Quantum mechanics of an anharmonic oscillator as 0+1 quantum field theory

Classical treatment:

Lagrangian

$$L = \frac{\dot{x}^2}{2} - V(x) = \frac{\dot{x}^2}{2} - \frac{\omega^2}{2}x^2 - \frac{\lambda}{4!}x^4$$

Euler-Lagrange equation $\Leftrightarrow$ Newton's 2nd law

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \qquad \Leftrightarrow \qquad \ddot{x} = -\omega^2 x - \frac{\lambda}{3!}x^3$$

Canonical momentum

$$p = \frac{\partial L}{\partial \dot{x}} = \dot{x}$$

Hamiltonian

$$H = p\dot{x} - L = \frac{p^2}{2} + \frac{\omega^2}{2}x^2 + \frac{\lambda}{4!}x^4$$

Change of names: $x \to \phi$, $p \to \pi$

$$L = \frac{\dot{\phi}^2}{2} - \frac{\omega^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

- Lagrangian of the 0+1 scalar field

$$H = \frac{\pi^2}{2} + \frac{\omega^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$$

Quantization:

$$\phi \quad \rightarrow \quad \widehat{\phi}, \qquad \pi \quad \rightarrow \quad \widehat{\pi}$$

Operators $\widehat{\phi}$ and $\widehat{\pi}$ act on the wave function $\Phi(\phi)$

$$\widehat{\phi}\Phi(\phi) = \phi\Phi(\phi), \qquad \widehat{\pi}\Phi(\phi) = -i\frac{\partial}{\partial\phi}\Phi(\phi)$$

$[\phi, \pi] = i$ - canonical commutation relation

$$H \rightarrow \widehat{H} = \frac{\widehat{\pi}^2}{2} + \frac{\omega^2}{2}\widehat{\phi}^2 + \frac{\lambda}{4!}\widehat{\phi}^4$$

Schrodinger picture: $\Phi(\phi, t), \widehat{\phi}, \widehat{\pi}$

Dynamics is governed by the Schrodinger equation

$$i\frac{\partial}{\partial t}\Phi(\phi, t) = \widehat{H}\Phi(\phi, t)$$

$\hbar = 1$

Heisenberg picture :

$$\begin{aligned} \Phi(\phi) &= \Phi_{\text{Schro}}(\phi, t = 0) \\ \widehat{\phi}(t) &= e^{i\widehat{H}t}\widehat{\phi}e^{-i\widehat{H}t} \\ \widehat{\pi}(t) &= e^{i\widehat{H}t}\widehat{\pi}e^{-i\widehat{H}t} \end{aligned}$$

Dynamics is determined by Heisenberg equations

$$\frac{d}{dt}\widehat{\phi}(t) = i[\widehat{H}, \widehat{\phi}(t)]$$
$$\frac{d}{dt}\widehat{\pi}(t) = i[\widehat{H}, \widehat{\pi}(t)]$$

Perturbation theory $\Leftrightarrow$ expansion in powers of a "coupling constant" λ.

Typical problem: find the dispersion ($\equiv$ mean ϕ^2) in the ground state of the anharmonic oscillator $\langle\Omega|\widehat{\phi}^2|\Omega\rangle$

QM solution:

$$\hat{H} = \hat{H}_0 + \hat{H}_{\rm int}, \quad \hat{H}_0 = \frac{\hat{\pi}^2}{2} + \frac{\omega^2}{2}\hat{\phi}^2, \quad \hat{H}_{\rm int} = \frac{\lambda}{4!}\hat{\phi}^4$$

Ground state of the harmonic oscillator ("perturbative vacuum"):

$$\hat{H}_0|\emptyset\rangle = E_0|\emptyset\rangle \to E_0 = \frac{\omega}{2}, \quad |\emptyset\rangle = e^{-\frac{\omega}{2}\phi^2}$$

Ground state of the anharmonic oscillator ("physical vacuum"):

$$\hat{H}|\Omega\rangle = E_{\rm vac}|\Omega\rangle$$

Perturbative solution

$$|\Omega\rangle = |\emptyset\rangle - \frac{\lambda}{4!}\sum{}' |n\rangle \frac{\langle n|\hat{\phi}^4|\emptyset\rangle}{E_n - E_0} + O(\lambda^2)$$

$|n\rangle$ - eigenstates of $\hat{H}_0$ (Hermit polynomials), $E_n = \omega(n + \frac{1}{2})$

$$\langle\Omega|\hat{\phi}^2|\Omega\rangle = \langle\emptyset|\hat{\phi}^2|\emptyset\rangle - \frac{\lambda}{12}\sum{}' \frac{\langle\emptyset|\hat{\phi}^2|n\rangle\langle n|\hat{\phi}^4|\emptyset\rangle}{E_n - E_0}$$

QFT solution ("interaction picture")

Some definitions:

Interaction representation:

$$\hat{\phi}_I(t) \equiv e^{i\hat{H}_0 t}\hat{\phi}e^{-i\hat{H}_0 t}$$
$$\hat{\pi}_I(t) \equiv e^{i\hat{H}_0 t}\hat{\pi}e^{-i\hat{H}_0 t}$$

T-product of operators

$$T\{\hat{\phi}(t)\hat{\phi}(t')\} \equiv \theta(t-t')\hat{\phi}(t)\hat{\phi}(t') + \theta(t'-t)\hat{\phi}(t')\hat{\phi}(t)$$

Evolution operator

$$\hat{U}(t,0) \equiv e^{i\hat{H}_0 t}e^{-i\hat{H}t} \quad \Rightarrow \quad \hat{U}^\dagger(t,0) = e^{i\hat{H}t}e^{-i\hat{H}_0 t}$$

$$\hat{U}(t_1,t_2) \equiv U(t_1,0)\hat{U}^\dagger(t_2,0)$$

Group property:

$$\hat{U}(t_1,t_2)\hat{U}(t_2,t_3) = \hat{U}(t_1,t_3)$$

Formula $(\hat{H}_I(t) = \frac{\lambda}{4!}\hat{\phi}_I^4(t))$.

$$\hat{U}(t,0) \;=\; T\,\mathrm{exp} - i\int_0^t dt'\hat{H}_I(t') \;=$$

$$1 - i\int_0^t dt'\hat{H}_I(t') + i^2\int_0^t dt'\int_0^{t'} dt''\,\hat{H}_I(t')\hat{H}_I(t'') + ...$$

Proof:

$$\frac{d}{dt}\hat{U}(t,0) = -ie^{i\hat{H}_0 t}\frac{\lambda}{4!}\hat{\phi}^4 e^{-i\hat{H}t} =$$

$$-ie^{i\hat{H}_0 t}\frac{\lambda}{4!}\hat{\phi}^4 e^{-i\hat{H}_0 t}\hat{U}(t,0) = -i\frac{\lambda}{4!}\hat{\phi}_I^4(t)\hat{U}(t,0)$$

$$\frac{d}{dt}(\mathrm{r.h.s.}) = -i\frac{\lambda}{4!}\hat{\phi}_I^4(t)\Big(1 - i\int_0^t dt'\hat{H}_I(t') + ...\Big)$$

$$= -i\frac{\lambda}{4!}\hat{\phi}_I^4(t)(\mathrm{r.h.s.})$$

Also, $\hat{U}(t,0)\Big|_{t=0} = (\mathrm{r.h.s.})|_{t=0} = 1$

$$\Rightarrow \qquad \hat{U}(t,0) = (\mathrm{r.h.s.})$$

Similarly one can prove that

$$\hat{U}(t_1,t_2) = T\,\mathrm{exp} - i\int_{t_2}^{t_1} dt'\hat{H}_I(t')$$

How to find $\langle\Omega|\hat{\phi}^2|\Omega\rangle$ using the evolution operator?

Key idea: if you take $|\emptyset\rangle$ and wait for a long time, you'll get $|\Omega\rangle$.

$$\lim_{T\to\infty(1-i\epsilon)} e^{-i\hat{H}T}|\emptyset\rangle = ?$$

(Strictly speaking, we must take $T(1-i\epsilon)$, then $T\to\infty$, and only then $\epsilon\to 0$).

$$\lim_{T\to\infty} e^{-i\hat{H}T(1-i\epsilon)}|\emptyset\rangle =$$
$$\lim_{T\to\infty} e^{-i\hat{H}T(1-i\epsilon)}\sum_N |N\rangle\langle N|\emptyset\rangle =$$
$$\lim_{T\to\infty} e^{-iE_{\text{vac}}T(1-i\epsilon)}\Big(|\Omega\rangle\langle\Omega|\emptyset\rangle +$$
$$\sum{}' |N\rangle\langle N|\emptyset\rangle e^{-\epsilon T(E_N-E_{\text{vac}})+iT(E_N-E_{\text{vac}})}\Big)$$

At $T\to\infty$

$e^{-\epsilon T(E_N-E_{\text{vac}})}\to 0$ because $E_N > E_{\text{vac}}$

$$\Rightarrow \lim_{T\to\infty} e^{-i\hat{H}T(1-i\epsilon)}|\emptyset\rangle = |\Omega\rangle\langle\Omega|\emptyset\rangle e^{-iE_{\text{vac}}T(1-i\epsilon)}$$

Thus

$$|\Omega\rangle = \lim_{T\to\infty}\left(e^{-iE_{\text{vac}}T}\langle\Omega|\emptyset\rangle\right)^{-1} e^{-i\hat{H}T}|\emptyset\rangle$$

In terms of evolution operators

$$|\Omega\rangle = \lim_{T\to\infty} (e^{-i(E_{\rm vac}-E_0)T}\langle\Omega|\,\emptyset\rangle)^{-1}\hat{U}(0,-T)|\,\emptyset\rangle$$
$$\langle\Omega| = \lim_{T\to\infty} (e^{-i(E_{\rm vac}-E_0)T}\langle\emptyset|\Omega\rangle)^{-1}\langle\emptyset|\hat{U}(T,0)$$

$$\to \langle\Omega|\hat{\phi}^2|\Omega\rangle = \lim_{T\to\infty} \frac{\langle\emptyset|\hat{U}(T,0)\hat{\phi}^2\hat{U}(0,-T)|\,\emptyset\rangle}{\langle\emptyset|\Omega|\rangle\langle\Omega|\,\emptyset\rangle e^{-2i(E_{\rm vac}-E_0)T}}$$

($T(1-i\epsilon)$ is always assumed).

By definition of the T-product

$$\hat{U}(T,0)\hat{\phi}^2\hat{U}(0,-T) = T\{\hat{\phi}_I^2(0)\hat{U}(T,-T)\} =$$
$$T\{\hat{\phi}_I^2(0)\exp -i\int_{-T}^{T} dt'\hat{H}_I(t')\}$$

Also,

$$\langle\emptyset|\hat{U}(T,-T)|\,\emptyset\rangle = \langle\emptyset|e^{i\hat{H}_0T}e^{-2i\hat{H}T}e^{i\hat{H}_0T}|\,\emptyset\rangle =$$
$$e^{2iE_0T}\langle\emptyset|e^{-2i\hat{H}T}|\,\emptyset\rangle \to \langle\emptyset|\Omega\rangle\langle\Omega|\,\emptyset\rangle e^{-2i(E_{\rm vac}-E_0)T}$$

$$\Rightarrow \langle\Omega|\hat{\phi}^2|\Omega\rangle = \lim_{T\to\infty} \frac{\langle\emptyset|T\{\hat{\phi}_I^2(0)e^{-i\int_{-T}^{T}dt'\hat{H}_I(t')}\}|\,\emptyset\rangle}{\langle\emptyset|T\{e^{-i\int_{-T}^{T}dt'\hat{H}_I(t')}\}|\,\emptyset\rangle}$$
$$= \frac{\langle 0|T\{\hat{\phi}_I^2(0)e^{-i\int_{-\infty}^{\infty}dt\hat{H}_I(t)}\}|\,\emptyset\rangle}{\langle\emptyset|T\{e^{-i\int_{-\infty}^{\infty}dt\hat{H}_I(t)}\}|\,\emptyset\rangle}$$

In general

$$\langle\Omega|T\{\hat\phi(t_1)...\hat\phi(t_n)\}|\Omega\rangle = \frac{\langle 0|T\{\hat\phi_I(t_1)...\hat\phi_I(t_n)e^{-i\int_{-\infty}^{\infty}dt\hat H_I(t)}\}|\not 0\rangle}{\langle\not 0|T\{e^{-i\int_{-\infty}^{\infty}dt\hat H_I(t)}\}|\not 0\rangle}$$

- master formula for calculations in the interaction representation.

Let us finish the calculation of the dispersion. In the first order in perturbation theory

$$e^{-i\int_{-\infty}^{\infty}dt\hat H_I(t)} \simeq 1 - i\frac{\lambda}{4!}\int_{-\infty}^{\infty}dt\hat\phi_I^4(t)$$

so

$$\begin{aligned}\langle\Omega|\hat\phi^2|\Omega\rangle &= \langle\not 0|\hat\phi^2|\not 0\rangle \\ &- i\frac{\lambda}{4!}\int_{-\infty}^{\infty}dt\Big[\langle\not 0|T\{\hat\phi_I^4(t)\hat\phi_I^2(0)\}|\not 0\rangle \\ &- \langle\not 0|\hat\phi_I^2(0)|\not 0\rangle\langle 0|\hat\phi_I^4(0)|\not 0\rangle\Big]\end{aligned}$$

The correlation functions of the type $\langle\not 0|T\{\hat\phi_I^4(t)\hat\phi_I^2(0)\}|\not 0\rangle$ are called Green functions. They are calculated using Feynman rules.

Feynman rules for the Green functions.

Consider the simplest Green function

$$G(t-t') = \langle \emptyset | T\{\hat{\phi}_I(t)\hat{\phi}_I(t')\} | \emptyset \rangle$$

which is called "propagator".
To find it, we use the ladder operator formalism for the harmonic oscillator:

$$\hat{a} = \frac{\omega\hat{\phi} + i\hat{\pi}}{\sqrt{2\omega}}$$
$$\hat{a}^\dagger = \frac{\omega\hat{\phi} - i\hat{\pi}}{\sqrt{2\omega}}$$

Properties of ladder operators:

$[\hat{\phi}, \hat{\pi}] = i \quad \Rightarrow$

$[\hat{a}, \hat{a}^\dagger] = 1$

– canonical commutation relation in terms of ladder operators.

$\hat{a}|\emptyset\rangle = 0$ $\qquad$ $\hat{a}$ – "annihilation operator"

$(\hat{a}^\dagger)^n|\emptyset\rangle \sim |n\rangle$ $\qquad$ $\hat{a}^\dagger$ – "creation operator"

$\hat{H}_0 = \omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$

Canonical commutation relation $\Rightarrow$
$[\widehat{H}_0, \widehat{a}] = -\omega\widehat{a}$, $[\widehat{H}_0, \widehat{a}^\dagger] = \omega\widehat{a}^\dagger \Rightarrow$

$$\left.\begin{array}{l} e^{i\widehat{H}_0 t}\widehat{a}e^{-i\widehat{H}_0 t} = \widehat{a}e^{-i\omega t} \\ e^{i\widehat{H}_0 t}\widehat{a}^\dagger e^{-i\widehat{H}_0 t} = \widehat{a}^\dagger e^{i\omega t} \end{array}\right\} \Rightarrow$$

$$\widehat{\phi}_I(t) = e^{i\widehat{H}_0 t}\widehat{\phi}e^{-i\widehat{H}_0 t} =$$

$$\frac{1}{\sqrt{2\omega}}e^{i\widehat{H}_0 t}(\widehat{a} + \widehat{a}^\dagger)e^{-i\widehat{H}_0 t} = \frac{\widehat{a}}{\sqrt{2\omega}}e^{-i\omega t} + \frac{\widehat{a}^\dagger}{\sqrt{2\omega}}e^{i\omega t}$$

Now we can find the "propagator"

$$\langle \emptyset | T\{\widehat{\phi}_I(t)\widehat{\phi}_I(t')\} | \emptyset\rangle =$$

$$\frac{\theta(t-t')}{2\omega}\langle \emptyset | (\widehat{a}e^{-i\omega t} + \widehat{a}^\dagger e^{i\omega t})(\widehat{a}e^{-i\omega t'} + \widehat{a}^\dagger e^{i\omega t'}) | \emptyset\rangle$$

$$+(t \leftrightarrow t') =$$

$$\frac{1}{2\omega}\langle \emptyset | [\widehat{a}, \widehat{a}^\dagger] e^{-i\omega(t-t')} | \emptyset\rangle + (t \leftrightarrow t') = \frac{1}{2\omega}e^{-i\omega|t-t'|}$$

(recall that $\widehat{a} | \emptyset\rangle = \langle 0 | \widehat{a}^\dagger = 0$).
Similarly

$$\langle \emptyset | T\{\widehat{\phi}_I^2(0)\widehat{\phi}_I^4(t)\} | \emptyset\rangle =$$

$$\frac{\theta(t)}{8\omega^3}\langle \emptyset | (\widehat{a}e^{-i\omega t} + \widehat{a}^\dagger e^{i\omega t})^4 (\widehat{a} + \widehat{a}^\dagger)^2 | \emptyset\rangle$$

$$\frac{\theta(-t)}{8\omega^3}\langle \emptyset | (\widehat{a} + \widehat{a}^\dagger)^2 (\widehat{a}e^{-i\omega t} + \widehat{a}^\dagger e^{i\omega t})^4 | \emptyset\rangle$$

The result of the calculation can be represented by Wick's theorem:

$$\langle \emptyset | T\{\hat{\phi}_I^2(0)\hat{\phi}_I^4(t)\}| \emptyset \rangle$$

$$= \sum_{\text{contractions}} \hat{\phi}_I(0)\hat{\phi}_I(0)\hat{\phi}_I(t)\hat{\phi}_I(t)\hat{\phi}_I(t)\hat{\phi}_I(t)$$

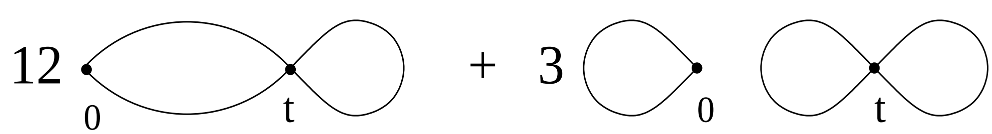
Each contraction is a propagator

$$\hat{\phi}_I(t)\hat{\phi}_I(0) = G(t) = \frac{1}{2\omega} e^{-i\omega|t|}$$

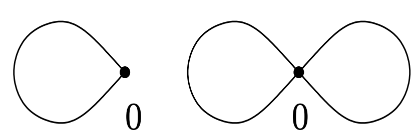
represented by a line in a Feynman diagram

The rest is combinatorics:

$$\langle \emptyset | T\{\hat{\phi}_I^4(t)\hat{\phi}_I^2(0)\}| \emptyset \rangle \;=$$

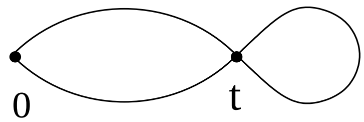


$$\langle \emptyset | \hat{\phi}_I^2(0)| \emptyset \rangle \langle \emptyset | \hat{\phi}_I^4(0)\}| \emptyset \rangle \;=\; 3$$



$\Rightarrow \langle \emptyset | T\{\hat{\phi}_I^4(t)\hat{\phi}_I^2(0)\}| \emptyset\rangle - \langle \emptyset|\hat{\phi}_I^4(0)| \emptyset\rangle\langle \emptyset|\hat{\phi}_I^2(0)\}| \emptyset\rangle =$

$= \sum$ of the connected Feynman diagrams $=$

$= 12$  $= 12G^2(t)\, G(0)$

Second term (coming from the denominator) cancels disconnected diagrams. This is a general property: any Green function is represented by the sum of the relevant connected Feynman diagrams.

$$\Rightarrow \langle \Omega|\hat{\phi}^2|\Omega\rangle = G(0) - \frac{i}{2}\lambda \int_{-\infty}^{\infty} dt G^2(t)G(0)$$

$$= \frac{1}{2\omega} - \frac{i\lambda}{8\omega^3}\int_{-\infty}^{\infty} dt e^{-2i\omega|t|}) =$$

$$\frac{1}{2\omega}(1 - \frac{\lambda}{8\omega^3}) + O(\lambda^2)$$

This may be a wierd way to calculate $\langle \phi^2 \rangle$ in quantum mechanics, but it generalizes to field theories.

QFT for the Klein-Gordon field

$\phi(\vec{x}, t)$ – Klein-Gordon field.
(if $m_\pi = 0$ it would be observable like electric field).
$(\partial^2 + m^2)\phi(x) = 0$ – Klein-Gordon equation
– analog of Maxwell's equations.

$$x \equiv (\vec{x}, t),\ \partial^2 \equiv \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\mu}$$

Classical theory:
Lagrangian $L = \int d^3x \mathcal{L}(\vec{x}, t)$,

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2$$

– Lagrangian density for the free KG field

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

– Lagrangian density for the self-interacting KG field .
Euler-Lagrange eqn reproduces the KG equation

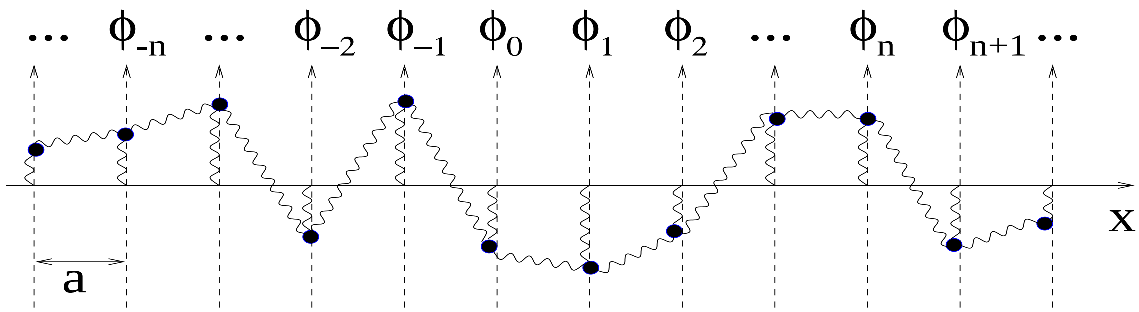
Quantization of the free KG field

For simplicity: one-dimensional KG field $\phi(x,t)$

$$L_0(t) = \int dx \left(\frac{\dot\phi^2(x,t)}{2} - \frac{(\phi'(x,t))^2}{2} - \frac{m^2}{2}\phi^2(x,t) \right)$$

Idea: KG field $\Leftrightarrow$ superposition of oscillators.

Lattice model of the KG field:



- harmonic oscillators at each point of the lattice with nearest-neighbor interaction.

$$L_0(t) = a \sum \left[\frac{\dot\phi_n^2(t)}{2} - \frac{(\phi_{n+1}(t) - \phi_n(t))^2}{2a^2} - \frac{m^2}{2}\phi_n^2(t) \right]$$

Change of the label: $\phi_n \to \phi(x_n,t) \Rightarrow$

$$L_0(t) = a \sum \left[\frac{\dot\phi^2(x_n,t)}{2} - \frac{(\phi(x_n+a,t) - \phi(x_n,t))^2}{2a^2} - \frac{m^2}{2}\phi^2(x_n,t) \right]$$

In the "continuum limit" $a \to 0$ this reproduces the above KG Lagrangian

Quantization of a set of oscillators.

Canonical momenta: $\pi_n = \frac{\partial L_0}{\partial \dot{\phi}_n} = a\dot{\phi}_n$

Define $\pi(x_n, t) \equiv \frac{1}{a}\pi_n = \dot{\phi}(x_n, t) \Rightarrow$

$H_0 = \sum \pi_n \dot{\phi}_n - L_0 =$

$$a \sum \left[\frac{\pi^2(x_n,t)}{2} + \frac{(\phi(x_n+a,t)-\phi(x_n,t))^2}{2a^2} + \frac{m^2}{2}\phi^2(x_n,t)\right]$$

In the continuum limit we get

$$H_0 = \int dx \left[\frac{\pi^2(x,t)}{2} + \frac{\phi'(x,t)^2}{2} + \frac{m^2}{2}\phi^2(x,t)\right]$$

As usual, for quantization of the set of oscillators we promote ϕ_n and π_n to operators satisfying canonical commutation relations

$$[\hat{\phi}_m, \hat{\pi}_n] = i\delta_{mn}, \quad [\hat{\phi}_m, \hat{\phi}_n] = [\hat{\pi}_m, \hat{\pi}_n] = 0$$

In terms of $\phi(x_n)$ and $\pi(x_n)$ this reads

$$[\hat{\phi}(x_m), \hat{\pi}(x_n)] = \frac{i}{a}\delta_{mn}$$

which reduces to

$$[\hat{\phi}(x), \hat{\pi}(y)] = i\delta(x-y),$$
$$[\hat{\phi}(x), \hat{\phi}(y)] = [\hat{\pi}(x), \hat{\pi}(y)] = 0$$

in the continuum limit.

The 3-dimensional KG field is the $a \to 0$ limit of the 3d lattice of harmonic oscillators with nearest-neighbor interaction $\Rightarrow$

The canonical commutation relations are

$$[\hat{\phi}(\vec{x}), \hat{\pi}(\vec{y})] = i\delta^3(x-y),$$
$$[\hat{\phi}(\vec{x}), \hat{\phi}(\vec{y})] = [\hat{\pi}(\vec{x}), \hat{\pi}(\vec{y})] = 0$$

Ladder operators

$$\hat{\phi}(\vec{x}) = \int \frac{d\!\!^-{}^3p}{\sqrt{2E_p}}\left(\hat{a}_{\vec{p}}e^{i\vec{p}\vec{x}} + \hat{a}^\dagger_{\vec{p}}e^{-i\vec{p}\vec{x}}\right)$$
$$\hat{\pi}(\vec{x}) = -i\int \frac{d\!\!^-3p}{\sqrt{2E_p}}E_p\left(\hat{a}_{\vec{p}}e^{i\vec{p}\vec{x}} - \hat{a}^\dagger_{\vec{p}}e^{-i\vec{p}\vec{x}}\right)$$

$(E_p = \sqrt{m^2 + \vec{p}^2})$.

