q}\vec{x}}\langle\Omega|[\hat\psi^+(\vec{x})\hat\psi(\vec{x}),\hat\psi^+(0)\gamma_0\gamma_\nu\hat\psi(0)]|\Omega\rangle =$$

$$= -\langle\Omega|\hat\psi^+(0)\gamma_0\gamma_\nu\hat\psi(0) - \hat\psi^+(0)\gamma_0\gamma_\nu\hat\psi(0)|\Omega\rangle = 0 \qquad (751)$$

$$\Rightarrow \tilde\Pi_{\mu\nu}(q) = (q_\mu q_\nu - q^2 g_{\mu\nu})\tilde\Pi(q^2)$$

It is easy to see that

$\tilde{\Pi}_{\mu\nu}(q) =$

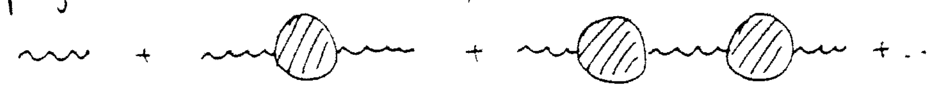


$\Rightarrow \quad \tilde{\Pi}(q^2) = \dfrac{\Pi(q^2)}{1-\Pi(q^2)}$

Photon propagator

$\frac{1}{q^2}(q_\alpha q_\beta - q^2 g_{\alpha\beta})(q^\beta q^\gamma - q^2 g^{\beta\gamma}) = -(q_\alpha q^\gamma - q^2\delta_\alpha^\gamma)$

$\mathcal{D}_{\mu\nu}(q) =$



$$= \frac{g_{\mu\nu}}{q^2} + \frac{g_{\mu\alpha}}{q^2}(q^\alpha q^\beta - q^2 g^{\alpha\beta})\Pi(q^2)\frac{g_{\beta\nu}}{q^2} = \frac{g_{\mu\nu}}{q^2} - \left(\frac{g_{\mu\nu}}{q^2} - \frac{q_\mu q_\nu}{q^4}\right)\frac{\Pi(q^2)}{1+\Pi(q^2)}$$

$$= \frac{g_{\mu\nu}}{q^2(1+\Pi(q^2))} + \frac{q_\mu q_\nu}{q^4}\frac{\Pi(q^2)}{1+\Pi(q^2)} \tag{752}$$

Due to Ward identity, this longitudinal part does not contribute to any of the S-matrix elements

$$\Rightarrow \mathcal{D}_{\mu\nu}(q) = \frac{g_{\mu\nu}}{q^2(1+\Pi(q^2))} \quad \text{- exact photon propagator} \tag{753}$$

At $q^2 \to 0$

$$\mathcal{D}_{\mu\nu}(q) \to \frac{Z_3 g_{\mu\nu}}{q^2} \quad \text{where } Z_3 = \frac{1}{1+\Pi(0)} \tag{754}$$

<u>NB</u>: $\mathcal{D}_{\mu\nu}(q)$ has a pole at $q^2 = 0$ $\Rightarrow$ photon remains massless in all orders of perturbation theory. This is due to the gauge invariance.

After regularization

$$\mathcal{D}^{reg}_{\mu\nu}(q) \to Z_3^{reg}\frac{g_{\mu\nu}}{q^2} \tag{755}$$

$$Z_3^{reg} = \frac{1}{1+\Pi^{reg}(0)} \simeq 1 - \Pi^{reg}(0) = 1 - \frac{e_0^2}{12\pi^2}\ln\frac{M^2}{m^2} = 1 - \frac{\alpha_0}{3\pi}\ln\frac{M^2}{m^2}$$

$$\Rightarrow Z_3^{reg} = 1 - \frac{\alpha_0}{3\pi}\ln\frac{M^2}{m^2} + O(\alpha_0^2) \tag{756}$$

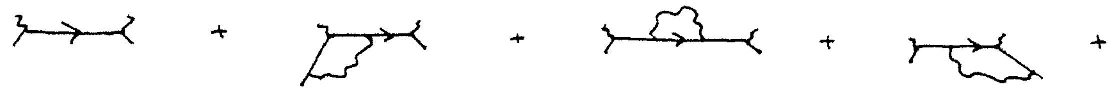
Now let us return to LSZ for the Compton scattering

