QCD

I. Lagrangian and gauge symmetry

Recall local gauge invariance in QED

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D}-m)\psi \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad (800)$$

$$\mathcal{D}_\mu = \partial_\mu + ie A_\mu$$

$$\left.\begin{array}{l}\psi(x) \to e^{i\alpha(x)}\psi(x)\\ \bar{\psi}(x) \to e^{-i\alpha(x)}\bar{\psi}(x)\\ A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x)\end{array}\right\} \quad \left.\begin{array}{l}\mathcal{D}_\mu\psi(x) \to e^{i\alpha(x)}\mathcal{D}_\mu\psi(x)\\ F_{\mu\nu}(x) \to F_{\mu\nu}(x)\end{array}\right\} \Rightarrow \mathcal{L}(x) \to \mathcal{L}(x)$$

„abelian gauge invariance"

(801)

Non-abelian gauge invariance in QCD

$$\mathcal{L}_{QCD} = -\frac{1}{2}\mathrm{Tr}\,G_{\mu\nu}G^{\mu\nu} + \bar{\psi}(i\not{D}-m)\psi$$

(„one-flavor" QCD)

(802)

Notations

A^a_μ - „gluon field" $a = 1,\dots 8$

$A_\mu = \sum A^a_\mu t^a$ $\quad t^a = \frac{\lambda^a}{2}$ $\quad \lambda^a$ - Gell-Mann $\overbrace{\text{matrices}}^{3\times3}$ $(a=1,\dots 8)$

(see textbook)

$G_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - ig[A_\mu(x), A_\nu(x)]$ $\qquad g$ - „coupling constant of QCD"

$G_{\mu\nu}(x)$ is 3×3 matrix as well as $A_\mu(x)$ is $\qquad G_{\mu\nu} = G^a_{\mu\nu}t^a$

ψ^k_ξ - „quark field" $\quad k = 1,2,3$ - „color index"

$\bar{\psi}^k_\xi \equiv \psi^{+k}_\eta(\gamma_0)_{\eta\xi}$ as usual $\quad \xi$ - Dirac bispinor index

$\mathcal{D}_\mu = \partial_\mu - igA_\mu$ - covariant derivative (803)

$\bar{\psi}(i\not{D}-m)\psi \equiv \bar{\psi}_k(i\not{\partial}\delta_{kl} + g\not{A}^a(t^a)_{kl} - m\delta_{kl})\psi_l$ ← explicit form of color structure

What one needs to know about matrices t^a

1. t^a are Hermitian $\quad (t^a)^+ = t^a$
2. $\mathrm{Tr}\, t^a t^b = \frac{1}{2}\delta^{ab}$
3. $[t^a, t^b] = i f_{abc} t^c$ $\qquad f_{abc}$ - „structure constants of Lie algebra of SU_3 group".
4. $\mathrm{Tr}\, t^a = 0$

f_{abc} is totally antysymmetric

explicit form - see textbook

$$G_{\mu\nu} = G^a_{\mu\nu}t^a \qquad G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu \qquad (804)$$

Non-abelian gauge transformations

$\psi(x) \to \Phi(x)\,\psi(x) \qquad \Phi(x) = e^{i\sum \omega^a t_a}$

$\omega^a(x)$ - real 8-vector

- arbitrary matrix fr SU_3 group

$\Phi^+(x) = e^{-i\sum\omega^a t_a}$

$\left(\det \mathcal{M} = e^{\operatorname{Tr}\ln \mathcal{M}} \Rightarrow \Rightarrow \det\Phi = 1\right)$

(*) $\bar\psi(x) \Rightarrow \psi^+(x)\,\Phi^+(x)\,\gamma_0 = \bar\psi(x)\,\Phi^+(x)$

$A_\mu \to \Phi(x) A_\mu(x) \Phi^+(x) + \frac{i}{g}\Phi(x)\,\partial_\mu \Phi^+(x)$

Let us demonstrate that

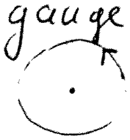
$\mathcal{L}_{QCD}(x) \to \mathcal{L}_{QCD}(x)$ under (*) (805)

1. $G_{\mu\nu}(x) \to \left[\partial_\mu(\Phi A_\nu \Phi^+ + \frac{i}{g}\Phi\partial_\nu\Phi^+) - \mu\leftrightarrow\nu\right] - ig\left[(\Phi A_\mu\Phi^+ + \frac{i}{g}\Phi\partial_\mu\Phi^+)(\Phi A_\nu \Phi^+ + \frac{i}{g}\Phi\partial_\nu\Phi^+) - \mu\leftrightarrow\nu\right] = \partial_\mu\Phi A_\nu\Phi^+ + \Phi\partial_\mu A_\nu \Phi^+ + \cancel{\Phi A_\nu\partial_\mu\Phi^+} + \frac{i}{g}\partial_\mu\Phi\,\partial_\nu\Phi^+ + \cancel{\frac{i}{g}\Phi\partial_\mu\partial_\nu\Phi^+} - ig\,\Phi A_\mu A_\nu\Phi^+ + \Phi(\partial_\mu\Phi^+)\Phi A_\nu\Phi^+ + \cancel{\Phi A_\mu\partial_\nu\Phi^+} + \frac{i}{g}(\Phi\partial_\mu\Phi^+)(\Phi\partial_\nu\Phi^+) - \mu\leftrightarrow\nu = \Phi(\partial_\mu A_\nu - ig A_\mu A_\nu)\Phi^+ + \partial_\mu\Phi A_\nu \Phi^+ + \underbrace{(\Phi\partial_\mu\Phi^+)}_{=\cancel{\partial_\mu(\Phi\Phi^+)} - (\partial_\mu\Phi)\Phi^+,\ \Phi\Phi^+=1}\Phi A_\nu\Phi^+ + \frac{i}{g}\partial_\mu\Phi\,\partial_\nu\Phi^+ + \frac{i}{g}\underbrace{(\Phi\partial_\mu\Phi^+)}_{=-(\partial_\mu\Phi)\cdot\Phi^+}(\Phi\partial_\nu\Phi^+) - \mu\leftrightarrow\nu =$

$= \Phi G_{\mu\nu}\Phi^+ \quad\Rightarrow\quad \operatorname{Tr} G_{\mu\nu}G^{\mu\nu} \to \operatorname{Tr}\Phi G_{\mu\nu}\underbrace{\Phi^+\Phi}_{=1} G^{\mu\nu}\Phi^+ = \operatorname{Tr}G_{\mu\nu}G^{\mu\nu}$

2. $\mathcal{D}_\mu\psi(x) \to \left(\partial_\mu - ig(\Phi A_\mu\Phi^+ + \frac{i}{g}\Phi\partial_\mu\Phi^+)\right)\Phi\psi = \cancel{(\partial_\mu\Phi)\psi} + \Phi(\partial_\mu\psi) - ig\,\Phi A_\mu\psi + \cancel{(\Phi\partial_\mu\Phi^+)\Phi\psi} = \Phi\mathcal{D}_\mu\psi \quad\Rightarrow\quad \bar\psi\mathcal{D}_\mu\psi \to \bar\psi\Phi^+\Phi\mathcal{D}_\mu\psi = \bar\psi\mathcal{D}_\mu\psi$

$\Rightarrow\ -\frac{1}{2}\operatorname{Tr}G_{\mu\nu}G^{\mu\nu} + \bar\psi(i\not{\mathcal{D}} - m)\psi \to -\frac{1}{2}\operatorname{Tr}G_{\mu\nu}G^{\mu\nu} + \bar\psi(i\not{\mathcal{D}} - m)\psi$ (806)

QCD is called non-Abelian gauge theory because SU_3 is non-Abelian group ($\Phi_1\Phi_2 \neq \Phi_2\Phi_1$ in general).
QED is called Abelian gauge theory since $e^{i\alpha}$ form the Abelian group $U(1)$  ($e^{i\alpha}e^{i\beta} = e^{i\beta}e^{i\alpha}$ of course)