It is easy to check that

$$\left.\begin{array}{l}\left[\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{p}'}\right] = (2\pi)^3\delta^3(\vec{p} - \vec{p}') \\ \left[\hat{a}_{\vec{p}}, \hat{a}_{\vec{p}'}\right] = \left[\hat{a}^\dagger_{\vec{p}}, \hat{a}^\dagger_{\vec{p}'}\right] = 0\end{array}\right\} \Rightarrow$$

$\Rightarrow$ canonical commutation relations.

Classical Hamiltonian

$$H_0 = \int d^3x \Big[\frac{\pi^2(\vec{x},t)}{2} + \frac{\vec{\nabla}\phi(\vec{x},t)^2}{2} + \frac{m^2}{2}\phi^2(\vec{x},t)\Big]$$

$$\Rightarrow \hat{H}_0 = \int d^3x \Big[\frac{\hat{\pi}^2(\vec{x})}{2} + \frac{\vec{\nabla}\hat{\phi}(\vec{x})^2}{2} + \frac{m^2}{2}\hat{\phi}^2(\vec{x})\Big]$$

In terms of ladder operators

$$\hat{H}_0 = \int d\!\!\!{}^-{}^3p \frac{E_p}{2}(\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}} + \hat{a}_{\vec{p}}\hat{a}^\dagger_{\vec{p}}) \Rightarrow \int d\!\!\!{}^-{}^3p E_p \hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}}$$

(we throw away the ∞ constant $E_0 = L^3 \int d\!\!\!{}^-{}^3p \frac{E_p}{2}$).

$\Rightarrow [\hat{H}_0, \hat{a}_{\vec{p}}] = -E_p\hat{a}_{\vec{p}}$, $[\hat{H}_0, \hat{a}^\dagger_{\vec{p}}] = E_p\hat{a}^\dagger_{\vec{p}}$. $\Rightarrow$

$$e^{i\hat{H}_0 t}\hat{a}_{\vec{p}}e^{-i\hat{H}_0 t} = \hat{a}_{\vec{p}}e^{-iE_pt}, \quad e^{i\hat{H}_0 t}\hat{a}^\dagger_{\vec{p}}e^{-i\hat{H}_0 t} = \hat{a}^\dagger_{\vec{p}}e^{iE_pt}$$

The Heisenberg operators are defined as usual

$$\hat{\phi}(\vec{x},t) = e^{i\hat{H}_0 t}\hat{\phi}(\vec{x})e^{-i\hat{H}_0 t}, \quad \hat{\pi}(\vec{x},t) = e^{i\hat{H}_0 t}\hat{\pi}(\vec{x})e^{-i\hat{H}_0 t}$$

$$\Rightarrow \quad \begin{aligned} \hat{\phi}(x) &= \int \frac{d\!\!\!{}^-{}^3p}{\sqrt{2E_p}}\Big(\hat{a}_{\vec{p}}e^{-ipx} + \hat{a}^\dagger_{\vec{p}}e^{ipx}\Big) \\ \hat{\pi}(x) &= -i\int \frac{d\!\!\!{}^-{}^3p}{\sqrt{2E_p}}E_p\Big(\hat{a}_{\vec{p}}e^{-ipx} - \hat{a}^\dagger_{\vec{p}}e^{ipx}\Big) \end{aligned}$$

where $x = (\vec{x},t)$ and $px = E_pt - \vec{p}\vec{x}$

Basic property

$$\hat{a}_{\vec{p}} | \emptyset \rangle = 0$$

where $|\emptyset\rangle$ is the ground state of the quantized field ($\equiv$ the lattice of oscillators).

Proof:

Suppose $\hat{a}_{\vec{p}}|\emptyset\rangle \neq 0$. Denote this state by $|X\rangle$.

$$\hat{H}_0|X\rangle = \hat{H}_0\hat{a}_{\vec{p}}|\emptyset\rangle = [\hat{H}_0, \hat{a}_{\vec{p}}]|\emptyset\rangle + \hat{a}_{\vec{p}}H_0|\emptyset\rangle =$$
$$-E_p\hat{a}_{\vec{p}}|\emptyset\rangle + E_0\hat{a}_{\vec{p}}|\emptyset\rangle = (E_0 - E_p)|X\rangle$$

We see, that the state $|X\rangle$ has energy less than the ground state energy which is impossible $\Rightarrow \hat{a}_{\vec{p}}|\emptyset\rangle = 0$.

$$|\vec{p}\rangle \equiv \hat{a}^{\dagger}_{\vec{p}}|\emptyset\rangle \quad -- \quad \text{one} - \text{particle state}$$

Proof: construct momentum operator and check that $\hat{\vec{P}}|\vec{p}\rangle = \vec{p}|\vec{p}\rangle$, $\hat{H}|\vec{p}\rangle = E_p|\vec{p}\rangle$.

Energy-momentum tensor of the classical KG field :

$$\Theta^{\mu\nu}(\vec{x},t) = \partial^\mu\phi(\vec{x},t)\partial^\nu\phi(\vec{x},t) - \frac{g^{\mu\nu}}{2}(\partial_\xi\phi\partial^\xi\phi - m^2\phi^2)$$

Hamiltonian:

$$H(t) = \int d^3x \Theta^{00}(\vec{x},t) = \int d^3x (\frac{\dot{\phi}^2}{2} + \frac{(\vec{\nabla}\phi)^2}{2} + \frac{m^2\phi^2}{2})$$

$$= \int d^3x (\frac{\pi^2}{2} + ...) \quad \Rightarrow \quad \hat{H} = \int d^3x (\frac{\hat{\pi}^2}{2} + ...)$$

Operator of the momentum of the KG field:

$$P^i(t) = \int d^3x \Theta^{0i}(\vec{x},t) = \int d^3x \dot{\phi}\partial^i\phi = \int d^3x \pi \partial^i \phi$$

$$\Rightarrow \quad \hat{P}^i = \int d^3x \hat{\pi}(\vec{x})\partial^i \hat{\phi}(\vec{x})$$

In terms of ladder operators

$$\hat{P}^i = \int đ^3p\, p^i \hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{p}} \quad \Rightarrow$$

$$[\hat{P}^i, \hat{a}^\dagger_{\vec{p}}] = p^i \hat{a}^\dagger_{\vec{p}} \quad \Rightarrow \quad \hat{\vec{P}}^i |p\rangle = \vec{p}|p\rangle$$

Similarly, $\hat{H}|p\rangle = E_p|p\rangle$.

$\hat{H}|p\rangle = E_p|p\rangle$, $\hat{\vec{P}}|p\rangle = \vec{p}|p\rangle$

$\Rightarrow |p\rangle$ is the one-particle state.

$\hat{a}^\dagger_{\vec{p}_1} ... \hat{a}^\dagger_{\vec{p}_n} |\,\emptyset\rangle \equiv |p_1, ..., p_n\rangle$ – n – particle state

Check: $\hat{H}|p_1,...,p_n\rangle = (E_{p_1} + ...\ E_{p_n})|p_1,...,p_n\rangle$,

$\hat{\vec{P}}|p_1,...,p_n\rangle = (\vec{p}_1 + ...\vec{p}_n)|p_1,...,p_n\rangle$

Free propagator $G_0(x-y) \equiv \langle \emptyset | T\{\hat{\phi}(x)\hat{\phi}(y)\} | \emptyset \rangle$

$$G_0(x-y) =$$
$$\theta(x_0 - y_0)\langle \emptyset | \int \frac{đ^3 p}{\sqrt{2E_p}} \left(\hat{a}_{\vec{p}} e^{-ipx} + \hat{a}^\dagger_{\vec{p}} e^{ipx}\right)$$
$$\int \frac{đ^3 p'}{\sqrt{2E'_p}} \left(\hat{a}_{\vec{p}'} e^{-ip'y} + \hat{a}^\dagger_{\vec{p}'} e^{+ip'y}\right) | \emptyset \rangle + (x \leftrightarrow y)$$

$\hat{a} | \emptyset \rangle = \langle \emptyset | \hat{a}^\dagger = 0 \Rightarrow$

$$G_0(x-y) =$$
$$\theta(x_0 - y_0) \int \frac{đ^3 p đ^3 p'}{2\sqrt{E_p E'_p}} e^{-ip(x-y)} \langle \emptyset | \hat{a}_{\vec{p}} \hat{a}^\dagger_{\vec{p}'} | \emptyset \rangle + (x \leftrightarrow y)$$

$[\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{p}'}] = (2\pi)^3 \delta^3(\vec{p} - \vec{p}') \quad \Rightarrow$

$$G_0(x-y) =$$
$$\theta(x_0 - y_0) \int \frac{đ^3 p}{2E_p} e^{-iE_p(x-y)_0 + i\vec{p}(\vec{x}-\vec{y})} + (x \leftrightarrow y)$$

It can be rewritten in the rel.-inv. form:

$$G_0(x-y) = \lim_{\epsilon \to 0} \int \frac{đ^4 p}{i} e^{-ip(x-y)} \frac{1}{m^2 - p^2 - i\epsilon}$$

Schrodinger equation in QFT

Harmonic oscillator: $\hat{H} = \frac{\hat{\pi}^2}{2} + \frac{\omega^2}{2}\hat{\phi}^2$.
Coordinate representation:
$\Psi(\phi)$ - wave function.

$$\hat{\phi}\Psi(\phi) = \phi\Psi(\phi), \quad \hat{\pi}\Psi(\phi) = -i\frac{\partial}{\partial\phi}\Psi(\phi)$$

- operators $\hat{\phi}$ and $\hat{\pi}$.
Stationary Schrodinger equation $\hat{H}\Psi(\phi) = E_0\Psi(\phi)$:

$$-\frac{1}{2}\frac{d^2}{d\phi^2}\Psi(\phi) + \frac{\omega^2\phi^2}{2}\Psi(\phi) = E_0\Psi(\phi)$$

Wave function for the (1-dim) lattice KG model in the coordinate representation

$$\Phi(\{\phi_k\}) \equiv \Psi(...\phi_{-n-1}, \phi_{-n}, ...\phi_0, \phi_1, ...\phi_n, \phi_{-n-1}, ..)$$

Canonical operators:

$$\begin{aligned} \hat{\phi}_n\Psi(\{\phi_k\}) &= \phi_n\Psi(\{\phi_k\}), \\ \hat{\pi}_n\Psi(\{\phi_k\}) &= -i\frac{\partial}{\partial\phi_n}\Psi(\{\phi_k\}) \end{aligned}$$

In the continuum limit $a \to 0$

$$\Psi(\{\phi_k\}) \to \Psi(\phi(x)) \text{ -- wave functional}$$

Canonical operators:

$\widehat{\phi}(x_n)\Psi(\{\phi_k\}) = \phi(x_n)\Psi(\{\phi_k\})$
$\Rightarrow \widehat{\phi}(x)\Psi(\phi) = \phi(x)\Psi(\phi)$,
$\widehat{\pi}(x_n)\Psi(\{\phi_k\}) = -i\frac{1}{a}\frac{\partial}{\partial\phi_n}\Psi(\{\phi_k\})$
$\Rightarrow \widehat{\pi}(x)\Psi(\phi) = ?$

For the lattice model

$$\begin{aligned}\Psi(\{\phi_k + h_k\}) &= \Psi(\{\phi_k\}) + i\sum h_k\widehat{\pi}_k\Psi(\{\phi_k\}) \\ &= \Psi(\{\phi_k\}) + ia\sum h(x_k)\widehat{\pi}(x_k)\Psi(\{\phi_k\})\end{aligned}$$

Compare to

$$\Psi(\phi(x) + h(x)) = \Psi(\phi(x)) + \int dx h(x)\frac{\delta\Psi(\phi)}{\delta\phi(x)}$$

$$\Rightarrow \quad \widehat{\pi}(x)\Psi(\phi) = -i\frac{\partial}{\partial\phi(x)}\Psi(\phi)$$

Check: CCR (canonical comm. relations)

$$[\widehat{\phi}(x), \widehat{\pi}(y)]\Psi(\phi) = -i\phi(x)\frac{\delta}{\delta\phi(y)}\Psi(\phi) +$$

$$i\frac{\delta}{\delta\phi(y)}(\phi(x)\Psi(\phi)) = i\Psi(\phi)\frac{\delta\phi(x)}{\delta\phi(y)} \ = \ i\delta(x-y)\Psi(\phi)$$

(For $F(\phi) = \phi(a)$ $F(\phi + h) = F(\phi) +$
$\int dy h(y)\delta(y-a) \Rightarrow \frac{\delta F(\phi)}{\delta\phi(y)} = \delta(y-a)$)

Stationary Schrodinger eqn $\hat{H}_0\Psi(\phi) = E_0\Psi(\phi)$:

$$\int dx\Big[\frac{\hat{\pi}^2(x)}{2}+\frac{\hat{\phi}'(x)^2}{2}+\frac{m^2}{2}\hat{\phi}^2(x)\Big]\Psi(\phi(x)) = E_0\Psi(\phi(x))$$

In the "coordinate basis"

$$\left.\begin{array}{r}\hat{\phi}(x)\Psi(\phi(x)) = \phi(x)\Psi(\phi(x)) \\ \hat{\pi}(x)\Psi(\phi(x)) = -i\frac{\delta}{\delta\phi(x)}\Psi(\phi(x))\end{array}\right\} \Rightarrow$$

$$\frac{1}{2}\int dx\Big[-(\frac{\delta}{\delta\phi(x)})^2+\phi'^2(x)+m^2\phi^2(x)\Big]\Psi(\phi(x)) = E_0\Psi(\phi(x))$$

$$\begin{array}{lrcl}\text{Solution :} & \Psi(\phi(x)) & = & e^{-\frac{1}{2}\int dx\ \phi(x)\hat{W}\phi(x)} \\ \text{Vacuum energy :} & E_0 & = & L^3\frac{1}{2}\int đp\ E_p\end{array}$$

Here

$$\begin{array}{rcl}\hat{W}\phi(x) & \equiv & \int dy W(x-y)\phi(y) \\ W(x-y) & \equiv & \int đp E_p e^{-iE_p(x-y)}\end{array}$$

The operator W is the analog of ω in the $e^{-\frac{\omega}{2}\phi^2}$ solution for the harmonic oscillator.

In the momentum space

$$\Psi(\phi(x)) = e^{-\frac{1}{2}\int đp\ E_p\phi(p)\phi(-p)}$$

Self-interacting Klein-Gordon field

$$L = \int d^3x \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

$$H = \int d^3x \Big[\frac{\pi^2(\vec{x},t)}{2} + \frac{\vec{\nabla}\phi(\vec{x},t)^2}{2} + \frac{m^2}{2}\phi^2(\vec{x},t) + \frac{\lambda}{4!}\phi^4\Big]$$

Quantization - same as for the free KG field:
$\phi \to \widehat{\phi}(\vec{x})$, $\pi \to \widehat{\pi}(\vec{x})$

$$[\widehat{\phi}(\vec{x}), \widehat{\pi}(\vec{y})] = i\delta^3(\vec{x}-\vec{y}), \qquad [\widehat{\phi}, \widehat{\phi}] = [\widehat{\pi}, \widehat{\pi}] = 0$$

$$\widehat{H} = \int d^3x \Big[\frac{\widehat{\pi}^2(\vec{x})}{2} + \frac{\vec{\nabla}\widehat{\phi}(\vec{x})^2}{2} + \frac{m^2}{2}\widehat{\phi}^2(\vec{x}) + \frac{\lambda}{4!}\widehat{\phi}^4\Big]$$

Heisenberg picture: $\widehat{\phi}(x)$ and $\widehat{\pi}(x)$ depend on time

$$\widehat{\phi}(x) \equiv \widehat{\phi}(\vec{x},t) = e^{i\widehat{H}t}\widehat{\phi}(\vec{x})e^{-i\widehat{H}t}$$
$$\widehat{\pi}(x) \equiv \widehat{\pi}(\vec{x},t) = e^{i\widehat{H}t}\widehat{\pi}(\vec{x})e^{-i\widehat{H}t}$$

Vectors of state (like the vector of ground state $|\Omega\rangle$) do not depend on time.

Heisenberg equations for the self-interacting Klein-Gordon field

$$[\phi(\vec{x},t),\pi(\vec{y},t)] = e^{i\hat{H}t}[\phi(\vec{x}),\pi(\vec{y})]e^{-i\hat{H}t} \ \Rightarrow$$

$$[\phi(\vec{x},t),\pi(\vec{y},t)] = i\delta^3(\vec{x}-\vec{y}) \quad \text{for any t}$$

- equal-time commutation relations.

$$\int d^3x\Big[\frac{\hat{\pi}^2(\vec{x},t)}{2} + \frac{\vec{\nabla}\hat{\phi}(\vec{x},t)^2}{2} + \frac{m^2}{2}\hat{\phi}^2(\vec{x},t) + \frac{\lambda}{4!}\hat{\phi}^4(\vec{x},t)\Big] =$$

$$\int d^3x e^{i\hat{H}t}\Big[\frac{\hat{\pi}^2(\vec{x})}{2} + \frac{\vec{\nabla}\hat{\phi}(\vec{x})^2}{2} + \frac{m^2}{2}\hat{\phi}^2(\vec{x}) + \frac{\lambda}{4!}\hat{\phi}^4(\vec{x})\Big]e^{-i\hat{H}t}$$

$$= \ e^{i\hat{H}t}\hat{H}e^{-i\hat{H}t} = \hat{H}$$

$$\frac{\partial}{\partial t}\hat{\phi}(\vec{x},t) = \frac{\partial}{\partial t}e^{i\hat{H}t}\hat{\phi}(\vec{x})e^{-i\hat{H}t} = i[\hat{H},\hat{\phi}(\vec{x},t)] =$$

$$i\int d^3x'\Big[\frac{\hat{\pi}^2(\vec{x}',t)}{2} + \frac{\vec{\nabla}\hat{\phi}(\vec{x}',t)^2}{2} + \frac{m^2}{2}\hat{\phi}^2(\vec{x}',t) + \frac{\lambda}{4!}\hat{\phi}^4(\vec{x}',t), \ \hat{\phi}(\vec{x},t)\Big] \overset{\text{equal-timeCCR}}{=}$$

$$\int d^3x'\hat{\pi}(\vec{x}',t)\delta(\vec{x}-\vec{x}') = \hat{\pi}(\vec{x},t)$$

$$\frac{\partial}{\partial t}\widehat{\pi}(\vec{x},t) = \frac{\partial}{\partial t}e^{i\widehat{H}t}\widehat{\pi}(\vec{x})e^{-i\widehat{H}t} = i[\widehat{H},\widehat{\pi}(\vec{x},t)] =$$

$$i\int d^3x'\Big[\frac{\widehat{\pi}^2(\vec{x}',t)}{2} + \frac{\vec{\nabla}\widehat{\phi}(\vec{x}',t)^2}{2} + \frac{m^2}{2}\widehat{\phi}^2(\vec{x}',t) + \frac{\lambda}{4!}\widehat{\phi}^4(\vec{x}',t),\ \widehat{\pi}(\vec{x},t)\Big] \overset{\text{equal−time CCR}}{=}$$

$$i\int d^3x'\delta(\vec{x}-\vec{x}')\{(\vec{\nabla}^2 - m^2)\widehat{\phi}(\vec{x}',t) - \frac{\lambda}{6}\widehat{\phi}^3(\vec{x}',t)\}$$

$$= (\vec{\nabla}^2 - m^2)\widehat{\phi}(\vec{x},t) - \frac{\lambda}{6}\widehat{\phi}^3(\vec{x},t)\widehat{\pi}(\vec{x},t)$$

$\Rightarrow$ Heisenberg equations are

$$\left.\begin{array}{lcl} \frac{\partial}{\partial t}\widehat{\phi}(\vec{x},t) & = & \widehat{\pi}(\vec{x},t) \\ \frac{\partial}{\partial t}\widehat{\pi}(\vec{x},t) & = & (\vec{\nabla}^2 - m^2)\widehat{\phi}(\vec{x},t) - \frac{\lambda}{6}\widehat{\phi}^3(\vec{x},t) \end{array}\right\} \Rightarrow$$

$$\frac{\partial^2}{\partial t^2}\widehat{\phi}(\vec{x},t) = (\vec{\nabla}^2 - m^2)\widehat{\phi}(\vec{x},t) - \widehat{\phi}^3(\vec{x},t) \Rightarrow$$

$$(\partial^2 + m^2)\widehat{\phi}(x) = -\frac{\lambda}{6}\widehat{\phi}^3(x)$$

Heisenberg operator $\widehat{\phi}(x)$ satisfies the same KG equation as the classical field $\phi(x)$.

Cross sections are determined by Green functions $\langle\Omega|T\{\widehat{\phi}(x_1)\widehat{\phi}(x_2)...\widehat{\phi}(x_n)\}|\Omega|\rangle$

$$S(p_1,p_1' \to p_2,p_2') = 1 + \delta(\sum p_1 - \sum p_2) T(p_1,p_1' \to p_2,p_2')$$

LSZ theorem:

$$S(p_1,p_1' \to p_2,p_2') = \lim_{p_i^2 \to m^2} \Pi(p_i^2 - m^2) \int d^4x_1 d^4x_1' d^4x_2 d^4x_2' e^{-ip_1x_1 - ip_1'x_1' + ip_2x_2 + ip_2'x_2'} \langle\Omega|T\{\widehat{\phi}(x_1)\widehat{\phi}(x_1')\widehat{\phi}(x_2)\widehat{\phi}(x_2')\}|\Omega\rangle$$

Perturbation theory: $\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$

$$\hat{H}_0 = \int d^3x \left[\frac{\hat{\pi}^2(\vec{x})}{2} + \frac{\vec{\nabla}\hat{\phi}(\vec{x})^2}{2} + \frac{m^2}{2}\hat{\phi}^2(\vec{x})\right]$$
$$\hat{H}_{\text{int}} = \int d^3x \frac{\lambda}{4!}\hat{\phi}^4(\vec{x})$$

Interaction representation defined as in QM

$$\hat{\phi}_I(\vec{x},t) \equiv e^{i\hat{H}_0 t}\hat{\phi}(\vec{x})e^{-i\hat{H}_0 t}, \;\; \hat{\pi}_I(\vec{x},t) \equiv e^{i\hat{H}_0 t}\hat{\pi}(\vec{x})e^{-i\hat{H}_0 t}$$

Literally repeating all the steps we get

$$\langle\Omega|\hat{\phi}(x_1)...\hat{\phi}(x_n)|\Omega\rangle = \frac{\langle 0|T\{\hat{\phi}_I(x_1)...\hat{\phi}_I(x_n)e^{-i\int_{-\infty}^{\infty}dt\hat{H}_I(t)}\}|\emptyset\rangle}{\langle\emptyset|T\{e^{-i\int_{-\infty}^{\infty}dt\hat{H}_I(t)}\}|\emptyset\rangle}$$

where $|\emptyset\rangle$ is the "perturbative vacuum" ($\equiv$ ground state of $\hat{H}_0$).

$\int_{-\infty}^{\infty}dt\hat{H}_I(t) = -\int d^4x\hat{\mathcal{L}}_I(x) \quad \Rightarrow$

$$\langle\Omega|\hat{\phi}(x_1)...\hat{\phi}(x_n)|\Omega\rangle = \frac{\langle\emptyset|T\{\hat{\phi}_I(x_1)...\hat{\phi}_I(x_n)e^{i\int d^4x\hat{\mathcal{L}}_I(x)}\}|\emptyset\rangle}{\langle\emptyset|T\{e^{i\int d^4x\hat{\mathcal{L}}_I(x)}\}|\emptyset\rangle}$$

This master formula is relativistic invariant (although the intermediate steps were not).

Cross section in the first order in λ.