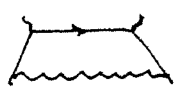
$$M(p_1 s_1; k_1 \lambda_1 \to p_2 s_2; k_2 \lambda_2) = \frac{1}{Z_2 Z_3} \lim_{\substack{p_i^2 \to m^2 \\ k_i^2 \to 0}} e^{\lambda_2}_{\mu}(k_2)\, k_2^2\, \bar u_\xi(p_2,s_2)(m - \not p_2)_{\xi\eta} \cdot$$

$$\cdot \underbrace{\mathcal{D}_{\mu\alpha}(k_2)}_{\frac{Z_3}{k_2^2}}\, \underbrace{G(p_2)_{\eta\delta}}_{\frac{Z_2(m+\not p_2)}{m^2-p_2^2}}\, G^{\mu\nu\, amp}_{\delta\rho}(p_1,k_1;p_2,k_2)\, \underbrace{\mathcal{D}_{\beta\nu}(k_1)}_{\frac{Z_3}{k_1^2}}\, \underbrace{G_{\rho\sigma}(p_1)}_{\frac{Z_2(m+\not p_1)}{m^2-p_1^2}}\, (m - \not p_1)_{\sigma\varphi}\, u_\varphi(p_1,s_1)\, k_1^2\, e^{\lambda_1}_\nu(k_1)$$

$$= Z_2 Z_3\, e^{\lambda_2}_{\mu}(k_2)\, \bar u_\xi(p_2,s_2)\, G^{amp\ \mu\nu}_{\xi\eta}(p_1,k_1;p_2,k_2)\, u_\eta(p_1,s_1)\, e^{\lambda_1}_\nu(k_1) \qquad (757)$$

where $G^{amp} =$

= 

\+  + (permutation $k_1 \leftrightarrow k_2$) + $O(e_0^6)$

Now we must find the physical charge in terms of e_0.

HW6.

Consider Yukawa theory with the Lagrangian

$$\mathcal{L} = \bar\psi(i\not\partial - m_0)\psi + \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - g_0\varphi\bar\psi\psi$$

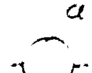
corresponding to $M_0 = 0$ (M_0 is a bare mass of π-meson) so the bare propagator of π-meson is

$G_0(p) = \frac{1}{-p^2 - i\epsilon}$ - - - - -

Show that the pole of exact propagator

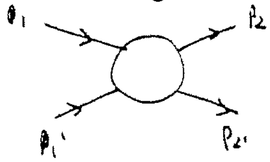
$G(p) =$ 

is no longer located at $p^2 = 0$.

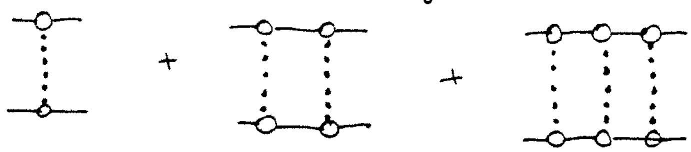
Hint: Calculate $\Sigma(p) =$  and show that $\Sigma(0) \neq 0$ so $\frac{1}{-p^2 + \Sigma(p)}$ will no longer have a pole at $p^2 = 0$, cf. eq. (311))

Physical charge

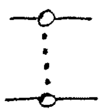
Electron - electron scattering in the NRL limit

$M(p_1 s_1, p_1' s_1' \to p_2 s_2, p_2' s_2') =$  $= \sum$ (relativistic) Feynman diagrams

$\xrightarrow[|\vec{p}_i| \ll m]{\text{NRL limit}}$ sum of the NRL diagrams for Coulomb potential:

 $+ \ldots - (p_2 \leftrightarrow p_2')$ (758)

where

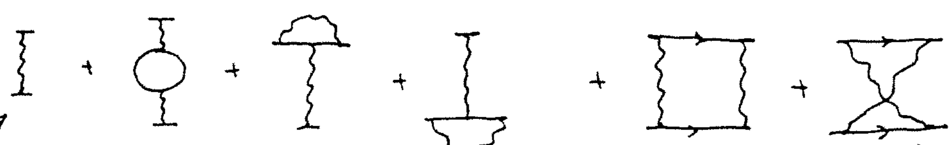
 $= -\frac{e^2}{\vec{q}^{\,2}} \, 4m^2 \delta_{s_1 s_2} \delta_{s_1' s_2'}$ $\leftarrow$ Fourier transform of Coulomb potential $\frac{e^2}{4\pi r}$ (759)