QCD equations of motion $\qquad \mathcal{L} = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \bar\psi(i\not{\mathcal{D}} - m)\psi$

$\frac{\partial\mathcal{L}}{\partial A^a_\nu} = \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu A^a_\nu} \qquad \frac{\partial\mathcal{L}}{\partial\partial_\mu A^a_\nu} = -G^{a\mu\nu} \qquad \frac{\partial\mathcal{L}}{\partial A^a_\nu} = g f_{abc}A^b_\mu G^{c\mu\nu}$

$\Rightarrow\ \partial_\mu G^{a\mu\nu} + g f_{abc}A^b_\mu G^{c\mu\nu} = -g\bar\psi t^a\gamma^\nu\psi \quad\Rightarrow\quad (\mathcal{D}_\mu G^{\mu\nu})^a = -g\bar\psi t^a\gamma^\nu\psi$

$(\mathcal{D}_\mu)_{ab} = \partial_\mu\delta_{ab} + g f_{abc}A^c_\mu$

$$\left.\begin{array}{l} \frac{\partial\mathcal{L}}{\partial\bar\psi} = \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\bar\psi} \Rightarrow (i\not{\mathcal{D}} - m)\psi = 0 \\ \text{Similarly,}\quad \bar\psi(i\overleftarrow{\not{\mathcal{D}}} - m) = 0 \\ \qquad (\mathcal{D}^\mu)^{ab}G^b_{\mu\nu} = -g\bar\psi t^a\gamma_\nu\psi \end{array}\right\}\ \text{QCD equations of motion} \qquad (807)$$

Tensor of energy-momentum

Noether theorem:

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\alpha} \partial^\nu A_\alpha + \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi} \partial_\nu \psi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \bar\psi} \partial_\nu \bar\psi - g^{\mu\nu} \mathcal{L} =$$

$$= -G^{a\mu\alpha} \partial^\nu A_\alpha + \bar\psi i \gamma^\mu \partial^\nu \psi - g^{\mu\nu}\mathcal{L} + \partial_\alpha (G^{a\mu\alpha} A^{a\nu})$$

does not contribute to H or P_i (after $\int d^3x$).

$$= \underbrace{-G^{a\mu\alpha}\partial^\nu A^a_\alpha} + \bar\psi i\gamma^\mu \partial^\nu \psi - g^{\mu\nu}\mathcal{L} + \underbrace{\partial_\alpha A^a G^{a\mu\alpha}} - \underbrace{g f_{abc} A^{b\nu} A^c_\alpha G^{a\mu\alpha}} +$$

$$+ g f_{abc} A^{b\nu} G^{a\mu\alpha} A^c_\alpha + A^{a\nu}\partial_\alpha G^{a\mu\alpha} = -G^{a\mu\alpha} G^{a\nu}{}_\alpha + \bar\psi i \gamma^\mu \partial^\nu \psi - g^{\mu\nu}\mathcal{L} +$$

$$+ A^{a\nu} \underbrace{(\mathcal{D}_\alpha G^{\mu\alpha})^a}_{=\bar\psi t^a \gamma^\mu \psi} = -G^{a\mu\alpha} G^{a\nu}{}_\alpha + \underbrace{\bar\psi i\gamma^\mu \partial^\nu \psi} - g^{\mu\nu}\mathcal{L} + \underbrace{\bar\psi \gamma^\mu A^\nu \psi} =$$

$$= -G^{a\mu\alpha} G^{a\nu}{}_\alpha + \bar\psi i \gamma^\mu \mathcal{D}^\nu \psi + \frac{g^{\mu\nu}}{4} G^a_{\alpha\beta} G^{a\alpha\beta} \qquad (808)$$

Symmetric form

$$T^{\mu\nu} = -G^{a\mu\alpha} G^{a\nu}{}_\alpha + \frac{g^{\mu\nu}}{4} G^a_{\alpha\beta} G^{a\alpha\beta} + \frac{i}{4}\bar\psi(\gamma_\mu \overleftrightarrow{\mathcal{D}}_\nu + \gamma_\nu \overleftrightarrow{\mathcal{D}}_\mu)\psi \qquad (809)$$

It is easy to see that $T^{\mu\nu}$ is gauge-invariant (roughly speaking, since it is constructed from $G_{\mu\nu}$, ψ, and covariant derivatives only)

Gauge links.

$$U(x,y) \overset{def}{\equiv} P\exp i \int_0^1 d\beta\, (x-y)^\mu A_\mu(\beta x + \bar\beta y) \qquad (810)$$

$$P\exp i\int_0^1 d\beta\,(x-y) A_\mu(\beta x + \bar\beta y) \equiv 1 + i\int_0^1 d\beta\,(x-y)^\mu A_\mu(\beta x+\bar\beta y) + i^2 \int_0^1 d\beta \int_0^\beta d\beta'$$

$$(x-y)^\mu A_\mu(\beta x + \bar\beta y)\,(x-y)^\nu A_\nu(\beta' x + \bar\beta' y) + i^3 \int_0^1 d\beta \int_0^\beta d\beta' \int_0^{\beta'} d\beta''\, A_\bullet(x_\beta) A_\bullet(x_{\beta'}) A_\bullet(x_{\beta''}) + \dots$$

$A_\bullet \equiv (x-y)^\mu A_\mu$

$x_\beta \equiv \beta x + \bar\beta y$

Group property: $U(x,y)\,U(y,z) = U(x,z)$

Gauge transformation

$$U(x,y) \to \Omega(x)\, U(x,y)\, \Omega^\dagger(y) \qquad (811)$$

Proof:

$U(\lambda x+y,y)$ satisfies the differential eqn $\frac{d}{d\lambda} U(\lambda x + y, y) = i x^\mu A_\mu(\lambda x + y) g \cdot U(\lambda x+y,y)$

Let us construct

$\Omega^\dagger(\lambda x+y)\, U^\Omega(\lambda x+y,y)\, \Omega(y)$ and check $\frac{d}{d\lambda}$

$$\frac{d}{d\lambda}\Omega^\dagger(\lambda x+y)\, U^\Omega(\lambda x+y,y)\,\Omega(y) = x^\mu \partial_\mu \Omega^\dagger(\lambda x+y)\, U^\Omega\, \Omega(y) + \Omega^\dagger(\lambda x+y)\, ig \underbrace{A^\Omega_\mu(\lambda x+y)}_{\Omega A_\mu \Omega^\dagger + \frac{i}{g}\Omega\partial_\mu\Omega} x^\mu$$

$$U^\Omega(\lambda x+y,y)\,\Omega(y) = x_\mu\big(\partial_\mu \Omega^\dagger(\lambda x+y) + ig A_\mu(\lambda x+y)\Omega^\dagger(\lambda x+y) -$$

$$- \partial_\mu \Omega^\dagger(\lambda x+y)\big)\, U^\Omega(\lambda x+y,y)\,\Omega(y) = ig\, x_\mu A_\mu(\lambda x+y)\, \Omega^\dagger(\lambda x+y)\, U^\Omega(\lambda x+y)\,\Omega(y)$$

$\Rightarrow \Omega^\dagger(\lambda x+y)\, U^\Omega(\lambda x+y,y)\,\Omega(y)$ satisfies the same diff. eqn. as $U(\lambda x+y,y)$

$\Rightarrow \Omega^\dagger(\lambda x+y)\, U^\Omega(\lambda x+y,y)\,\Omega(y) = U(\lambda x+y) \Rightarrow U^\Omega(x,y) = \Omega(x)\, U(x,y)\, \Omega^\dagger(y)$

Quantization $\qquad F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$

$$\mathcal{L} = \underbrace{-\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu}}_{\substack{\text{eight issues of}\\ \text{free electrodynamicses}}} + \underbrace{\bar\psi(i\partial\!\!\!/ - m)\psi}_{\substack{\text{3 free}\\ \text{Dirac fields}}} + \underbrace{i\,Tr F_{\mu\nu}[A^\mu, A^\nu] + \frac{Tr}{2}[A_\mu, A_\nu][A^\mu, A^\nu] + \bar\psi A\!\!\!/\psi}_{\mathcal{L}_{int}} \qquad (812)$$