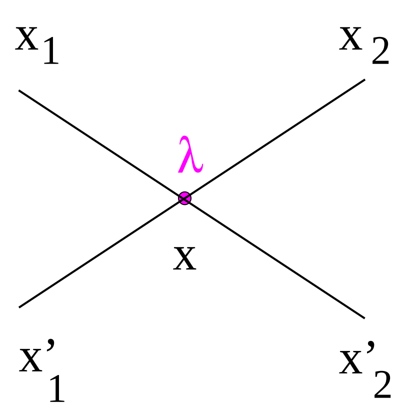
$$i\frac{\lambda}{4!}\langle \emptyset | T\{\hat{\phi}_I(x_1)\hat{\phi}_I(x'_1)\hat{\phi}_I(x_1)\hat{\phi}_I(x'_2)\int d^4x \hat{\phi}_I^4(x)\}| \emptyset\rangle$$
$$= i\frac{\lambda}{4!}\sum_{\text{contractions}}$$

$$\hat{\phi}_I(x_1)\hat{\phi}_I(x'_1)\hat{\phi}_I(x_1)\hat{\phi}_I(x'_2)\int d^4x \hat{\phi}_I(x)\hat{\phi}_I(x)\hat{\phi}_I(x)\hat{\phi}_I(x)$$

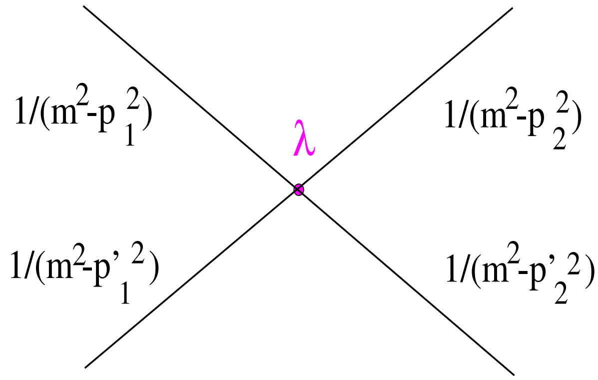
Each contraction is a free propagator

$$\hat{\phi}_I(x_1)\hat{\phi}_I(x) = G_0(x_1 - x)$$

$\Rightarrow$ Feynman diagram for the four-point Green function



Feynman diagram in the momentum representation



$$G(p_1, p_1' \to p_2, p_2') =$$
$$\int d^4x_1 d^4x_1' d^4x_2 d^4x_2' e^{-ip_1x_1 - ip_1'x_1' + ip_2x_2 + ip_2'x_2'}$$
$$\langle \Omega | T\{\hat{\phi}(x_1)\hat{\phi}(x_1')\hat{\phi}(x_2)\hat{\phi}(x_2')\} | \Omega \rangle =$$
$$\frac{-i\lambda(2\pi)^4 \delta(p_1 + p_1' - p_2 - p_2')}{(m^2 - p_1^2)(m^2 - p'^2_1)(m^2 - p_2^2)(m^2 - p'^2_2)} + O(\lambda^2)$$

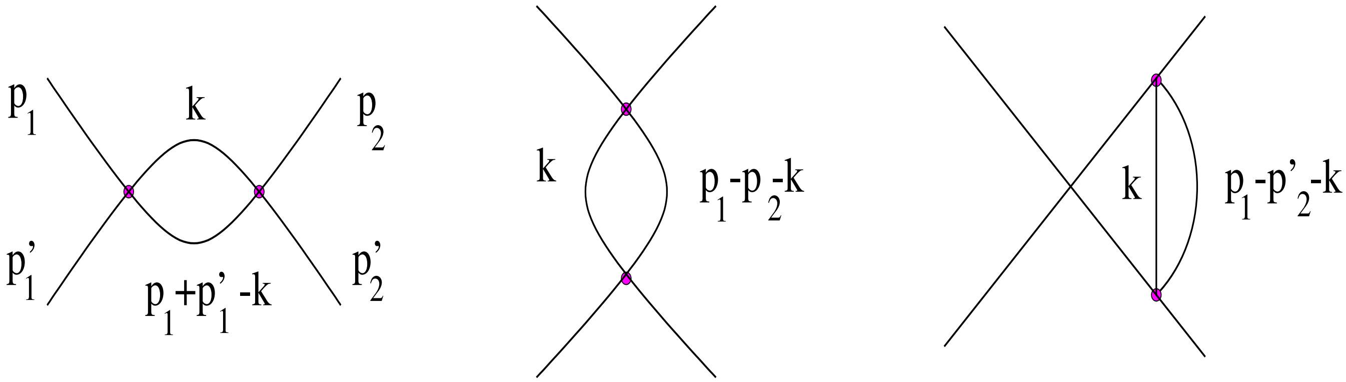
LSZ theorem $\Rightarrow$

$$T(p_1, p_1' \to p_2, p_2') = -\lambda + O(\lambda^2) \Rightarrow$$

The meson-meson cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\lambda^2}{64\pi^2 s} + O(\lambda^4)$$

In higher orders in perturbative expansion in powers of λ we have more complicated diagrams such as

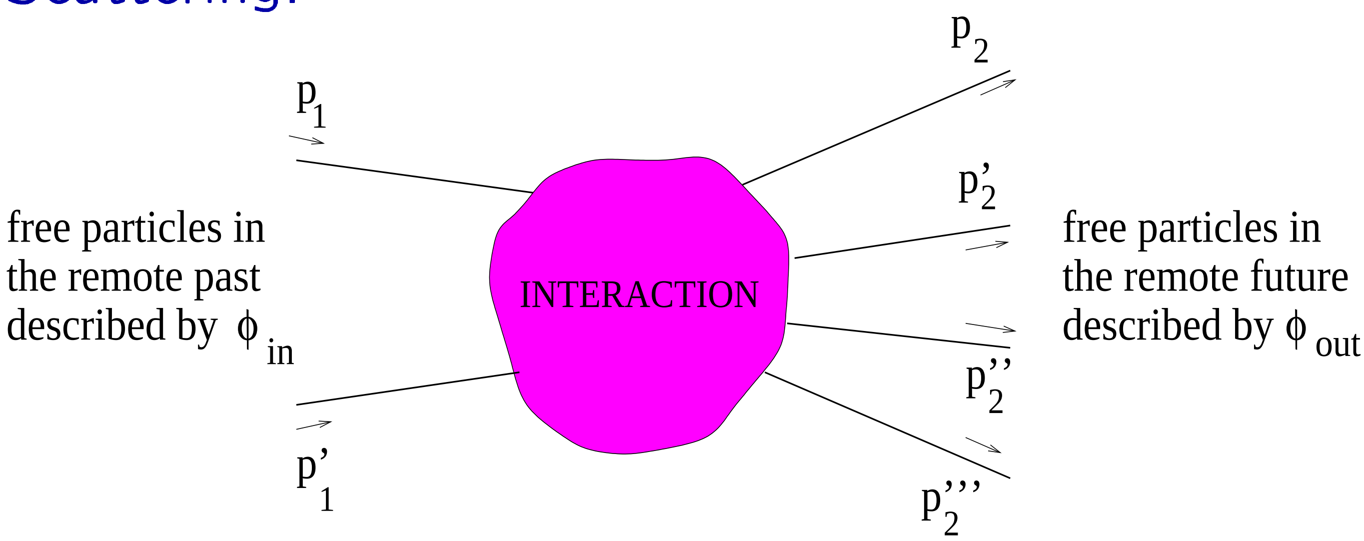


Feynman rules for the Green functions in momentum representation:

- $\frac{1}{m^2-p^2-i\epsilon}$ for each propagator with momentum p.
- $-\lambda$ for each vertex
- $\int \frac{d^4k}{16\pi^4 i}$ for each "loop" momentum k.

LSZ reduction formula

Scattering:



$$\hat{\phi}_{\text{in}}(x) \equiv \hat{\phi}(x) + i\frac{\lambda}{3!}\int d^4z G^0_R(x-z)\hat{\phi}^3(z)$$

$G^0_R(x-y)$ - "retarded" Green function

$$G^0_R(x-y) \equiv \int \frac{d\!\!{}^-{}^4p}{i}\frac{e^{-ip(x-y)}}{m^2-p^2+i\epsilon p_0}$$

$$(\partial^2+m^2)G^0_R(x-y) = -i\delta^{(4)}(x-y);$$
$$G^0_R(x-y) = 0 \text{ if } x_0 < y_0$$

$$(\partial^2+m^2)\hat{\phi}_{\text{in}}(x) = 0$$

$\Rightarrow$ ϕ_{in} describes free particles at $t \to -\infty$

$$\hat\phi_{\rm out}(x) \;\equiv\; \hat\phi(x) + i\frac{\lambda}{3!}\int d^4z G^0_A(x-z)\hat\phi^3(z)$$

$G^0_R(x-y)$ - "advanced" Green function

$$G^0_A(x-y) \;\equiv\; \int \frac{d\!\!\!{}^-{}^4p}{i}\frac{e^{-ip(x-y)}}{m^2-p^2-i\epsilon p_0}$$

$(\partial^2+m^2)G^0_A(x-y) = -i\delta^{(4)}(x-y)$;
$G^0_A(x-y) = 0$ if $x_0 > y_0$

$$(\partial^2+m^2)\hat\phi_{\rm out}(x) \;=\; 0$$

$\Rightarrow$ $\phi_{\rm out}$ describes free particles at $t\to\infty$
Ladder operators:

$$\hat\phi_{\rm in} = \int \frac{d\!\!\!{}^-{}^3p}{\sqrt{2E_p}}(\hat a_{\rm in}(p)e^{-ipx} + \hat a^\dagger_{\rm in}(p)e^{ipx})\Big|_{p_0=E_p}$$

$[\hat a_{\rm in}(p), \hat a^\dagger_{\rm in}(p')] = (2\pi)^3\delta(\vec p-\vec p'),$
$\hat a_{\rm in}(p)|0_{\rm in}\rangle \;= 0 \qquad |0_{\rm in}\rangle \;\equiv$ ground state of $\hat H^0_{\rm in}$.

$$\hat\phi_{\rm out} = \int \frac{d\!\!\!{}^-{}^3p}{\sqrt{2E_p}}(\hat a_{\rm out}(p)e^{-ipx} + \hat a^\dagger_{\rm out}(p)e^{ipx})\Big|_{p_0=E_p}$$

$[\hat a_{\rm out}(p), \hat a^\dagger_{\rm out}(p')] = (2\pi)^3\delta(\vec p-\vec p'),$
$\hat a_{\rm out}|0_{\rm out}\rangle \;= 0 \;\; |0_{\rm out}\rangle \;\equiv$ ground state of $\hat H^0_{\rm out}$.

Hypothesis: $|0_{\text{in}}\rangle = |0_{\text{out}}\rangle = |\Omega\rangle$

"In" and "out" states:

$$|p_1, ...p_n\rangle_{\text{in}} \equiv \Pi\sqrt{2E_{p_k}}a^\dagger_{p_k}|\emptyset\rangle_{\text{in}}$$
$$|p_1, ...p_n\rangle_{\text{out}} \equiv \Pi\sqrt{2E_{p_k}}a^\dagger_{p_k}|\emptyset\rangle_{\text{out}}$$

The amplitude of the $m \to n$ transition is given by

$$S(p_1, p_1', ...p_1^{(m)} \to p_2, p_2', ...p_2^{(n)})$$
$$= {}_{\text{out}}\langle p_2, p_2', ...p_2^{(n)}|p_1, p_1', ..., p_1^{(m)}\rangle_{\text{in}}$$

- matrix element of the S-matrix.

LSZ reduction formula (for 2→2 scattering)

$$S(p_1, p_1' \to p_2, p_2') =$$
$$i^4 \lim_{p_i^2 \to m^2} (m^2 - p_1^2)(m^2 - p_2^2)(m^2 - p'^2_1)(m^2 - p'^2_2)$$
$$\times \int dx dx' dy dy' e^{-ip_1x_1 - ip_1'x' + ip_2y + ip_2'y'}$$
$$\times \langle\Omega|T\{\hat{\phi}(x)\hat{\phi}(x')\hat{\phi}(y)\hat{\phi}(y')\}|\Omega\rangle$$

$$\hat{\phi}_{\text{in}}(x) \equiv \hat{\phi}(x) + i\int d^4z G_R^0(x-z)\hat{\phi}^3(z)$$

$$\Rightarrow \hat{\phi}(x) \stackrel{t\to-\infty}{\to} \hat{\phi}_{\text{in}}(x).$$

Similarly,

$$\hat{\phi}_{\text{out}}(x) \equiv \hat{\phi}(x) + i\int d^4z G_A^0(x-z)\hat{\phi}^3(z)$$

$$\Rightarrow \hat{\phi}(x) \stackrel{t\to\infty}{\to} \hat{\phi}_{\text{out}}(x).$$

Proof of the LSZ.

For any free KG field

$$\sqrt{2E_p}\hat{a}(p) = i\int d^3x e^{-i\vec{p}\vec{x}+iE_pt} \overleftrightarrow{\partial_0} \hat{\phi}(x)$$

$$\sqrt{2E_p}\hat{a}^\dagger(p) = -i\int d^3x e^{i\vec{p}\vec{x}-iE_pt} \overleftrightarrow{\partial_0} \hat{\phi}(x) \Rightarrow$$

$$_{\text{out}}\langle p_2, p_2'|p_1, p_1'\rangle_{\text{in}} = {}_{\text{out}}\langle p_2, p_2'|\hat{a}^\dagger_{\text{in}}(p_1)|p_1'\rangle_{\text{in}}\sqrt{2E_1} =$$
$$_{\text{out}}\langle p_2, p_2'|\hat{a}^\dagger_{\text{out}}(p_1) + (\hat{a}^\dagger_{\text{in}}(p_1) - \hat{a}^\dagger_{\text{out}}(p_1))|p_1'\rangle_{\text{in}}\sqrt{2E_1} =$$
$$_{\text{out}}\langle p_2, p_2'|\int d^3x e^{i\vec{p}_1\vec{x}-iE_1t} i \overleftrightarrow{\partial_0} (\hat{\phi}_{\text{out}}(x) - \hat{\phi}_{\text{in}}(x))|p_1'\rangle_{\text{in}}$$

The l.h.s. does not depend on $t \Rightarrow$

$$ {}_{\text{out}}\langle p_2, p_2'| \int d^3x e^{i\vec{p}_1\vec{x} - iE_1 t} i \overset{\leftrightarrow}{\partial_0} \hat{\phi}_{\text{out}}(x) |p_1'\rangle_{\text{in}} $$

$$ = \text{ take } t \to \infty = $$

$$ {}_{\text{out}}\langle p_2, p_2'| \int d^3x e^{i\vec{p}_1\vec{x} - iE_1 t} i \overset{\leftrightarrow}{\partial_0} \hat{\phi}(x) |p_1'\rangle_{\text{in}} \Big|_{t=\infty} $$

$$ {}_{\text{out}}\langle p_2, p_2'| \int d^3x e^{i\vec{p}_1\vec{x} - iE_1 t} i \overset{\leftrightarrow}{\partial_0} \hat{\phi}_{\text{in}}(x) |p_1'\rangle_{\text{in}} $$

$$ = \text{ take } t \to -\infty = $$

$$ {}_{\text{out}}\langle p_2, p_2'| \int d^3x e^{i\vec{p}_1\vec{x} - iE_1 t} i \overset{\leftrightarrow}{\partial_0} \hat{\phi}(x) |p_1'\rangle_{\text{in}} \Big|_{t=-\infty} $$

Using the formula

$$ \int d^3x g_1(t, \vec{x}) \overset{\leftrightarrow}{\partial_0} g_2(t, \vec{x}) \Big|_{t=-\infty}^{t=\infty} $$

$$ = \int d^4x [g_1(x) \frac{\partial^2}{\partial t^2} g_2(x) - g_2(x) \frac{\partial^2}{\partial t^2} g_1(x)] $$

we get ($g_1(x) \equiv e^{i\vec{p}_1\vec{x} - iE_1 t}$, $g_2 \equiv \hat{\phi}(x)$)

$$ {}_{\text{out}}\langle p_2, p_2'|p_1, p_1'\rangle_{\text{in}} = $$

$$ {}_{\text{out}}\langle p_2, p_2'| i \int d^4x e^{-ip_1 x} (E_1^2 + \partial_0^2) \hat{\phi}(x) |p_1'\rangle_{\text{in}} = $$

$$ \lim_{p_1^2 \to m^2} (m^2 - p_1^2) i \int d^4x e^{-ip_1 x} {}_{\text{out}}\langle p_2, p_2'|\hat{\phi}(x)|p_1'\rangle_{\text{in}} $$

$${}_{\text{out}}\langle p_2, p_2' | \widehat{\phi}(x) | p_1' \rangle_{\text{in}} =$$

$${}_{\text{out}}\langle p_2' | a_{\text{out}}(p_2)\widehat{\phi}(x) - \widehat{\phi}(x) a_{\text{in}}(p_2) \sqrt{2E_2} =$$

$$i \int d^3y e^{-i\vec{p}_2\vec{y} + iE_2 t} \langle p_2' | \widehat{\phi}_{\text{out}}(t, \vec{y}) \widehat{\phi}(x) | p_1' \rangle_{\text{in}}$$

$$-i \int d^3y e^{-i\vec{p}_2\vec{y} + iE_2 t} {}_{\text{out}}\langle p_2' | \widehat{\phi}(x) \widehat{\phi}_{\text{in}}(t, \vec{y}) | p_1' \rangle_{\text{in}}$$

$$= i \int d^3y e^{-i\vec{p}_2\vec{y} + iE_2 t} {}_{\text{out}}\langle p_2' | \widehat{\phi}(t, \vec{y}) \widehat{\phi}(x) | p_1' \rangle_{\text{in}} \Big|_{t=\infty}$$

$$- i \int d^3y e^{-i\vec{p}_2\vec{y} + iE_2 t} {}_{\text{out}}\langle p_2' | \widehat{\phi}(x) \widehat{\phi}(t, \vec{y}) | p_1' \rangle_{\text{in}} \Big|_{t=-\infty}$$

$$= i \int d^3y e^{-i\vec{p}_2\vec{y} + iE_2 t} {}_{\text{out}}\langle p_2' | T\{\widehat{\phi}(x) \widehat{\phi}(t, \vec{y})\} | p_1' \rangle_{\text{in}} \Big|_{t=-\infty}^{t=\infty}$$

$$g_1(y) \equiv e^{-i\vec{p}_2\vec{y} + iE_2 t}, \; g_2(y) \equiv T\{\widehat{\phi}(x)\widehat{\phi}(t, \vec{y})\} \;\Rightarrow$$

$${}_{\text{out}}\langle p_2, p_2' | \widehat{\phi}(x) | p_1' \rangle_{\text{in}}$$

$$= i \int d^4y e^{ip_2 y} (E_2^2 + \partial_0^2) \langle p_2' | T\{\widehat{\phi}(y) \widehat{\phi}(x) | p_1' \rangle_{\text{in}}$$

$$= \lim_{p_2^2 \to m^2} (m^2 - p_2^2) i \int d^4y e^{ip_2 y} {}_{\text{out}}\langle p_2' | T\{\widehat{\phi}(y) \widehat{\phi}(x) | p_1' \rangle_{\text{in}}$$

$${}_{\text{out}}\langle p_2, p_2' | p_1, p_1' \rangle_{\text{in}} = \lim_{p_1^2, p_2^2 \to m^2} (m^2 - p_1^2)(m^2 - p_2^2)$$

$$i^2 \int d^4x d^4y e^{-ip_1 x + ip_2 y} {}_{\text{out}}\langle p_2' | T\{\widehat{\phi}(x) \widehat{\phi}(y)\} | p_1' \rangle_{\text{in}}$$

Repeating this trick two more times, we get

$$S(p_1, p'_1 \to p_2, p'_2) =$$
$$i^4 \lim_{p_i^2 \to m^2} (m^2 - p_1^2)(m^2 - p_2^2)(m^2 - p'^2_1)(m^2 - p'^2_2)$$
$$\times \int dx dx' dy dy' e^{-ip_1 x - ip'_1 x' + ip_2 y + ip'_2 y'}$$
$$\times \ \langle \Omega | T\{\hat{\phi}(x)\hat{\phi}(x')\hat{\phi}(y)\hat{\phi}(y')\} | \Omega \rangle$$

- LSZ reduction formula for 2→2 scattering

In the general case,

$$S(p_1, p'_1, ...p_1^{(m)} \to p_2, p'_2, ...p_2^{(n)})$$
$$= \ {}_{\text{out}}\langle p_2, p'_2, ...p_2^{(n)} | p_1, p'_1, ..., p_1^{(m)} \rangle_{\text{in}}$$

$$= \ i^{m+n} \lim_{p_i^2 \to m^2} \Pi (m^2 - p_i^2)$$
$$\times \int \Pi dx_1^{(i)} \Pi dx_2^{(j)} e^{-i \sum p_1^{(i)} x_1^{(i)} + i \sum p_2^{(j)} x_2^{(j)}}$$
$$\times \ \langle \Omega | T\{\hat{\phi}(x_1)...\hat{\phi}(x_1^{(m)})\hat{\phi}(x_2)...\hat{\phi}(x_2^{(n)})\} | \Omega \rangle$$

Path integrals in quantum mechanics

Anharmonic oscillator

$$\hat{H} = \hat{H}_0 + \hat{V}, \quad \hat{H}_0 = \frac{\hat{\pi}^2}{2}, \hat{V} = \frac{\omega^2}{2}\hat{\phi}^2 + \frac{\lambda}{4!}\hat{\phi}^4$$

(note that here $\hat{H}_0$ does not include $\frac{\omega^2}{2}\hat{\phi}^2$).
Wave function in Dirac notations is $\Psi(\phi, t) = \langle\phi|\Psi(t)\rangle$
where $|\phi\rangle$- eigenstates of the coordinate operator $\hat{\phi}$: $\quad \hat{\phi}|\phi\rangle = \phi|\phi\rangle$.
Evolution is described by the operator $e^{-i\hat{H}t}$:

$$|\Psi(t_f)\rangle = e^{-i\hat{H}(t_f - t_i)}|\Psi(t_i)\rangle$$

In terms of wave functions

$$\Psi(\phi_f, t_f) = \langle\phi_f|\Psi(t)\rangle = \int d\phi_i \langle\phi_f|e^{-i\hat{H}(t_f-t_i)}|\phi_i\rangle\Psi(\phi_i, t_i)$$

$\Rightarrow K(\phi_f, t_f; \phi_i, t_i) = \langle\phi_f|e^{-i\hat{H}(t_f-t_i)}|\phi_i\rangle$ -
kernel of the evolution operator (propagation amplitude).
Physical meaning: we release the oscillating particle in the point $\phi = \phi_i$ at time $t = t_i$ and $K(\phi_f, t_f; \phi_i, t_i)$ is a probabilty to find this particle in the point $x = \phi_f$ at later time $t = t_f$.

Path integral for the evolution kernel

Idea: insert $1 = \int d\phi |\phi\rangle\langle\phi|$ n times

$$\langle\phi_f|e^{-i\hat{H}(t_f-t_i)}|\phi_i\rangle =$$

$$\int d\phi_1 ... \int d\phi_n \langle\phi_f|e^{-i\hat{H}\Delta t}|\phi_n\rangle ... \langle\phi_2|e^{-i\hat{H}\Delta t}|\phi_1\rangle\langle\phi_1|e^{-i\hat{H}\Delta t}|\phi_i\rangle$$

where $\Delta t = \frac{t_f - t_i)}{n+1}$.
For small Δt

$$\langle\phi_k|e^{-i\hat{H}\Delta t}|\phi_{k-1}\rangle =$$

$$\int \frac{dp}{2\pi}\langle\phi_k|p\rangle\langle p|e^{-i\hat{H}\Delta t}|\phi_{k-1}\rangle =$$

$$\int \frac{dp}{2\pi} e^{ip(\phi_k-\phi_{k-1})} e^{-i\frac{p^2}{2}\Delta t - iV(\phi_k)\Delta t} =$$

$$\frac{1}{\sqrt{2\pi i\Delta t}} e^{i\frac{(\phi_k-\phi_{k-1})^2}{2\Delta t} - iV(\phi_k)\Delta t}$$

$$\langle\phi_f|e^{-i\hat{H}(t_f-t_i)}|\phi_i\rangle =$$

$$\left(\frac{1}{2\pi i\Delta t}\right)^{\frac{n+1}{2}} \int d\phi_1 ... \int d\phi_n e^{i\Delta t\sum\left[\frac{(\phi_k-\phi_{k-1})^2}{2(\Delta t)^2} - V(\phi_k)\right]}$$

$$\Rightarrow N^{-1}\int D\phi(t) e^{i\int_{t_i}^{t_f} dt\left(\frac{\dot{\phi}^2}{2} - V(\phi)\right)}$$

Exercize: propagation amplitude of a free non-relativistic particle (with mass = 1)

$$\langle\phi_f|e^{-i\hat{H}_0(t_f-t_i)}|\phi_i\rangle =$$

$$\left(\frac{1}{2\pi i\Delta t}\right)^{\frac{n+1}{2}}\int d\phi_1 \ldots \int d\phi_n e^{i\Delta t\sum_{k=1}^{n+1}\frac{(\phi_k-\phi_{k-1})^2}{2(\Delta t)^2}}$$

$$\frac{1}{\sqrt{2\pi i\Delta t}}\int d\phi_1 \exp\left[\frac{i}{2\Delta t}(\phi_2-\phi_1)^2+\frac{i}{2\Delta t}(\phi_1-\phi_i)^2\right]$$
$$=\frac{1}{\sqrt{2}}\exp\left[\frac{i}{4\Delta t}(\phi_2-\phi_i)^2\right],$$

$$\frac{1}{\sqrt{2\pi i\Delta t}}\frac{1}{\sqrt{2}}\int d\phi_2 \exp\left[\frac{i}{2\Delta t}(\phi_3-\phi_2)^2+\frac{i}{4\Delta t}(\phi_2-\phi_i)^2\right]$$
$$=\frac{1}{\sqrt{3}}\exp\left[\frac{i}{6\Delta t}(\phi_3-\phi_i)^2\right], \ldots$$

$$\frac{1}{\sqrt{2\pi i\Delta t}}\frac{1}{\sqrt{n}}\int d\phi_n \exp\left[\frac{i}{2\Delta t}(\phi_f-\phi_n)^2+\frac{i}{2n\Delta t}(\phi_n-\phi_i)^2\right]$$
$$=\frac{1}{\sqrt{n+1}}\exp\left[\frac{i(\phi_f-\phi_i)^2}{2(n+1)\Delta t}\right]$$

$$\Rightarrow \langle\phi_f|e^{-i\hat{H}_0(t_f-t_i)}|\phi_i\rangle = \frac{1}{\sqrt{2\pi i(t_f-t_i)}}e^{\frac{i(\phi_f-\phi_i)^2}{2(t_f-t_i)}}$$

Restore $\hbar$ for a minute

$$K(\phi_f, t_f; \phi_i, t_i) = N^{-1} \int_{\phi(t_i)=\phi_i}^{\phi(t_f)=\phi_f} D\phi(t) e^{\frac{i}{\hbar}S(\phi(t))}$$

This formula can be used as a postulate of quantum mechanics instead of Schrodinger equation.