By definition, the coefficient in front of $\frac{1}{4\pi r}$ is the physical charge of the electron

Let us compute $M(p_1 s_1, p_1' s_1' \to p_2 s_2, p_2' s_2')$ in terms of e_0 (and m)
After the first step we get (see eq. (741)):

$$M(p_1 s_1, p_1' s_1' \to p_2 s_2, p_2' s_2') = Z_2^2 \, \bar{u}_3(p_2, s_2) \, \bar{u}_2(p_2', s_2') \, G^{amp}_{33'22'}(p_1, p_1', p_2, p_2') \, u(p_1, s_1) \, u(p_1', s_1') \Big|_{p_i^2 = m^2}$$

where (760)

$G^{amp} =$  $- (p_2 \leftrightarrow p_2')$

Coulomb exchange in the lowest order

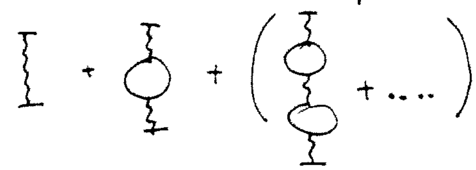
Corrections to first-order Coulomb exchange

Second-order Coulomb exchange (see second term in (758))

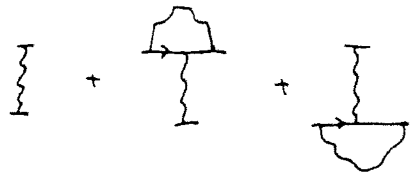
small in the NRL limit

For simplicity, take c.m. frame, then $q^2 = -\vec{q}^{\,2} = -4|p|^2 \sin^2\frac{\theta}{2}$

$\Rightarrow$ in the NRL limit $q^2 \ll m^2 \Rightarrow$

$\Rightarrow$  $= \gamma_\mu \mathcal{D}^{\mu\nu}(q)\gamma_\nu \underset{q^2 \ll m^2}{\longrightarrow} \gamma_\mu \dfrac{g^{\mu\nu} Z_3}{q^2}\gamma_\nu$ (761)

Thus,

$\mathcal{G}^{amp} = Z_3$  + iterations of Coulomb potential $-(p_2 \leftrightarrow p_2')$ (762)

Consider now 

Definition:

$\Lambda_\mu(p_1, p_2) =$  $+ \ldots =$

= sum of all the 3-point 1PI diagrams without pure γ_μ

Property

$$\Lambda_\mu(p_1, p_1) = -\frac{\partial}{\partial p_{1\mu}}\Sigma(p_1) \qquad (763)$$

Proof in the leading order in pert. theory:

$\Lambda_\mu(p_1, p_2) \equiv$  $= e_0^2 \int \frac{d^4p}{i}\, \gamma_\alpha \frac{m + \not p_1 - \not p}{m^2 - (p_1-p)^2} \gamma_\mu \frac{m + \not p_2 - \not p}{m^2 - (p_2-p)^2} \gamma_\beta \frac{g^{\alpha\beta}}{p^2}$ (764)

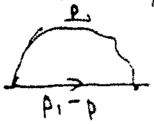
$$\Rightarrow \Lambda_\mu(p_1, p_1) = e_0^2 \int \frac{dp}{i} \frac{\gamma_\alpha (m + \not p_1 - \not p)\gamma_\mu (m + \not p_1 - \not p)\gamma^\alpha}{(m^2 - (p_1-p)^2)^2 p^2} \qquad (765)$$

On the other hand

$$-\frac{\partial}{\partial p_{1\mu}}\Sigma(p_1) = +\frac{\partial}{\partial p_{1\mu}} \int \frac{dp}{i}\, \gamma_\alpha \frac{m + \not p_1 - \not p}{m^2 - (p_1-p)^2}\gamma^\alpha e_0^2 = \int \frac{dp}{i} e_0^2 \gamma_\alpha \Big\{ \frac{\gamma_\mu}{m^2 - (p_1-p)^2} +$$