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int} \Rightarrow H = H_0 + H_{int} \qquad H_{int} = -\mathcal{L}_{int} \qquad (813)$$

H_0 = eight electrodynamixes + three free Dirac fields $\Rightarrow$

$\Rightarrow$ everything is the same as in QED (Coulomb gauge quantization)

$$\hat{\vec{A}}^a(\vec{x}) = \int \frac{d^3k}{\sqrt{2|k|}} \sum_{\lambda=1,2} e^\lambda(k)\left(\hat{a}^{\lambda a}_k e^{+i\vec{k}\vec{x}} + \hat{a}^{+\lambda a}_k e^{-i\vec{k}\vec{x}}\right)$$

$$[\hat{a}^{\lambda a}_k, \hat{a}^{+\lambda' b}_{k'}] = (2\pi)^3\delta(\vec{k}-\vec{k}')\,\delta_{\lambda\lambda'}\,\delta^{ab} \qquad [a_k, a_{k'}] = [a^+_k, a^+_{k'}] = 0. \qquad (814)$$

$$\psi^n(\vec{x}) = \int \frac{d^3p}{\sqrt{2E_p}} \sum_s \left(\hat{a}^{sn}_p u(p,s) e^{i\vec{p}\vec{x}} + \hat{b}^{sn+}_p v(p,s) e^{-i\vec{p}\vec{x}}\right)$$

$$\bar\psi^n(\vec{x}) = \int \frac{d^3p}{\sqrt{2E_p}} \sum_s \left(b^{sn}_p \bar v(p,s) e^{i\vec{p}\vec{x}} + \hat{a}^{+sn}_p \bar u(p,s) e^{-i\vec{p}\vec{x}}\right)$$

$$\{a^{sm}_p, a^{+s'n}_{p'}\} = (2\pi)^3\delta(\vec{p}-\vec{p}')\,\delta_{ss'}\,\delta_{mn} \qquad \{b^{sm}_p, b^{+s'n}_{p'}\} = (2\pi)^3\delta(\vec{p}-\vec{p}')\,\delta_{ss'}\,\delta_{mn}$$

Other $\{\ \}$'s are 0. $\qquad (815)$

Perturbative vacuum $|0\rangle \qquad \hat{a}^{\lambda a}_k|0\rangle = \hat{a}^{sn}_p|0\rangle = \hat{b}^{sn}_p|0\rangle = 0$

Propagators are the same as in QED

$$\overline{A^a_\mu(x)\,A^b_\nu(y)} = \int\frac{d^4k}{i}\,\frac{e^{-ik(x-y)}}{k^2+i\epsilon}\,\delta_{ab}\,g_{\mu\nu}\ ; \quad \overline{\psi^m(x)\,\bar\psi^n(y)} = \int\frac{d^4p}{i}\,\frac{e^{-ip(x-y)}}{m-p\!\!\!/}\,\delta^{mn} \qquad (816)$$

However, there is a subtle point in the transition from the propagator of physical degrees of freedom

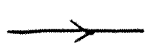
$$D^{tr\,ab}_{\mu\nu} = \int\frac{d^4k}{i}\,\frac{e^{-ik(x-y)}}{k^2+i\epsilon}\left(g_{\mu\nu} + \frac{k_\mu k_\nu}{\vec{k}^2} - \frac{k_0}{\vec{k}^2}(k_\mu\eta_\nu + k_\nu\eta_\mu) - \frac{k^2\eta_\mu\eta_\nu}{\vec{k}^2}\right)$$

to $\qquad D^{F\,ab}_{\mu\nu} = \int\frac{d^4k}{i}\,\frac{e^{-ik(x-y)}}{k^2+i\epsilon}\,g_{\mu\nu} \qquad \eta = (1,0,0,0) \qquad (817)$

In QED, we used Ward identity to prove that one can use D^F instead of D^{tr} in computation of physical cross sections. In QCD, this Ward identity is slightly different $\Rightarrow$ we will need so-called ghost particles to eliminate the unphysical gluons from loops. (See below).

So, propagators are


k $\qquad \dfrac{g_{\mu\nu}\,\delta^{ab}}{k^2+i\epsilon}$ – gluon


p $\qquad \dfrac{m+p\!\!\!/}{m^2-p^2-i\epsilon}\,\delta^{mn}$ – quark

Let us proceed with perturbation theory as in QED case

For the Green functions, everything is Ok

We still have the condition $e^{-iHT}|0\rangle \xrightarrow[T\to\infty]{} e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle$ ← physical vacuum

(vacuum is still the state with lowest energy) so we can get usual formula

$$\langle\Omega|T\{\hat{\Psi}^m(x)\hat{\bar\Psi}^n(y)\hat{A}^a(z)\dots\}|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\{\hat\Psi^m_I(x)\hat{\bar\Psi}^n_I(y)\hat{A}^a_I(z)\dots e^{-i\int_{-T}^{T}dt\,\hat{H}_I(t)}\}|0\rangle}{\langle 0|T\{\exp -i\int_{-T}^{T}dt\,\hat{H}_I(t)\}|0\rangle}$$

$$= \frac{\langle 0|T\{\hat\Psi^m_I(x)\hat{\bar\Psi}^n_I(y)\hat{A}^a_I(z)\dots \exp(i\int\hat{\mathcal{L}}_I(z)d^4z)\}|0\rangle}{\langle 0|T\{\exp i\int\hat{\mathcal{L}}_I(z)d^4z\}|0\rangle}$$

$T\to\infty(1-i\epsilon) \Rightarrow \frac{1}{m^2-p^2-i\epsilon}$

$$\left.\begin{aligned}
\hat\Psi^m_I(x) &= \int\frac{d^3p}{\sqrt{2E_p}}\sum_s\left(\hat{a}^{s\,m}_p \underbrace{u(p,s)}e^{-ipx} + \hat{b}^{+s\,m}_p \underbrace{v(p,s)} e^{ipx}\right)\Big|_{p_0=E_p}\\
\hat{\bar\Psi}^m_I(x) &= \int\frac{d^3p}{\sqrt{2E_p}}\sum_s\left(\hat{b}^{s\,m}_p\bar{v}(p,s)e^{-ipx} + \hat{a}^{+s\,m}_p\bar{u}(p,s)e^{ipx}\right)\Big|_{p_0=E_p}\\
A^a_{\mu I}(x) &= \int\frac{d^3k}{\sqrt{2k_0}}\sum_{\lambda=1,2}e^\lambda(k)\left(\hat{a}^{\lambda a}_k e^{-ikx} + \hat{a}^{+\lambda a}_k e^{ikx}\right)\Big|_{k_0=|k|}
\end{aligned}\right\}\text{ as in the free theory}$$

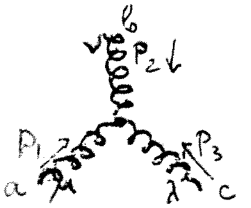
$$\mathcal{L}_I(z) = g\bar\Psi_I A\!\!\!/_I\Psi_I + igTr\,F^I_{\mu\nu}[A^I_\mu, A^I_\nu] + \frac{g^2}{2}Tr[A^I_\mu, A^I_\nu]^2 \qquad (818)$$

$$\Rightarrow \overline{\Psi^m_I(x)\bar\Psi^n_I(y)} = \int\frac{dp}{i}\,\frac{\delta^{mn}(m+p\!\!\!/)}{m^2-p^2-i\epsilon}e^{-ip(x-y)}$$

$$\overline{A^a_{\mu I}(x)A^b_{\nu I}(y)} = \int\frac{dp}{i}\,\frac{\delta^{ab}e^{-ip(x-y)}}{p^2+i\epsilon}\left(g_{\mu\nu} + \frac{p_\mu p_\nu}{|\vec p^2|} - \frac{p_0}{|\vec p|^2}(p_\mu\eta_\nu + p_\nu\eta_\mu) - \frac{p^2\eta_\mu\eta_\nu}{|\vec p|^2}\right) \to \underbrace{\int\frac{dp}{i}\,\frac{g_{\mu\nu}e^{-ip(x-\dots}}{p^2+i\epsilon}}_{\text{Feynman propagator}}$$