Classical limit.
At $\hbar \to 0$ (classical limit) this integral is determined by a stationary phase point corresponding to minimum of the action $S(\phi(t)) \Rightarrow$
least action principle - given the initial and final points, the classical path is a path with minimal action.

In quantum mechanics, all trajectories are possible. Each trial path is weighted with $e^{\frac{i}{\hbar}S}$ and we have to sum over trial configurations due to the superposition principle of quantum mechanics (for undistinguishable paths, we must sum the amplitudes).

Consider the two-point Green function

$$G(t_1,t_2) = \langle\Omega|T\{\hat{\phi}(t_1)\hat{\phi}(t_2)\}|\Omega\rangle$$

At first, we prove that ($t_{fi} \equiv t_f - t_i$)

$$G(t_1,t_2) =$$

$$\theta(t_{12}) \lim_{t_f\to\infty, t_i\to-\infty} \frac{\langle\emptyset|e^{-i\hat{H}t_{f1}}\hat{\phi}e^{-i\hat{H}t_{12}}\hat{\phi}e^{-i\hat{H}t_{2i}}\}|\emptyset\rangle}{\langle\emptyset|e^{-i\hat{H}t_{fi}}|\emptyset\rangle} + (t_1 \leftrightarrow t_2$$

Proof:

Take $t_1 > t_2$. Consider the numerator

$$\text{Num}(t_1,t_2) = \langle\emptyset|e^{-i\hat{H}t_{f1}}\hat{\phi}e^{-i\hat{H}t_{12}}\hat{\phi}e^{-i\hat{H}t_{2i}}|\emptyset\rangle$$

Recall that

$$\left.\begin{array}{c} e^{-i\hat{H}t_{2i}}|\emptyset\rangle \overset{t_i\to-\infty}{\to} e^{-iE_{\text{vac}}t_{2i}}|\Omega\rangle\langle\Omega|\emptyset\rangle \\ \langle\emptyset|e^{-i\hat{H}t_{f1}} \overset{t_f\to\infty}{\Rightarrow} \langle\emptyset|\Omega\rangle\langle\Omega|e^{-iE_{\text{vac}}t_{f1}} \end{array}\right\} \Rightarrow$$

$$\lim_{t_f\to\infty, t_i\to-\infty} \text{Num}(t_1,t_2) =$$

$$|\langle\emptyset|\Omega\rangle|^2 e^{-iE_{\text{vac}}(t_{f1}+t_{2i})}\langle\Omega|\hat{\phi}e^{i\hat{H}t_{12}}\hat{\phi}|\Omega\rangle$$

Similarly, the denominator is

$$\text{Den} = \langle \emptyset | e^{-i\hat{H}t_{fi}} | \emptyset \rangle \Rightarrow |\langle \emptyset | \Omega \rangle|^2 e^{-iE_{\text{vac}} t_{fi}}$$

$$\Rightarrow \frac{\text{Num}}{\text{Den}} = e^{iE_{\text{vac}} t_{12}} \langle \Omega | \hat{\phi} e^{-i\hat{H}t_{12}} \hat{\phi} | \Omega \rangle$$

On the other hand

$$\langle \Omega | e^{i\hat{H}t_1} \hat{\phi} e^{-i\hat{H}t_{12}} \hat{\phi} e^{-i\hat{H}t_2} | \Omega \rangle = e^{iE_{\text{vac}} t_{12}} \langle \Omega | \hat{\phi} e^{i\hat{H}t_{12}} \hat{\phi} | \Omega \rangle$$

$$\Rightarrow \lim_{t_f \to \infty, t_i \to -\infty} \frac{\text{Num}}{\text{Den}} = G(t_1, t_2)$$

Path integral for $G(t_1, t_2)$.

We know that

$$\text{Den} = \langle \emptyset | e^{-i\hat{H}t_{fi}} | \emptyset \rangle = \int_{\emptyset}^{\emptyset} D\phi \; e^{i \int_{t_i}^{t_f} L(t) dt}$$

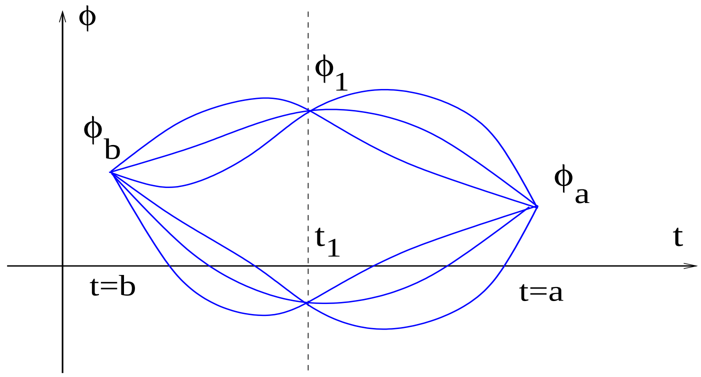
$$\equiv \int d\phi_f d\phi_i e^{-\frac{\omega}{2}\phi_f^2} e^{-\frac{\omega}{2}\phi_i^2} \int_{\phi(t_i)=\phi_i}^{\phi(t_f)=\phi_f} D\phi e^{iS_0}$$

The numerator

$$\text{Num} = \langle \emptyset | e^{-i\hat{H}t_{f1}} \hat{\phi} e^{-i\hat{H}t_{12}} \hat{\phi} e^{-i\hat{H}t_{2i}} | \emptyset \rangle =$$

$$\int d\phi_1 d\phi_2 \langle 0 | e^{-i\hat{H}t_{f1}} \hat{\phi} | \phi_1 \rangle \langle \phi_1 | e^{-i\hat{H}t_{12}} \hat{\phi} | \phi_2 \rangle \langle \phi_2 | e^{-i\hat{H}t_{2i}} | \emptyset \rangle$$

$$\int d\phi_1 d\phi_2 \langle \emptyset | e^{-i\hat{H}t_{f1}} | \phi_1 \rangle \phi_1 \langle \phi_1 | e^{-i\hat{H}t_{12}} | \phi_2 \rangle \phi_2 \langle \phi_2 | e^{-i\hat{H}t_{2i}} | \emptyset \rangle$$



Each path is weighted with $F(\phi_1) = F(\phi(t_1))$

Property

$$\int d\phi_1 \int_{\phi(t_1)=\phi_1}^{\phi(a)=\phi_a} D\phi e^{i\int_{t_1}^{a} L(t)dt} F(\phi_1)$$

$$\int_{\phi(b)=\phi_b}^{\phi(t_1)=\phi_1} D\phi e^{i\int_{b}^{t_1} L(t)dt}$$

$$= \int_{\phi(b)=\phi_b}^{\phi(a)=\phi_a} D\phi e^{i\int_{b}^{a} L(t)dt} F(\phi(t_1)) \qquad \Rightarrow$$

$$\text{Num} = \int_{\emptyset}^{\emptyset} D\phi\ \phi(t_1)\phi(t_2) e^{i\int_{t_i}^{t_f} L(t)dt}$$

$$\Rightarrow G(t_1, t_2) = \frac{\int_{\emptyset}^{\emptyset} D\phi\ \phi(t_1)\phi(t_2) e^{i\int_{-\infty}^{\infty} L(t)dt}}{\int_{\emptyset}^{\emptyset} D\phi\ e^{i\int_{-\infty}^{\infty} L(t)dt}}$$

In general

$$\langle\Omega|T\{\phi(t_1)...\phi(t_n)\}|\Omega\rangle = \frac{\int_{\emptyset}^{\emptyset} D\phi\ \phi(t_1)...\phi(t_n)\ e^{iS(\phi)}}{\int_{\emptyset}^{\emptyset} D\phi\ e^{iS(\phi)}}$$

Euclidean path integral

Analytic continuation of the evolution operator to the imaginary time $t = -i\tau$ gives

$$\langle\phi_f|e^{-\widehat{H}(\tau_f-\tau_i)}|\phi_i\rangle = N^{-1}\int D\phi(\tau)e^{-\int_{\tau_i}^{\tau_f} d\tau\left[\frac{\dot{\phi}^2(\tau)}{2}+V(\phi(\tau))\right]}$$

($\dot{\phi}(\tau)$ denotes now the derivative with respect to τ.)

The finite-sum version of this formula

$$\langle\phi_f|e^{-\widehat{H}(\tau_f-\tau_i)}|\phi_i\rangle =$$

$$\left(\frac{1}{2\pi\Delta\tau}\right)^{\frac{n+1}{2}}\int d\phi_1 ... \int d\phi_n e^{-\Delta\tau\sum\left[\frac{(\phi_k-\phi_{k-1})^2}{2(\Delta\tau)^2}+V(\phi_k)\right]}$$

is very convenient for practical calculations since the integrals of this type can be computed using the Monte-Carlo methods. This is the simplest example of a lattice calculation in a quantum theory.

Feynman diagams from path integrals

Consider the two-point Green function

$$\langle\Omega|T\{\phi(t_1)\phi(t_2)\}|\Omega\rangle = \frac{\int_{\emptyset}^{\emptyset} D\phi\ \phi(t_1)\phi(t_2)e^{iS(\phi)}}{\int_{\emptyset}^{\emptyset} D\phi\ e^{iS(\phi)}}$$

for the anharmonic oscillator.

At small λ

$$S = S_0 + S_{\text{int}}$$
$$S_0 = \int dt \Big[\frac{\dot{\phi}^2(t)}{2} - \frac{\omega^2}{2}\phi^2(t)\Big]$$
$$S_{\text{int}} = -\int dt \frac{\lambda}{4!}\phi^4(t)$$

Note that unlike p.39, we include $\frac{\omega^2}{2}\phi^2$ in S_0.

$$\langle\Omega|T\{\phi(t_1)\phi(t_2)\}|\Omega\rangle = \frac{\int D\phi\ \phi(t_1)\phi(t_2)e^{iS_0(\phi)+iS_{\text{int}}(\phi)}}{\int D\phi\ e^{iS_0(\phi)+iS_{\text{int}}(\phi)}}$$

Perturbative expansion $\Leftrightarrow$ expansion of $e^{iS_{\text{int}}}$ in the numerator and denominator.

In the first order in perturbation theory we get

$$-i\frac{\lambda}{4!}\frac{\int dt \int D\phi\ \phi(t_1)\phi(t_2)\phi^4(t)e^{iS_0}}{\int D\phi\ e^{iS_0}}$$
$$+\ i\frac{\lambda}{4!}\frac{\int D\phi\ \phi(t_1)\phi(t_2)e^{iS_0}}{\int D\phi e^{iS_0}}\ \ \frac{\int dt \int D\phi\ \phi^4(t)}{\int D\phi\ e^{iS_0}}$$

It is the same expansion as in interaction representation picture (see p.9):

$$\langle\Omega|T\{\hat{\phi}(t_1)\hat{\phi}(t_2)\}|\Omega\rangle = \langle\emptyset|T\{\hat{\phi}_I(t_1)\hat{\phi}_I(t_2)\}|\emptyset\rangle$$
$$-\ \ i\frac{\lambda}{4!}\int dt\Big[\langle\emptyset|T\{\hat{\phi}_I(t_1)\hat{\phi}_I(t_2)\hat{\phi}_I^4(t)\}|\emptyset\rangle$$
$$-\langle\emptyset|T\{\hat{\phi}_I(t_1)\hat{\phi}_I(t_2)\}|\emptyset\rangle\langle\emptyset|\hat{\phi}_I^4(t)|\emptyset\rangle\Big]$$

because

$$\frac{\int D\phi\ \phi(t_1)...\phi(t_n)e^{iS_0}}{\int D\phi\ e^{iS_0}} = \langle\emptyset|T\{\hat{\phi}_I(t_1)...\hat{\phi}_I(t_n)\}|\emptyset\rangle$$

Proof: this is simply the master formula for path-integral representation of Green functions (see p.32) applied for the unperturbed harmonic oscillator. (Recall that $\hat{\phi}_I$ is a Heisenberg operator for $\hat{H} = \hat{H}_0$).

Path-integral derivation of Wick's theorem

$$G_0(t_1, ...t_n) = \frac{\int_{\emptyset}^{\emptyset} D\phi \;\phi(t_1)...\phi(t_n) e^{iS_0}}{\int_{\emptyset}^{\emptyset} D\phi \; e^{iS_0}} = \sum_{\text{contractions}} =$$

$$G_0(t_1 - t_2) G_0(t_3 - t_4)...G_0(t_{n-1} - t_n) + \text{permutations}$$

Define the functional

$$Z(J) = \frac{\int_{\emptyset}^{\emptyset} D\phi \; e^{iS_0 + i\int dt J(t)\phi(t)}}{\int_{\emptyset}^{\emptyset} D\phi \; e^{iS_0}}$$

Expansion of this generating functional $Z(J)$ in powers of the source $J(t)$ generates the set of Green functions G_0:

$$Z(J) =$$

$$1 + i\int dt G_0(t) J(t) + \frac{i^2}{2}\int dt_1 dt_2 J(t_1) J(t_2) G_0(t_1, t_2)$$

$$+ \frac{i^3}{3!}\int dt_1 dt_2 dt_3 J(t_1) J(t_2) J(t_3) G_0(t_1, t_2, t_3) + ...$$

The generating functional $Z(J)$ is a gaussian integral (albeit a path one) so we can try to calculate by appropriate shift of variable.

Accurately

$Z(J) =$

$$\frac{\int d\phi_f d\phi_i e^{-\frac{\omega}{2}\phi_f^2} e^{-\frac{\omega}{2}\phi_i^2} \int_{\phi(t_i)=\phi_i}^{\phi(t_f)=\phi_f} D\phi e^{iS_0 + i\int_{t_i}^{t_f} dt J(t)\phi(t)}}{\int d\phi_f d\phi_i e^{-\frac{\omega}{2}\phi_f^2} e^{-\frac{\omega}{2}\phi_i^2} \int_{\phi(t_i)=\phi_i}^{\phi(t_f)=\phi_f} D\phi e^{iS_0}}$$

Let us make a shift $\phi(t) \to \phi(t) + \bar{\phi}(t)$ in the path integral in the numerator. The exponential in the numerator will turn to

$$-\frac{\omega}{2}(\phi_f + \bar{\phi}_f)^2 - \frac{\omega}{2}(\phi_i + \bar{\phi}_i)^2 + iS_0(\phi + \bar{\phi}) + i\int_{t_i}^{t_f} dt J(t)(\phi(t) + \bar{\phi}(t)) =$$

$$-\frac{\omega}{2}\bar{\phi}_f^2 - \frac{\omega}{2}\bar{\phi}_i^2 + iS_0(\bar{\phi}) + i\int_{t_i}^{t_f} dt J(t)\bar{\phi}(t)$$

$$-\omega(\phi_f\bar{\phi}_f + \phi_i\bar{\phi}_i) + i\int_{t_i}^{t_f} dt(\dot{\phi}\dot{\bar{\phi}} - \omega^2\phi\bar{\phi})$$

$$-\frac{\omega}{2}\phi_f^2 - \frac{\omega}{2}\phi_i^2 + iS_0(\phi) + i\int_{t_i}^{t_f} dt J(t)(\phi(t)) =$$

$$\int_{t_i}^{t_f} dt(\dot{\phi}\dot{\bar{\phi}} - \omega^2\phi\bar{\phi}) \Rightarrow \text{by parts} \Rightarrow$$

$$\dot{\bar{\phi}}\phi\Big|_{t_i}^{t_f} + \int_{t_i}^{t_f} dt\phi[(-\partial_t^2 - \omega^2)\bar{\phi}(t) + J(t)] \quad \Rightarrow$$

$$= -\frac{\omega}{2}\bar{\phi}_f^2 - \frac{\omega}{2}\bar{\phi}_i^2 + iS_0(\bar{\phi}) + i\int_{t_i}^{t_f} dt J(t)\bar{\phi}(t)$$

$$-\frac{\omega}{2}\phi_f^2 - \frac{\omega}{2}\phi_i^2 + iS_0(\phi)$$

$$+(i\dot{\bar{\phi}}(t_f) - \omega\bar{\phi}_f)\phi_f - (i\dot{\bar{\phi}}(t_i) + \omega\bar{\phi}_i)\phi_i$$

$$+\int_{t_i}^{t_f} dt\phi[(-\partial_t^2 + \omega^2)\bar{\phi}(t) + J(t)] =$$

We choose $\bar{\phi}(t)$ in such a way that it eliminates the linear (black) term in the exponential $\Rightarrow$ we get the differential equation

$$(\partial_t^2 - \omega^2)\bar{\phi}(t) = J(t)$$

with boundary conditions

$$i\dot{\bar{\phi}}(t_f) = \omega\bar{\phi}_f, \qquad i\dot{\bar{\phi}}(t_i) = -\omega\bar{\phi}_i$$

The solution of this equation is

$$\bar{\phi}(t) = i\int_{t_i}^{t_f} G(t-t')J(t'), \qquad G_0(t-t') = \frac{1}{2\omega}e^{-i\omega|t-t'|}$$

where $G_0(t-t')$ is the "propagator" for harmonic oscillator (see p. 11).

$$= -\frac{1}{2}\int_{t_i}^{t_f} dtdt' J(t)G(t-t')J(t') - \frac{\omega}{2}\phi_f^2 - \frac{\omega}{2}\phi_i^2 + iS_0(\phi)$$

$\Rightarrow$ the numerator reduces to

$$\int d\phi_f d\phi_i \int D\phi e^{-\frac{\omega}{2}(\phi_f^2+\phi_i^2)} e^{iS_0+i\int_{t_i}^{t_f} dt J(t)\phi(t)}$$

$$= \quad e^{-\frac{1}{2}\int_{t_i}^{t_f} dt dt' J(t) G(t-t') J(t')}$$
$$\times \quad \int d\phi_f d\phi_i \int D\phi e^{-\frac{\omega}{2}(\phi_f^2+\phi_i^2)} e^{iS_0}$$

$$\Rightarrow \quad Z(J) = e^{-\frac{1}{2}\int_{t_i}^{t_f} dt dt' J(t) G(t-t') J(t')}$$

As $t_f \to \infty, t_i \to -\infty$, we get

$$Z(J) = e^{-\frac{1}{2}\int dt dt' J(t) G(t-t') J(t')}$$

Expanding this in powers of J we obtain

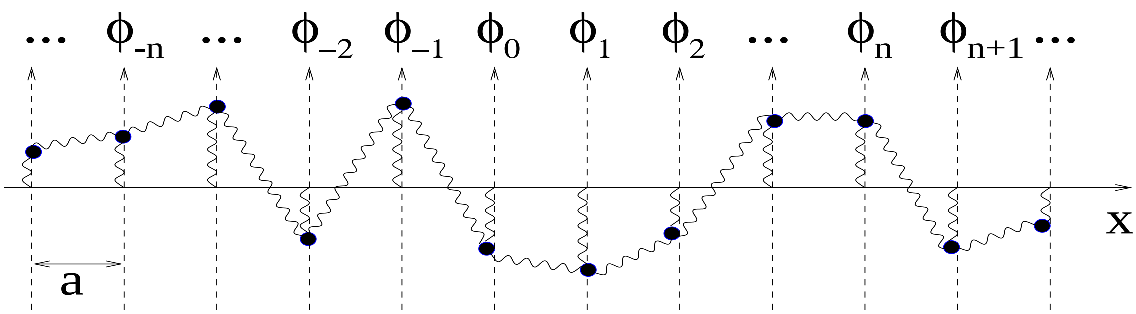
$$G_0(t_1,t_2) = G_0(t_1-t_2)$$
$$G_0(t_1,t_2,t_3,t_4) = G_0(t_1-t_2)G_0(t_3-t_4) +$$
$$G_0(t_2-t_3)G_0(t_1-t_4) + G_0(t_1-t_3)G_0(t_2-t_4)$$
$$G_0(t_1,t_2,t_3,t_4,t_5,t_6) =$$
$$G_0(t_1-t_2)G_0(t_3-t_4)G_0(t_5-t_6) + \ldots$$

while all the G_n with odd n vanish

$\Rightarrow$ Wick's theorem (see p. 12).

Functional integrals

Consider again lattice model for 1+1 Klein-Gordon field



$$\hat{H} = a \sum \Big[\frac{\hat{\pi}_k^2}{2a^2} + \frac{(\hat{\phi}_{k+1} - \hat{\phi}_k)^2}{2a^2} + V(\hat{\phi}_k)\Big]$$

where $V(\hat{\phi}) = \frac{m^2}{2}\hat{\phi}^2$ for the free KG field of $V(\hat{\phi}) = \frac{m^2}{2}\hat{\phi}^2 + \frac{\lambda}{4!}\hat{\phi}^4$ for the self-interacting field.

For one oscillator, we found the path integral representation for the evolution kernel

$$\langle \phi_f | e^{-i\hat{H}t} | \phi_i \rangle$$

where $|\phi\rangle$ were the eigenstates of the coordinate operator $\hat{\phi}$.

For 2N oscillators of our lattice model, the eigenstates of the coordinate operator $\hat{\phi}_n$ are

$$|\{\phi_K\}\rangle \equiv |\phi_{-N}\rangle|\phi_{-N+1}\rangle...|\phi_{-1}\rangle|\phi_0|\phi_1\rangle...|\phi_N\rangle$$

By construction, $|\{\phi_K\}\rangle$ are eigenstates of "field operator" $\hat{\phi}_k$:

$$\hat{\phi}_k|\{\phi_K\}\rangle = \phi_k|\{\phi_K\}\rangle$$

The evolution kernel is

$$\langle\{\phi_K\}^f|e^{-i\hat{H}t_{fi}}|\{\phi_K\}^i\rangle$$

As in the case of harmonic oscilator, in order to find the path integral representation for the evolution kernel we divide t_{fi} into $n+1$ small intervals Δt and insert

$$1 = \int d\phi_{-N}|\phi_{-N}\rangle\langle\phi_{-N}|...\int d\phi_{-N}|\phi_{-N}\rangle\langle\phi_{-N}|$$
$$= \int \Pi d\phi_k|\{\phi_K\}\rangle\langle\{\phi_K\}|$$

n times.

We get:

$$\langle\{\phi_K\}^f|e^{-i\hat{H}t_{fi}}|\{\phi_K\}^i\rangle = \int \Pi d\phi_k^l\langle\{\phi_K\}^f|e^{-i\hat{H}\Delta t}|\{\phi_K\}^l\rangle$$
$$\langle\{\phi_K\}^l|e^{-i\hat{H}\Delta t}|\{\phi_K\}^{l-1}\rangle...\langle\{\phi_K\}^1|e^{-i\hat{H}\Delta t}|\{\phi_K\}^i\rangle$$

For small Δt the evolution kernel for our lattice Hamiltonian is simply a product of evolution kernels for individual oscillators:

$$\langle\{\phi_K\}^{l+1}|e^{-i\hat{H}\Delta t}|\{\phi_K\}^l\rangle =$$
$$\left(\frac{a}{2\pi i\Delta t}\right)^{N+\frac{1}{2}} e^{ia\Delta t\sum_k\left[\frac{(\phi_k^{l+1}-\phi_k^l)^2}{2\Delta t^2}-\frac{(\phi_{k+1}^l-\phi_k^l)^2}{2a^2}-V(\phi_k^l)\right]}$$

$\Rightarrow$ functional integral for the KG field on the lattice

$$\langle\{\phi_K\}^f|e^{-i\hat{H}t_{fi}}|\{\phi_K\}^i\rangle = \left[\frac{a}{2\pi i\Delta t}\right]^{(n+1)(N+\frac{1}{2})}$$
$$\int \Pi d\phi_k^l e^{ia\Delta t\sum_{k,n}\left[\frac{(\phi_k^{l+1}-\phi_k^l)^2}{2\Delta t^2}-\frac{(\phi_{k+1}^l-\phi_k^l)^2}{2a^2}-V(\phi_k^l)\right]}$$

As in the case of one oscillator, it is convenient to label the integration variables by the time t_l and position x_k rather than by l and k

$$\phi_k^l \to \phi(x_k, t_l) \qquad \Rightarrow$$

$$\langle\{\phi_K\}^f|e^{-i\hat{H}t_{fi}}|\{\phi_K\}^i\rangle = \left[\frac{a}{2\pi i\Delta t}\right]^{(n+1)N}\int \Pi d\phi(x_k,t_l)$$

$$\exp\left\{ia\Delta t\sum_{k,n}\left[\frac{(\phi(x_k,t_l+\Delta t)-\phi(x_k,t_l))^2}{2\Delta t^2}\right.\right.$$

$$\left.\left.-\ \frac{(\phi(x_k+a,t_l)-\phi(x_k,t_l))^2}{2a^2}-V(\phi(x_k,t_l))\right]\right\}$$

In the "continuum limit" $a, \Delta t \to 0$ we get

$$\langle\{\phi\}^f|e^{-i\hat{H}t_{fi}}|\{\phi\}^i\rangle =$$
$$\int D\phi\ e^{i\int_{t_i}^{t_f} dt\int_{-L/2}^{L/2}\left[\frac{(\dot{\phi}(x,t))^2}{2}-\frac{(\phi'(x,t))^2}{2}-V(\phi(x,t))\right]}$$

where L is the size of the "box" in x dimension.
Here $|\{\phi\}^i\rangle$ is a wave functional describing the state where the field is equal to $\phi(x,t)$.

The final form of the functional integral for the evolution kernel is

$$\langle\{\phi\}^f|e^{-i\hat{H}t_{fi}}|\{\phi\}^i\rangle = \int_{\phi(t_i)=\phi_i}^{\phi(t_f)=\phi_f} D\phi\ e^{iS(\phi)}$$

Because of the complicated structure of the wave functional $|\{\phi\}^i\rangle$ it is more convenient to work in terms of Green functions where the initial and final states are simple (perturbative vacua).