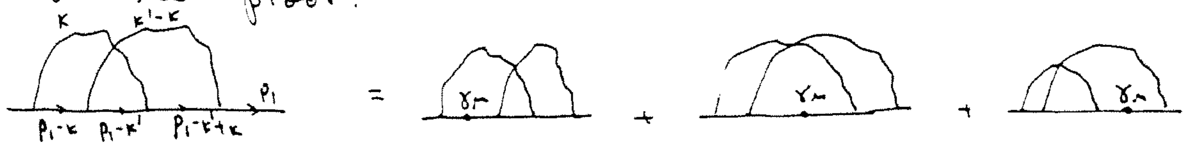
$$+ \frac{2(p_1-p)_\mu\, m + \not p_1 - \not p}{(m^2 - (p_1-p)^2)^2} \Big\} \gamma^\alpha = e_0^2 \int \frac{dp}{i}\, \gamma_\alpha \Big\{ \frac{(m + \not p_1 - \not p)\gamma_\mu (m + \not p_1 - \not p)}{(m^2 - (p_1-p)^2)^2} \Big\} \gamma^\alpha = (765)$$

Graphically $\frac{\partial}{\partial p_{1\mu}}$  (766)

$\Rightarrow \frac{\partial}{\partial p_{1\mu}}$  $=$  $\Rightarrow \frac{\partial}{\partial p_{1\mu}}(-\Sigma) = \Lambda^\mu(p_1, p_1)$

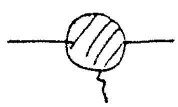
Using (766), the property (763) can be easily proved in arbitrary order in pert. theory

Outline of the proof:

$\frac{\partial}{\partial p_{1\mu}}$ 

= diagrams for $\Lambda_\mu(p_1, p_1)$

Definition

$\Gamma_\mu(p_1,p_2)$ =  = sum of all the 3-point 1PI diagram

$$= \gamma_\mu + \Lambda_\mu(p_1,p_1) \qquad (767)$$

$$\Gamma_\mu(p_1,p_1) = \gamma_\mu - \frac{\partial}{\partial p_1^\mu}\Sigma(p_1) = \gamma_\mu - \frac{\partial}{\partial p_1^\mu}\left(m\Sigma_1(p_1^2) - \not{p}_1 \Sigma_2(p_1^2)\right) =$$

$$= \gamma_\mu\left(1 + \Sigma_2(p_1^2)\right) - 2p_{1\mu}\left(m\Sigma_1'(p_1^2) - \not{p}_1\Sigma_2'(p_1^2)\right) \qquad (768)$$

$$\Rightarrow \bar{u}(p_1,s_2)\Gamma_\mu(p_1,p_1)u(p_1,s_1)\Big|_{p_1^2=m^2} = (1+\Sigma_2(m^2))\underbrace{\bar{u}(p_1,s_2)\gamma_\mu u(p_1,s_1)}_{2p_{1\mu}\delta_{s_1s_2}} - 2p_{1\mu}m\left(\Sigma_1'(m^2) - \Sigma_2'(m^2)\right) \cdot \underbrace{\bar{u}(p_1,s_2)u(p_1,s_1)}_{2m\delta_{s_1s_2}}$$

$$= 2p_{1\mu}\delta_{s_1s_2}\left(1 + \Sigma_2(m^2) - 2m^2(\Sigma_1'(m^2) - \Sigma_2'(m^2))\right) = 2p_{1\mu}\delta_{s_1s_2}Z_2^{-1} \qquad (769)$$

Historically, the constant of proportionality between Γ_μ and γ_μ on the mass shell wass called Z_1^{-1}:

$$\bar{u}(p_1,s_2)\Gamma_\mu(p_1,p_1)u(p_1,s_1) = Z_1^{-1}\bar{u}(p_1,s_2)\gamma_\mu u(p_1,s_1) \qquad (770)$$

and therefore we obtained the result $Z_1 = Z_2$. (It is also called „Ward identity").