Vertices

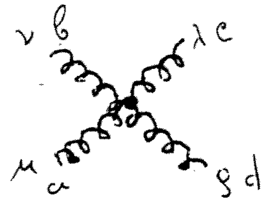
$g\bar\Psi A\!\!\!/\Psi \to$  quark-gluon vertex ($g\gamma_\mu t^a$ in the rules for modified Green functions $\mathcal{G}$, $ig\gamma_\mu t^a$ in the rules for G).



$$-ig[(p_1-p_2)_\lambda g_{\mu\nu} + (p_2-p_3)_\mu g_{\nu\lambda} + (p_3-p_1)_\nu g_{\lambda\mu}]f_{abc}$$

(in the rules for $\mathcal{G}$ $\qquad (819)$

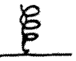
three-gluon vertex

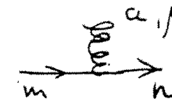


$$-g^2[f^{abm}f^{cdm}(g^{\mu\lambda}g^{\nu\rho} - g^{\mu\rho}g^{\nu\lambda}) + f^{acm}f^{bdm}(g^{\mu\nu}g^{\lambda\rho} - g^{\mu\rho}g^{\nu\lambda}) + f^{adm}f^{bcm}(g^{\mu\nu}g^{\lambda\rho} - g^{\mu\lambda}g^{\nu\rho})] \qquad (820)$$

four-gluon vertex.

Quark-gluon vertex can be found by inspection of the corresponding QED vertex

$\mathcal{L}_I \in -e\bar\Psi\gamma^\mu A_\mu\Psi \Rightarrow$ vertex is $-e\gamma_\mu$ 

$\mathcal{L}_I = g\bar\Psi_m(t^a)_{mn}A^a_\mu\gamma^\mu\Psi_n \Rightarrow$ vertex is $g(t^a)_{mn}\gamma^\mu$ 

Let us find the three-gluon vertex

$$G(p_1,p_2,p_3) = \int dx_1 dx_2 dx_3\, e^{-i(p_1x_1+p_2x_2+p_3x_3)} \langle \Phi | T\{A^a_\mu(x_1) A^b_\nu(x_2) A^c_\lambda(x_3)\} | \Phi \rangle$$

$$\langle \Phi | T\{A^a_\mu(x_1) A^b_\nu(x_2) A^c_\lambda(x_3)\} | \Phi \rangle = \langle 0 | T\{A^a_\mu(x_1) A^b_\nu(x_2) A^c_\lambda(x_3)\, i\int dz\, \mathcal{L}_I(z)\} | 0 \rangle =$$

$$\langle 0 | T\{A^a_\mu(x_1) A^b_\nu(x_2) A^c_\lambda(x_3) \int dz\, Tr\{F_{\alpha\beta}[A^\alpha, A^\beta](z)\} | 0 \rangle =$$

$$= \cdot A^a_\mu(x_1) A^b_\nu(x_2) A^c_\lambda(x_3) \int dz(-ig) f_{mnl}\, \partial_\alpha A^m_\beta A^n_\alpha A^l_\beta(z) + (\ \leftrightarrow\ )$$

$$+ A^a_\mu(x_1) A^b_\nu(x_2) A^c_\lambda(x_3)(-ig)\int dz\, f_{mnl}\, \partial_\alpha A^m_\beta A^n_\alpha A^l_\beta(z) + (\ \leftrightarrow\ ) +$$

$$+ A^a_\mu(x_1) A^b_\nu(x_2) A^c_\lambda(x_3)(-ig)\int dz\, f_{mnl}\, \partial_\alpha A^m_\beta A^n_\alpha A^l_\beta(z) + (\ \leftrightarrow\ )$$

$$A^a_\mu(x_1) A^m_\beta(z) = \int \frac{dk_1}{i} \frac{g_{\mu\beta}\delta_{am}}{k_1^2+i\epsilon} e^{-ik_1(x_1-z)} \qquad A^a_\mu(x_1)\, \partial_\alpha A^m_\beta(z) = \int dk_1 \frac{k_{1\alpha} g_{\mu\beta} e^{-ik_1(x-z)}}{k_1^2+i\epsilon}\delta_{am}$$

$$A^b_\nu(x_2) A^l_\beta(z) = \int \frac{dk_2}{i} \frac{g_{\nu\beta}\delta^{bl}}{k_2^2} e^{-ik_2(x_2-z)} \qquad A^c_\lambda(x_3) A^n_\alpha(z) = \int \frac{dk_3}{i} \frac{g_{\lambda\alpha} e^{-ik_3(x_3-z)}}{k_3^2}\delta_{cn}$$

$$G^{abc}_{\mu\nu\lambda}(p_1,p_2,p_3) = \int dx_1 dx_2 dx_3\, e^{-i(p_1x_1+p_2x_2+p_3x_3)} \int dk_1 dk_2 dk_3 \frac{-ig}{i^2}\int dz\, e^{i(k_1+k_2+k_3)z}$$

$$\cdot e^{-ik_1x_1-ik_2x_2-ik_3x_3} \{ f_{acb} k_{1\lambda} g_{\mu\nu} + f_{abc} k_{1\nu} g_{\mu\lambda} + f_{bca} k_{2\lambda} g_{\mu\nu} +$$

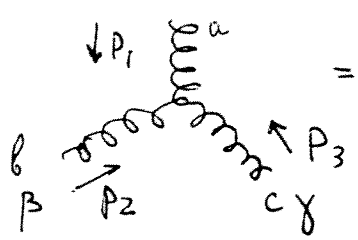
$$+ f_{bac} k_{2\mu} g_{\nu\lambda} + f_{cba} k_{3\nu} g_{\mu\lambda} + f_{cab} k_{3\mu} g_{\nu\lambda}] \frac{1}{k_1^2 k_2^2 k_3^2} = \qquad p_i = -k_i$$

$$= +ig(2\pi)^4 \delta(p_1+p_2+p_3) \frac{f_{abc}}{p_1^2 p_2^2 p_3^2} [(p_1-p_2)_\lambda g_{\mu\nu} + (p_2-p_3)_\mu g_{\nu\lambda} + (p_3-p_1)_\nu g_{\lambda\mu}]$$

$$= (2\pi)^4 \delta(p_1+p_2+p_3) \frac{g_{\mu\alpha}\delta^{am}}{i p_1^2} \frac{g_{\nu\beta}\delta^{bn}}{i p_2^2} \frac{g_{\lambda\gamma}\delta^{cl}}{i p_3^2} \underbrace{(+g) f_{mnl} [(p_1-p_2)^\gamma g^{\alpha\beta} + (p_2-p_3)^\alpha g^{\beta\gamma} + (p_3-p_1)^\beta g^{\gamma\alpha}]}_{\text{vertex in the Feynman rules for } G}$$

(821)

Thus, for modified Green functions

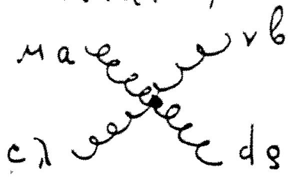


$$= -ig f_{abc} \{(p_1-p_2)_\gamma g_{\alpha\beta} + (p_2-p_3)_\alpha g_{\beta\gamma} + (p_3-p_1)_\beta g_{\gamma\alpha}\}$$

(822)