Repeating the steps which lead us from the path integral for evolution kernel to path integrals for the Green functions, we get

$$\langle\Omega|T\{\hat{\phi}(x_1,t_1)...\hat{\phi}(x_n,t_n)|\Omega\rangle = \frac{\int D\phi \;\phi(x_1,t_1)...\phi(x_n,t_n)\; e^{iS(\phi)}}{\int D\phi(x,t)e^{iS(\phi)}}$$

In four dimensions everything is the same (except we must start from 3-dimensional lattice) $\Rightarrow$

$$\langle\Omega|T\{\hat{\phi}(x_1)...\hat{\phi}(x_n)|\Omega\rangle = \frac{\int D\phi(x)\;\phi(x_1)...\phi(x_n)e^{iS}}{\int D\phi(x)e^{iS}}$$

The generating functional for Green functions

$$Z(J) = \frac{\int D\phi e^{iS_0 + i\int dx J(x)\phi(x)}}{\int D\phi e^{iS_0}}$$

As in the QM case, linear term in the numerator must vanish after the shift $\phi(x) \to \phi(x) + \bar{\phi}(x)$

$\Rightarrow$ we get the differential equation

$$(\partial^2 + m^2)\bar{\phi}(x) = J(x)$$

with the boundary conditions

$$i\frac{\partial}{\partial t}\bar{\phi}(\vec{p},t) \stackrel{t\to\infty}{\Rightarrow} \omega_p \bar{\phi}(\vec{p},t)$$
$$i\frac{\partial}{\partial t}\bar{\phi}(\vec{p},t) \stackrel{t\to-\infty}{\Rightarrow} -\omega_p \bar{\phi}(\vec{p},t),$$

reflecting the perurbative vacua at $t \to \pm\infty$.
$(\phi(\vec{p},t) \equiv \int d^3x e^{i\vec{x}\vec{p}}\phi(\vec{x},t))$
Solution:

$$\bar{\phi}(x) = i\int dx' G_0(x-x')J(x')$$

where $G_0(x-x') = \int \frac{d^4p}{16\pi^4 i}\frac{1}{m^2-p^2-i\epsilon}e^{-ip(x-x')}$ is a free propagator $\Rightarrow$

$$Z(J) = e^{-\frac{1}{2}\int dx dx' J(x) G_0(x-x') J(x')}$$

Expanding this generating functional in powers of J one obtains Wick's theorem, just like for the anharmonic oscillator $\Rightarrow$

$$\langle\Omega|T\{\hat{\phi}(x_1)...\hat{\phi}(x_n)|\Omega\rangle = \frac{\int D\phi(x)\phi(x_1)...\phi(x_n)e^{iS}}{\int D\phi(x)e^{iS}}$$

= sum of Feynman diagrams

The Euclidean version of the functional integral for Green functions is

$$\langle\hat{\phi}(x_1)...\hat{\phi}(x_n)\rangle = \frac{\int D\phi(x)\phi(x_1)...\phi(x_n)e^{-S}}{\int D\phi(x)e^{-S}}$$

where the the boundary conditions are $\phi(\vec{x},t) \to 0$ at $t \to \pm\infty$. The correlation function $\langle\hat{\phi}(x_1)...\hat{\phi}(x_n)\rangle$ is the analytical continuation of the Green function $\langle\Omega|T\{\hat{\phi}(x_1)...\hat{\phi}(x_n)|\Omega\rangle$ to imaginary times $t_i \to -it_i$. The lattice form of the Euclidean functional integral is very convenient for computer calculations.

QED

Classical electrodynamics is a theory of electromagnetic field (described by $F_{\mu\nu} = (\vec{E}, \vec{B})$ - field strength tensor) interacting with charged Dirac bispinor field $\psi(x)$.

First pair of Maxwell's eqs:

$$\partial_\mu F^{\mu\nu}(x) = -g j^\nu(x), \qquad g = |e| - \text{positron charge}$$

$g = |e|$ - positron charge,

$j_\mu = \bar{\psi}\gamma_\mu\psi$ - 4-vector of the electomagnetic current

($\rho(x) \equiv g j_0 = g\psi^\dagger\psi$ - charge density)

Second pair of Maxwell's eqns $\Leftarrow$ description in terms of potentials

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$,

$F^{0i} = E_x, ..., F^{12} = -B_z$

$A_\mu = (\Phi, \vec{A})$ - scalar and vector potentials ("electromagnetic field").

The choice of potential is ambiguous $\Rightarrow$ gauge invariance: one can add $A_\mu \to A_\mu + \partial_\mu\alpha$ with an arbitrary $\alpha(x)$ and $\vec{E}$ and $\vec{B}$ will not change.

Dirac equation in an external electromagnetic field:

$$i\,\not{\mathcal{D}}\psi(x) = m\psi(x)$$

$\mathcal{D}_\mu = \partial_\mu - igA_\mu$ - covariant derivative.
The electomagnetic coupling constant $-g = e$ is the charge of the electron.

Electrodynamics Lagrangian:

$$\mathcal{L}(x) = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) + \bar{\psi}(x)(i\,\not{\mathcal{D}} - m)\psi(x)$$

Euler-Lagrange equation $\Rightarrow$ Maxwell's eqns + Dirac eqn.

Gauge invariance:

$$\left.\begin{array}{l} \psi(x) \to e^{i\alpha(x)}\psi(x) \\ \bar{\psi}(x) \to e^{-i\alpha(x)}\bar{\psi}(x) \\ A_\mu(x) \to A_\mu(x) + \frac{1}{g}\partial_\mu\alpha(x) \end{array}\right\} \Rightarrow \qquad \mathcal{L}(x) \to \mathcal{L}(x)$$

Coulomb gauge:

$$\partial_k A_k = 0, \qquad\qquad k = 1,2,3$$

In Coulomb gauge Maxwell's eqns turn to

$$\vec{\partial}^2 \Phi(\vec{x},t) = \rho((\vec{x},t)), \qquad \partial^2 A_k(x) = j_k(x)$$

$\Rightarrow$ $A_0 \equiv \Phi$ is *not* an independent dynamical variable:

$$\Phi(\vec{x},t) \;=\; -\int d^3x' \frac{\rho(\vec{x}',t)}{4\pi|\vec{x}-\vec{x}'|} \;-\; \text{Coulomb potential}$$

The electromagnetic coupling constant e ($\equiv$ charge of the electon) is small

$$\frac{e^2}{4\pi} = \frac{1}{137} \qquad \left(\frac{e^2}{4\pi\hbar c} = \frac{1}{137}\right)$$

$\Rightarrow$ we can use perturbative expansion

$$\mathcal{L} = \mathcal{L}_F + \mathcal{L}_D + \mathcal{L}_{\text{int}}$$

$\mathcal{L}_F = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x)$ – free e.m. Lagrangian
$\mathcal{L}_D = \bar{\psi}(x)(i\,\not{\partial} - m)\psi(x)$ – free Dirac Lagrangian
$\mathcal{L}_{\text{int}} = g\bar{\psi}(x)\not{A}\psi(x)$ – interaction Lagrangian

Quantization of the free e.m. field

$$\mathcal{L}_F = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) = \frac{1}{2}(\vec{E}^2 - \vec{B}^2)$$

We will quantize the electromagnetic field in the Coulomb gauge.

(+) - Non-physical degrees of freedom are absent.

(-) - Intermediate steps are not Lorentz invariant.

Without sources $A_0 \equiv 0 \Rightarrow$
we can try $A_k(x)$ as canonical coordinates
Canonical momenta:

$$\left.\begin{array}{l} \pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} \quad - \quad \text{does not exist} \\ \pi^k = \frac{\partial \mathcal{L}}{\partial \dot{A}_k} = -\dot{A}^k - \frac{\partial A_0}{\partial x_k} = -\dot{A}^k = E^k \end{array}\right\} \Rightarrow$$

$$H_F = \int d^3x(\pi^k \dot{A}_k - \mathcal{L}) = \int d^3x \frac{1}{2}(\vec{E}^2 + \vec{B}^2)$$

(recall that $\vec{E}^2 + \vec{B}^2$ is the energy density of e.m. field).

Quantization: we promote $A_k(\vec{x},t)$ and $\pi_k(\vec{x},t)$ to operators $\hat{A}_k(\vec{x})$ and $\hat{\pi}_k(\vec{x})$ satisfying the CCR

$$[\hat{A}_i(\vec{x}), \hat{A}_j(\vec{y})] = 0$$
$$[\hat{\pi}_i(\vec{x}), \hat{\pi}_j(\vec{y})] = 0$$
$$[\hat{\pi}_i(\vec{x}), \hat{A}_j(\vec{y})] \equiv [\hat{E}_i(\vec{x}), \hat{A}_j(\vec{y})] = i\delta_{ij}\delta^3(\vec{x}-\vec{y})$$

A problem: last line contradicts to Maxwell's eqs.

We want to have Gauss law $\vec{\nabla}\cdot\vec{\hat{E}} = 0$ as in classical physics, but

$$[\hat{E}_i(\vec{x}), \hat{A}_j(\vec{y})] = i\delta_{ij}\delta^3(\vec{x}-\vec{y}) \Rightarrow$$
$$[\nabla\cdot\vec{\hat{E}}(\vec{x}), \hat{A}_j(\vec{y})] = i\partial_j\delta^3(\vec{x}-\vec{y}) \neq 0$$

A trick that works in QED (but not in QCD):

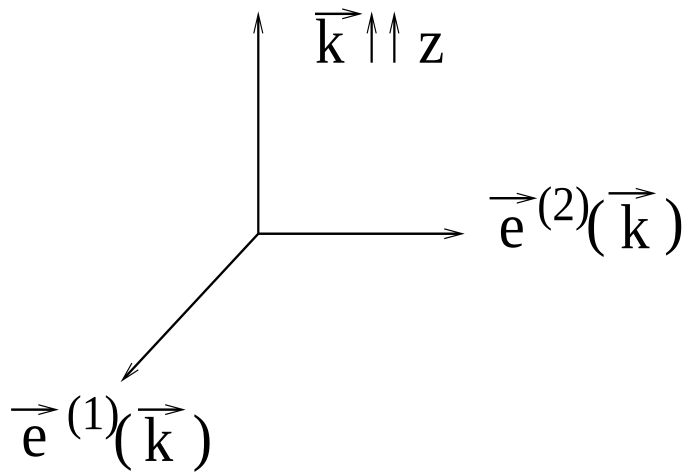
$$[\hat{\pi}_i(\vec{x}), \hat{A}_j(\vec{y})] \;=\; i\delta^{\text{tr}}_{ij}\delta^3(\vec{x}-\vec{y})$$
$$\delta^{\text{tr}}_{ij}\delta^3(\vec{x}-\vec{y}) \;\stackrel{\text{def}}{\equiv}\; \int d^3k \left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2}\right) e^{-i\vec{k}(\vec{x}-\vec{y})}$$

Expansion in ladder operators

$$\vec{A}(\vec{x}) = \int \frac{đ^3k}{\sqrt{2E_k}} \sum_{\lambda=1,2} \vec{e}^{\lambda}(\vec{k})\left(\hat{a}_k^{\lambda} e^{i\vec{k}\vec{x}} + \hat{a}_k^{\dagger\lambda} e^{-i\vec{k}\vec{x}}\right)$$

$$\vec{\pi}(\vec{x}) = \int \frac{đ^3k}{\sqrt{2E_k}} iE_k \sum_{\lambda=1,2} \vec{e}^{\lambda}(\vec{k})\left(\hat{a}_k^{\lambda} e^{i\vec{k}\vec{x}} - \hat{a}_k^{\dagger\lambda} e^{-i\vec{k}\vec{x}}\right)$$

where $e^{\lambda}(\vec{k})$ - polarization vectors and $E_k = |\vec{k}|$



$$\left.\begin{array}{rcl} \left[\hat{a}_k^{\lambda}, \hat{a}_{k'}^{\lambda'}\right] & = \left[\hat{a}_k^{\dagger\lambda}, \hat{a}_{k'}^{\dagger\lambda'}\right] & = 0 \\ \left[\hat{a}_k^{\lambda}, \hat{a}_{k'}^{\dagger\lambda'}\right] & = & (2\pi)^3 \delta^3(\vec{k} - \vec{k}')\delta_{\lambda\lambda'} \end{array}\right\} \Rightarrow \text{CCR}$$

$$\hat{H}_F = \frac{1}{2} \int d^3x(\vec{\hat{E}}^2 + \vec{\hat{B}}^2) = \sum_{\lambda=1,2} \int đ^3k E_k \hat{a}_k^{\dagger\lambda} \hat{a}_k^{\lambda}$$

(Again, we throw away $E_0 = L^3 \int đ^3p E_p/2$).

$$\text{CCR} \Rightarrow \left.\begin{array}{rcl} \left[\hat{H}_F, \hat{a}_k^{\dagger\lambda}\right] &=& E_k \hat{a}_k^{\dagger\lambda} \\ \left[\hat{H}_F, \hat{a}_k^{\lambda}\right] &=& -E_k \hat{a}_k^{\lambda} \end{array}\right\} \Rightarrow$$

$\Rightarrow \hat{a}^\lambda |0_F\rangle = 0$

$|0_F\rangle$ – the perturbative vacuum for free e.m. field (ground state of the Hamiltonian $\hat{H}_F$).

Heisenberg operators:

$$\vec{\hat{A}}(\vec{x}, t) = e^{i\hat{H}_F t} \vec{\hat{A}}(\vec{x}) e^{-i\hat{H}_F t}$$
$$\vec{\hat{\pi}}(\vec{x}, t) = e^{i\hat{H}_F t} \vec{\hat{\pi}}(\vec{x}) e^{-i\hat{H}_F t}$$

$$\Rightarrow \left.\begin{array}{rcl} e^{i\hat{H}_F t} \hat{a}_k^{\lambda} e^{-i\hat{H}_F t} &=& \hat{a}_k^{\lambda} e^{-iE_k t} \\ e^{i\hat{H}_F t} \hat{a}_k^{\dagger\lambda} e^{-i\hat{H}_F t} &=& \hat{a}_k^{\dagger\lambda} e^{iE_k t} \end{array}\right\} \Rightarrow$$

$$\vec{\hat{A}}(x) = \int \frac{đ^3 k}{\sqrt{2E_k}} \sum_{\lambda=1,2} \vec{e}^{\lambda}(\vec{k}) \left(\hat{a}_k^{\lambda} e^{-ikx} + \hat{a}_k^{\dagger\lambda} e^{ikx} \right) \Big|_{k_0 = E_k}$$

$\partial^2 \vec{\hat{A}} = 0, \nabla \cdot \vec{\hat{A}} = 0 \Rightarrow$ $\vec{\hat{A}}(x)$ satisfies Maxwell's eqs.

Propagator of the transverse photons

$$D_{ij}^{\text{tr}} \equiv \langle 0_F | T\{\hat{A}_i(x) \hat{A}_j(y)\} | 0_F \rangle =$$
$$\int \frac{đ^4 k}{i} e^{-ik(x-y)} \frac{1}{-k^2 - i\epsilon} \left(\delta_{ik} - \frac{k_i k_j}{\vec{k}^2} \right)$$

Quantization of the free Dirac field

$$\mathcal{L}_D = \bar{\psi}(x)(i\,\partial\!\!\!/ - m)\psi(x)$$

Canonical coordinate: $\psi(\vec{x}, t)$
$\Rightarrow$ canonical momentum $\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^\dagger$

$$H = \int d^3x \bar{\psi}(-i\vec{\gamma}\vec{\nabla} + m)\psi$$

Quantization: $\psi(\vec{x}, t) \to \hat{\psi}(\vec{x})$, $\pi(\vec{x}, t) \to \hat{\pi}(\vec{x})$
CAR (canonical anticommutation relations):

$$\{\hat{\psi}(\vec{x}), \hat{\psi}(\vec{y})\} = \{\hat{\psi}^\dagger(\vec{x}), \hat{\psi}^\dagger(\vec{y})\} = 0$$
$$\{\hat{\psi}(\vec{x}), \hat{\psi}^\dagger(\vec{y})\} = \delta^3(\vec{x} - \vec{y})$$

Ladder operators

$$\hat{\psi}(\vec{x}) = \int \frac{d\!\!\!^{-3}p}{\sqrt{2E_p}} \left[\sum_{s=\uparrow,\downarrow} (\hat{a}^s_{\vec{p}} u(\vec{p}, s) e^{i\vec{p}\vec{x}} + \hat{b}^{\dagger s}_{\vec{p}} v(\vec{p}, s) e^{-i\vec{p}\vec{x}} \right]$$
$$\hat{\bar{\psi}}(\vec{x}) = \int \frac{d\!\!\!^{-3}p}{\sqrt{2E_p}} \left[\sum_{s=\uparrow,\downarrow} (\hat{b}^s_{\vec{p}} \bar{v}(\vec{p}, s) e^{i\vec{p}\vec{x}} + \hat{a}^{\dagger s}_{\vec{p}} \bar{u}(\vec{p}, s) e^{-i\vec{p}\vec{x}} \right]$$

$u(\vec{p}, s)$ – Dirac bispinor for the electron,
$v(\vec{p}, s)$ – for the positron.

CAR for the ladder operators

$$\left.\begin{array}{ll} \{\hat{a}^s_{\vec{p}}, \hat{a}^{s'}_{\vec{p}}\} = 0 & \{\hat{b}^s_{\vec{p}}, \hat{b}^{s'}_{\vec{p}}\} = 0 \\ \{\hat{a}^{\dagger s}_{\vec{p}}, \hat{a}^{\dagger s'}_{\vec{p}'}\} = 0 & \{\hat{b}^{\dagger s}_{\vec{p}}, \hat{b}^{\dagger s'}_{\vec{p}'}\} = 0 \\ \{\hat{a}^s_{\vec{p}}, \hat{a}^{\dagger s'}_{\vec{p}'}\} = 8\pi^3\delta^3(\vec{p}-\vec{p}')\delta_{ss'}, & \{\hat{b}^s_{\vec{p}}, \hat{b}^{\dagger s'}_{\vec{p}'}\} = 8\pi^3\delta^3(\vec{p}-\vec{p}')\delta_{ss'} \end{array}\right\}$$

$$\{\hat{\psi}, \hat{\psi}\} = \{\hat{\bar{\psi}}, \hat{\bar{\psi}}\} = 0, \quad \{\hat{\psi}(\vec{x}), \hat{\bar{\psi}}(\vec{y})\} = \delta^3(\vec{x}-\vec{y})$$

$$\hat{H}_D = \int d^3x \hat{\bar{\psi}}(\vec{x})(-i\vec{\gamma}\vec{\nabla} + m)\psi(\vec{x})$$

$$= \int d^3p E_p \sum_s \left(\hat{a}^{\dagger s}_{\vec{p}} \hat{a}^s_{\vec{p}} + \hat{b}^{\dagger s}_{\vec{p}} \hat{b}^s_{\vec{p}}\right)$$

CAR $\leftrightarrow$ CCR would lead to $+ \leftrightarrow -$

$\Rightarrow$ Hamiltonian would not have the ground state.

$$\left.\begin{array}{ll} \left[\hat{H}, \hat{a}^s_{\vec{p}}\right] = -E_p \hat{a}^s_{\vec{p}} & \left[\hat{H}, \hat{b}^s_{\vec{p}}\right] = -E_p \hat{b}^s_{\vec{p}} \\ \left[\hat{H}, \hat{a}^{\dagger s}_{\vec{p}}\right] = E_p \hat{a}^{\dagger s}_{\vec{p}} & \left[\hat{H}, \hat{b}^{\dagger s}_{\vec{p}}\right] = E_p \hat{b}^{\dagger s}_{\vec{p}} \end{array}\right\} \Rightarrow$$

$\hat{a}^s_{\vec{p}}|0_D\rangle = \hat{b}^s_{\vec{p}}|0_D\rangle = 0$

$|0_D\rangle \equiv$ perturbative vacuum for the Dirac field.

$$\begin{array}{rcll} \sqrt{2E_p}\hat{a}^{\dagger s}_{\vec{p}}|0_D\rangle & = & |p,s\rangle_e & - \text{ one} - \text{electron state} \\ \sqrt{2E_p}\hat{b}^{\dagger s}_{\vec{p}}|0_D\rangle & = & |p,s\rangle_p & - \text{ one} - \text{positron state} \end{array}$$

Heisenberg operators are defined as usual

$$\hat{\psi}(\vec{x},t) = e^{i\hat{H}_D t}\hat{\psi}(\vec{x})e^{-i\hat{H}_D t}, \quad \hat{\bar{\psi}}(\vec{x},t) = e^{i\hat{H}_D t}\hat{\bar{\psi}}(\vec{x})e^{-i\hat{H}_D t}$$

$$\left.\begin{array}{c} e^{i\hat{H}_D t}\hat{a}^s_{\vec{p}}e^{-i\hat{H}_D t} = \hat{a}^s_{\vec{p}}e^{-iE_p t} \\ e^{i\hat{H}_D t}\hat{a}^{\dagger s}_{\vec{p}}e^{-i\hat{H}_D t} = \hat{a}^{\dagger s}_{\vec{p}}e^{iE_p t} \\ \text{same for } \hat{b} \text{ and } \hat{b}^\dagger \end{array}\right\} \Rightarrow$$

$$\hat{\psi}(x) = \int \frac{d\!\!\!{}^-{}^3p}{\sqrt{2E_p}}\left[\sum_{s=\uparrow,\downarrow}(\hat{a}^s_{\vec{p}}u(\vec{p},s)e^{-ipx} + \hat{b}^{\dagger s}_{\vec{p}}v(\vec{p},s)e^{ipx}\right]$$

$$\hat{\bar{\psi}}(x) = \int \frac{d\!\!\!{}^-{}^3p}{\sqrt{2E_p}}\left[\sum_{s=\uparrow,\downarrow}(\hat{b}^s_{\vec{p}}\bar{v}(\vec{p},s)e^{-ipx} + \hat{a}^{\dagger s}_{\vec{p}}\bar{u}(\vec{p},s)e^{ipx}\right]$$

Propagator of the Dirac particle

$$S(x-y) = \langle 0_D|T\{\psi(x)\bar{\psi}(y)\}|0_D\rangle$$

$$T\{\psi(\hat{x})\hat{\bar{\psi}}(y)\} \equiv \theta(x_0-y_0)\psi(\hat{x})\hat{\bar{\psi}}(y) - \theta(y_0-x_0)\hat{\bar{\psi}}(y)\hat{\psi}(x)$$

CAR +

$$\sum_s u(\vec{p},s)\bar{u}(\vec{p},s) = \not{p}+m, \quad \sum_s v(\vec{p},s)\bar{v}(\vec{p},s) = \not{p}-m$$

$$\Rightarrow \qquad S(x-y) = \int \frac{d\!\!\!{}^-{}^4p}{i}\frac{m+\not{p}}{m^2-p^2-i\epsilon}e^{-ip(x-y)}$$

Quantization of electrodynamics in the Coulomb gauge

$$\mathcal{L}(x) = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) + \bar{\psi}(x)(i\not{D} - m)\psi(x)$$

Canonical coordinates: A_i, ψ, canonical momenta: $\pi^k = E^k, \pi = \psi^\dagger$.

In a free EM theory $\Phi \equiv A_0$ was 0. Now

$$\Phi(\vec{x}, t) \;=\; -\int d^3x' \frac{g\psi^\dagger(\vec{x}', t)\psi(\vec{x}', t)}{4\pi|\vec{x} - \vec{x}'|}$$

- Coulomb potential due to the charge density $\rho(\vec{x}', t) = g\psi^\dagger(\vec{x}', t)\psi(\vec{x}', t)$

$\Rightarrow$ A_0 is not an independent dynamical variable.

Hamiltonian $H(t) = \int d^3x \mathcal{H}(\vec{x}, t)$

$$\mathcal{H}(\vec{x}, t) = \frac{\partial \mathcal{L}}{\partial \dot{\psi}}\dot{\psi} + \frac{\partial \mathcal{L}}{\partial \dot{A}^k}\dot{A}^k - \mathcal{L} =$$

$$\bar{\psi}(-i\vec{\gamma}\cdot\vec{\nabla} + m)\psi + \frac{1}{2}(\dot{\vec{A}}^2 + \vec{B}^2) - \frac{1}{2}(\vec{\nabla}\Phi)^2 - gA^\mu\bar{\psi}\gamma_\mu\psi$$

$$\left.\begin{array}{lcl}\int d^3x \frac{1}{2}(\dot{\vec{A}}^2 + \vec{B}^2) & = & \text{free EM Hamiltonian} \\ \int d^3x \bar{\psi}(-i\vec{\gamma}\cdot\vec{\nabla} + m)\psi & = & \text{free Dirac Hamiltonian}\end{array}\right\} \Rightarrow$$

$$H = H_D + H_F + H_{\text{int}} + H_{\text{Coulomb}}$$

$$H_{\text{int}}(t) = -g \int d^3x A^\mu(\vec{x},t)\bar{\psi}(\vec{x},t)\gamma_\mu\psi(\vec{x},t)$$

$$H_{\text{Coulomb}}(t) = \frac{1}{2}\int d^3x \Phi(\vec{x},t)\nabla^2\Phi(\vec{x},t)$$

$$= \int d^3x d^3y \rho(\vec{x},t)\frac{1}{4\pi|\vec{x}-\vec{y}|}\rho(\vec{y},t)$$

(Recall that $\Phi(\vec{x},t) = -g\int d^3x' \frac{\rho(\vec{x}',t)}{4\pi|\vec{x}-\vec{x}'|}$)

Quantization: $A_i(\vec{x},t) \to \hat{A}_i(\vec{x})$, $\psi(\vec{x},t) \to \hat{\psi}(\vec{x})$, $\pi_i(\vec{x},t) \to \hat{\pi}_i(\vec{x})$, $\pi(\vec{x},t) \to \hat{\pi}(\vec{x})$.

$$\hat{A}_0(\vec{x}) = -g\int d^3x' \frac{\hat{\rho}(\vec{x}')}{4\pi|\vec{x}-\vec{x}'|}, \quad \hat{\rho}(\vec{x}) \equiv \hat{\psi}^\dagger(\vec{x})\hat{\psi}(\vec{x})$$

Canonical (anti) commutaton relations – same as in a free theory:

$$[\hat{\pi}_i(\vec{x}), \hat{A}_j(\vec{y})] = \delta^{\text{tr}}_{ij}\delta(\vec{x}-\vec{y}), \; \{\hat{\psi}^\dagger_\xi(\vec{x}), \hat{\psi}_\eta(\vec{y})\} = \delta_{\xi\eta}\delta(\vec{x}-\vec{y})$$

All other (anti)commutators (including $[\psi, A_k]$) vanish.