Now we can assemble the answer for the matrix element $M(p_1s_1, p_1's_1' \to p_2,s_2; p_2's_2')$ in the NRL limit

(762):

$\mathcal{G}^{amp}(p_1, p_1', p_2, p_2') = Z_3$  $+$  $+$  $+$ iterations of Coulomb potential $-(p_2 \leftrightarrow p_2')$

Since $q^2 << m^2$ (NRL limit)

 $\simeq Z_1^{-1} \frac{\gamma_\mu}{\mathrm{T}}$ and  $\simeq Z_1^{-1} \frac{\wr}{\gamma_\mu}$ (up to $\frac{v}{c}$ corrections)

$\Rightarrow$ $\mathcal{G}^{amp} = Z_3 Z_1^{-2}$  $+$ iterations of Coulomb potential $-(p_2 \leftrightarrow p_2')$ (771)

Substituting this in eq. (760) we get

$M(p_1 s_1, p_1' s_1' \to p_2 s_2, p_2' s_2') \overset{NRL}{=} Z_2^2 Z_3 Z_1^{-2}$  $-(p_2 \leftrightarrow p_2')$ + iterations of the potential

$= Z_2^2 Z_1^{-2} Z_3\, \bar{u}(p_2 s_2)\gamma_\mu u(p_1 s_1)\, \bar{u}(p_2' s_2')\gamma^\mu u(p_1', s_1')\, \frac{e_0^2}{q^2} - (p_2 \leftrightarrow p_2') + \text{iterations} =$

$= \frac{Z_2^2 Z_1^{-2} Z_3 e_0^2}{-\vec{q}^{\,2}} 4m^2 \delta_{s_1 s_2} \delta_{s_1' s_2'} - (\vec{q} \leftrightarrow \vec{p}_1 - \vec{p}_2')$ + iterations of Coulomp potential (772)

The coefficient in front of the transform of Coulomb potential is the physical charge so we get

$$e^2 = Z_2^2 Z_1^{-2} Z_3 e_0^2 = Z_3 e_0^2 \qquad (773)$$

where we used the property $Z_1 = Z_2$, see eq. (770)

In explicit form

$$e^2 = e_0^2\left(1 - \frac{e_0^2}{12\pi^2} \ln \frac{\mu^2}{M^2}\right) + O(e_0^6) \qquad (774)$$

If we now express e_0 in terms of e

$$e_0^2 = e^2\left(1 + \frac{e^2}{12\pi^2} \ln \frac{\mu^2}{M^2}\right) \qquad (775)$$

and plug in the expression (775) for e_0 into (regularized) Feynman diagrams we will see that all the dependence on μ disappears.

Effective coupling constant.

Let us calculate physical electric charge for a certain heavy particle like the muon. If the mass of this particle is M we get eq. (774) in the form

$$e^2(M) = e_0^2\left(1 - \frac{e_0^2}{12\pi^2}\ln\frac{\mu^2}{M^2}\right). \qquad (776)$$

It is instructive to compare $e^2(M)$ to physical charge of the electron $e^2(m)$ rather than to e_0^2. We get

$$e^2(M) = e^2\left(1 + \frac{e^2}{12\pi^2}\ln\frac{\mu^2}{m^2}\right)\left(1 - \frac{e^2}{12\pi^2}\ln\frac{\mu^2}{M^2}\right) \simeq e^2\left(1 + \frac{e^2}{12\pi^2}\ln\frac{M^2}{m^2}\right) \qquad (777)$$

It can be demonstrated that a more accurate version of eq. (777) looks like ($e(m) \equiv e$ - electron charge)

$$e^2(M) = \frac{e^2(m)}{1 - \frac{e^2(m)}{12\pi^2}\ln\frac{M^2}{m^2}} \qquad (778)$$

We see that the strength of the electromagnetic interaction increases with the mass of interacting particle.

„Asymptotic freedom": for QCD (and for the theory of weak interactions) the situation is opposite - the strength of the interaction decreases with the mass of interacting particles („quarks")

$$g^2(M) = \frac{g^2(M_0)}{1 + \frac{g^2(M_0)}{16\pi^2}\, b\, \ln\frac{M^2}{M_0^2}} \rightarrow 0 \quad \text{as} \quad M \to \infty \qquad (779)$$

„asymptotic freedom"

$$b = 11 - \tfrac{2}{3}(\#\text{ of quarks}) = 9$$

The only asymptotically free theories are those with non-Abelian gauge symmetry.

Grand Unification