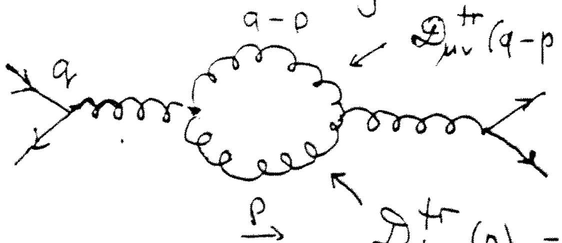
Similarly

(823)



$$= -g^2 [f^{abm} f^{cdm} (g^{\mu\lambda} g^{\nu\rho} - g^{\mu\rho} g^{\nu\lambda}) + f^{acm} f^{bdm} \cdot (g^{\mu\nu} g^{\lambda\rho} - g^{\mu\rho} g^{\nu\lambda}) + f^{adm} f^{bcm} (g^{\mu\nu} g^{\lambda\rho} - g^{\mu\lambda} g^{\nu\rho})]$$

Subtle point : ghosts



$$\mathcal{D}^{tr}_{\mu\nu}(q-p) = \frac{1}{(q-p)^2}\left(g_{\mu\nu} + \frac{(q-p)_\mu (q-p)_\nu}{|\vec{q}-\vec{p}|^2} - \frac{(q-p)_0\left((q-p)_\mu \eta_\nu + \mu\to\nu\right)}{|\vec{q}-\vec{p}|^2} - \frac{(q-p)^2 \eta_\mu \eta_\nu}{|\vec{q}-\vec{p}|^2}\right)$$

$$\mathcal{D}^{tr}_{\mu\nu}(p) = \frac{1}{p^2}\left(g_{\mu\nu} + \frac{p_\mu p_\nu}{\vec{p}^{\,2}} - \frac{p_0}{\vec{p}^{\,2}}(p_\mu \eta_\nu + p_\nu \eta_\mu) - \frac{p^2 \eta_\mu \eta_\nu}{\vec{p}^{\,2}}\right)$$

We want to replace both of them by $\mathcal{D}^F_{\mu\nu}$

In QED, we used Ward identity to eliminate longitudinal terms

In QCD, Ward identity is more complicated $\Rightarrow$

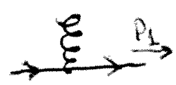
$\Rightarrow$ some contributions of longitudinal terms remain (in loop integrals).

It was proved (using functional integrals) that a good way to memorize these longitudinal remnants is to add a new term in the Lagrangian

$$\Delta\mathcal{L} = \bar{c}^m(-\partial_\mu \mathcal{D}^\mu)_{mn} c^n = \bar{c}^n(-\partial^2)c^n + ig\,\bar{c}^m t^a_{mn} A^a_\mu \partial^\mu c^n \qquad (824)$$

scalar fermions (ghosts)
Live only in loops!

Propagator ——— $\frac{1}{-p^2 - i\epsilon}$ − massless scalar

Vertex  $ig f_{abc} p_\mu$

(−1) for each ghost loop!

QCD Lagrangian for practitioners ($\mathcal{L}_{QCD}$ + gauge-fixing term + ghosts)

$$\mathcal{L}_F = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \sum_f \bar{q}_f (i\not{D} - m) q_f - \frac{1}{2}(\partial_\mu A^\mu)^2 + \bar{c}^a(-\partial_\mu \mathcal{D}^\mu)^{ab} c^b =$$

$\uparrow$ Feynman gauge

$q_f \equiv \psi_{q_f}$ - quark field of flavor f

$$= +\frac{1}{2} A^a_\mu \partial^2 A^{a\mu} + \sum_f \bar{q}_f (i\not{\partial} - m) q_f + \bar{c}^a(-\partial^2) c^a -$$

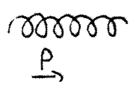
$$- g f_{abc} (\partial_\mu A^a_\nu) A^{b\mu} A^{c\nu} - \frac{g^2}{4} f_{mab} f_{mcd} A^a_\mu A^b_\nu A^c_\mu A^d_\nu + g \sum_f \bar{q}_f \not{A} q_f -$$

$$- g f_{abc} \bar{c}^a \partial_\mu (A^{b\mu} c) \qquad (825)$$

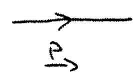
$$= \mathcal{L}_0 + \mathcal{L}_{int}.$$

For modified Green functions $\mathcal{G}$:

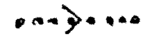
$\mathcal{L}_0 \to$ propagators



$= \dfrac{g_{\mu\nu}\delta^{ab}}{p^2 + i\epsilon}$



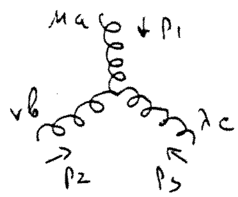
$= \dfrac{(m_q + \not{p})\delta_{kl}}{m_q^2 - p^2 - i\epsilon}$



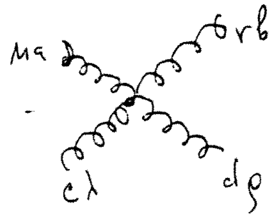
$= \dfrac{\delta^{ab}}{-p^2 - i\epsilon}$

$$\overbrace{c^a(x)\,\bar{c}^b(y)} = \int \frac{dk}{i} \frac{\delta^{ab}}{-k^2 - i\epsilon} e^{-ik(x-}$$

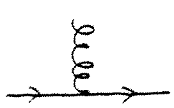
$\mathcal{L}_{int} \to$ vertices



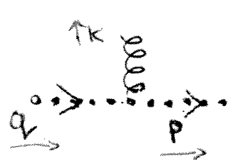
$$= -ig f_{abc} \{(p_1 - p_2)_\lambda g_{\mu\nu} + (p_2 - p_3)_\mu g_{\nu\lambda} + (p_3 - p_1)_\nu g_{\lambda\mu}\}$$



$$= -g^2 [f^{abm} f^{cdm} (g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda}) + \ldots]$$



$$= \gamma_\mu (t^a)_{kl}$$



$$= +ig f_{abc} p_\mu$$

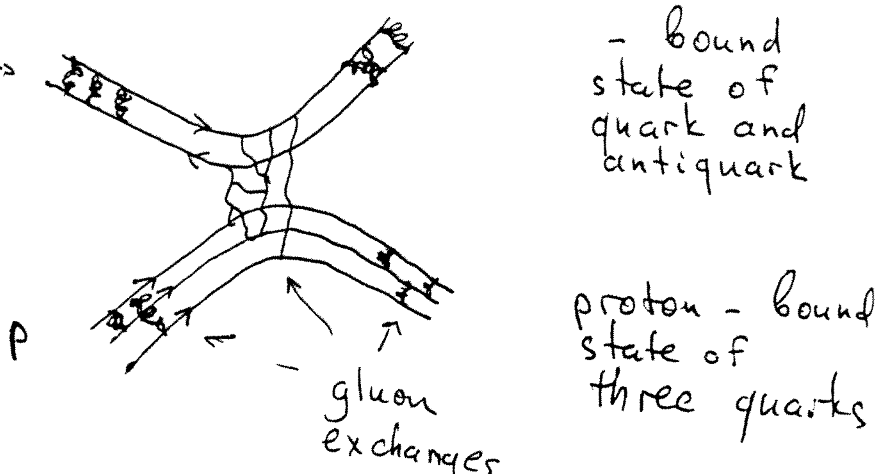
As usual,
$\int \frac{d^4p}{i}$ for each loop and (-1) for every fermion (and ghost) loop.

So, we have perturbation theory for the QCD Green functio[ns]
just as in QED case

Unlike QED, we do not observe quarks and gluons (in the initial of final states of a scattering process).
Instead, we see only hadrons.

Confinement – only the composite particles without color can be observed. (Color particles are confined within the interaction range).

Typical scattering



How to calculate the mass of, say, ρ-meson (hope)

$$\int dx \langle \Omega | T\{ j_\mu(x)\, j_\nu(0) \} | \Omega \rangle e^{ipx} =$$

current with quantum numbers of ρ-meson
($\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d$ for ρ^0)

$$= (p_\mu p_\nu - p^2 g_{\mu\nu})\, \Pi(p^2)$$

$\Pi(p^2)$ must have a pole at $p^2 \to m_\rho^2$

$$\Pi(p^2) \to \frac{\text{const}}{m_\rho^2 - p^2\ i\epsilon}$$

If one some day will do this calculation, he (or she) will find the pole at $p^2 = 0.6\,\text{GeV}^2$ (m_ρ^2).