QED Hamiltonian

$$\hat{H} = \hat{H}_F + \hat{H}_D + \hat{H}_{\text{int}} + \hat{H}_{\text{Coulomb}}$$

$$\hat{H}_{\text{int}} = -g\int d^3x \hat{A}_\mu(\vec{x})\hat{\bar{\psi}}(\vec{x})\gamma_\mu\hat{\psi}(\vec{x})$$

$$\hat{H}_{\text{Coulomb}} = \frac{1}{2}\int d^3x d^3y \hat{\rho}(\vec{x})\frac{1}{4\pi|\vec{x}-\vec{y}|}\hat{\rho}(\vec{y})$$

Heisenberg operators - as usually.

$$\hat{A}_\mu(\vec{x},t) \equiv e^{i\hat{H}t}\hat{A}_\mu(\vec{x})e^{-i\hat{H}t}$$
$$\hat{\psi}^{(\dagger)}(\vec{x},t) \equiv e^{i\hat{H}t}\hat{\psi}^{(\dagger)}(\vec{x})e^{-i\hat{H}t}$$

Equal-time (anti)commutators

$$[\hat{\pi}_i(\vec{x},t),\hat{A}_j(\vec{y},t)] = \delta^{\text{tr}}_{ij}\delta(\vec{x}-\vec{y})$$
$$\{\hat{\psi}^\dagger_\xi(\vec{x},t),\hat{\psi}_\eta(\vec{y},t)\} = \delta_{\xi\eta}\delta(\vec{x}-\vec{y})$$

$|\Omega\rangle$ - physical vacuum of the interacting theory (ground state of $\hat{H}$).

Green functions:

$$\langle\Omega|TA^{\mu_1}(x_1),...A^{\mu_m}(x_m)\psi(x_1)\psi(x_2)...\bar{\psi}(x_n)|\Omega\rangle$$

Green functions determine physical cross sections via the LSZ theorem.

Perturbation theory: $\hat{H} = \hat{H}_0 + \tilde{H}_{\text{int}}$,
$\hat{H}_0 \equiv \hat{H}_F + \hat{H}_D$, $\tilde{H}_{\text{int}} \equiv \hat{H}_{\text{int}} + \hat{H}_{\text{Coulomb}}$.
Perturbative vacuum: $|\,\emptyset\rangle = |\,\emptyset\rangle_F |\,\emptyset\rangle_D$

Interaction representation:

$$\psi_I^{(\dagger)}(\vec{x},t) = e^{i\hat{H}_0 t}\psi^{(\dagger)}(\vec{x})e^{-i\hat{H}_0 t}$$
$$A_I^{\mu}(\vec{x},t) = e^{i\hat{H}_0 t}A^{\mu}(\vec{x})e^{-i\hat{H}_0 t}$$

$$\vec{\hat{A}}_I(x) = \sum_{\lambda=1,2}\int \frac{đ^3 k}{\sqrt{2E_k}}\vec{e}^{\lambda}(\vec{k})[\hat{a}^{\lambda}_{\vec{k}}e^{-ikx} + \hat{a}^{\dagger\lambda}_{\vec{k}}e^{ikx}]$$
$$\hat{\psi}_I(x) = \sum_{s=\uparrow,\downarrow}\int \frac{đ^3 p}{\sqrt{2E_p}}[\hat{a}^{s}_{\vec{p}}u(\vec{p},s)e^{-ipx} + \hat{b}^{\dagger s}_{\vec{p}}v(\vec{p},s)e^{ipx}]$$
$$\hat{\bar{\psi}}_I(x) = \sum_{s=\uparrow,\downarrow}\int \frac{đ^3 p}{\sqrt{2E_p}}[\hat{b}^{s}_{\vec{p}}\bar{v}(\vec{p},s)e^{-ipx} + \hat{a}^{\dagger s}_{\vec{p}}\bar{u}(\vec{p},s)e^{ipx}]$$

Master formula ($\tilde{H}_I(t) \equiv \tilde{H}_{\text{int}}(\psi_I(t), A_I(t))$)

$$\langle\Omega|\hat{\psi}(x_1)\hat{\bar{\psi}}(x_2)\ldots\hat{A}^{\mu}(x_n)|\Omega\rangle = \frac{\langle\emptyset|T\{\hat{\psi}_I(x_1)\hat{\bar{\psi}}_I(x_2)\ldots\hat{A}^{\mu}_I(x_n)e^{-i\int dt\tilde{H}_I(t)}\}|\,\emptyset\rangle}{\langle\emptyset|T\{e^{-i\int dt\tilde{H}_I(t)}\}|\,\emptyset\rangle}$$

$\widehat{H}_F$ leads to the propagator of the transverse photons $D^{\rm tr}_{ij}$ ($\eta = (1,0,0,0)$):

$$D^{\rm tr}_{\mu\nu} = \int \frac{d^4k}{(2\pi)^4 i} \frac{e^{-ik(x-y)}}{-k^2 - i\epsilon}$$

$$\left(-g_{\mu\nu} - \frac{k_\mu k_\nu - k_0(k_\mu \eta_\nu + \mu \leftrightarrow \nu)}{\vec{k}^2} - k^2 \frac{\eta_\mu \eta_\nu}{\vec{k}^2} \right)$$

- First (red) term is a Feynman propagator for the photon.
- Second (black) term does not contribute to physical matrix elements due to Ward identity
  Ward identity: Multiplication of the amplitude of the emission of the photon with momentum k by k_μ vanishes provided all the electrons and positrons are on the mass shell($\equiv p_i^2 = m^2$)
- Third (blue) term $= i\eta_\mu \eta_\nu \frac{\delta(x_0 - y_0)}{4\pi |\vec{x} - \vec{y}|}$ is the instantaneous interaction which cancels the contribution coming from $\widehat{H}_{\rm Coulomb}$

$\Rightarrow$ One can omit $\hat{H}_{\text{Coulomb}}$ from the Hamiltonian and use the rel.-inv. Feynman propagator

$$D^{\text{F}}_{\mu\nu} = \int \frac{d^4k}{(2\pi)^4 i} e^{-ik(x-y)} \frac{g_{\mu\nu}}{k^2 + i\epsilon}$$

$\Rightarrow$ New master formula

$$\langle\Omega|T\{\hat{\psi}(x_1)...\hat{\bar{\psi}}(x_n)A_\mu(y_1)...A_\nu(y_1)\}|\Omega\rangle = \frac{\langle\,\emptyset|T\{\hat{\psi}_I(x_1)...\hat{\bar{\psi}}_I(x_n)A^I_\mu(y_1)...A^I_\nu(y_n)e^{i\int d^4x\hat{\mathcal{L}}_I dx}\}|\,\emptyset\rangle}{\langle\,\emptyset|T\{e^{i\int d^4x\hat{\mathcal{L}}_I dx}\}|\,\emptyset\rangle}$$

Expanding in powers of $\hat{\mathcal{L}}_I = g\hat{A}^I_\mu\hat{\bar{\psi}}_I\gamma^\mu\hat{\psi}_I$ one obtains a set of the correlation functions of the type

$$\langle\,\emptyset|T\{\hat{\psi}_I(x_1)...\hat{\bar{\psi}}_I(x_n)A^I_\mu(y_1)...A^I_\nu(y_n)\}|\,\emptyset\rangle = \sum_{\text{contractions}} \hat{\psi}_I(x_1)...\hat{\bar{\psi}}_I(x_n)A^I_\mu(y_1)...A^I_\nu(y_n)$$

where

$$\hat{\psi}_I(x)\hat{\bar{\psi}}_I(x') = S(x-x') \quad \text{and} \quad A^I_\mu(y)A^I_\nu(y') = D^F_{\mu\nu}(y-y')$$

$\Rightarrow \langle\Omega|T\{\hat{\psi}(x_1)...\hat{\bar{\psi}}(x_n)A_\mu(y_1)...A_\nu(y_1)\}|\Omega\rangle =$

sum of Feynman diagrams with the photon propagator $D^F_{\mu\nu}(x-y)$, Dirac propagator $S(x-y)$, and the vertex $g\gamma_\mu$.

Fourier transformation $\Rightarrow$

Set of Feynman rules for the QED Green functions in the momentum representation $G(p_1, ...k_n)$:

- $\frac{m+\not{p}}{i(m^2-p^2-i\epsilon)}$ for a Dirac propagator
- $\frac{g_{\mu\nu}}{i(k^2+i\epsilon)}$ for a photon propagator
- $ig\gamma_\mu 16\pi^4\delta(p_1+p_2+k)$ for a vertex with incoming momenta p_1, p_2, and k.
- $\int \frac{d^4p}{16\pi^4}$ for each internal line.
- extra (-1) for each fermion loop .

As usually, it is convenient to introduce the reduced Green functions $\mathcal{G}(p_1,..,k_n)$:

$$G(p_1,..,k_n) = (2\pi)^4\delta(\sum p_i + \sum k_i)(-i)^{n-1}\mathcal{G}(p_1,..,k_n)$$

Set of Feynman rules for the reduced Green functions in the momentum representation $\mathcal{G}(p_1,...k_n)$:

- $\frac{m+\not p}{m^2-p^2-i\epsilon}$ for a Dirac propagator
- $\frac{g_{\mu\nu}}{k^2+i\epsilon}$ for a photon propagator
- $g\gamma_\mu$ for a vertex.
- $\int \frac{d^4p}{16\pi^4 i}$ for each loop momentum.
- extra (-1) for each fermion loop .

The matrix elements of the S-matrix are obtained using the LSZ theorem.

LSZ reduction formula for QED

Example:

LSZ theorem:

$$S(p_1, s_1; p'_1, s'_1; k_1, \lambda_1 \to p_2, s_2; p'_2, s'_2; k_2, \lambda_2)$$
$$= \; i^6 \lim_{k_i^2 \to 0} k_1^2 e_\mu^{\lambda_1}(k_1) k_2^2 e_\nu^{\lambda_2}(k_2) \lim_{p_i^2 \to m^2}$$
$$\times \; [\bar{u}(p_2, s_2)(m - \not{p}_2)]_{\xi'} [\bar{v}(p'_1, s'_1)(m + \not{p}'_1)]_\eta$$
$$\times \; G^{\mu\nu}_{\xi'\xi,\eta\eta'}(p_1, p'_1, k_1, p_2, p'_2, k_2)$$
$$\times \; [(m - \not{p}_1) u(p_1, s_1)]_\xi [(m + \not{p}'_2) v(p'_2, s'_2)]_{\eta'}$$

$$G^{\mu\nu}_{\xi'\xi,\eta\eta'}(p_1, p'_1, k_1, p_2, p'_2, k_2) =$$
$$\int dx_1 dx'_1 dy_1 dx_2 dx'_2 dy_2 e^{-ip_1x_1 - ip'_1x'_1 - ik_1y_1} e^{ip_2x_2 + ip'_2x'_2 + ik_2y_2}$$
$$\langle \Omega | T\{\hat{\psi}_{\xi'}(x_2) \hat{\bar{\psi}}_\xi(x_1) \hat{\psi}_\eta(x'_1) \hat{\bar{\psi}}_{\eta'}(x'_2) \hat{A}^\nu(y_2) \hat{A}^\mu(y_1)\} \Omega \rangle$$

$$\left.\begin{array}{rcl} S(p_1,...k_n) &=& (2\pi)^4\delta(\sum p_i + \sum p_i) iT(p_1,...k_n) \\ G(p_1,...k_n) &=& (2\pi)^4\delta(\sum p_i + \sum p_i) i^{-n-1}\mathcal{G}(p_1,...k_n) \end{array}\right\} \Rightarrow$$

$$\begin{aligned} & T(p_1,s_1;p_1',s_1';k_1,\lambda_1 \to p_2,s_2;p_2',s_2';k_2,\lambda_2) \\ = & \lim_{k_i^2\to 0} k_1^2 e_\mu^{\lambda_1}(k_1) k_2^2 e_\nu^{\lambda_2}(k_2) \lim_{p_i^2\to m^2} \\ \times & \; [\bar{u}(p_2,s_2)(m - \not{p}_2)]_{\xi'} [\bar{v}(p_1',s_1')(m + \not{p}_1')]_\eta \\ \times & \; \mathcal{G}^{\mu\nu}_{\xi'\xi,\eta\eta'}(p_1,p_1',k_1,p_2,p_2',k_2) \\ \times & \; [(m - \not{p}_1)u(p_1,s_1)]_\xi [(m + \not{p}_2')v(p_2',s_2')]_{\eta'} \end{aligned}$$

$$\begin{aligned} & \mathcal{G}^{\mu\nu}_{\xi'\xi,\eta\eta'}(p_1,p_1',k_1,p_2,p_2',k_2) = \\ & \frac{(m + \not{p}_2)_{\xi'\alpha'}}{m^2 - p_2^2} \frac{(m - \not{p}_2')_{\beta'\eta'}}{m^2 - p_2'^2} \frac{(m + \not{p}_1)_{\alpha\xi}}{m^2 - p_1^2} \frac{(m - \not{p}_1')_{\eta\beta}}{m^2 - p_1'^2} \\ & \frac{1}{k^2}\frac{1}{k'^2} \left(\mathcal{G}^{\text{amp}}\right)^{\mu\nu}_{\alpha'\alpha,\beta\beta'} (p_1,p_1',k_1,p_2,p_2',k_2) \end{aligned}$$

The factors $m \pm \not{p}_i$ amputate the legs from $\mathcal{G}$, for example

$$\begin{aligned} & [\bar{u}(p_2,s_2)(m - \not{p}_2)]_{\xi'} \frac{(m + \not{p}_2)_{\xi'\alpha'}}{m^2 - p_2^2} \left(\mathcal{G}^{\text{amp}}\right)^{\mu\nu}_{\alpha'\alpha,\beta\beta'} \\ = & \; \bar{u}(p_2,s_2)_{\alpha'} \left(\mathcal{G}^{\text{amp}}\right)^{\mu\nu}_{\alpha'\alpha,\beta\beta'} \end{aligned}$$

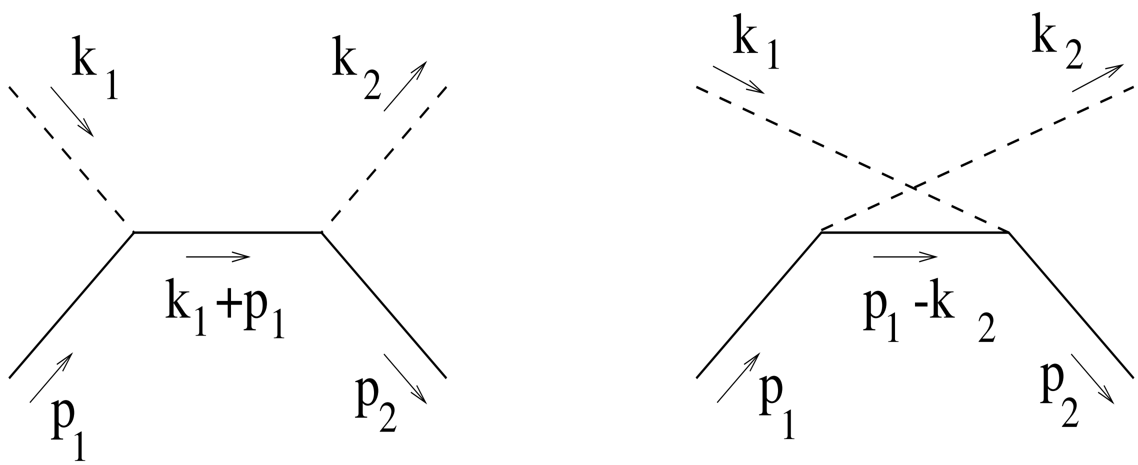
$\Rightarrow$

$$
\begin{aligned}
&T(p_1, s_1; p_1', s_1'; k_1, \lambda_1 \to p_2, s_2; p_2', s_2'; k_2, \lambda_2) \\
&\quad = e_\mu^{\lambda_1}(k_1) e_\nu^{\lambda_2}(k_2) \bar{u}(p_2, s_2)_{\alpha'} \bar{v}(p_1', s_1')_\beta \\
&\quad \times \left(\mathcal{G}^{\text{amp}}\right)^{\mu\nu}_{\alpha'\alpha,\beta\beta'} (p_1, p_1', k_1, p_2, p_2', k_2) \\
&\qquad \times u(p_1, s_1)_\alpha v(p_2', s_2')_{\beta'}
\end{aligned}
$$

Set of rules for the matix elements of the T-matrix:

- $u(p, s)$ for the incoming electron
- $\bar{v}(p, s)$ for the incoming positron
- $\bar{u}(p, s)$ for the outgoing electron
- $v(p, s)$ for the outgoing positron
- $e_\mu^\lambda(k)$ for the (initial or final) photon.
- Multiply by $\mathcal{G}^{\text{amp}}$

Compton scattering



$$G_{\mu\nu}(p_2, k_2; p_1, k_1) =$$

$$\frac{e^2}{k_1^2 k_2^2} \frac{m + \not{p}_2}{(m^2 - p_2^2)}$$

$$\left(\gamma_\mu \frac{m + \not{p}_1 + \not{k}_1}{m^2 - (p_1 + k_1)^2} \gamma_\nu + \gamma_\nu \frac{m + \not{p}_1 - \not{k}_2}{m^2 - (p_1 - k_2)^2} \gamma_\mu \right)$$

$$\frac{m + \not{p}_1}{(m^2 - p_1^2)}$$

LSZ:

$$T(p_1, k_1; \lambda_1, s_1 \to p_2, k_2; \lambda_1, s_2) =$$
$$\lim_{k_i^2 \to 0} \lim_{p_i^2 \to m^2} k_1^2 e_\nu^{\lambda_1} k_2^2 e_\mu^{\lambda_2}$$
$$\bar{u}(p_2, s_2)(m - \not{p}_2) G_{\mu\nu}(p_2, k_2; p_1, k_1)(m - \not{p}_1) u(p_1, s_1)$$
$$= e^2 \bar{u}(p_2, s_2) \left(\not{e}^{\lambda_2} \frac{m + \not{p}_1 + \not{k}_1}{m^2 - (p_1 + k_1)^2} \not{e}^{\lambda_1} \right.$$
$$\left. + \quad \not{e}^{\lambda_1} \frac{m + \not{p}_1 - \not{k}_2}{m^2 - (p_1 - k_2)^2} \not{e}^{\lambda_2} \right) u(p_1, s_1)$$

Cross section is $\frac{\partial \sigma}{\partial \Omega} = \frac{|T|^2}{64\pi^2 s}$.

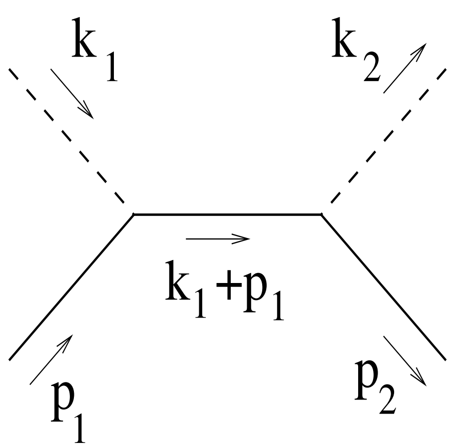
The Yukawa model

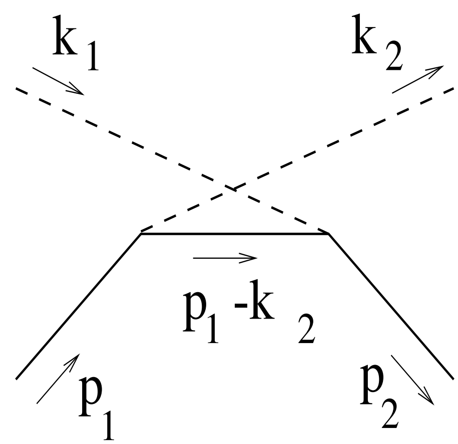
$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m_\pi^2}{2}\phi^2 + \bar{\psi}(i\not\partial - m)\psi - g\phi\bar{\psi}\psi$$

Feynman rules:

- $(m + \not p)(m^2 - p^2)^{-1}$ for the Dirac propagator
- $(m_\pi^2 - p^2)^{-1}$ for the meson propagator
- $-g$ for the $ee\pi$ vertex.

"Compton scattering" in the Yukawa model





$$G(p_2, k_2; p_1, k_1) = \frac{g^2}{(m_\pi^2 - k_1^2)(m_\pi^2 - k_2^2)} \frac{m + \not p_2}{(m^2 - p_2^2)}$$

$$\left(\frac{m + \not p_1 + \not k_1}{m^2 - (p_1 + k_1)^2} + \frac{m + \not p_1 - \not k_2}{m^2 - (p_1 - k_2)^2} \right) \frac{m + \not p_1}{(m^2 - p_1^2)}$$

LSZ:

$$T(p_1,k_1;s_1 \to p_2,k_2;s_2) =$$
$$\lim_{k_i^2 \to m_\pi^2} \lim_{p_i^2 \to m^2} (m_\pi^2 - k_1^2)(m_\pi^2 - k_2^2)$$
$$\bar{u}(p_2,s_2)(m - \not{p}_2) G(p_2,k_2;p_1,k_1)(m - \not{p}_1) u(p_1,s_1)$$
$$= g^2 \bar{u}(p_2,s_2) \left(\frac{m + \not{p}_1 + \not{k}_1}{m^2 - s} + \frac{m + \not{p}_1 - \not{k}_2}{m^2 - u} \right) u(p_1,s_1)$$
$$= g^2 \bar{u}(p_2,s_2) \left(\frac{2m + \not{k}_1}{m^2 - s} + \frac{2m - \not{k}_2}{m^2 - u} \right) u(p_1,s_1)$$

Cross section of the unpolarized $e\pi$ scattering (in the c.m. frame) is

$$\frac{\partial\sigma}{\partial\Omega} = \frac{g^4}{64\pi^2 s} \times$$
$$\Big[\frac{1}{(m^2-s)^2} \text{Tr}\{(2m+\not{k}_1)(m+\not{p}_1)(2m+\not{k}_1)(m+\not{p}_2)\} +$$
$$\frac{2}{(m^2-s)(m^2-u)} \text{Tr}\{(2m+\not{k}_1)(m+\not{p}_1)(2m-\not{k}_2)(m+\not{p}_2)\}$$
$$+\frac{1}{(m^2-u)^2} \text{Tr}\{(2m-\not{k}_2)(m+\not{p}_1)(2m-\not{k}_2)(m+\not{p}_2)\}\Big]$$

A try on CCR in QED

$$[\hat{A}_i(\vec{x}), \hat{A}_j(\vec{y})] = [\hat{\pi}_i(\vec{x}), \hat{\pi}_j(\vec{y})] = 0$$
$$[\hat{\pi}_i(\vec{x}), \hat{A}_j(\vec{y})] \equiv [\hat{E}_i(\vec{x}), \hat{A}_j(\vec{y})] = i\delta_{ij}\delta^3(\vec{x}-\vec{y})$$

A problem: we want to have Gauss law $\vec{\nabla} \cdot \vec{\hat{E}} = 0$ as in classical physics, but

$$[\vec{\nabla} \cdot \vec{\hat{E}}(\vec{x}), \hat{A}_j(\vec{y})] = i\partial_k \delta^3(\vec{x}-\vec{y}) \neq 0$$

The trick that works for both QED and QCD (and for other gauge theories as well):

We impose the Gauss law on physical states instead of imposing it on the operators.

$$[\hat{\pi}_i(\vec{x}), \hat{A}_j(\vec{y})] \;=\; i\delta_{ij}\delta^3(\vec{x}-\vec{y})$$

but

$$\vec{\nabla} \cdot \vec{\hat{E}} |\Psi_{\text{physical}}\rangle = 0$$

This still appears to contadict to CCR since

$$\langle \Psi_{\text{phys}} | [\nabla \cdot \vec{\hat{E}}(\vec{x}), \hat{A}_j(\vec{y})] | \Psi_{\text{phys}} \rangle = i\partial_k \delta^3(\vec{x}-\vec{y}) \neq 0$$
$$\langle \Psi_{\text{phys}} | \nabla \cdot \vec{\hat{E}}(\vec{x}) \hat{A}_j(\vec{y}) - \hat{A}_j(\vec{y}) \nabla \cdot \vec{\hat{E}}(\vec{x}) | \Psi_{\text{phys}} \rangle = 0$$

but actually there is no contradiction since the l.h.s is ill-defined (see the QM examle below).