Lattice simulations:
approximate calculation in Euclidean region of p^2

$$\Pi(p^2 = -P^2) = \frac{\text{const}}{m_\rho^2 + P^2} \qquad \Pi(x) \sim e^{-m_\rho |x|} \qquad |x| = \sqrt{x^2}$$

(in Euclidean region)

Techique: functional integral at discrete space-time (lattice).

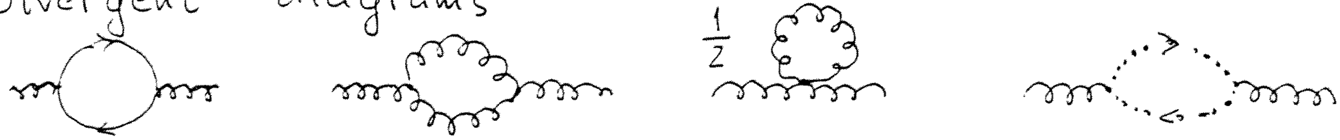
Asymptotic freedom

Renormalization (one-loop)

symmetry coefficient

1. Divergent diagrams

$\frac{1}{2}$



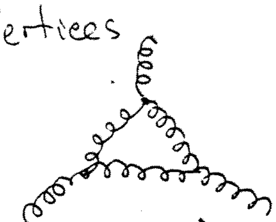
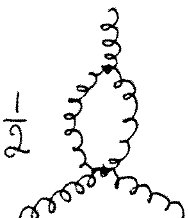
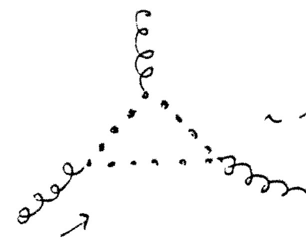
$\ln\mu$ $\qquad$ $\underbrace{a\mu^2 + b\ln\mu \qquad cM^2 + d\ln\mu \qquad -(a+c)\mu^2 + e\ln\mu}_{(b+d+e)\ln\mu}$ $\qquad$ (see below)

gluon self-energy

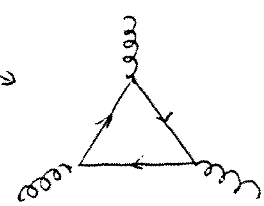
 – ghost self-energy $\qquad$  – quark self-energy

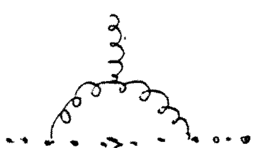
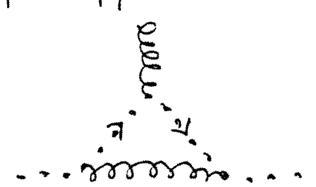
$\ln\mu$ $\qquad$ $\ln\mu$.

Vertices

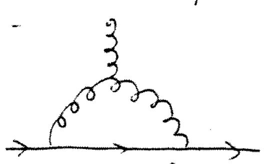
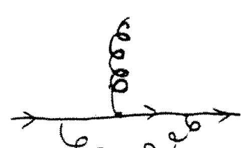
 $\sim\ln\mu$ $\qquad$ $\frac{1}{2}$  $\sim\ln\mu$ $\qquad$  $\sim\ln\mu$.

three-gluon vertex (in one loop approximation)



 ← ghost-ghost-gluon → 

$\ln\mu$

 ← quark-gluon vertex → 

$\sim\ln\mu$ $\qquad$ $\sim\ln\mu$.

2. Renormalization program: same as in QED.

Subtraction at $p^2 = -M^2$

$$\mathcal{L} = \mathcal{L}_F(\bar{\psi}_{(0)}, \psi_{(0)}, A_{(0)}, \bar{c}_{(0)}, c_{(0)}, g_0, m_{\psi_0})$$

$$\psi_{(0)} = Z_2^{1/2}\psi \qquad A^{\mu}_{(0)} = Z_3^{1/2} A^{\mu} \qquad c_{(0)} = \sqrt{Z_2^c}\, c$$

← renormalized fields

$$\mathcal{L} = \mathcal{L}_{basic} + \mathcal{L}_{counterterm}$$

$\mathcal{L}_{basic} \to \mathcal{L}^F(\psi, A, c)$

$\mathcal{L}_{counterterm}$ ↓ to be calculated at 1-loop level (after that, at 2-loop etc).

$m(M)$ ← renormalized mass

$g(M)$ ← renormalized coupling constant (to be defined below)

$$\mathcal{L}_B^F = +\frac{1}{2}A^a_\mu \partial^2 A^{a\mu} + \Sigma\, \bar{q}(i\not{\partial} - m)q + \bar{c}^a(-\partial^2)c^a - g f_{abc}(\partial_\mu A^a_\nu)A^{b\mu}A^{c\nu} -$$
$$- \frac{g^2}{4} f_{abm} f_{cdm} A^a_\mu A^b_\nu A^{c\mu}A^{d\nu} + g\,\Sigma\,\bar{q}\not{A}q - g f_{abc}\,\bar{c}^a \partial_\mu(A^{b\mu}c) \qquad (826)$$

Renormalized fields + renormalized masses and coupling constant

$$\mathcal{L}_{c.t.} = \frac{1}{2}\delta_3 A^a_\mu \partial^2 A^{a\mu} + \bar{\psi}(\delta_2 i\not{\partial} - \delta m)\psi - \delta_2^c\, \bar{c}^a\partial^2 c^a + g\delta_1 \bar{\psi}\not{A}\psi -$$
$$- g\,\delta_1^g f_{abc}(\partial_\mu A^a_\nu)A^{b\mu}A^{c\nu} - \frac{g^2}{4}\tilde{\delta}_1^g f_{mab}A^a_\mu A^b_\nu f_{mcd}A^{c\mu}A^{d\nu} - g\,\delta_1^c f_{abc}\,\bar{c}^a\partial_\mu(A^{\mu b}c^c) \qquad (827)$$

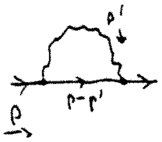
Counterterms

$$\delta_2 = Z_2 - 1\,, \quad \delta_3 = Z_3 - 1\,, \quad \delta_2^c = Z_2^c - 1\,, \quad \delta m = Z_2 m_{(0)} - m \qquad (828)$$

$$\delta_1 = \underbrace{g_{(0)}g^{-1}Z_2 Z_3^{1/2}}_{Z_1} - 1 \qquad \delta_1^g = \underbrace{g_0 g^{-1}(Z_3)^{3/2}}_{Z_1^g} - 1 \qquad \tilde{\delta}_1^g = \underbrace{g_0^2 g^{-2} Z_3^2}_{\tilde{Z}_1^g} - 1 \qquad \delta_1^c = \underbrace{g_0 g^{-1} Z_2^c Z_3^{1/2}}_{Z_1^c}$$

Let us calculate these guys at one-loop order (Gross and Wilszek, Politzer (1973))

Z_2: we have actually done that



$$-\Sigma(\not{p}) = \ldots = g^2\mu^{4-d}\int\frac{d^dp}{i}\,\frac{t^a\gamma_\alpha(m+\not{p}-\not{p}')\gamma_\alpha t^a}{(m^2-(p-p')^2-i\epsilon)(p'^2+i\epsilon)} = c_F\cdot(-\Sigma)_{QED}$$
$$c_F = t^a t^a = \frac{4}{3}$$

$$\Sigma(p) = c_F A(p^2)(m-\not{p}) + c_F B(p^2)m$$

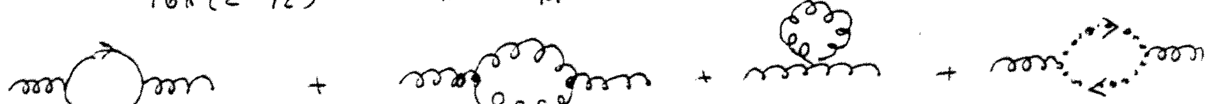
$$\Rightarrow \quad G(p) = \frac{1}{m - \not{p} + \Sigma(p) - \delta m + \delta_2(m-\not{p})} \xrightarrow[p^2\to -M^2]{} \frac{1}{m-\not{p}} \quad \Rightarrow$$