Baby version of a gauge theory

Consider a mechanical model with the Lagrangian ($x_{12} \equiv x_1 - x_2$)

$$L(A(t), x_1(t), x_2(t)) = \frac{\dot{x}_1^2}{2} + \frac{\dot{x}_2^2}{2} + A^2 + (\dot{x}_1 + \dot{x}_2)A - \frac{\omega}{2} x_{12}^2$$

The Euler-Lagrange equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = \frac{\partial L}{\partial x_1} \quad \Rightarrow \quad \ddot{x}_1 + \dot{A} = -\omega x_{12}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} = \frac{\partial L}{\partial x_1} \quad \Rightarrow \quad \ddot{x}_2 + \dot{A} = \omega x_{12}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{A}} = \frac{\partial L}{\partial A} \quad \Rightarrow \quad 0 = \dot{x}_1 + \dot{x}_2 + 2A$$

This Lagrangian is invariant under the following "gauge transformation"

$$\begin{aligned} x_1(t) &\rightarrow x_1(t) + \alpha(t) \\ x_2(t) &\rightarrow x_2(t) + \alpha(t) \\ A(t) &\rightarrow A(t) - \dot{\alpha}(t) \end{aligned}$$

We can use this freedom to get rid of the variable A by choosing $\alpha(t) = \int dt A(t)$.

$$\Rightarrow \begin{cases} \ddot{x}_1 = -\omega x_{12}, \qquad \ddot{x}_2 = \omega x_{12} \\ \dot{x}_1 + \dot{x}_2 = 0 \end{cases}$$

First two equations describe two particles with $m = 1$ connected by a spring and the last one means that the sum of their momenta is 0.

This problem may be described by the Lagrangian

$$L(x_1(t), x_2(t)) = \frac{\dot{x}_1^2}{2} + \frac{\dot{x}_2^2}{2} - \frac{\omega}{2} x_{12}^2$$

PLUS

the additional requirement that the total momentum of the two particles vanishes:

$p_1(t) + p_2(t) = 0$.

This is an example of the constrained canonical system.

At first, let us forget about the constraint $p_1(t) + p_2(t) = 0$.

New canonical coordinates:

$X = (x_1 + x_2)/2$ - coordinate of the c.m.

$x = x_{12}$ - separation

$$L(X(t), x(t)) = \dot{X}^2 + \frac{\dot{x}^2}{4} - \frac{\omega}{2}x^2$$

New canonical momenta:

$P = 2\dot{X} = p_1 + p_2$, $p = \frac{\dot{x}}{2} = \frac{1}{2}(p_1 - p_2)$

$$\Rightarrow H = \frac{P^2}{4} + p^2 - \frac{\omega}{2}x^2$$

Quantization:

$$\hat{H} = \frac{\hat{P}^2}{4} + \hat{p}^2 - \frac{\omega}{2}\hat{x}^2$$

Solutions of the Schrodinger eqn are

$$\Psi(X, x) = e^{iPX}\psi_n(x)$$

$\psi_n(x)$ - wavefunction of the n-th level of harmonic oscillator (Hermit polynomial).

Q: How to generalize the classical constraint that the observable $P = p_1 + p_2$ vanishes to quantum mechanics?

Wrong A: Require that the operator corresponding to this observable $\hat{P} = \hat{p}_1 + \hat{p}_2$ vanishes - contradicts to CCR $[p_i, x_j] = -i\delta_{ij}$.

Right A: Require that we consider only the "physical" states $\Psi_{\text{phys}} = \sum a_n \psi_n(x)$ with total momentum $P = 0 \to F(\hat{P})\Psi_{\text{phys}} = 0 \Rightarrow$we will observe $P = 0$ in all experiments.

Apparent "contradiction"

$$\langle \Psi_{\text{phys}} | [\hat{P}, \hat{X}] | \Psi_{\text{phys}} \rangle \quad = \quad 0 \text{ or } i?$$

In explicit form ($|\Psi_{\text{phys}}\rangle = \Psi_{\text{vac}}$ for simplicity).

$$\int dx_1 dx_2 e^{-\frac{\omega}{2}x_{12}^2} \left[\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, x_1 + x_2 \right] e^{-\frac{\omega}{2}x_{12}^2} =$$
$$\int dx_1 dx_2 e^{-\frac{\omega}{2}x_{12}^2} \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) (x_1 + x_2) e^{-\frac{\omega}{2}x_{12}^2}$$

which is 0 or ∞ depending on your taste (the integral is divergent along the coordinate $X = x_1 + x_2$).

Path integral for the constrained system

$$\langle \Psi^f_{\text{phys}} | e^{-i\hat{H}t} | \Psi^i_{\text{phys}} \rangle \quad = \quad ?$$

If we know how to solve the constraint,

$$\langle \Psi^f_{\text{phys}} | e^{-i\hat{H}t} | \Psi^i_{\text{phys}} \rangle =$$

$$\int d\phi_f d\phi_i \Psi^f_{\text{phys}}(\phi_f) \Psi^i_{\text{phys}}(\phi_i) \int_{x(t_i)=\phi_i}^{x(t_f)=\phi_f} Dx(t) e^{i\int_{t_i}^{t_f} dt \left(\frac{\dot{x}^2}{4} - \frac{\omega^2}{2} x^2\right)}$$

If we do not know how to resolve the constraint(like in QCD), we can try the path integral in terms of both coordinates x_1 and x_2.
Insert

$$\hat{1} = \int \bar{d}p_1 \bar{d}p_2 \int dx_1 dx_2 e^{-ip_1x_1 - ip_2x_2} |p_1\rangle |p_2\rangle \langle x_2| \langle x_1|$$

n times (for simplicity, n=2 and $|\Psi_{\text{phys}}\rangle = |\emptyset\rangle$

$$\langle \emptyset | e^{-i\hat{H}t} | \emptyset \rangle$$
$$= \int dx_1^f dx_2^f \; dx_1 dx_2 \; \bar{d}p_1 \bar{d}p_2 \; dx_1^i dx_2^i \; \bar{d}p_1^i \bar{d}p_2^i$$
$$\langle \emptyset | x_1^f \rangle | x_2^f \rangle \langle x_1^f | \langle x_2^f | e^{-i\hat{H}\Delta t} | p_1 \rangle | p_2 \rangle e^{-ip_1x_1 - ip_2x_2}$$
$$\langle x_1 | \langle x_2 | e^{-i\hat{H}\Delta t} | p_1^i \rangle | p_2^i \rangle e^{-ip_1^i x_1^i - ip_2^i x_2^i} \langle x_1^i | \langle x_2^i | \emptyset \rangle$$

$$
\begin{aligned}
= & \int dX_f dx_f \; dXdx \; đP đp \; dX_i dx_i \; đP_i đp_i \\
& e^{-\frac{\omega}{2}x_f^2} e^{-i\Delta t(\frac{P^2}{4}+p^2-\frac{\omega}{2}x_f^2)} e^{iP(X_f-X)+ip(x_f-x)} \\
& e^{-i\Delta t(\frac{P_i^2}{4}+p_i^2-\frac{\omega}{2}x^2)} e^{iP_i(X-X_i)+ip_i(x-x_i)} e^{-\frac{\omega}{2}x_i^2} \\
= & \int dX_f \; dx_f \; dx \; dx_i \; e^{-\frac{\omega}{2}x_f^2-\frac{\omega}{2}x_i^2} \frac{1}{\pi\Delta t} \\
& e^{\frac{i}{2}\Delta t\left[\frac{(x_f-x)^2}{\Delta t^2}-\frac{\omega^2}{2}x_f^2\right]} e^{\frac{i}{2}\Delta t\left[\frac{(x-x_i)^2}{\Delta t^2}-\frac{\omega^2}{2}x^2\right]}
\end{aligned}
$$

$\Rightarrow$ integral over X_f is divergent. To avoid the divergence along the non-dynamical variables X, we should write the path integral with intermediate integrations over the dynamical coordinates only. To this end, we repeat the derivation of the path integral, inserting at each t_k the projector on physical states

$$\hat{1}_{\text{phys}} = \sum |\Psi_{\text{phys}}\rangle\langle\Psi_{\text{phys}}|$$

instead of the projector on the total set of states $\hat{1}$.

$$\hat{1}_{\text{phys}} = \int dp_1 dp_2 \delta(p_1 + p_2)|p_1\rangle|p_2\rangle\langle p_2|\langle p_1|$$
$$\int dx_1 dx_2 \delta(x_1 + x_2 - a)|x_1\rangle|x_2\rangle\langle x_2|\langle x_1|$$

where a is an arbitrary number. Check:

$$\langle y_1|\langle y_2|\hat{1}_{\text{phys}}|\Psi_{\text{phys}}\rangle =$$
$$\int \not{d} p_1 \not{d} p_2 \delta(p_1 + p_2) e^{ip_1 y_1 + ip_2 y_2} \int dx_1 dx_2$$
$$e^{-ip_1 x_1 - ip_2 x_2} \delta(x_1 + x_2 - a)\rangle \Psi_{\text{phys}}(x_{12}) = \Psi_{\text{phys}}(y_{12})$$

$$\hat{1}_{\text{phys}} = \int \tilde{\not{d}}^2 p \tilde{d}^2 x e^{-ip_1 x_1 - ip_2 x_2} |p_1\rangle|p_2\rangle\langle x_1|\langle x_2|$$

$$\tilde{\not{d}}^2 p \equiv \not{d} p_1 dp_2 \delta(p_1 + p_2), \;\; \tilde{d}^2 x \equiv dx_1 dx_2 \delta(x_1 + x_2 - a)$$

Insert $\hat{1}_{\text{phys}}$ n times:

$$\langle \Psi^f_{\text{phys}}|e^{-i\hat{H}t\text{fi}}|\Psi^i_{\text{phys}}\rangle =$$
$$\langle \Psi^f_{\text{phys}}|e^{-i\hat{H}\Delta t}\hat{1}_{\text{phys}} e^{-i\hat{H}\Delta t}\hat{1}_{\text{phys}} \ldots e^{-i\hat{H}\Delta t}|\Psi^i_{\text{phys}}\rangle$$

$$e^{-i\hat{H}\Delta t}\hat{1}_{\text{phys}} =$$
$$\int \tilde{d}^2 x^k \tilde{\not{d}}^2 p^k e^{-ip_1^k x_1^k - ip_2^k x_2^k} e^{-iH^k \Delta t}|p_1\rangle|p_2\rangle\langle x_1|\langle x_2|$$

$$H^k \equiv \frac{(p_1^k)^2}{2} + \frac{(p_2^k)^2}{2} - \frac{\omega^2}{2} x_k^2$$

$$e^{-i\hat{H}\Delta t}\hat{1}_{\text{phys}}e^{-i\hat{H}\Delta t}\hat{1}_{\text{phys}} =$$
$$\int \tilde{d}x^{k+1}\tilde{d}p^{k+1}|p_1^{k+1}\rangle|p_2^{k+1}\rangle e^{-i(p_1^{k+1}x_1^{k+1}+p_2^{k+1}x_2^{k+1}+H^{k+1}\Delta t)}$$
$$\int \tilde{d}x^k\tilde{d}p^k e^{i(p_1^k(x_1^{k+1}-x_1^k)+p_2^k(x_2^{k+1}-x_2^k)-H^k\Delta t)}\langle x_1^k|\langle x_2^k|$$

In the end of the day ($x_0 \equiv x^i$)

$$\langle\Psi^f_{\text{phys}}|e^{-i\hat{H}t_{fi}}|\Psi^i_{\text{phys}}\rangle =$$
$$\int \tilde{d}x_f\Psi^f_{\text{phys}}(x^f_{12})\Psi^f_{\text{phys}}(x^i_{12})\ \Pi^n_{k=0}\tilde{d}x^k\tilde{d}p^k$$
$$e^{-i\sum_{k=0}^n\left(p_1^{k+1}(x_1^{k+1}-x_1^k)+p_2^{k+1}(x_2^{k+1}-x_2^k)+H^{k+1}\Delta t\right)}$$

In the continuum limit this gives

$$\langle\Psi^f_{\text{phys}}|e^{-i\hat{H}t_{fi}}|\Psi^i_{\text{phys}}\rangle =$$
$$\int \tilde{d}x_f\Psi^f_{\text{phys}}(x^f_{12})\tilde{d}x_i\Psi^f_{\text{phys}}(x^i_{12})\int \tilde{d}p_i$$
$$\int Dp_1(t)Dp_2(t)\ \Pi_t\delta(p_1(t)+p_2(t))$$
$$\int Dx_1(t)Dx_2(t)\ \Pi_t\delta(x_1(t)+x_2(t)-a(t))$$
$$\exp\left\{i\int_{t_i}^{t_f}dt\,(p_1(t)\dot{x}_1(t)+p_2(t)\dot{x}_2(t)-H(t))\right\}$$

The meaning of the $\delta(X_k - a_k)$ is to restrict the integral over non-dynamical variables X_k:

$$\int \Pi dX_k \delta(X_k - a_k) = 1 \to \int \Pi dX_k dx_k \delta(X_k - a_k)\Psi(x_k) = \Psi(x_k)$$

If the explicit form of this variables is unknown, one can use the arbitrary functions $f_k(X, x)$ ($X = X_1, ...X_n$, $x = x_1, ...x_n$) because

$$\int \Pi dX_k \delta(f_k(X, x)) \det \left| \frac{df_i(X, x)}{dX_j} \right| = 1$$

provided the equation $f_k(X, x) = 0$ has no multiple roots.

In terms of Poisson brackets

$$\frac{df_i(X, x)}{dX_j} = \{P_i, f_j\} \qquad \text{where}$$

$$\{F_i, G_j\} \stackrel{\text{def}}{\equiv} \sum_k \frac{\partial F_i}{\partial P_k}\frac{\partial G_j}{\partial X_k} + \frac{\partial F_i}{\partial p_k}\frac{\partial G_j}{\partial x_k} - (F \leftrightarrow G)$$

- Poisson brackets for arbitrary F and G.

Poisson brackets are invariant with respect to change of canonical coordinates $\Rightarrow$

$$\frac{df_i(X, x)}{dX_j} = \{(p_1 + p_2)_i, f_j(x_1, x_2)\}$$

In the continuum limit

$f_k(x_1,x_2) \to f(x_1(t),x_2(t))$ and

$\{p_1+p_2)_i, f_k(x_1,x_2)\} \Rightarrow \{(p_1+p_2)(t), f(x_1(t'),x_2(t')\}$

Functional Poisson bracket $\{F(t),G(t')\}$ is a variational derivative:

$$\{F(t),G(t')\} \equiv \int dt \left(\frac{\delta F}{\delta p_1(t)}\frac{\delta G}{\delta x_1(t')}+\frac{\delta F}{\delta p_2(t)}\frac{\delta G}{\delta x_2(t')}\right)-(F \leftrightarrow G)$$

The final form of the path integral for the constrained system:

$$\langle \Psi^f_{\text{phys}}|e^{-i\hat{H}t_{fi}}|\Psi^i_{\text{phys}}\rangle =$$
$$\int \tilde{d}x_f \Psi^f_{\text{phys}}(x^f)\tilde{d}x_i \Psi^f_{\text{phys}}(x^i)\int \tilde{\bar{d}}p^i$$
$$\int Dp_1(t)Dp_2(t)\Pi_t\delta(p_1(t)+p_2(t))\int Dx_1(t)Dx_2(t)$$
$$\det\{p_1(t)+p_2(t), f(x_1(t),x_2(t)\}\Pi_t\delta(f(x_1(t),x_2(t))$$
$$\exp\left\{i\int_{t_i}^{t_f} dt\,(p_1(t)\dot{x}_1(t)+p_2(t)\dot{x}_2(t)-H(t))\right\}$$

$\delta(p_1+p_2)$ - Gauss law

$\delta(f(x_1(t),x_2(t))$ - “choice of gauge”

QCD

QCD is a theory of interacting quarks and gluons.

8 gluons are described by A^a_μ - 8 real massless vector fields (like 8 different photons)
Convenient matrix notation: $A_\mu \equiv A^a_\mu t^a$.
t^a are 8 Gell-Mann matrices - Hermitian matrices with properties $\mathrm{Tr} t^a = 0$, $\mathrm{Tr} t^a t^b = \frac{1}{2}\delta^{ab}$ (for the explicit form, see any textbook).

Matrices t^a are the generators of SU(3)group - the group of unitary 3×3 matrices S with $\det S = 1$. An arbitrary SU(3) matrix can be parametrized as $\exp(i\sum_1^8 \omega^a t^a)$ where ω^a are real numbers. (cf. the parametrization $U = e^{\frac{i}{2}\sum_1^3 \omega_k \sigma_k}$ for an arbitrary $SU(2)$ matrix U)

3 quarks are described by the three-component SU(3) spinor ψ^k_ξ (the quark of each color k has the additional Lorentz (bi)spinor index ξ similarly to the gluon which has color index a and vector index μ).

QCD Lagrangian

$$\mathcal{L} = -\frac{1}{2}\text{Tr}G_{\mu\nu}G^{\mu\nu} + \sum_{\text{flavors}} \bar{\psi}_q(i\not{D} - m_q)\psi_q$$

$G_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\mu A_\nu - ig[A_\mu, A_\nu]$, $D_\mu \equiv \partial_\mu - igA_\mu$

g - coupling constant.

$[t^a, t^b] = if^{abc}t^c$ (f^{abc} - SU(3) structure constants) $\Rightarrow G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$

Non-Abelian gauge invariance:

$$\left.\begin{array}{l} \psi(x) \to S(x)\psi(x) \\ \bar{\psi}(x) \to \bar{\psi}(x)S^\dagger(x) \\ A_\mu(x) \to S(x)A_\mu(x)S^\dagger(x) + \frac{i}{g}S(x)\partial_\mu S^\dagger(x) \end{array}\right\} \Rightarrow \mathcal{L}(x) \to \mathcal{L}(x)$$

(Cf. the gauge invariance of the QED Lagrangian $\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi$ under the Abelian gauge transformations $\psi \to e^{i\alpha(x)}\psi$, $A_\mu \to A_\mu + \frac{1}{g}\partial_\mu\alpha$.)

Classical theory: non-linear equations

$$(D^\mu G_{\mu\nu})^a = -\sum \bar{\psi}_q t^a \gamma_\nu \psi, \qquad (i\not{D} - m_q)\psi_q(x) = 0$$

(where $(D^\mu G_{\mu\nu})^a \equiv \partial^\mu G^a_{\mu\nu} + gf^{abc}A^{b\mu}G^c_{\mu\nu}$).

Quantum theory = perturbation theory (pQCD) + lattice simulations.

Proof of the gauge invariance of the Lagrangian

First, we prove that $G_{\mu\nu}(x) \to S(x)G_{\mu\nu}(x)S^\dagger(x)$:

$$G^{\mu\nu}(x) \;\to\; \partial_\mu(SA_\nu S^\dagger + ig^{-1}S\partial_\nu S^\dagger)$$
$$-ig(SA_\mu S^\dagger + ig^{-1}S\partial_\mu S^\dagger)(SA_\nu S^\dagger + ig^{-1}S\partial_\nu S^\dagger) - (\mu \leftrightarrow \nu)$$
$$= \;(\partial_\mu S)A_\nu S^\dagger + S(\partial_\mu A_\nu)S^\dagger + SA_\nu\partial_\mu S^\dagger + ig^{-1}(\partial_\mu S)\partial_\nu S^\dagger$$
$$+\; ig^{-1}S\partial_\mu\partial_\nu S^\dagger - igSA_\mu A_\nu S^\dagger + S(\partial_\mu S^\dagger)SA_\nu S^\dagger$$
$$+\; SA_\mu\partial_\nu S^\dagger + ig^{-1}S(\partial_\mu S^\dagger)S\partial_\nu S^\dagger - (\mu \leftrightarrow \nu)$$
$$= \;S(\partial_\mu A_\nu - igA_\mu A_\nu)S^\dagger - (\mu \leftrightarrow \nu) = SG_{\mu\nu}S^\dagger$$

(we used the property $S\partial_\mu S^\dagger = -(\partial_\mu S)S^\dagger$)

Next, we prove that $D_\mu\psi(x) \to S(x)D_\mu\psi(x)$:

$$D_\mu\psi(x) \to (\partial_\mu - igSA_\mu S^\dagger + S(\partial_\mu S^\dagger))S\psi$$
$$= S\partial_\mu\psi - igSA_\mu\psi = SD_\mu\psi$$

Finally,

$$\mathrm{Tr}G^{\mu\nu}(x)G_{\mu\nu}(x) \to$$
$$\mathrm{Tr}S(x)G^{\mu\nu}(x)S^\dagger(x)S(x)G_{\mu\nu}(x)S^\dagger(x) = \mathrm{Tr}G^{\mu\nu}(x)G_{\mu\nu}(x),$$
$$\bar\psi(x)\not{D}\psi(x) \to \bar\psi(x)S^\dagger(x)\gamma^\mu S(x)D_\mu\psi(x) = \bar\psi(x)\not{D}\psi(x)$$
$$m\bar\psi(x)\psi(x) \to m\bar\psi(x)S(x)S^\dagger(x)\psi(x) = m\bar\psi(x)\psi(x)$$

Perturbation theory - like QED:

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_F + \mathcal{L}_D + \mathcal{L}_{\text{int}} \\
\mathcal{L}_F &= -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \\
\mathcal{L}_D &= \sum \bar{\psi}(i \not{\partial} - m_q)\psi_q \\
\mathcal{L}_{\text{int}} &= ig \text{Tr} \partial^\mu A^\nu [A_\mu, A_\nu] + g \sum \bar{\psi}_q \not{A} \psi_q \\
&- g^2 \text{Tr}[A_\mu, A_\nu][A^\mu, A^\nu]
\end{aligned}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$.
Another representaion of $\mathcal{L}_{\text{int}}$

$$\begin{aligned}
\mathcal{L}_{\text{int}} &= g \sum \bar{\psi}_q \not{A} \psi_q - g f^{abc} \partial^\mu A^{a\nu} A^b_\mu A^c_\nu \\
&- g^2 f^{abn} f^{cdn} A^a_\mu A^b_\nu A^{c\mu} A^{d\nu}
\end{aligned}$$

(Recall $[t^a, t^b] = i f^{abc} t^c$).

Free Lagrangian $\equiv$ 8 issues of electrodynamics labeled by $a = 1 \div 8$

$\Rightarrow$

Feynman rules are the same, except now we have the self-interaction of gluons.

This is almost true - Ward identity in QCD is different $\Rightarrow$ ghosts.

Functional integral for QCD

First, we consider the gluodynamics $\equiv$ QCD without quarks.

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu}$$

Classical equations:

$$\frac{\partial \mathcal{L}}{\partial A^{a\alpha}} = g f^{abc} G^b_{\alpha\beta} A^{c\beta}, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A^a_\beta)} = -G^a_{\alpha\beta}$$

$$\frac{\partial}{\partial x_\mu} \frac{\partial \mathcal{L}}{\partial \partial^\mu A^{a\nu}} = \frac{\partial \mathcal{L}}{\partial A^a_\nu} \Rightarrow$$

$$-\partial^\mu G^a_{\mu\nu} = g f^{abc} G^b_{\nu\beta} A^{c\beta} \equiv (D^\mu G_{\mu\nu})^a = 0$$

Coulomb gauge: $\partial_k A^a_k = 0$.
$E^a_x \equiv G^{a01}, \ldots$ - chromoelectric field,
$B^a_x \equiv -G^{a23}, \ldots$ - chromomagnetic field.

Canonical coordinates: $A^{ak}(x)$.
Canonical momenta: $\pi^{ak}(x) = \frac{\partial \mathcal{L}}{\partial \dot{A}^a_k} = E^{ak}$.

Hamiltonian

$$H = \int d^3x (\pi^{ak} \dot{A}^a_k - \mathcal{L}) = \int d^3x \frac{1}{2} (\vec{E}^a \cdot \vec{E}^a + \vec{B}^a \cdot \vec{B}^a)$$

Similarly to QED, the Ampere's law

$$D^i G_{ik} = 0 \;\leftrightarrow\; \vec{\nabla} \times \vec{B} = -ig[\vec{A}, \times \vec{B}]$$

can be generalized to the operator form
$\vec{\nabla} \times \hat{\vec{B}} = -ig[\hat{\vec{A}}, \times \hat{\vec{B}}]$,
but the Gauss law

$$D^i G_{0i} = 0 \leftrightarrow \vec{\nabla} \cdot \vec{E} = -ig[\vec{A}, \cdot \vec{E}]$$

cannot be generalized since it would contradict CCR

$\Rightarrow$ we impose Gauss law on physical states

$$\hat{C}(\vec{x})|\Psi_{\text{phys}}\rangle = 0$$

(here $C \equiv -\vec{\nabla} \cdot \vec{E} - ig[\vec{A}, \cdot \vec{E}]$).

The Gauss law is a constraint on the canonical momenta $\Rightarrow$ we use the analogy with QM constrained system to obtain the functional integral for QCD.