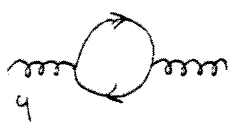
$$\delta_2 = -A(-M^2)c_F = -\frac{c_F g^2}{16\pi^2(2-d/2)} - \frac{c_F g^2}{8\pi^2}\int_0^1 d\beta\left[\bar{\beta}\,\ln\frac{\mu^2}{m^2\beta + M^2\bar{\beta}\beta} - \frac{1}{2}\right] \qquad (829)$$
$$\delta m = c_F m B(-M^2) = \frac{3c_F m}{16\pi^2(2-d/2)} + \frac{c_F g^2 m}{8\pi^2}\int_0^1 d\beta\left[(2m-\bar{\beta})\,\ln\frac{\mu^2}{m^2\beta + M^2\bar{\beta}\beta} - \frac{1}{2}\right]$$

At $M^2 \gg m^2$

$$\delta_2 = -\frac{c_F g^2}{16\pi^2(2-d/2)} - \frac{c_F g^2}{\pi^2}\ln\frac{\mu^2}{M^2} \qquad (830)$$

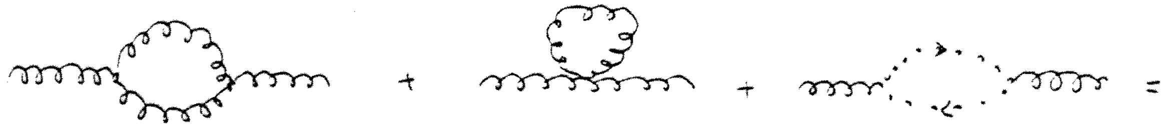
Z_3:





$$= \mu^{4-d}g^2\int\frac{d^dp}{i}(-1)\,\frac{\mathrm{Tr}\,t^a\gamma_\mu(m+\not{p})\gamma_\nu t^b(m+\not{p}-\not{q})}{(m^2-p^2-i\epsilon)(m^2-(q-p)^2-i\epsilon)} = \frac{\delta_{ab}}{2}(QED)$$

$$= \frac{\delta_{ab}}{2}(q_\mu q_\nu - q^2 g_{\mu\nu})\left(\frac{g^2}{12\pi^2(2-d/2)} + \frac{g^2}{2\pi^2}\int_0^1 d\beta\,\bar{\beta}\beta\,\ln\frac{\mu^2}{m^2-q^2\bar{\beta}\beta}\right) =$$
$$= (q_\mu q_\nu - q^2 g_{\mu\nu})\,\frac{\delta_{ab}}{2}\left(\frac{g^2}{12\pi^2(2-d/2)} + \frac{g^2}{2\pi^2}\int_0^1 d\beta\,\bar{\beta}\beta\,\ln\frac{\mu^2}{m^2-q^2\bar{\beta}\beta}\right).$$



$$= \text{calculation (see e.g. Peskin)} = \qquad (831)$$

$$= (q_\mu q_\nu - g_{\mu\nu} q^2)\, \frac{N_c g^2 \delta_{ab}}{16\pi^2}\left[-\frac{5}{3}\frac{1}{2-d/2} + \int_0^1 d\beta\, \left((1-2\beta)^2 - 2\right) \ln\frac{\mu^2}{-q^2\bar\beta\beta + i\epsilon}\right]$$

↑ due to Ward identity in QCD

$N_c = 3$ – „number of colors"

$$\Rightarrow \Pi(q^2) = -\frac{5}{3}\frac{N_c g^2}{16\pi^2(2-d/2)} + \frac{g^2}{12\pi^2(2-d/2)} - \frac{g^2 N_c}{16\pi^2}\int_0^1 d\beta\, \left(2-(1-2\beta)^2\right)\ln\frac{\mu^2}{-q^2\bar\beta\beta}$$

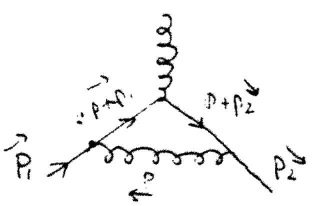
$$+ \frac{g^2}{2\pi^2}\int_0^1 d\beta\, \bar\beta\beta \ln\frac{\mu^2}{m^2 - q^2\bar\beta\beta}$$

$$\Rightarrow \quad \mathcal{D}^{ab}_{\mu\nu} = \frac{g_{\mu\nu}\delta_{ab}}{p^2(1+\Pi(p^2)+\delta_3)} \xrightarrow[p^2\to -M^2]{} \frac{g_{\mu\nu}\delta_{ab}}{p^2} \quad \Rightarrow$$

$$\Rightarrow \delta_3 = -\Pi(-M^2) = \frac{5}{3}\frac{N_c g^2}{16\pi^2(2-d/2)} - \frac{g^2}{12\pi^2(2-d/2)} + \frac{g^2 N_c}{16\pi^2}\int_0^1 d\beta\, \left(2-(1-2\beta)^2\right)\ln\frac{\mu^2}{M^2\bar\beta\beta}$$

$$- \frac{g^2}{2\pi^2}\int_0^1 d\beta\, \bar\beta\beta \ln\frac{\mu^2}{m^2+M^2\bar\beta\beta} \longrightarrow \frac{5}{3}\frac{N_c g^2}{16\pi^2}\left(\frac{1}{2-d/2} + \ln\frac{\mu^2}{M^2}\right) - \frac{g^2}{2\cdot 12\pi^2}\left(\frac{1}{2-d/2} + \ln\frac{\mu^2}{M^2}\right) \qquad (832)$$

Renormalization of quark - gluon vertex (δ_1)



$$= g^3 \mu^{4-d}\int\frac{dp}{i}\, \underbrace{t^b t^a t^b}_{-\frac{1}{2N_c}t^a}\, \frac{\gamma_\alpha(m+\not p+\not p_1)\gamma_\mu(m+\not p+\not p_2)\gamma^\alpha}{p^2(m-(p+p_1)^2)(m-(p+p_2)^2)}$$

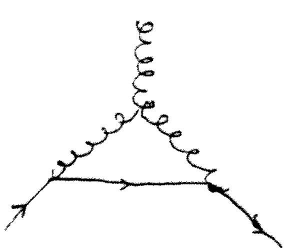
$$= \text{calculation (Peskin)} = \qquad (833)$$

$$= \frac{g^3}{16\pi^2}\left(-\frac{1}{2N_c}\right)t^a\gamma_\mu\left(\frac{1}{2-d/2} + \ln\frac{\mu^2}{\text{physical momenta}}\right)$$

(cf. eq. (787a))