Path integral for the Green functions of the constrained system in QM:

$$\langle\Omega|T\{\hat{\mathcal{O}}(t_1)\hat{\mathcal{O}}(t_2)...\hat{\mathcal{O}}(t_n)\}|\Omega\rangle =$$
$$\int \tilde{d}\phi_f \Psi^0_{\text{vac}}(\phi^f)\tilde{d}\phi_i \Psi^0_{\text{vac}}(\phi^i)$$
$$\int D\pi_1(t)D\pi_2(t)\Pi_t\delta(\pi_1(t)+\pi_2(t))\int D\phi_1(t)D\phi_2(t)$$
$$\det\{\pi_1(t)+\pi_2(t), f(\phi_1(t),\phi_2(t)\}\Pi_t\delta(f(\phi_1(t),\phi_2(t))$$
$$\mathcal{O}(t_1)\mathcal{O}(t_2)...\mathcal{O}(t_n)$$
$$\exp\left\{i\int_{t_i}^{t_f} dt\left(\pi_1(t)\dot{\phi}_1(t)+\pi_2(t)\dot{\phi}_2(t)-H(t)\right)\right\}$$

$\delta(\pi_1+\pi_2)$ - "Gauss law"

$\delta(f(\phi_1(t),\phi_2(t))$ - "choice of gauge"

$\hat{\mathcal{O}}(t)=\mathcal{O}(\hat{\phi}_{12}(t))$ - "local gauge-invariant operator"

The functional integral for the Green functions in the gluodynamics looks like in QM:

$\phi_i(t)$	$\rightarrow$	$A^a_i(\vec{x},t)$		
$\pi_i(t)$	$\rightarrow$	$E^a_i(\vec{x},t)$		
$\delta(\pi_1+\pi_2)(t)$	$\rightarrow$	$\delta(C^a(\vec{x},t))$	$-$	Gauss law
$\delta(f(\phi_1(t),\phi_2(t))$	$\rightarrow$	$\delta(\partial_k A^{ak}(\vec{x},t))$	$-$	choice of gauge

$$\langle\Omega|T\{\hat{\mathcal{O}}(x_1)\hat{\mathcal{O}}(x_2)...\hat{\mathcal{O}}(x_n)\}|\Omega\rangle =$$

$$\int DE_i^a(x)DA_i^a(x)\delta(\partial_k A^{ak}(x))\delta(C^a(x))$$
$$\Psi^0_{\rm vac}(A^f)\Psi^0_{\rm vac}(A^i)\mathcal{O}(x_1)\mathcal{O}(x_2)...\mathcal{O}(x_n)$$
$$\det\{C^a(x), \partial_k A^{bk}(y)\}e^{-i\int d^4x\left(\vec{E}^a(x)\cdot\dot{\vec{A}}^i(x)+\frac{1}{2}(\vec{E}^{a2}+\vec{H}^{a2})\right)}$$

where $\mathcal{O}(x)$ is an arbitrary gauge-invariant local operator (e.g. $G^a_{\mu\nu}(x)G^{a\mu\nu}(x)$).
$|\Psi^0_{\rm vac}\rangle$ is the "perturbative vacuum" of QCD ($\equiv$ product of 8 perturbative vacua for 8 issues of QED).

The Poisson bracket is

$$\{C^a(x), \partial_k A^{bk}(y)\} =$$
$$\int dz\ \frac{\delta(-\vec{\nabla}\cdot\vec{E}^a + gf^{amn}\vec{A}^m\cdot\vec{E}^n)}{\delta E^{dk}(z)}\ \frac{\delta(\vec{\nabla}\cdot\vec{A}^b)}{\delta A^{dk}(z)} - (A^{dk}\leftrightarrow E^{dk})$$
$$= \int dz(-\partial_k\delta^{ad} - gf^{amd}A^m_k(x))\delta(x-z)\ \partial_k\delta(y-z)\delta^{bd}$$
$$= \ M^{ab}(x)\delta^4(x-y)$$

where $M^{ab}(x)\equiv\nabla^2\delta^{ab}+gf^{abc}\vec{\nabla}\cdot\vec{A}^c(x)$.

The constrained δ-function can be written as a (functional) phase integral

$$\Pi\delta(C^a(x)) = \int DA_0^a e^{i\int d^4x A_0^a C^a(x)}$$

$\Rightarrow$ the Gaussian integration over $E_i^a(x)$ can be performed by shift $E_i^a(x) \to E_i^a(x) + G_{i0} \Rightarrow$

$$\langle\Omega|T\{\hat{\mathcal{O}}(x_1)\hat{\mathcal{O}}(x_2)...\hat{\mathcal{O}}(x_n)\}|\Omega\rangle =$$
$$\mathcal{N}^{-1}\int DA_\mu^a(x)\delta(\partial_k A^{ak}(x))\,\det\{M\delta(x-y)\}\Psi^0_{\text{vac}}(A^f)\Psi^0_{\text{vac}}(A$$
$$\times\ \mathcal{O}(x_1)\mathcal{O}(x_2)...\mathcal{O}(x_n)e^{-\frac{i}{4}\int d^4x G^a_{\mu\nu}(x)G^{a\mu\nu}(x)}$$

This is the final form of the functional integral for the gluodynamics in the Coulomb gauge.

Without the constraint, the integral

$$\int DA_\mu^a(x)\mathcal{O}(x_1)\mathcal{O}(x_2)...\mathcal{O}(x_n)e^{-\frac{i}{4}\int d^4x G^a_{\mu\nu}G^{a\mu\nu}}$$

would contain the additional integration over gauge group at each point x since the integrand is invariant under
$A_\mu(x) \to S(x)A_\mu(x)S^\dagger(x) + \frac{i}{g}S(x)\partial_\mu(x)S^\dagger(x)$.
The constraint $\Pi\delta(\partial_k A^{ak}(x))$ eliminates this integrations in the same way as the constraint $f(x_1,x_2) = 0$ eliminates the integration over the coordinate of c.m. X in our QM toy model.

Functional integral for QCD in the Feynman gauge

First, we go to the generalized Lorentz gauge

$$\partial_\mu A^{a\mu}(x) = c^a(x)$$

with arbitrary functions $c^a(x)$.

In QM, we saw that the gauge fixing condition $f(\phi_1(t), \phi_2(t) = 0$ may be arbitrary as long as we put the appropriate Poisson bracket $\{\pi_1(t) + \pi_2(t), f(\phi_1(t), \phi_2(t)\}$.
$\Rightarrow \delta(\partial_k A^{ak}(x))$ may be replaced by $\delta(\partial_\mu A^{a\mu}(x) - c^a(x))$ if we change the operator M to $M_L = \partial^2 \delta^{ab} - g f^{abc} \partial_\mu A^{c\mu}(x)$

We get

$$\begin{aligned}
&\langle \Omega | T\{\hat{\mathcal{O}}(x_1)\hat{\mathcal{O}}(x_2)...\hat{\mathcal{O}}(x_n)\}|\Omega\rangle \\
&= \mathcal{N}^{-1} \int DA^a_\mu(x) \Pi_x \delta(\partial_\mu A^{a\mu}(x) - c^a(x)) \\
&\times \ \det\{M_L \delta(x-y)\} \Psi^0_{\text{vac}}(A^f)\Psi^0_{\text{vac}}(A^i) \\
&\times \ \mathcal{O}(x_1)\mathcal{O}(x_2)...\mathcal{O}(x_n) e^{-\frac{i}{4}\int d^4x G^a_{\mu\nu}(x) G^{a\mu\nu}(x)}
\end{aligned}$$

R.h.s. does not depend on $c^a(x) \Rightarrow$

We can integrate over all possible $c^a(x)$ with the Gaussian weight $\exp\{-\frac{i}{2}\int d^4x\, c^a(x)c^a(x)\}$ $\Rightarrow$

$$\langle\Omega|T\{\hat{\mathcal{O}}(x_1)\hat{\mathcal{O}}(x_2)...\hat{\mathcal{O}}(x_n)\}|\Omega\rangle =$$
$$\mathcal{N}^{-1}\int_{\emptyset}^{\emptyset} DA^a_\mu(x)\,\det\{M_L\delta(x-y)\}\mathcal{O}(x_1)\mathcal{O}(x_2)...\mathcal{O}(x_n)$$
$$\times\; e^{i\int d^4x\{-\frac{1}{4}G^a_{\mu\nu}(x)G^{a\mu\nu}(x)+\frac{1}{2}(\partial_\mu A^{a\mu}(x))(\partial_\nu A^{a\nu}(x))\}}$$

The integrand in the exponent is

$$\frac{1}{2}A^{a\mu}\partial^2 A^{a\mu} + \mathcal{L}_{\text{int}}(A)$$

Using the general formula

$$\det A(x,y) = N^{-1}\int D\bar{c}(x)Dc(x)e^{-\int d^4x d^4y \bar{c}(x)A(x,y)c(y)}$$

the determinant can be written down as a Grassman functional integral

$$\det M_L\delta(x-y) = N^{-1}\int D\bar{c}(x)Dc(x)e^{-i\int d^4x\bar{c}^a(x)M_L^{ab}c^b(x)}$$

The Grassman variables $c(x)$ are called ghosts. They are scalar fermions (there is no violation of Pauli spin-statistics theorem because ghosts are auxiliary fields rather than physical degrees of freedom).

The final formula for the functional integral for the gluodynamics in the Feynman gauge is

$$\langle\Omega|T\{\hat{\mathcal{O}}(x_1)\hat{\mathcal{O}}(x_2)...\hat{\mathcal{O}}(x_n)\}|\Omega\rangle =$$
$$\mathcal{N}^{-1}\int_{\emptyset}^{\emptyset} DA_\mu^a(x)D\bar{c}(x)Dc(x)\mathcal{O}(x_1)\mathcal{O}(x_2)...\mathcal{O}(x_n)$$
$$\times\ e^{i\int d^4x\{\frac{1}{2}A^{a\mu}\partial^2A^{a\mu}+\mathcal{L}_{\text{int}}(A)-\bar{c}^a\partial^2c^a+gf^{abc}\bar{c}^a\partial_\mu c^bA^{c\mu}\}}$$

Adding integration over quark fields we obtain the QCD functional integral in the Feynman gauge

$$\langle\Omega|T\{\hat{\mathcal{O}}(x_1)\hat{\mathcal{O}}(x_2)...\hat{\mathcal{O}}(x_n)\}|\Omega\rangle$$
$$=\ \mathcal{N}^{-1}\int_{\emptyset}^{\emptyset} DA_\mu^aD\bar{c}DcD\bar{\psi}D\psi\ \mathcal{O}(x_1)\mathcal{O}(x_2)...\mathcal{O}(x_n)$$
$$\times\ e^{i\int d^4x\{\frac{1}{2}A^{a\mu}\partial^2A^{a\mu}+\bar{\psi}(i\partial\!\!\!/-m)\psi-\bar{c}^a\partial^2c^a+\mathcal{L}_{\text{int}}(A,\psi,c)\}}$$

where

$$\begin{aligned}\mathcal{L}_{\text{int}}(A,\psi,c) &= g\sum\bar{\psi}_q A\!\!\!/\psi_q - gf^{abc}(\partial_\mu\bar{c}^a)c^bA^{c\mu}\\ &- gf^{abc}\partial^\mu A^{a\nu}A^b_\mu A^c_\nu - g^2f^{abn}f^{cdn}A^a_\mu A^b_\nu A^{c\mu}A^{d\nu}\end{aligned}$$

($\mathcal{N}\equiv$ same integral without $\mathcal{O}(x_1)...\mathcal{O}(x_n)$).

The free Lagrangian describes propagation of massless gluons

$$A^a_\mu(x) A^b_\nu(y) = \int \frac{đ^4 k}{i} \frac{g_{\mu\nu}\delta^{ab}}{k^2 + i\epsilon} e^{-ik(x-y)}$$

$(a, b = 1 \div 8)$

quarks with mass m

$$\psi^k_\xi(x) \bar{\psi}^l_\eta(y) = \int \frac{đ^4 k}{i} \frac{\delta^{kl}(m + \not{p})_{\xi\eta}}{m^2 - p^2 - i\epsilon} e^{-ip(x-y)}$$

$(k, l = 1 \div 3)$

and massless ghosts

$$c^a(x) \bar{c}^b(y) = \int \frac{đ^4 q}{i} \frac{\delta^{ab}}{-q^2 - i\epsilon} e^{-iq(x-y)}$$

$(a, b = 1 \div 8)$

Ghosts are auxiliary fermion scalar particles which interact with gluons. Ghosts live in only in loops.

Physical meaning: ghosts cancel the contributions of the non-physical gluons from gluon loops.

Let us find the ghost-gluon vertex.

$$\langle\Omega|T\{\hat{c}^a(x)\hat{\bar{c}}^b(y)\hat{A}^c_\lambda(z)\}|\Omega\rangle$$

$$= \mathcal{N}^{-1}\int_{\emptyset}^{\emptyset} DA^a_\mu D\bar{c}DcD\bar{\psi}D\psi\; c^a(x)\bar{c}^b(y)A^c_\lambda(z)$$

$$\times\; e^{i\int d^4w\{\mathcal{L}_0(w)+\mathcal{L}_{\text{int}}(w)\}}$$

$$= \mathcal{N}^{-1}\int_{\emptyset}^{\emptyset} DA^a_\mu D\bar{c}Dc\; c^a(x)\bar{c}^b(y)A^c_\lambda(z)\; e^{i\int d^4w\mathcal{L}_0(w)}$$

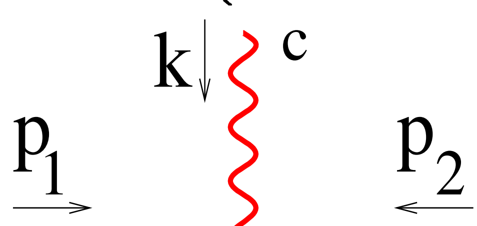
$$\times\;\left[1 - igf^{mnl}\int d^4w(\partial^\mu\bar{c}^m(w))c^n(w)A^l_\mu(w)\right] + O(g^2) =$$

$$gf^{abc}\!\int d^4w\frac{\not{d}^4p_1}{i}\frac{\not{d}^4p_2}{i}\frac{\not{d}^4k}{i}\left(\frac{\partial}{\partial w^\lambda}\frac{e^{-ip_1(x-w)}}{-p_1^2-i\epsilon}\right)\frac{e^{-ip_1(y-w)}}{-p_2^2-i\epsilon}\frac{e^{-ik(z-w)}}{k^2+i\epsilon}$$

$$\Rightarrow\quad G(p_1,p_2,k) = \frac{(2\pi)^4\delta(p_1+p_2+k)igf^{abc}p_{1\mu}}{(k^2+i\epsilon)(p_1^2+i\epsilon)(p_2^2+i\epsilon)}\quad\Rightarrow$$

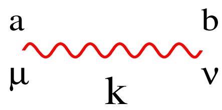
$$\mathcal{G}(p_1,p_2,-p_1-p_2) = \frac{-igf^{abc}p_{1\mu}}{((p_1+p_2)^2+i\epsilon)(p_1^2+i\epsilon)(p_2^2+i\epsilon)}$$

Vertex $\equiv$ ($\mathcal{G}^{\text{amp}}$ in the lowest order in g) $\Rightarrow$



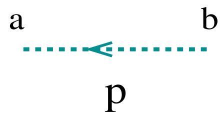
$= \;-igf^{abc}\,p_{1\mu}$

Feynman rules for QCD



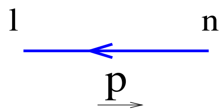
$$= \frac{g^{\mu\nu}\,\delta^{ab}}{k^2 + i\,\epsilon}$$

gluon propagator



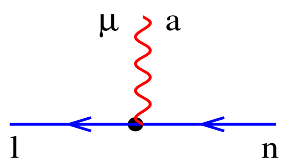
$$= \frac{\delta^{ab}}{-p^2 - i\,\epsilon}$$

ghost propagator



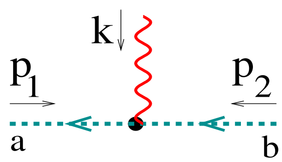
$$= \frac{(m + \not p)\,\delta^{ln}}{m^2 - p^2 - i\,\epsilon}$$

quark propagator



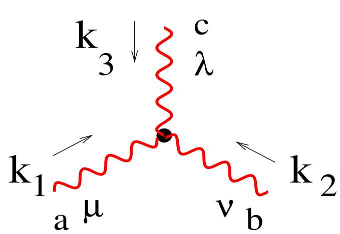
$$= g\,(t^a)_{ln}\,\gamma^\mu$$

quark-gluon vertex



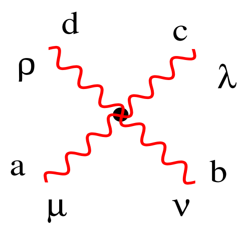
$$= -\,igf^{abc}\,p_1^\mu$$

ghost-gluon vertex



$$= -ig\,f^{abc}\,[(k_1 - k_2)_\lambda g_{\mu\nu} + (k_2 - k_3)_\mu\, g_{\nu\lambda} + (k_3 - k_1)_\nu\, g_{\lambda\mu}]$$

three-gluon vertex



$$= \begin{aligned} &-g^2 f^{mab} f^{mcd} (g_{\mu\lambda}\, g_{\nu\rho} - g_{\mu\rho}\, g_{\nu\lambda}) \\ &-g^2 f^{mac} f^{mbd} (g_{\mu\lambda}\, g_{\lambda\rho} - g_{\mu\rho}\, g_{\nu\lambda}) \\ &-g^2 f^{mad} f^{mbc} (g_{\mu\lambda}\, g_{\nu\rho} - g_{\mu\lambda}\, g_{\nu\rho}) \end{aligned}$$

four-gluon vertex

As usual,

- $\int \frac{d^4p}{16\pi^4 i}$ for each loop momentum.
- extra (-1) for each quark and ghost loop.
- ghosts live only in loops.

Set of rules for the "matix elements of the T-matrix":

- $u^l(p,s)$ for the incoming quark with color index $l = 1 \div 3$
- $\bar{v}^l(p,s)$ for the incoming antiquark
- $\bar{u}^l(p,s)$ for the outgoing quark
- $v^l(p,s)$ for the outgoing antiquark
- $e^{a\lambda}_{\mu}(k)$ for the (initial or final) gluon with color index $a = 1 \div 8$.
- Multiply by $\mathcal{G}^{\text{amp}}$

Appendix 1: Notations.

Metric tensor: $g_{\mu\nu}$ = diag(1,-1,-1,-1)

$\Rightarrow \quad a^\mu b_\mu = a_0 b_0 - \vec{a}\cdot\vec{b}$

$$\int đ^n p \;\equiv\; \int \frac{d^n p}{(2\pi)^n}$$

$$f(t)\,\frac{\overleftrightarrow{\partial}}{\partial t}\,g(t) \;\equiv\; f(t)\frac{\partial}{\partial t}g(t) - \left(\frac{\partial}{\partial t}f(t)\right)g(t)$$

Connection between the S-matrix and the T-matrix

$$S^{\text{connected}}(p_1, ...p_1^{(m)} \to p_2, ...p_2^{(n)}) =$$
$$(2\pi)^4 i\delta(\sum p_1 - \sum p_2) T(p_1, ...p_1^{(m)} \to p_2, ...p_2^{(n)})$$

(Note that $S = \Pi\sqrt{2E_k}S^{\text{AQM}}$, see eq. (4.9.1) of the AQM course).

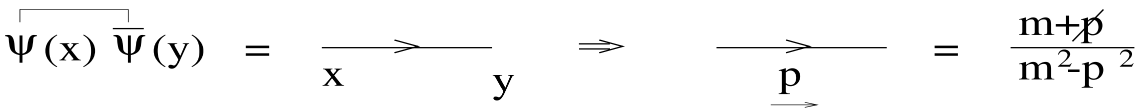
Definition of the reduced Green functions $\mathcal{G}(p_1, p_2, ..., p_n)$

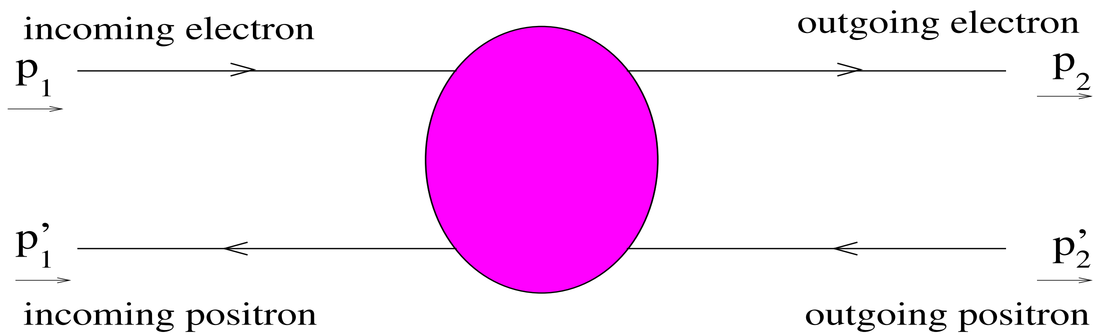
$$G(p_1, p_2, ..., p_n) =$$
$$(2\pi)^4 \delta(\sum p_i)(-i)^{n-1}\mathcal{G}(p_1, p_2..., -p_1 - ... - p_{n-1})$$

where $G(p_1, p_2, ..., p_n)$ is the Fourier transform of $\langle\Omega|$T-product of the corresponding operators$|\Omega\rangle$.

Arrows on the Dirac lines:

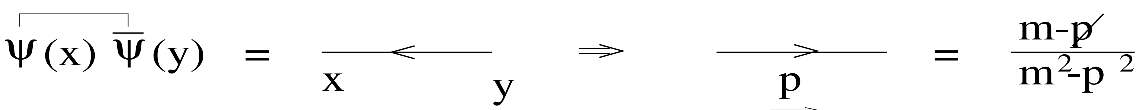
Peskin's textbook:

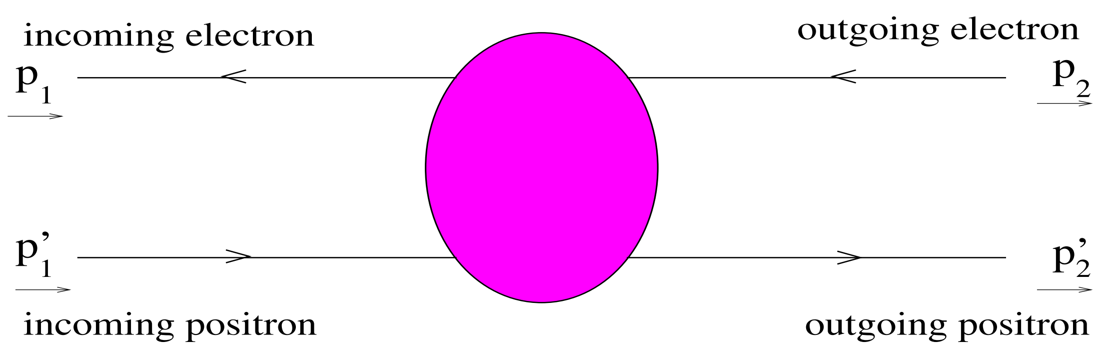




arrow $\downharpoonleft\upharpoonright$ charge flow

AQM lecture notes:





arrow $\upharpoonleft\upharpoonright$ charge flow

Appendix 2: Variational derivatives.

$F(\phi(x))$ - functional (say, F : $L_2 \to R$).
Example: KG action
$S(\phi(x)) = \frac{1}{2}\int d^4x(\partial_\mu(x)\phi\partial^\mu(x)\phi - m^2\phi^2(x))$

"Taylor expansion" for functionals:

$$F(\phi(x)+h(x)) = F(\phi(x)) + \int dx\, h(x)\, \frac{\delta F(\phi)}{\delta\phi(x)} + ...$$

$\frac{\delta F(\phi)}{\delta\phi(x)}$ is called the first variational derivative of $F(\phi)$. Example:

$$S(\phi(x) + h(x)) =$$
$$\frac{1}{2}\int d^4x(\partial_\mu(\phi + h)\partial^\mu(\phi + h) - m^2(\phi + h)^2) =$$
$$S(\phi) + \int d^4x(\partial_\mu h(x)\partial^\mu\phi(x) - m^2 h\phi) + O(h^2)$$
$$\stackrel{\text{by parts}}{=} S(\phi) + \int d^4x(-h(x)\partial^2\phi(x) - m^2h(x)\phi(x))$$

$$\Rightarrow \frac{\delta S(\phi)}{\delta\phi(x)} = -(\partial^2 + m^2)\phi(x).$$

Least action principle: $\frac{\delta S(\phi)}{\delta\phi(x)} = 0$ - classical eqn of motion. (KG eqn in our case).

HW1: Find $\langle\Omega|T\{\hat{\phi}(t)\hat{\phi}(0)\}|\Omega\rangle$ in the first non-trivial order in λ.

HW2: Find $i\int dx e^{ipx}\langle\Omega|T\{\hat{\phi}(x)\hat{\phi}(0)\}|\Omega\rangle$ in the first nontrivial order in λ (in the KG theory). Leave the answer in terms of integral(s) over 4-momenta.

HW3: Check the boundary conditions at $t \to \pm\infty$ for the function $\bar{\phi}(x)$ at page 50.

HW4: Using the expresasion for $\hat{\vec{E}} = \hat{\vec{\pi}}$ in terms of ladder operators (see p. 66) and the formula

$$\hat{\vec{B}}(\vec{x}) = -\vec{\nabla} \times \hat{\vec{E}}(\vec{x}) =$$
$$-\sum_{\lambda=1,2}\int \frac{d\!\!^{-3}k}{\sqrt{2E_k}} i\vec{k} \times \vec{e}^{\lambda}(\vec{k})\left(\hat{a}_k^{\lambda} e^{i\vec{k}\vec{x}} - \hat{a}_k^{\dagger\lambda} e^{-i\vec{k}\vec{x}}\right),$$

prove the last eqn. on page 66.

HW5.

Prove that

$$\int d\bar{\xi}_1 d\xi_1 \int d\bar{\xi}_2 d\xi_2 \int d\bar{\xi}_3 d\xi_3 \, e^{-\sum_{i,j=1}^{3} \bar{\xi}_i M_{ij} \xi_j} = \det M$$

where $\bar{\xi}_i$ and ξ_i are independent Grassman variables.

HW6.

Find the differential cross section for the unpolarized electron-electron scattering in the Yukawa model (in the lowest order in perturbation theory).

HW7.

Find the three-gluon vertex in QCD (consider $\langle\Omega|T\{\hat{A}^a_\mu(x)\hat{A}^b_\nu(y)\hat{A}^c_\lambda(z)\}|\Omega\rangle$ and follow the derivation of the ghost-gluon vertex).