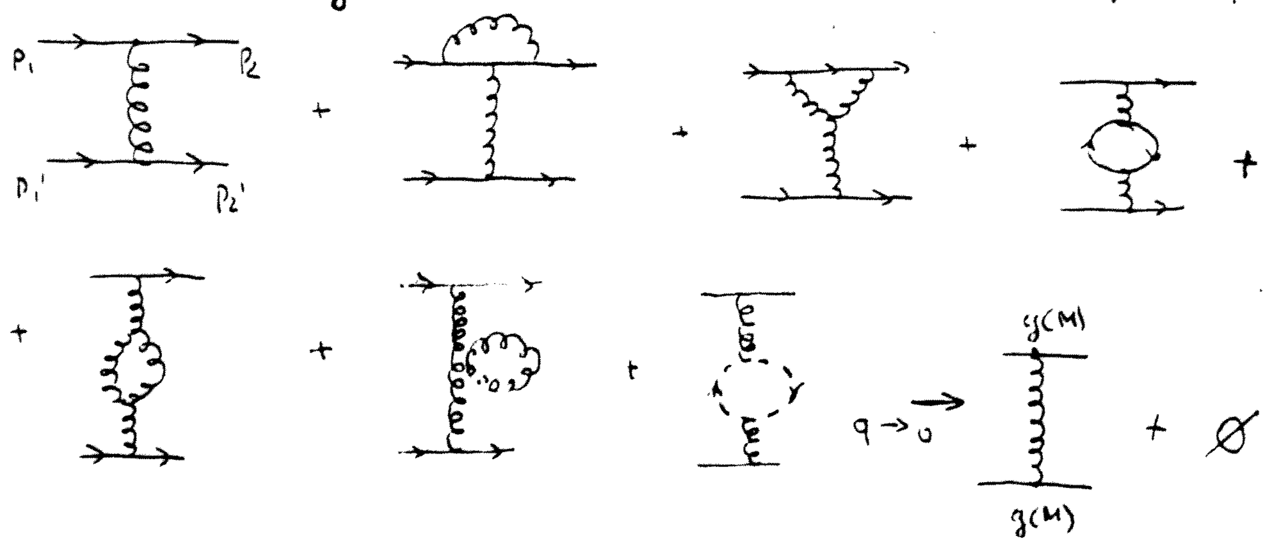
↑ like p_1^2, p_2^2, or $(p_1-p_2)^2$.

Similarly,



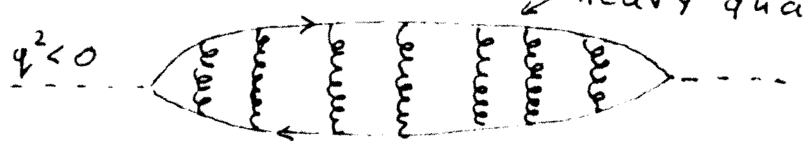
$$= \frac{3g^3}{32\pi^2}N_c\, t^a\gamma_\mu\left(\frac{1}{2-d/2} + \ln\frac{\mu^2}{\text{momenta}}\right) \qquad (834)$$

Definition of $g(M)$ $\qquad p_1^2 = p_2^2 = -M^2 \qquad q = p_1 - p_2$



$g(M)$ describes "Coulomb potential between two virtial quarks with $p^2 = -M^2$"

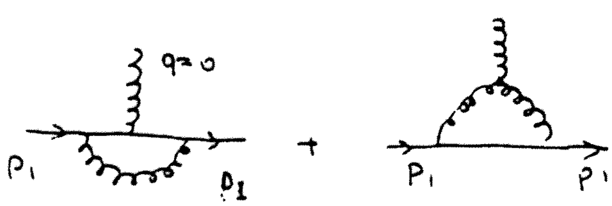
For example



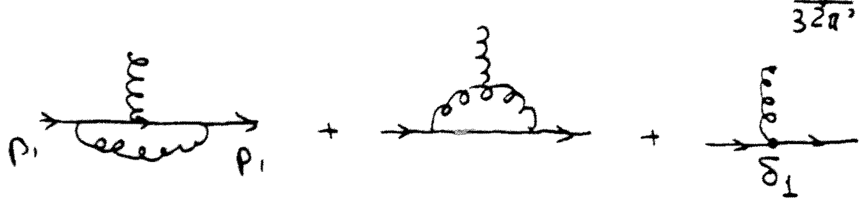
heavy quark and antiquark interact by potential $\frac{g^2(m_q)}{4\pi r}$

Thus,

$p_1^2 =$



$$= g t^a \gamma_\mu \left[-\frac{g^2}{16\pi^2} \frac{1}{2N_c} \left(\frac{1}{2-d/2} + \ln \frac{\mu^2}{M^2} \right) + \frac{3g^2}{32\pi^2} N_c \left(\frac{1}{2-d/2} + \ln \frac{\mu^2}{M^2} \right) \right]$$



$= 0 \quad \Rightarrow$

$$\Rightarrow \delta_1 = -\frac{g^2}{32\pi^2} \left(3N_c - \frac{1}{N_c} \right) \left(\frac{1}{2-d/2} + \ln \frac{\mu^2}{M^2} \right) \qquad (835)$$

Now, let us collect our pieces of asymptotic freedom

$$g(M) = g_0 Z_1^{-1} Z_2 Z_3^{1/2} = g_0 (1 - \delta_1 + \delta_2 + \tfrac{1}{2}\delta_3)$$

$\delta_i \equiv Z_i - 1$

$$= g_0 \left\{ 1 + \frac{g^2(M)}{16\pi^2} \left(+ \frac{3}{2} N_c - \frac{1}{2N_c} - \left(\frac{N_c}{2} - \frac{1}{2N_c} \right) + \frac{5}{6} N_c - \frac{1}{3} \right) \left(\frac{1}{2 - d/2} + \ln \frac{\mu^2}{M^2} \right) \right\} =$$

$$= g_0 \left\{ 1 + \frac{g^2(M)}{16\pi^2} \left(\frac{11}{6} N_c - \frac{1}{3} \right) \right. \qquad (836)$$

With n_f flavours, only the diagram  contributin to Z_2 is different (simply multiplied by n_f), so

$$g(M) = g_0 \left\{ 1 + \frac{g^2(M)}{16\pi^2} \left(\frac{11}{6} N_c - \frac{n_f}{3} \right) \left(\frac{1}{2 - d/2} + \ln \frac{\mu^2}{M^2} \right) \right\}$$

$\frac{11}{3} N_c - \frac{2}{3} n_f \equiv b$ common notation

Let us compare now

$$g(M_1) = g_0 \left\{ 1 + \frac{g^2(M_1)}{16\pi^2} \frac{b}{2} \left(\frac{1}{2 - d/2} + \ln \frac{\mu^2}{M_1^2} \right) \right\}$$

$$g(M_2) = g_0 \left\{ 1 + \frac{g^2(M_2)}{16\pi^2} \frac{b}{2} \left(\frac{1}{2 - d/2} + \ln \frac{\mu^2}{M_2^2} \right) \right\}$$

$\Rightarrow$ eliminate g_0 which we do not know

$$\Rightarrow g(M_1) = \frac{1 + \frac{g^2(M_1)}{16\pi^2} \frac{b}{2} \left(\frac{1}{2 - d/2} + \ln \mu^2/M_1^2 \right)}{1 + \frac{g^2(M_2)}{16\pi^2} \frac{b}{2} \left(\frac{1}{2 - d/2} + \ln \mu^2/M_2^2 \right)} g(M_2) \simeq g(M_2) \left(1 + \frac{g^2(M_2)}{16\pi^2} \frac{b}{2} \ln \frac{M_2^2}{M_1^2} \right)$$

$$\Rightarrow g^2(M_1) = \frac{g^2(M_2)}{1 + \frac{g^2(M_2)}{16\pi^2} b \ln \frac{M_1^2}{M_2^2}} \quad \left(= g(M_2) + \frac{g^3(M_2)}{16\pi^2} b \ln \frac{M_1}{M_2} - \frac{g^5(M_2)}{(16\pi^2)^2} b^2 \ln^2 \frac{M_1}{M_2} + \dots \right)$$

We have obtained only the first term of this arithmetic progression, but it is easy to see that the RG equation

$$M \frac{dg(M)}{dM} = - \frac{g_0 g^2(M)}{8\pi^2} \frac{b}{2} + O(g^3) \simeq - \frac{g^3(M)}{16\pi^2} b + O(g^3(M)) \qquad (837)$$

has formal solution

$$g^2(M) = \frac{g^2(M_2)}{1 + \frac{g^2(M_2)}{16\pi^2} b \ln \frac{M^2}{M_2^2}} \qquad (838)$$

"first coefficient of Gell-Mann-Low function" $\beta(g(M))$

$M \to \infty \quad g(M) \to 0 \quad \Rightarrow$ asymptotic freedom!